J. Astrophys. Astr. (1985) **6,** 131-135

Higher-Dimensional Bianchi Type-VI$_h$ Cosmologies

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Received 1985 February 18; accepted 1985 May 28

Abstract. We present the higher-dimensional perfect fluid generalizations of the extended Bianchi type-VI$_h$ vacuum space-times discussed recently by Demaret & Hanquin (1985). It is shown that the Chodos-Detweiler mechanism of cosmological dimensional-reduction is possible in these cases.

Key words: cosmology, Bianchi type—cosmology, higher dimensional

1. Introduction

The topic of higher-dimensional cosmologies has recently received much attention (Chodos & Detweiler 1980; Freund 1982; Sahdev 1984; Lorenz-Petzold 1984a, b; 1985a-d). It has been suggested that such cosmologies play an important role in the very early universe (Chodos 1984; Chodos & Myers 1984; Appel & Dresden 1984). In the course of time, the influence and effect of extra dimensions $D \geqslant 1$ ($N=1+d+D$) must have diminished or changed. An easy way to visualize this process is that in the course of time, the usual $d = 3$ dimensions kept on expanding, while the extra dimensions $D \geqslant 1$ contracted very rapidly to distances of the order of the Planck length. Such $MN=|R \otimes M^d \otimes M^D$ space-time models were first constructed by Chodos & Detweiler (1980) (see also Belinskii & Khalatnikov 1972; Forgacs & Horvath 1979; Barrow 1983; Tosa 1984; Mann & Vincent 1985).

An interesting paper in this respect has been recently published by Demaret & Hanquin (1985). Most of the studies done so far are of the higher-dimensional generalization of the anisotropic (1 + 3)-dimensional Bianchi type-I model (see Lorenz-Petzold 1985c). However, no generalization of the Petrov-Bianchi classification exist in (1 + n)-dimensions ($n = N - 1$) (Petrov 1969; Sahdev 1984). A simple definition was given by Demaret & Hanquin (1985). A N-dimensional spatially homogeneous model is defined as a N-dimensional space-time possessing a (N – 1)-dimensional group of isometry acting simply transitively on (N – 1)-dimensional space-like hyper-surfaces. This definition does not probably recover all possible cases of spatially homogeneous models in (1+3)-dimensions (Ryan & Shepley 1975; Kramer *et al.* 1980). However, the definition given by Demaret & Hanquin is already sufficiently rich as to recover a large set of spatially homogeneous models.

In this paper we discuss the higher-dimensional perfect fluid field equations of a generalization of the (1 + 3)-dimensional Bianchi type-VI$_h$ space-time, which includes also the Bianchi type-V and the Bianchi type-Ill as special cases. Only the vacuum case was discussed by Demaret & Hanquin (1985).

2. Field equations and solutions

We consider the metrics

$$ds^2 = -dt^2 + R_1^2 (dx^1)^2 + \exp(2p_i x^1) R_i^2 (dx^i)^2, \tag{1}$$

where $R_j = R_j(t)(j = 1, \ldots, N-1;\ n = N-1)$ are the cosmic scale functions and P_i = const. The corresponding perfect fluid field equations to be solved are given by

$$H_1' + H_1 (\ln u)' - \sum p_i^2 = \varepsilon(2-\gamma)R_1^2/(N-2), \tag{2a}$$

$$H_i' + H_i (\ln u)' - p_i \sum p_i = \varepsilon(2-\gamma)R_1^2/(N-2), \tag{2b}$$

$$H_1 \sum p_i - (\ln g)' = 0, \tag{2c}$$

$$(\ln u)'^2 + 2(\ln u)' H_1 - \sum H_i^2 - \sum p_i^2 - (\sum p_i)^2 = 2\varepsilon R_1^2, \tag{2d}$$

$$\varepsilon' + \gamma\varepsilon(\ln R_1 u)' = 0, \tag{2e}$$

$$p = (\gamma - 1)\varepsilon, \qquad 1 \leqslant \gamma \leqslant 2, \tag{2f}$$

where H_j = (In R_j)’ are the Hubble parameters, $u = R_2 \ldots R_{N-1}$, g $= R_2^{p_2} \ldots R_{N-1}^{p_{N-1}}$, $dt = R_1\, d\eta$, $(\ )' = d/d\eta$, $i = 2, \ldots, N-1$ and ρ and ε are, respectively, the pressure and energy density of the perfect fluid matter. In cosmology, we are mainly interested in the cases $\varepsilon = 0$ (vacuum), $y = 1$ (dust), $y = 2$ (stiff matter) and $\gamma = N/(N-1)$ (radiation). From Equation (2e) we obtain the conservation equation $\varepsilon = M(R_1 \ldots R_{N-1})^{-y}$, M = const.

