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Equilibrium Structure for a Plasma Magnetosphere Around Compact Objects

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Abstract. Starting from a set of general equations governing the dynamics of a magneto-fluid around a compact object on curved space time, a fairly simple analytical solution for a test disc having only azimuthal component of velocity has been obtained. The electromagnetic field associated has a modified dipole configuration which admits a reasonable pressure profile for the case of fully relativistic treatment of Keplerian type of velocity distribution.

Key words: magnetosphere—disc configuration—general relativity—equilibrium

1. Introduction

One of the outstanding problems in pulsar modelling has been the structure and dynamics of the magnetosphere containing plasma around compact objects. Since the first model of the pulsar electrodynamics by Goldreich and Julian, there have been several attempts to understand the evolution and dynamics of the pulsar magnetosphere, but still no comprehensive picture has come out (Michel 1987). In most of the discussions of the dynamics of magnetospheres, be it for pulsars or for any other high energy source, the emphasis has always been only on the possible plasma processes led and sustained by electromagnetic fields alone. The role of gravity has largely been ignored with the underlying assumption that it is always weak.

However, it has been shown in earlier discussions that even the most intense magnetic field of 10^{12} gauss associated with pulsar carries an energy which is very small compared to the gravitational potential energy on the surface of a neutron star of 1 solar mass (Prasanna 1980). Thus, in our opinion a fully realistic discussion should take into account the role of gravity in inducing possible electromagnetic effects due to induced currents and drifts in plasmas surrounding compact objects. Uchida & Low (1982) considered equilibrium configuration of the magnetosphere of a star loaded with accreted magnetised mass and noticed that the mass slides down along the field lines to the point closest to the star and is stratified in hydrostatic equilibrium to form a disc in the equatorial plane. The picture obtained was encouraging enough to look for more detailed analysis wherein one would also consider the relativistic equations through curved space formalism.

With this in mind we now take up the study of the dynamical equilibrium and stability of magnetospheric plasma around a non-rotating compact object including the effects of general relativity through the analysis of fluid as well as Maxwell's equations self-consistently. This obviously will not describe a model of pulsar magnetospheres. However, in future, this will help us in studying a more realistic model of pulsar magnetosphere by including rotation and gravity.

2. Formalism

The general equations of motion for a plasma (magneto-fluid) surrounding a central compact object in the test disc approximation on a general curved space are obtained through the conservation laws

$$T^{j}_{i;j} = 0 \tag{2.1}$$

with

$$T^{j}_{i} = \left(\rho + \frac{p}{c^2}\right) U_i U^j - \left(\frac{p}{c^2}\right)\delta^j_i - \frac{1}{4\pi c^2}\left(F_{ik}F^{jk} - \frac{1}{4}\delta^j_i F_{kl}F^{kl}\right) \tag{2.2}$$

alongwith the Maxwell's equations

$$F^{ij}_{;j} = -\frac{4\pi}{c} J^i, \tag{2.3}$$

$$F_{[ij;k]} = 0. \tag{2.4}$$

The conservation law (2.1) when expressed in terms of currents is given by the equation of continuity

$$\begin{aligned}
&\left(\rho + \frac{p}{c^2}\right)\left[V^{\alpha}_{,\alpha} + c\Gamma^{\alpha}_{0\alpha} - \left(\Gamma^{0}_{0\alpha} - \Gamma^{\beta}_{\beta\alpha}\right)V^{\alpha} - \Gamma^{0}_{\alpha\beta}\frac{V^{\alpha}V^{\beta}}{c}\right] + \frac{\partial}{\partial t}\left(\rho - \frac{p}{c^2}\right) \\
&\quad + V^{\alpha}\frac{\partial}{\partial x^{\alpha}}\left(\rho - \frac{p}{c^2}\right) + \frac{1}{c^2(U^0)^2}\left(g^{00}\frac{\partial p}{\partial t} + c g^{0\alpha}\frac{\partial p}{\partial x^{\alpha}}\right) \\
&\quad + \frac{1}{c^2(U^0)^2}\left[F^{0}{}_{k}J^k - 2F_{ik}J^k U^i U^0\right] = 0,
\end{aligned} \tag{2.5}$$

and the equation of momentum balance

$$\begin{aligned}
&\left(\rho + \frac{p}{c^2}\right)(U^0)^2\left[\frac{\partial V^{\alpha}}{\partial t} + V^{\beta}\frac{\partial V^{\alpha}}{\partial x^{\beta}} + c^2\left(\Gamma^{\alpha}_{00} - \frac{V^{\alpha}}{c}\Gamma^{0}_{00}\right) + 2cV^{\beta}\left(\Gamma^{\alpha}_{0\beta} - \frac{V^{\alpha}}{c}\Gamma^{0}_{0\beta}\right)\right. \\
&\quad \left. + V^{\beta}V^{\gamma}\left(\Gamma^{\alpha}_{\beta\gamma} - \frac{V^{\alpha}}{c}\Gamma^{0}_{\beta\gamma}\right)\right] + \left(g^{0i}\frac{V^{\alpha}}{c} - g^{\alpha i}\right)\frac{\partial p}{\partial x^i} + \left(F^{0}{}_{k}\frac{V^{\alpha}}{c} - F^{\alpha}{}_{k}\right)\frac{J^k}{c} = 0
\end{aligned} \tag{2.6}$$

wherein V^{α}, ρ, p denote the fluid 3-velocity, the density and the pressure respectively, with greek indices taking values 1, 2, 3 while the latin ones take values 0, 1, 2, 3 ($x^0 = ct$).

