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Higher-Dimensional Vacuum Bianchi-Mixmaster Cosmologies

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Abstract. We derive some new exact 7-dimensional cosmological solutions $|R \otimes \mathrm{I} \otimes N$, where N = I, II, VI_0, VII_0, VIII and IX are the various 3-dimensional Bianchi models. The solutions given are higher-dimensional generalizations of the mixmaster cosmologies. There is a strong influence of the extra spaces N, which results in a fundamental change of the 3-dimensional cosmology.

Key words: cosmology, vacuum Bianchi type — cosmology, higher dimensional

1. Introduction

The topic of higher-dimensional cosmologies is of much interest in view of the modern Kaluza–Klein picture of the universe (Lee 1984). In this approach the basic world manifold is of type $M^n = |R \otimes P^d \otimes Q^D$, where P^d, Q^D are some d, D-dimensional spaces. By taking $d = 3$, $D \geqslant 1$, the three-space P^3 should be identified with one of the isotropic Friedmann–Robertson–Walker (FRW) models or with one of their anisotropic generalizations of the various Bianchi types I–IX (Ryan & Shepley 1975; Kramer *et al.* 1980). The internal (or extra) space Q^D must be some higher-dimensional generalization of the Kaluza–Klein S^1 sphere. For instance, in $d = 11$ supergravity a natural candidate for Q^D is one of the various S^7-spheres (Lorenz-Petzold 1985; Alvarez 1984; Fujii & Okada 1984; Gleiser, Rajpoot & Taylor 1984). However, there are an embarassingly large number of other solutions with other topologies (Bais, Nicolai & van Nieuwenhuizen 1983; Castellani, Romans & Warner 1984).

Recently, some 1 + 3 + 3 = 7-dimensional Bianchi–mixmaster cosmologies of types I $\otimes$ IX (Furusawa & Hosoya 1984) and IX $\otimes$ IX (Tomimatsu & Ishihara 1984) have been constructed on the basis of higher-dimensional gravity. In (1+3)-dimensions, type-I leads to the well-known Kasner solution while type-IX is known as the mixmaster model (Misner 1969; Barrow 1984). It is well known that the original mixmaster model shows a chaotic behaviour near the initial singularity (Barrow & Tipler 1979; Barrow 1981; 1982; 1984; Chernoff & Barrow 1983; Elskens 1983; Zardecki 1983; Lifshitz *et al.* 1983). However, there are also some controversial results concerning the possibility of ‘mixing’ (Doroshkevich & Novikov 1970a, b; MacCallum 1971; Doroshkevich, Lukash & Novikov 1971). It is now interesting to see that the influence of the extra dimensions may prevent the chaotic behaviour near the initial singularity (Furusawa & Hosoya 1984; Tomimatsu & Ishihara 1984).

In view of this it becomes interesting to study some more general higher-dimensional cosmologies of type I $\otimes$ N, where N denotes one of the Bianchi types I, II, VI_0, VII_0, VIII and IX with different topologies (The (1 + 3)-dimensional type-VIII has been first considered by Lifshitz & Khalatnikov 1970; for types VI_0, VII_0 see Khalatnikov & Pokrovski 1972; Lukash 1974; Ruban 1978; Belinskii, Khalatnikov & Lifshitz 1982; Lorenz-Petzold 1984; Jantzen 1984). In this paper we solve the corresponding field equations in 7-dimensions.

2. Field equations and solutions

In choosing a local orthonormal basis σ^μ, we can put the metric on $\mathbb{R} \otimes$ I $\otimes N$ in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\nu\sigma^\mu, \tag{1a}$$

where $\eta_{\mu\nu} = (-1, 1, \ldots, 1)$ is the seven-dimensional Minkowski metric tensor. We have

$$\sigma^0 = \omega^0 = dt, \quad \sigma^i = r_i\omega^i, \quad \sigma^j = R_j\omega^j \text{ (no sum)}, \tag{1b}$$

where $r_i = r_i(t)$ are the cosmic scale functions on type-I, $R_i = R_i(t)$ are defined on type-N, ω^i — dx^i, $\omega^j(i,j = 1, 2, 3)$ are time-independent differential forms for the Bianchi types I, II, VI_0, VII_0, VIII and IX (see Kramer *et al.* 1980). The corresponding vacuum field equations to be solved are given by

$$(\ln r_i)'' = 0, \tag{2a}$$

$$(\ln R_i^2)'' = r^6[(n_jR_j^2 - n_kR_k^2)^2 - n_i^2R_i^4], \tag{2b}$$

$$\begin{aligned}9hH + h_1h_2 + h_1h_3 + h_2h_3 + H_1H_2 + H_1H_3 + H_2H_3 \\ = (1/4R^6)[n_1^2R_1^4 + n_2^2R_2^4 + n_3^2R_3^4 \\ -2\{n_1n_2(R_1R_2)^2 + n_1n_3(R_1R_3)^2 + n_2n_3(R_2R_3)^2\}],\end{aligned} \tag{2c}$$