We first consider the cases $\varepsilon = 0$ and $y = 2$. From the linear combination of Equations (2b) we obtain

$$u'' - p^2 u = 0, \qquad p = \sum p_i, \tag{3}$$

with the general solutions

(i)
$$u = \sinh p\eta, \qquad p \neq 0, \tag{4a}$$

(ii)
$$u = \eta, \qquad p = 0, \tag{4b}$$

and the special solution

(iii)
$$u = \exp p\eta, \quad p \neq 0. \tag{4c}$$

Equations (2a) and (2b) can now be easily integrated (Equation 2d is nothing but the constraint equation on the constants of integration) to give:

(i)
$$R_1 = (\sinh p\eta)^{(1/p^2)\Sigma p_i^2} \left(\tanh \frac{p}{2}\eta\right)^{(1/p)\Sigma c_i p_i},$$

$$R_i = (\sinh p\eta)^{p_i/p} \left(\tanh \frac{p}{2}\eta\right)^{c_i},$$

$$\sum c_i = 0, \qquad p^2 + \sum p_i^2 = p^2 \sum c_i^2 + 2M, \tag{5a}$$

(ii)
$$R_1 = \eta^a \exp\left(\frac{1}{4}\sum p_i^2\right),$$

$$R_i = \eta^{c_i},$$

$$\sum c_i = 1, \qquad \sum c_i p_i = 0, \qquad 2a = \sum c_i^2 - 1 + 2M, \tag{5b}$$

(iii)
$$R_1 = \exp\left(\frac{1}{p}\sum p_i^2 \eta\right),$$
$$R_i = \exp p_i\eta, \qquad M = 0, \tag{5c}$$
where c_j, a = const.

The stiff matter solutions ($M \neq 0$) are new. By setting $M = 0$ we rediscover the vacuum solutions ($p = 1$, if $p \neq 0$ without loss of generality) discussed by Demaret & Hanquin (1985). The four-dimensional solutions (5a) ($i = 2, 3$) are due to Collins (1971) (in a somewhat different form) and Ruban (1978a) (see also Wainwright, Ince & Marshman 1979), which include the Bianchi type-VI_h vacuum solutions of Ellis & MacCallum (1969) and the Bianchi type-V solution given by Joseph (1966) ($p_i = 1$). The special solution (5b) has no analogue in (1 + 3)-dimensions. Our solution (5c) has been discussed in four-dimensions by Lifshitz & Khalatnikov (1963a, b), Collins (1971, 1977), Evans (1974, 1978), Siklos (1981), Ruban (1978a, b), Siklos (1981), Ruban, Ushakov & Chernin (1981), Belinskii, Khalatnikov & Lifshitz (1982), Wainwright (1983) and Wainwright & Anderson (1984) (see also the recent paper by Rosquist & Jantzen 1985).

We next consider the special power law perfect fluid solutions:
$$R_1 = \exp a\eta, \qquad R_i = \exp c_i\eta, \tag{6}$$
where a, c_i — const. It follows that
$$a = c\gamma/(2-\gamma), \qquad \gamma \neq 2, \qquad c = \sum c_i,$$
$$c_i = a + [\gamma/a(2-\gamma)][pp_i - \sum p_i^2], \tag{7a}$$
and thus a can be re-expressed in terms of p_i:
$$a^2 = \gamma^2[p^2 - (N-2)\sum p_i^2]/[(2-\gamma)(2-\gamma(N-1))], \tag{7b}$$
which yields in addition to $\gamma \neq 2$ the condition $y \neq 2 \ (N-1)$. According to Equation (2c) we have also

(i)
$$p \neq 0:$$
$$a = \sum c_i p_i/p, \tag{8a}$$
(ii)
$$p = 0:$$
$$\sum c_i p_i = 0, \tag{8b}$$
and the value of M is determined by Equation (2d).