If we consider the background space-time due to the compact object to be the Schwarzschild geometry (static, spherically symmetric) and the electromagnetic field as well as the matter distribution to be stationary and axisymmetric the equations reduce to the form

$$\left(\rho+\frac{p}{c^2}\right)\left[\frac{\partial V^r}{\partial r}+\frac{\partial V^\theta}{\partial\theta}+\frac{2V^r}{r}\left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{3m}{r}\right)+V^\theta\cot\theta\right]$$
$$+V^r\frac{\partial}{\partial r}\left(\rho-\frac{p}{c^2}\right)+V^\theta\frac{\partial}{\partial\theta}\left(\rho-\frac{p}{c^2}\right)$$
$$+\frac{1}{c^2}\left(1-\frac{2m}{r}\right)\left(1-\frac{V^2}{c^2}\right)\left[F^0{}_k-2F_{ik}U^iU^0\right]J^k=0, \tag{2.7}$$

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{V^2}{c^2}\right)^{-1}\left[V^r\frac{\partial V^r}{\partial r}+V^\theta\frac{\partial V^r}{\partial\theta}+\frac{MG}{r^2}\left(1-\frac{2m}{r}\right)\right.$$
$$\left.-\frac{3m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}(V^r)^2-(r-2m)\{(V^\theta)^2+\sin^2\theta(V^\phi)^2\}\right]$$
$$+\left(1-\frac{2m}{r}\right)\frac{\partial p}{\partial r}+\left(F^0{}_k\frac{V^r}{c}-F^r{}_k\right)\frac{J^k}{c}=0, \tag{2.8}$$

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{V^2}{c^2}\right)^{-1}\left[V^r\frac{\partial V^\theta}{\partial r}+V^\theta\frac{\partial V^\theta}{\partial\theta}+\frac{2}{r}\left(1-\frac{2m}{r}\right)^{-1}\right.$$
$$\left.\times\left(1-\frac{3m}{r}\right)V^rV^\theta-\sin\theta\cos\theta(V^\phi)^2\right]+\frac{1}{r^2}\frac{\partial p}{\partial\theta}$$
$$+\left(F^0{}_k\frac{V^\theta}{c}-F^\theta{}_k\right)\frac{J^k}{c}=0, \tag{2.9}$$

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{V^2}{c^2}\right)^{-1}\left[V^r\frac{\partial V^\phi}{\partial r}+V^\theta\frac{\partial V^\phi}{\partial\theta}+\frac{2}{r}\left(1-\frac{2m}{r}\right)^{-1}\right.$$
$$\left.\times\left(1-\frac{3m}{r}\right)V^rV^\phi+2\cot\theta V^\theta V^\phi\right]+\left(F^0{}_k\frac{V^\phi}{c}-F^\phi{}_k\right)\frac{J^k}{c}=0, \tag{2.10}$$

$$\frac{\partial}{\partial\theta}(r^2\sin\theta F^{r\theta})=-\frac{4\pi}{c}r^2\sin\theta J^r,$$
$$\frac{\partial}{\partial r}(r^2\sin\theta F^{\theta r})=-\frac{4\pi}{c}r^2\sin\theta J^\theta,$$
$$\frac{\partial}{\partial r}(r^2\sin\theta F^{\phi r})+\frac{\partial}{\partial\theta}(r^2\sin\theta F^{\phi\theta})=-\frac{4\pi}{c}r^2\sin\theta J^\phi,$$
$$\frac{\partial}{\partial r}(r^2\sin\theta F^{tr})+\frac{\partial}{\partial\theta}(r^2\sin\theta F^{t\theta})=-\frac{4\pi}{c}r^2\sin\theta J^t, \tag{2.11}$$

and
$$F_{ij,k}+F_{ki,j}+F_{jk,i}=0, \tag{2.12}$$

where $m = MG/c^2$, M being the mass of the compact object, G the gravitational constant and c the velocity of light.

Expressing in terms of the electric and magnetic field vectors E and B, through their local Lorentz components $E_{(\alpha)}$, $B_{(\alpha)}$ using the tetrad

$$\lambda^i_{(a)} = \operatorname{diag}\left[\left(1-\frac{2m}{r}\right)^{-1/2}, \left(1-\frac{2m}{r}\right)^{1/2}, \frac{1}{r}, \frac{1}{r\sin\theta}\right] \tag{2.13}$$

alongwith

$$\begin{aligned} F_{(a)(b)} &= \lambda^i_{(a)}\,\lambda^k_{(b)}\,F_{ik}, \\ J^{(a)} &= \lambda^{(a)}_i\,J^i, \end{aligned} \tag{2.14}$$

$$\begin{aligned} E_{(\alpha)} &= F_{(\alpha)(t)}, \\ B_{(\alpha)} &= \varepsilon_{\alpha\beta\gamma}\,F_{(\beta)(\gamma)} \ \text{(no summation)}, \end{aligned} \tag{2.15}$$