where $r_i = ri(t)$, $R_i = R_i(t)$, $h_i = (\ln r_i)^{\cdot}$, $H_i = (\ln R_i)^{\cdot}$, $3h = \Sigma h_i$, $3H = \Sigma H_i$, $r^3 = r_1r_2r_3$, $R^3 = R_1R_2R_3$, $dt = (rR)^3\, d\eta$, $(\ )^{\cdot} = d/dt$, $(\ )' = d/d\eta$, n_i are the structure constants of the various Bianchi types given by

n_1	n_2	n_3	
0	0	0,	I
1	0	0,	II
1	−1	0,	VI_0
1	1	0,	VII_0
1	1	−1,	VIII
1	1	1,	IX

and i, j, k are in cyclic order.

The general solutions of Equation (2a) are of the Kasner-type:

$$r_i = r_{i0} \exp(k_i\eta), \quad \Sigma k_i = k, \tag{3}$$

where r_{i0}, k_i, k = const. We obtain the following results:

(1) N = I:

$$r_i = \tilde{r}_{i0} t^{p_i}, \quad R_i = \tilde{R}_{i0} t^{q_i}, \quad \tilde{r}_{i0}, \quad \tilde{R}_{i0} = \text{const}, \tag{4a}$$

$$[\Sigma(p_i + q_i)]^2 = \Sigma(p_i^2 + q_i^2) = 1, \; p_i, \; q_i = \text{const}. \tag{4b}$$

This is the seven-dimensional generalization of the Kasner-solution in four dimensions. Equation (2b) yields $R_i = R_{i0} \exp(K_i \eta)$ and (4a) is obtained by setting $p_i = k_i / (k + K)$, $q_i = K_i / (k + K)$, where $\Sigma K_i = K$. Our solution (4a) turns out to be identical with the IX ⊗ IX solution (Tomimatsu & Ishihara 1984) when the spatial curvature terms of the right-hand side of (2b) are ineffective, which is characteristic for the original Bianchi type-IX mixmaster cosmology.

(2) N = II:

$$R_1 R_2 = R_{12} \exp(p\eta), \quad R_1 R_3 = R_{13} \exp(q\eta), \tag{5a}$$

$$r^3 = a \exp(k\eta), \tag{5b}$$

$$\check{H}_1^2 + k\check{H}_1 - b^2 + (1/4) r^6 R_1^4 = 0, \tag{5c}$$

$$2b^2 = k^2 + 2[k(p+q) + pq] - \Sigma k_i^2, \tag{5d}$$

where R_{12}, R_{13}, p, q, a = const, and $H_i = (\ln R_i)'$. We obtain two different kinds of solutions:

(i) the general solution with $k = 0$;

$$R_1^2 = (2b/a)(\cosh 2b\eta)^{-1},$$
$$R_2^2 = \tilde{R}_2 \exp(2p\eta)(\cosh 2b\eta)^2,$$
$$R_3^2 = \tilde{R}_3 \exp(2q\eta)(\cosh 2b\eta)^2,$$
$$r_i = \tilde{r}_{i0} \exp(k_i \eta), \tag{6a}$$

(ii) the special power-type solution

$$r_i = \tilde{r}_{i0} t^{p_i}, \quad R_i = \tilde{R}_{i0} t^{q_i},$$
$$p_i = \frac{2k_i}{2(p+q) + 3k}, \quad q_1 = \frac{-k}{2(p+q) + 3k},$$
$$q_2 = \frac{2p + k}{2(p+q) + 3k}, \quad q_3 = \frac{2q + k}{2(p+q) + 3k},$$
$$a^2 c^4 = 4b^2 + k^2, \tag{6b}$$

where $\tilde{R}_2$, $\tilde{R}_3$, $\tilde{R}_{i0}$, c = const.

Our solution (6a) is the generalization of the vacuum Bianchi type-II solution in four dimensions first given by Taub (1951) (see also Lorenz 1980a). Our solution (6b) obeys the relation $q_1 + 1 = q_2 + q_3$, from which it follows that no Kasner conditions are satisfied if $k \neq 0$.