Our solutions (6–8) are the higher-dimensional special perfect fluid generalizations of the (1 + 3)-dimensional solutions first given by Collins (1971). Such solutions are of much interest in cosmology and have been discussed recently in (1 + 3)-dimensions by Ruban, Ushakov & Chernin (1981), Barrow (1982, 1984), Wainwright (1983, 1984) and Wainwright & Anderson (1984) (see also the recent paper by Lorenz-Petzold 1985e). By setting $a = c_2 = c_3$, $c_4 = b$, we obtain the isotropic solution in (1+4)-dimensions with
$$b = a(2-3\gamma)/\gamma, \qquad a > 0, \tag{9}$$
leading thus to a contraction of the physical unobservable dimension and an expansion of the three physical dimensions.

We finally consider the space-times defined by $p_i = 1$, which are the higher-dimensional generalization of the (1 + 3)-dimensional Bianchi type-V cosmologies. Exact solutions (besides the ones given by Equation 5) can be obtained in the cases $y = N/(N-1)$ (radiation) und $\gamma = 1$ (dust). Without going into all the details we present only the results (see Lorenz-Petzold 1985f for derivation):

$$u = a\sinh^2 q\eta + b\sinh 2q\eta, \tag{10a}$$

$$\begin{aligned} R_1 &= u^{1/(N-2)} \\ R_i &= u^{1/(N-2)}[1 + (2b/a)\coth q\eta]^{c_i/(N-2)}, \\ a &= 2M/(N-2)(N-1), \qquad q = (N-2)/2, \\ \sum c_i &= 0, \qquad \sum c_i^2 = (N-1)(N-2), \end{aligned} \tag{10}$$

where b, c_i = const. By setting $i = 2, 3$, we rediscover the (1 + 3)-dimensional Bianchi type-V radiation solution first given by Ruban (1977a, b) (Note that these papers are not quoted by Kramer *et al.* 1980).

The dust case can be reduced to the solution of a generalized Friedmann equation:

$$\begin{aligned} R'^2 &= R^2 + (a^2/(N-1)(N-2))\sum c_i^2\, R^{2(1-N)} + (2M/(N-1)(N-2))R^{5-N}, \\ S &= \exp\left[a \int R^{1-N} \mathrm{d}\eta\right], \end{aligned}$$

where a = const., $R = R_1$ $R_i = RS^{c_i}$, $\sum c_i = 0$, c_i = const. The case $N = 4$ is nothing but the (1 + 3)-dimensional Bianchi type-V dust solution first given by Schucking & Heckmann (1958), which can be solved explicitly in terms of elliptic functions. This completes our study of the higher-dimensional Bianchi type-VI$_h$ perfect fluid field equations. According to the relations Σc_i = const., it is always possible to construct solutions such that d = 3-dimensions are expanding while the extra dimensions $D \geqslant 1$ are contracting to an unobservable scale.

References

Appel, D., Dresden, M. 1984, in *Proc. Johns Hopkins Workshop*; *Particles and Gravity*, Eds G. Domokos & S. Kovesi-Domokos, World Scientific, Singapore, p. 229.

Barrow, J. D. 1982, *Q. J. R. astr. Soc*, **23,** 344.

Barrow, J. D. 1983, *Phil. Tram. R. Soc. London*, **310,** 337.

Barrow, J. D. 1984, *Mon. Not. R. astr. Soc*, **211,** 221.

Belinskii, V. A., Khalatnikov, I. M. 1972, *Zh. Eksp. Teor. Fiz.*, **63,**1121; 1973, *Sov. Phys. JETP*, **36,** 591.

Belinskii, V. A., Khalatnikov, I. M. 1972, *Adv. Phys.*, **31,** 639.

Chodos, A. 1984, in *Workshop on Kaluza-Klein Theories: An Introduction to Kaluza Klein Theories*, Ed. H. C. Lee, World Scientific, Singapore, p. 256.

Chodos, A., Detweiler, S. 1980, *Phys. Rev.* **D21,** 2167.

Chodos, A, Myers, E. 1984, *Ann. Phys.*, New York, **156,** 412.