the Maxwell's equations reduce to

$$\begin{aligned} &\frac{\partial}{\partial\theta}(\sin\theta\, B_{(\phi)}) = -\frac{4\pi r}{c}\sin\theta\, J^{(r)}, \\ &\frac{\partial}{\partial r}\left[r\left(1-\frac{2m}{r}\right)^{1/2} B_{(\phi)}\right] = \frac{4\pi r}{c}J^{(\theta)}, \\ &\frac{\partial}{\partial r}\left[r\left(1-\frac{2m}{r}\right)^{1/2} B_{(\theta)}\right] - \frac{\partial B_{(r)}}{\partial\theta} = -\frac{4\pi r}{c}J^{(\phi)}, \\ &\left(1-\frac{2m}{r}\right)^{1/2}\frac{\partial}{\partial r}[r^2\sin\theta\, E_{(r)}] + \frac{\partial}{\partial\theta}[r\sin\theta\, E_{(\theta)}] = -\frac{4\pi}{c}r^2\sin\theta\, J^{(t)}, \\ &\frac{\partial}{\partial r}[r^2\sin\theta\, B_{(r)}] + \frac{\partial}{\partial\theta}\left[r\left(1-\frac{2m}{r}\right)^{-1/2}\sin\theta\, B_{(\theta)}\right] = 0, \\ &\frac{\partial}{\partial r}\left[r\left(1-\frac{2m}{r}\right)^{1/2} E_{(\theta)}\right] - \frac{\partial E_{(r)}}{\partial\theta} = 0, \\ &\frac{\partial}{\partial r}\left[r\left(1-\frac{2m}{r}\right)^{1/2}\sin\theta\, E_{(\phi)}\right] = 0, \\ &\frac{\partial}{\partial\theta}[\sin\theta\, E_{(\phi)}] = 0. \end{aligned} \tag{2.16}$$

3. Possible solutions

One admissible solution set of the Maxwell's equations (2.16) is

$$\begin{aligned} E_{(r)} &= E_0\left(\frac{R}{r}\right)^3\cos\theta, \\ E_{(\theta)} &= \frac{E_0}{2}\left(\frac{R}{r}\right)^3\left(1-\frac{2m}{r}\right)^{-1/2}\sin\theta, \\ E_{(\phi)} &= \frac{k_1}{r\sin\theta}\left(1-\frac{2m}{r}\right)^{-1/2}, \end{aligned} \tag{3.1}$$

$$B_{(r)} = B_0 \left(\frac{R}{r}\right)^3 \cos\theta,$$

$$B_{(\theta)} = \frac{B_0}{2}\left(1 - \frac{2m}{r}\right)^{1/2} \left(\frac{R}{r}\right)^3 \sin\theta,$$

$$B_{(\phi)} = \frac{k_2}{r\sin\theta}\left(1 - \frac{2m}{r}\right)^{-1/2}, \tag{3.2}$$

giving rise to the currents

$$J^{(r)} = 0,$$

$$J^{(\theta)} = 0,$$

$$J^{(\phi)} = \frac{-3mc}{4\pi}\frac{B_0}{r^2}\left(\frac{R}{r}\right)^3 \sin\theta,$$

$$J^{(t)} = \frac{-2mc}{4\pi}\frac{E_0}{r^2}\left(\frac{R}{r}\right)^3 \left(1 - \frac{2m}{r}\right)^{-1/2} \cos\theta, \tag{3.3}$$

where R represents the radius of the compact object with B_0 and E_0 being the surface field strengths, and k_1 and k_2 are arbitrary constants ($k_2 = 0$ in this case). Using the same tetrad (2.11) one can express the 3-velocity V also in terms of local Lorentz components as given by

$$V^r = \left(1 - \frac{2m}{r}\right) V^{(r)},$$

$$V^\theta = \left(1 - \frac{2m}{r}\right)^{1/2} \frac{V^{(\theta)}}{r},$$

$$V^\phi = \left(1 - \frac{2m}{r}\right)^{1/2} \frac{V^{(\phi)}}{r\sin\theta}. \tag{3.4}$$

As the currents in the r and θ directions are zero, we can look for a self-consistent solution of the fluid equations for a purely rotating fluid having only the azimuthal component of V^a to be nonzero. Thus with V^r and V^0 equal to zero the Equations (2.7) – (2.10) reduce to

$$\left[F^0{}_k J^k - 2F_{ik} J^k U^i U^0 \right] = 0, \tag{3.5}$$

$$\left(\rho + \frac{p}{c^2}\right)\left(1 - \frac{V^{(\phi)2}}{c^2}\right)^{-1} \left[\frac{MG}{r^2} - \left(1 - \frac{2m}{r}\right)\frac{V^{(\phi)2}}{r}\right] + \left(1 - \frac{2m}{r}\right)\frac{\partial p}{\partial r}$$
$$+ \frac{1}{c}\left(1 - \frac{2m}{r}\right)^{1/2} [E_{(r)} J^{(t)} - B_{(\theta)} J^{(\phi)} + B_{(\phi)} J^{(\theta)}] = 0. \tag{3.6}$$

$$\left(\rho + \frac{p}{c^2}\right)\left(1 - \frac{V^{(\phi)2}}{c^2}\right)^{-1} \left(-\cot\theta \frac{V^{(\phi)2}}{r}\right) + \frac{1}{r}\frac{\partial p}{\partial\theta}$$
$$+ \frac{1}{c}[E_{(\theta)} J^{(t)} - B_{(\phi)} J^{(r)} + B_{(r)} J^{(\phi)}] = 0, \tag{3.7}$$

$$\left(F^0_k \frac{V^{(\phi)}}{c} - F^\phi_k\right) J^k = 0. \tag{3.8}$$

If we consider the case when the toroidal electric field is zero, then (3.5) and (3.8) are identically satisfied and thus we are left with two equations for $V^{(\phi)}$, p and p.