We now turn to the spaces I ⊗ VI_0 and I ⊗ VII_0. In considering first the LRS case (see Ellis & MacCallum 1969) $R = R_1 = R_2$, $S = R_3$, the Bianchi type-VII_0 model reduces to a special Bianchi type-I model. We thus consider only the Bianchi type-VI_0

space. The corresponding field equations to be solved are

(3) $N = \mathrm{VI}_0$:

$$(\ln R^2)'' = 0, \tag{7a}$$

$$(\ln S^2)'' - 4r^6 R^4 = 0. \tag{7b}$$

From (7a) we obtain the solution

$$R^2 = \exp b\eta, \tag{8a}$$

where b = const, and (7 b) gives now

$$(\ln S^2)'' = 4\alpha^2 \exp 2(k+b)\eta \tag{8b}$$

It can be shown that the case $k + b = 0$ is not compatible with Equation (2c). For $k + b \neq 0$ it is more convenient to consider Equation (2c) instead of (8b). The field equation to be solved is given by

$$\tilde{H}_3 = [1/(b+k)]r^6R^4 - [1/4(b+k)][b(b+4k) - 2(\Sigma k_i^2 - k^2)], \tag{9}$$

where $\tilde{H}_3 = (\ln S)'$, ( ) = d/dη. The solutions can now be easily completed in terms of the generalized Ellis-MacCallum (1969) parameter $u = r^3R^2$:

$$\begin{aligned} R^2 &= ur^{-3}, \\ S^2 &= u^{-A^2/[2(b+k)]} \exp((a^2/(b+k))u^2, \\ r_i &= r_{i0}(ur^{-3})^{k_i/b}, \\ r^3 &= a \exp k\eta, \end{aligned} \tag{10}$$

where

$$A^2 = b(b+4k) - 2(\Sigma k_i^2 - k^2). \tag{11}$$

By setting $k_i = 0$, $a = b = 1$, we rediscover the (1 + 3) dimensional solution first given by Ellis & MacCallum (1969) (Note that this solution is incorrectly given by Kramer *et al.* 1980; in Ellis & MacCallum (1969) q_0 should be replaced by q_0^2).

We next consider the non-LRS case $R_1 \neq R_2 \neq R_3$. Introducing the new variables $u_i = u_i(\eta)$ by

$$R_i = \exp u_i, \quad u = 2(u_1 - u_2), \tag{12}$$

the corresponding field equations can be decoupled and partially integrated to give

$$u_1 + u_2 = b(\eta - \eta_0), \tag{13a}$$

$$u'' + 4a^2 \exp[2(k+b)\eta - 2b\eta_0] \sinh u = 0, \tag{13b}$$

$$u_3'(k+b) = -u_1'u_2' - kb + \tfrac{1}{2}(\Sigma k_i^2 - k^2) + \tfrac{1}{4}[\exp 2u_1 - \delta \exp 2u_2]^2, \tag{13c}$$

where b, η_0 = const. and $\delta = (n_2) = -$ (VI_0), $\delta = 1$ (VII0). After solving Equation (13b) to give $u = u(\eta)$ the most general Bianchi type-VI_0 and type-VII_0 solutions would arise. We will now show how the solutions can be expressed in terms of a particular form of the third Painleve transcendents (Ince 1956). Introducing the time variable ζ by

$$\zeta = \frac{2a}{k+b} \exp[(k+b)\eta - b\eta_0], \tag{14}$$

can transform the system (13) to obtain

$$\ddot{u} + \frac{1}{\zeta}\dot{u} + \sinh u = 0, \tag{15a}$$

$$u_1 = \ln\left[\frac{k+b}{2a}\zeta \exp(-k\eta_0)\right]^{b/2(k+b)} + \frac{u}{4}, \tag{15b}$$

$$u_2 = \ln\left[\frac{k+b}{2a}\zeta \exp(-k\eta_0)\right]^{b/2(k+b)} - \frac{u}{4}, \tag{15c}$$

$$\dot{u}_3 = \frac{\zeta}{16}(\dot{u}^2 - 4b^2) - \frac{1}{(k+b)^2\zeta}\left[kb + \frac{1}{2}(k^2 - \Sigma k_i^2)\right]$$

$$+ \frac{1}{4a(k+b)}\left[\frac{(k+b)}{2a}\zeta\right]^{(b-k)/(b+k)}[\cosh u + \delta]\exp\left[-\frac{2kb\eta_0}{k+b}\right], \tag{15d}$$

where $(\ )^{\cdot} = d/C.\zeta$ In the limit $k = k_i = 0$ we rediscover the field equations first given by Belinskii & Khalatnikov (1969) (for type-IX) and Lifshitz & Khalatnikov (1970) (for type-VIII) and later by Khalatnikov & Pokrovski (1972). The connection with the Bianchi type-VI_0 and type-VII_0 spaces has been first observed by Lorenz-Petzold (1984) and independently by Jantzen (1984) (Note that there are some errors in the papers of Belinskii & Khalatnikov, Lifshitz & Khalatnikov, and Lorenz-Petzold).