Collins, C. B. 1971, *Commun. Math. Phys.*, **23,** 137.

Collins, C. B. 1977, *Phys. Lett.*, **60A,** 397.

Demaret, J., Hanquin, J.-L. 1985, *Phys. Rev.* **D31,** 258.

Ellis, G. F. R., MacCallum, M. A. H. 1969, *Commun. Math. Phys.*, **12,** 108.

Evans, A. B. 1974, *Nature*, **252,** 109.

Evans, A. B. 1978, *Mon. Not. R. astr. Soc*, **183,** 727.

Forgäcs, P., Horväth, Z. 1979, *Gen. Relativ. Gravitation*, **11,** 205.

Freund, P. G. O. 1982, *Nucl. Phys.*, **B209,** 146.

Joseph, V. 1966, *Proc. Cambridge Phil. Soc*, **62,** 87.
Kramer, D., Stephani, H., MacCallum, M. A. H., Herlt, E.1980, *Exact Solutions of Einstein's Field Equations*, Cambridge Univ. Press.
Lifshitz, E. M., Khalatnikov, I. M. 1963a, *Usp. Fiz. Nauk*, **80,** 391; 1964, *Sov. Phys. Usp.*, **6,**495.
Lifshitz, E. M., Khalatnikov, I. M. 1963b, *Adv. Phys.*, **12,** 185.
Lorenz-Petzold, D. 1984a, *Phys. Lett.*, **148B,** 43.
Lorenz-Petzold, D. 1984b, *Phys. Lett.*, **149B,** 79.
Lorenz-Petzold, D. 1985a, *Phys. Lett. B.*, **151B,** 105.
Lorenz-Petzold, D. 1985b, *Phys. Lett. B.*, **153B,** 134.
Lorenz-Petzold, D. 1985c, *Phys. Rev.*, **31D,** 929.
Lorenz-Petzold, D. 1985d, *Prog. Theor. Phys.*, **73,** 533.
Lorenz-Petzold, D. 1985e, *Acta phys. pol.*, **B16,** 171.
Lorenz-Petzold, D. 1985f, *Lett. Math. Phys.* (Submitted).
Mann, R. B., Vincent, D. E. 1985, *Phys. Lett.*, **107A,** 75.
Petrov, A. Z. 1969, *Einstein Spaces,* Pergamon Press, Oxford.
Rosquist, K., Jantzen, R. T. 1985, *Phys. Lett.*, **107A,** 29.
Ruban, V. A. 1977a, *Zh. Eksp. Teor. Fiz.*, **72,** 1201; 1977, *Sov. Phys. JETP*, **45,** 629.
Ruban, V. A. 1977b, Leningrad Institute of Nuclear Phys., B. P. Konstantinova, Preprint No. 327.
Ruban, V. A. 1978a, Leningrad Institute of Nuclear Phys., B. P. Konstantinova, Preprint No. 412.
Ruban, V. A. 1978b, Leningrad Institute of Nuclear Phys., B. P. Konstantinova, Preprint No. 411.
Ruban, V.A., Ushakov, Yu. A., Chernin, A. D.1981, *Zh. Eksp. Teor. Fiz.*, **80,** 816; 1981, *Sov. Phys. JETP*, **53,** 413.
Ryan, Jr. M. P., Shepley, L. C. 1975, *Homogeneous Relativistic Cosmologies*, Princeton Univ. Press.
Sahdev, D. 1984, *Phys. Rev.*, **D30,** 339.
Schucking, E., Heckmann, O. 1958, in *Institute International de Physique, Onzieme Conseil de Physique, Editions Stoops*, Brussels.
Siklos, S. T. C. 1981, *J. Phys. A.: Math. Gen.*, **14,** 395.
Tosa, Y. 1984, *Phys. Rev.*, **D30,** 339.
Wainwright, J. 1983, *Phys. Lett.*, **99A,** 301.
Wainwright, J. 1984, *Gen. Relativ. Gravitation*, **16,** 657.
Wainwright, J., Anderson, P. J. 1984, *Gen. Relativ. Gravitation*, **16,** 609.
Wainwright, J., Ince, W. C W., Marshman, B. J. 1979, *Gen. Relativ. Gravitation*, **10,** 259.