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{V^{(\phi)2}}{c^2}\right)^{-1}\left[\frac{MG}{r^2}-\left(1-\frac{2m}{r}\right)\frac{V^{(\phi)2}}{r}\right]+\left(1-\frac{2m}{r}\right)\frac{\partial p}{\partial r}$$
$$=\frac{-m}{4\pi r^2}\left(\frac{R}{r}\right)^6\left(1-\frac{2m}{r}\right)\left[\frac{3}{2}B_0^2\sin^2\theta-2E_0^2\left(1-\frac{2m}{r}\right)^{-1}\cos^2\theta\right], \tag{3.9}$$

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{V^{(\phi)2}}{c^2}\right)^{-1}V^{(\phi)2}\cot\theta-\frac{\partial p}{\partial\theta}$$
$$=\frac{-m}{4\pi r}\left(\frac{R}{r}\right)^6\left[3B_0^2+E_0^2\left(1-\frac{2m}{r}\right)^{-1}\right]\sin\theta\cos\theta. \tag{3.10}$$

Thus one would require an equation of state to close the system.

3.1 *Thin Disc*

As a test case if one restricts the discussion to matter confined to the equatorial plane $\theta=\pi/2$, then one has

$$\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{V^{(\phi)2}}{c^2}\right)^{-1}\left[\frac{MG}{r^2}\left(1-\frac{2m}{r}\right)^{-1}-\frac{V^{(\phi)2}}{r}\right]+\frac{\partial p}{\partial r}=\frac{-3m}{8\pi}\frac{B_0^2R^6}{r^8},$$

and

$$\frac{\partial p}{\partial\theta}=0. \tag{3.11}$$

Case 1:

Considering the motion to be almost Keplerian as expressed by

$$V^{(\phi)}=\sqrt{\frac{2MG}{r}}$$

(3.11) reduces to

$$\frac{dp}{dr}-\frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right)^{-2}\left(1-\frac{4m}{r}\right)\left(\rho+\frac{p}{c^2}\right)=-\frac{3m}{8\pi}\frac{B_0^2R^6}{r^8}. \tag{3.12}$$

In order to look for an exact analytic solution in closed form, one takes the Newtonian limit of this equation and gets

$$\frac{dp}{dr}-\frac{m}{r^2}\left(\rho c^2+p\right)=-\frac{3m}{8\pi}\frac{B_0^2R^6}{r^8}, \tag{3.13}$$

whose solution for ρ = constant is given by

$$\left(\rho c^2+p\right)=De^{-m/r}+\frac{3B_0^2R^6}{8\pi m^6}\left[\left(\frac{m}{r}-6\right)\left(\frac{m}{r}\right)^5+\left(\frac{m}{r}-4\right)30\left(\frac{m}{r}\right)^3\right.$$
$$\left.+\frac{360m}{r}\left(\frac{m}{r}-2\right)+720\right]. \tag{3.14}$$

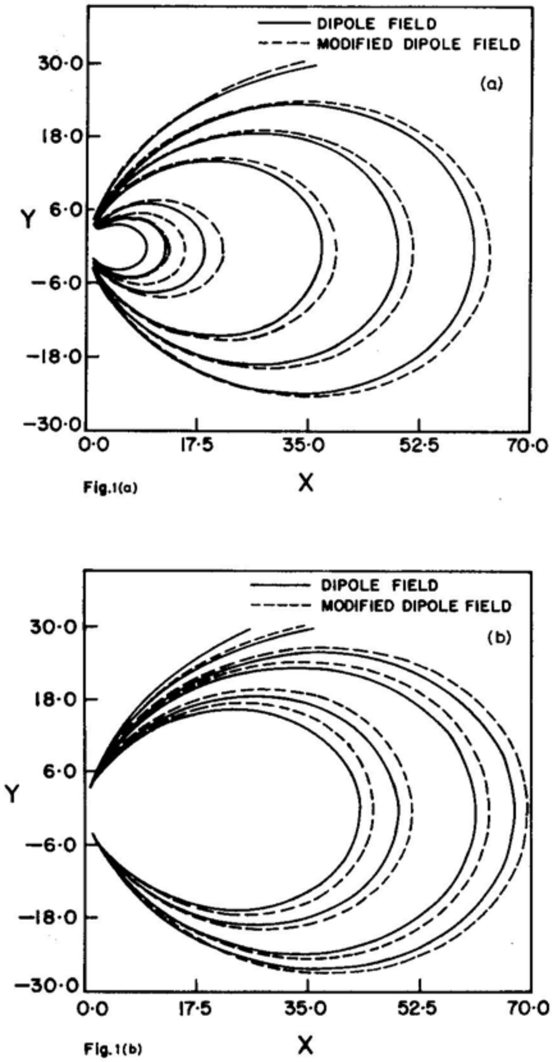

Figure 1. Magnetic field configuration in the meridional plane for a compact object having radius (a) $R = 3m$ and (b) $R = 12m$ for relativistic (broken line) and Newtonian case (solid line).

The constant of integration D may be obtained through using the boundary condition that at the inner edge $r = r_a$ the hydrostatic pressure p equals the magnetic pressure at that surface as given by $p_m = \frac{B_0^2}{8\pi}\left(\frac{R}{r_a}\right)^6$. This gives the constant

$$D = 10^3 \rho + 3n^6 \left\{ \frac{1}{x_a^5} - \frac{5}{x_a^4} + \frac{20}{x_a^3} - \frac{60}{x_a^2} + \frac{120}{x_a} - 120 \right\} \tag{3.15}$$

where $n = R/m$ and $x_a = r_a / m$. Fig. 2 shows the profiles of pressure in terms of p_m the magnetic pressure for the two cases with $R = 3m$ and $R = 15m$ for $B = 10^{12}$ gauss and $\rho = 10.0$.