If we put

$$w = \exp u, \quad z = \frac{\zeta^2}{4}, \quad w = w(z), \quad (\ )' = d/dz, \tag{16}$$

Equation (15a) becomes

$$w'' = \frac{w'^2}{w} - \frac{1}{z}\left[w' + \frac{1}{2}(w^2 - 1)\right]. \tag{17}$$

This equation is a particular form of the nonlinear equation of second order which defines the third Painleve transcendent (Ince 1956). The Bianchi types-VI_0, VII_0 solutions are completed by Equations (15b), (15c) and (15d) to give $u_i = u_i$ (w(z)). A solution of Equation (15a) in terms of elliptic function was given by Khalatnikov & Pokrovski (1972). The scale functions r_i are given by

$$r_i = r_{i0}\left[\frac{k+b}{2a}\zeta \exp(b\eta_0)\right]^{k_i/(k+b)} \tag{18}$$

We finally consider the spaces I ⊗ VIII and I⊗ IX. By setting $R = R_1 = R_2, S = R_3$, $g = RS$, $f = (RV$, $d = n_3$, $z = S^2$, the field equations (2a-2c) can be decoupled to give

(4) N = VIII, IX:

(i) $k = 0$:

$$\ddot{g} + \delta a^2 g = 0, \tag{19a}$$

$$z'^2 - 2[2(\dot{g}^2 + \delta a^2 g^2) - \Sigma k_i^2]z^2 + a^2 z^4 = 0, \tag{19b}$$

where d $\tau = gd\eta$, ( )'. = d / dτ, ( )' =d/dη and

(ii) $k \neq 0$:

$$f'' = \frac{f'^2}{f} - \frac{1}{\zeta}(f' + \delta c^2 f^2), \tag{20a}$$

$$z'^2 + \frac{1}{\zeta} zz' + \frac{c^2}{2} z^4 + 2[(\ln f)' - \frac{1}{2(k\zeta)^2}(k^2 - \Sigma k_i^2)] z^2 = 0, \tag{20b}$$

where $\zeta = \exp(2k\eta)$, $d\zeta = 2k\zeta d\eta$, ( )' = d/dζ. From Equation (19a) we obtain the solutions

$$g = A \sin(a\tau), \quad \delta = 1, \tag{21a}$$

$$g = A \sinh(a\tau), \quad \delta = -1, \tag{21b}$$

where A = const. It is now an easy matter to solve Equation (19b) to give $S = S(\tau)$. The results are

$$R^2 = (A^2 a/2D) \sin^2(a\tau) \cosh\left[\ln\left(\tan\frac{a}{2}\tau\right)^{2D/Aa}\right],$$

$$S^2 = (2D/a) \cosh^{-1}\left[\ln\left(\tan\frac{a}{2}\tau\right)^{2D/Aa}\right],$$

$$r_i = r_{i0}\left(\tan\frac{a}{2}\tau\right)^{k_i/Aa}, \qquad \text{type-VIII}, \tag{22a}$$

$$R^2 = (A^2 a/2D) \sinh^2(a\tau) \cosh\left[\ln\left(\tanh\frac{a}{2}\tau\right)^{2D/Aa}\right],$$

$$S^2 = (2D/a) \cosh^{-1}\left[\ln\left(\tanh\frac{a}{2}\tau\right)^{2D/Aa}\right],$$

$$r_i = r_{i0}\left(\tanh\frac{a}{2}\tau\right)^{k_i/Aa}, \qquad \text{type-IX}, \tag{22b}$$

where

$$2D^2 = 2A^2a^2 - \Sigma k_i^2.$$

Our solutions (22) are the generalizations of the (1 + 3)-dimensional vacuum solutions first given by Taub (1951) (only the type-IX solution was given explicitly by Taub; for type-VIII see Lorenz 1980b). No such explicit solutions are possible in the more general case $k \neq 0$. Equation (20a) defines a special kind of a third Painleve transcendental function (Ince 1956) $f = f(\zeta)$, which also determines $z = z(\zeta)$ *via* Equation (20b).

3. Conclusions

We have given a complete discussion of the higher-dimensional vacuum Bianchi-mixmaster cosmologies of types $|R \otimes I \otimes N$, N = I, II, VI$_0$, VII$_0$, VIII, IX. Only the

Kasner solution I⊗I (4) was known (Tomimatsu & Ishihara 1984). There is a strong influence of the spaces *N* on the Bianchi type-I model and vice versa. This can be seen explicitly by our new solutions of types-II (Equation 6b), VI_0 (Equations 10, 15), VII_0 (Equation 15), VIII and IX (Equation 20). However, due to the great numbers of solutions it remains a problem for the near future to discuss our solutions in adequate detail. A next step into some more general cosmologies would be to construct some perfect fluid solutions. It is also worth investigating the mixmaster cosmologies of type-$N \otimes N$ (besides the IX ⊗ IX model of Tomimatsu & Ishihara 1984).

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