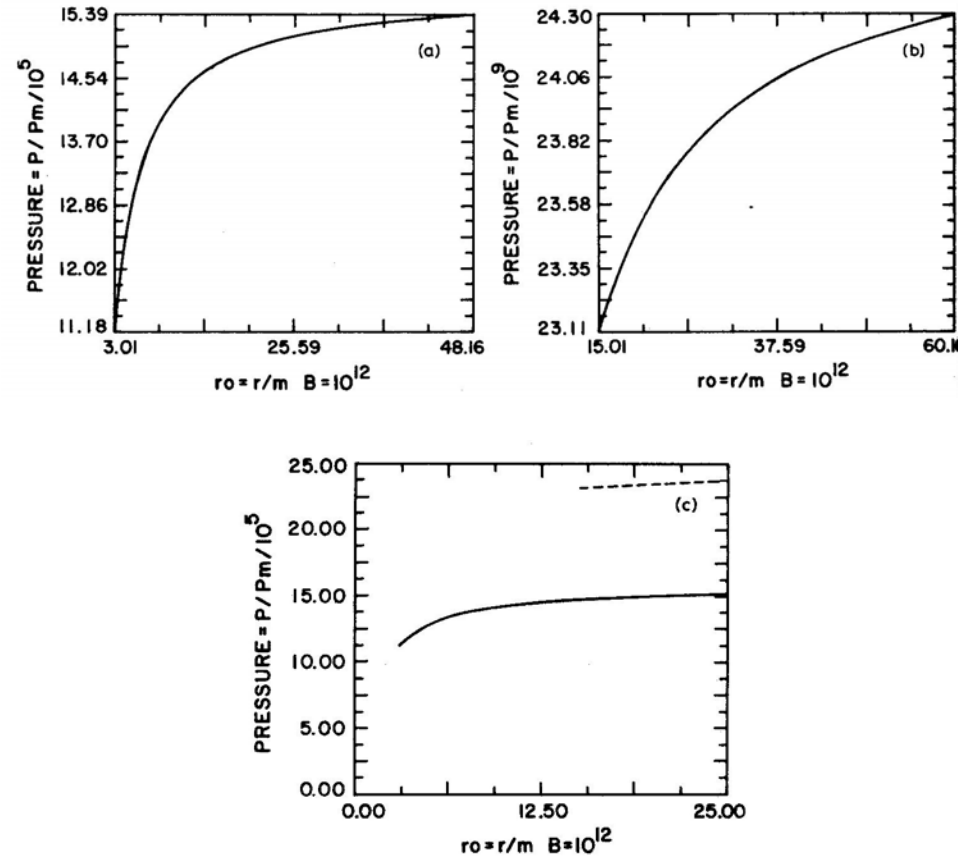

Figure 2. Pressure profiles in the Newtonian limit for thin disc with $V^{(\phi)} = \sqrt{2GM/r}$ for a compact object with (a) $R = 3m$, (b) $R = 15m$; comparison of the two pressure profiles is shown in (c) by solid ($R = 3m$), and broken ($R = 15m$) lines.

One can in fact solve (3.12) exactly numerically and doing so for the same values of ρ, B, and n one gets the pressure profiles as in Fig. 3.

Case 2:

$$V^{(\phi)} = \sqrt{\left(1-\frac{2m}{r}\right)^{-1}\frac{MG}{r}}. \tag{3.16}$$

With this value of $V_{(\varphi)}$ the centrifugal force is completely balanced by the gravitational force and thus one has the pressure gradient to balance the magnetic stress, giving the equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{3m}{8\pi}\frac{B_0^2 R^6}{r^8} \tag{3.17}$$

Whose solution is

$$p = D + \frac{3m}{56\pi}\frac{B_0^2 R^6}{r^7}. \tag{3.18}$$

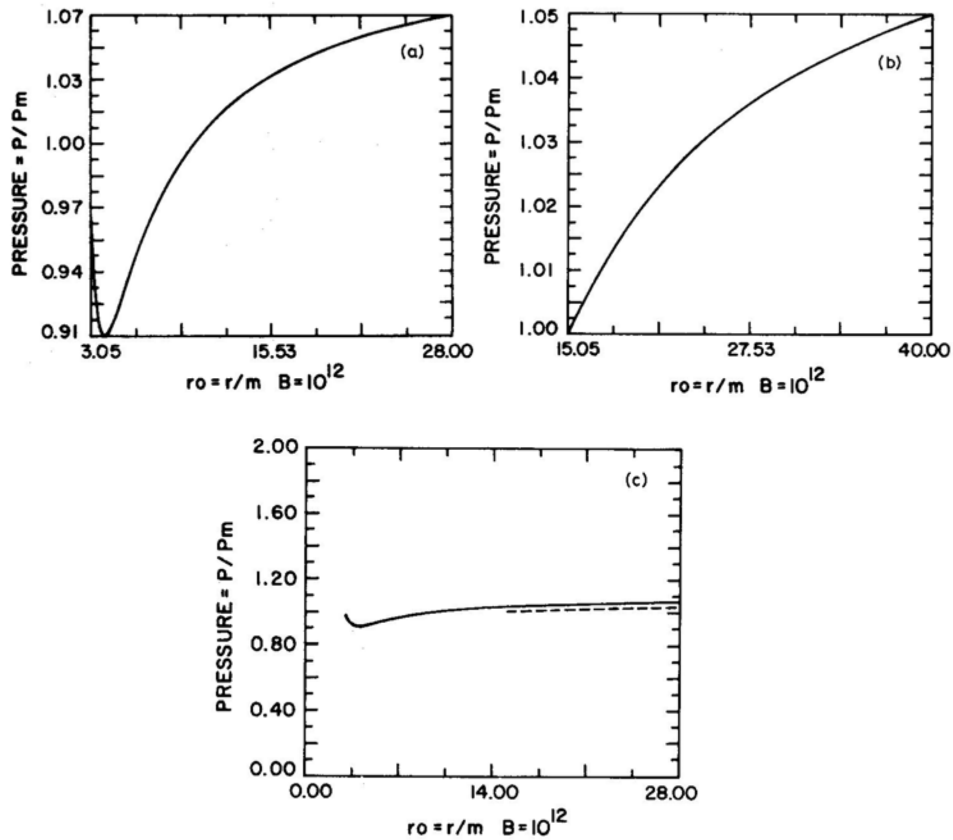

Figure 3. Pressure profiles, for thin disc with $V^{(\phi)} = \sqrt{2GM/r}$ for a compact object with (a) $R = 3m$, and (b) $R = 15m$ with no approximation; comparison of the two pressure profiles is shown in (c) by solid ($R = 3m$), and broken ($R = 15m$) lines.

Using the same boundary condition as above one can calculate D and thus gets the pressure

$$p = \frac{B_0^2 R^6}{8\pi r_a^6}\left[1 - \frac{3m}{7r_a}\left\{1 - \left(\frac{r_a}{r}\right)^7\right\}\right]. \tag{3.19}$$

Fig. 5 gives the pressure profile as a function of r/m wherein again at the inner boundary the magnetic pressure $p\text{m} = \dfrac{B_0^2 R^6}{8\pi r_a^6}$ matches the hydrostatic pressure.

3.2 *Thick Disc*

Case 1:

$$V^{(\phi)} = \sqrt{\frac{2MG}{r}}.$$

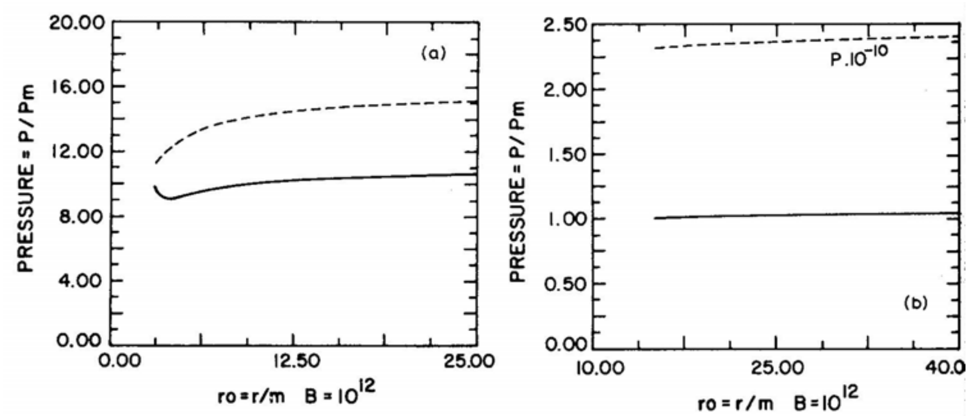

Figure 4. Comparison of pressure profiles for $V^{(\phi)} = \sqrt{2GM/r}$ with (broken line) and without (solid line) approximation for (a) $R = 3m$, (b) $R = 15m$.

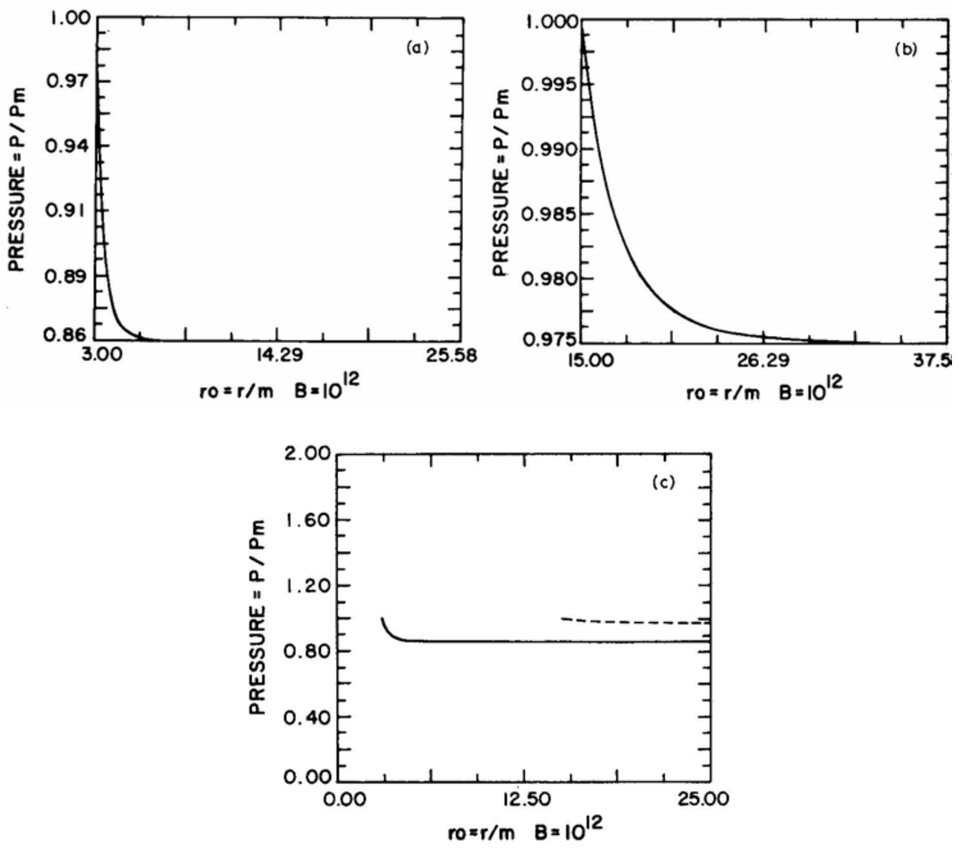

Figure 5. Pressure profile for thin disc with $V_{(f)} = \sqrt{(1 - 2m/r)^{-1}\ GM/r}$ for a compact object with (a) $R = 3m$; (b) $R = 15m$; comparison of the two pressure profiles is made in (c) by solid ($R = 3m$), and broken (R = $15m$) lines.

The momentum equation would now give

$$\frac{\partial p}{\partial r} = (\rho c^2 + p)\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-2}\left(1 - \frac{4m}{r}\right)$$
$$- \frac{mR^6}{4\pi r^8}\left[\frac{3}{2}B_0^2 \sin^2\theta - 2E_0^2\left(1 - \frac{2m}{r}\right)^{-1}\cos^2\theta\right], \qquad (3.20)$$

$$\frac{\partial p}{\partial \theta} = (\rho c^2 + p)\frac{2m}{r}\left(1 - \frac{2m}{r}\right)^{-1}\cot\theta + \frac{mR^6}{4\pi r^7}\left[3B_0^2 + E_0^2\left(1 - \frac{2m}{r}\right)^{-1}\right]\sin\theta\cos\theta. \qquad (3.21)$$

Integrability of these require, putting $(pc^2 + p) = f(r)\sin^2\theta$ with ρ being constant, the equation for f

$$\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{2f}{r} = \frac{R^6}{8\pi r^7}\left[18B_0^2\left(1 - \frac{2m}{r}\right) + 3E_0^2\left(1 - \frac{2m}{r}\right)^{-1}\left(1 - \frac{4m}{3r}\right)\right], \qquad (3.22)$$

whose exact solution is

$$f = Dr^2 + \frac{R^6 r^2}{8\pi}\left[B_0^2\left(\frac{4m}{r^9} - \frac{9}{4r^8}\right) + E_0^2\left\{-\frac{1}{4r^8} + \frac{1}{14mr^7}\right.\right.$$
$$+ \frac{1}{24m^2r^6} + \frac{1}{40m^3r^5} + \frac{1}{64m^4r^4} + \frac{1}{96m^5r^3} + \frac{1}{128m^6r^2}$$
$$\left.\left. + \frac{1}{128m^7r} + \frac{1}{256m^8}\ln\left(1 - \frac{2m}{r}\right)\right\}\right], \qquad (3.23)$$

D being the constant of integration. Thus one has from

$$(\rho c^2 + p) = f(r)\sin^2\theta \qquad (3.24)$$

the pressure profile for every given density distribution once the constant D is determined. One can use the same boundary condition as in the other case, *viz.,* at the inner edge the hydrostatic pressure is equivalent to the magnetic pressure $\frac{B^2}{8\pi} = \frac{B_0^2}{8\pi}\left(\frac{R}{r_a}\right)^6$. The pressure profile as a function of r for the case $\theta = \pi/2$ is as shown in Fig. 6(a).

Case 2:

$$V^{(\phi)} = \sqrt{\left(1 - \frac{2m}{r}\right)^{-1}\frac{MG}{r}}.$$

With this the momentum equation would give

$$\frac{\partial p}{\partial r} = \frac{-mR^6}{4\pi r^8}\left[\frac{3}{2}B_0^2\sin^2\theta - 2E_0^2\left(1 - \frac{2m}{r}\right)^{-1}\cos^2\theta\right], \qquad (3.25)$$

$$\frac{\partial p}{\partial \theta} = \frac{MG}{r}\left(1 - \frac{2m}{r}\right)^{-1}\left(\rho + \frac{p}{c^2}\right)\cot\theta + \frac{mR^6}{4\pi r^7}\left\{3B_0^2 + E_0^2\left(1 - \frac{2m}{r}\right)^{-1}\right\}\sin\theta\cos\theta. \qquad (3.26)$$

Integrability requires with $\rho c^2 + p = f(r)\sin^2\theta$, the equation for f to be

$$\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{f}{r}\left(1 - \frac{3m}{r}\right)^{-1} = \frac{R^6}{4\pi r^7}\left(1 - \frac{3m}{r}\right)\left\{18B_0^2 + 3E_0^2\left(1 - \frac{2m}{r}\right)^{-2}\left(1 - \frac{4m}{3r}\right)\right\}, \tag{3.27}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r - 3m}\right) = \frac{R^6}{4\pi}\left\{\frac{18B_0^2}{r^8} + \frac{3E_0^2}{r^8}\left(1 - \frac{2m}{r}\right)^{-2}\left(1 - \frac{4m}{r}\right)\right\}. \tag{3.28}$$

The solution therefore is given by

$$\begin{aligned}(\rho c^2 + p) = \Bigg[& D(r - 3m) - (r - 3m)\left\{\frac{18B_0^2 R^6}{28\pi r^7}\right\} \\ & - (r - 3m)\frac{E_0^2 R^6}{60\pi}\left\{\frac{-5}{2mr^6} - \frac{3}{4m^2r^5} + \frac{5}{16m^4r^3} + \frac{15}{32m^5r^2}\right. \\ & \left. + \frac{45}{64m^6r} + \frac{15}{128m^7}\left(1 - \frac{2m}{r}\right)^{-1} + \frac{60}{128m^7}\ln\left(1 - \frac{2m}{r}\right)\right\}\Bigg]\sin^2\theta \end{aligned} \tag{3.29}$$

Using the same boundary condition one can again get the pressure profile for $\theta = \pi/2$, as shown in Fig. 6(b).

Fig. 6(c) shows the comparison of the pressure profiles for both the above cases.

4. Discussion and conclusions

Looking at the structure of magnetic field lines (Fig. 1) one finds that due to the presence of the gravitational field of the compact object, the field structure is modified to the effect that the field strength at every point increases. The pressure profiles show clearly the difference in the results that could arise due to varied approximation. Particularly in the case of Keplerian type distribution $v^{(\phi)} = \sqrt{2MG/r}$ the difference between the Newtonian and the general relativistic treatment is very interesting. In the Newtonian treatment the solution presents a pressure profile which is increasing from the inner edge outwards for a short distance and then stays almost constant. On the other hand, in the case when no approximation was made (Fig. 3) the pressure first decreases, reaches a minimum and then increases just as in the earlier case. The minimum occurs at $r = 4m$ which is because of the general relativistic term $(1 - 4m/r)$ in the equation. In contrast to this when we consider the velocity distribution to be relativistic Keplerian $V_{(\phi)} = \sqrt{\frac{2MG}{r}(1 - 2m/r)^{-1}}$, the pressure profile is more physical decreasing outwards as one would normally expect. In our opinion this shows that the assumption of nearly Keplerian velocity distribution is not consistent with the fully relativistic equations we are dealing with, and one will have to consider the relativistic contribution into the velocity field for getting the pressure profile.

In the case of thick discs, since the constant of integration gets multiplied by r^2 for the Keplerian distribution and by $(r - 3m)$ for the relativistic Keplerian, this term dominates over the rest as could be seen in the pressure profiles (Fig. 6) which are monotonically increasing. If one were to choose the constant of integration as zero,

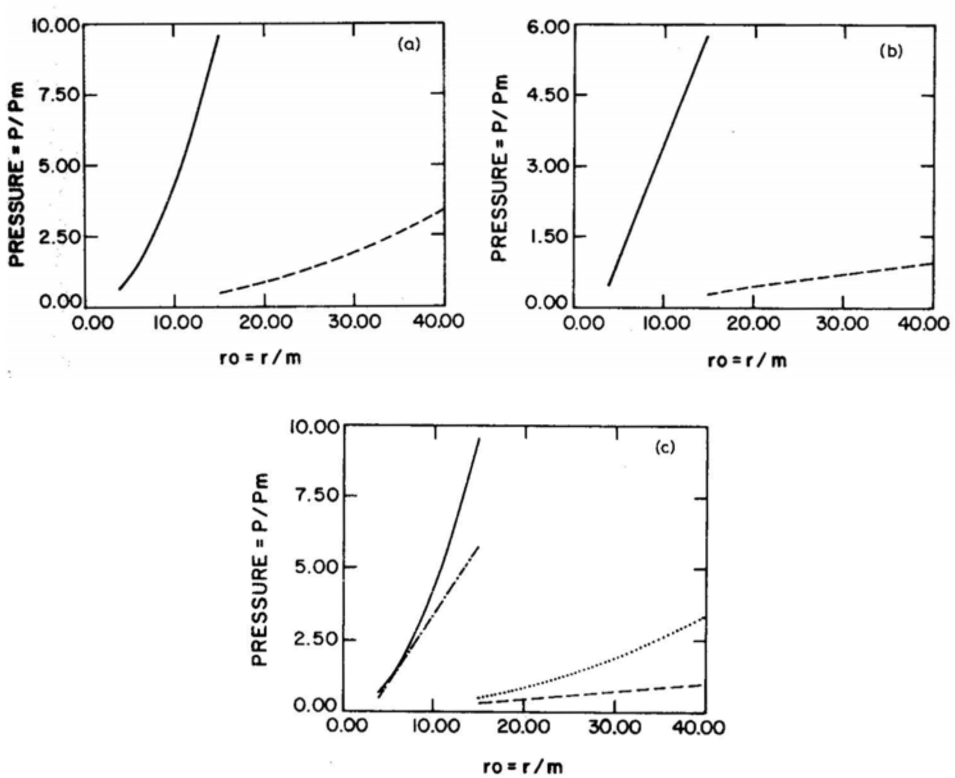

Figure 6. Pressure profile for thick disc with (a) $V^{(\phi)} = \sqrt{2GM/r}$, and (b) $V^{(\phi)} = \sqrt{(1-2m/r)^{1} GM/r}$ for a compact object with R = 3.5m (solid line) and R = 12m (broken line); comparison of profiles is made in (c) for the two cases: (a) R= 3.5m (———), and R= 12m (—·—·—), (b) R = 3.5m (......), and 12m (----).

this would lead to negative ($\rho c^2 + p$) thus making the solution totally unphysical. The increasing pressure profile also indicates an unstable (runaway) configuration and thus one concludes that with the type of electromagnetic field that one has considered, no physically meaningful equilibrium solution exists for prescribed velocity distribution as chosen above.

In spite of the reasonable pressure profile obtained in the fully relativistic treatment of the thin discs, there may be still an unsatisfactory element as far as Ohm's law is considered, since we have not specifically made use of it It is worth noting that if one were to consider Ohm's law

$$J^i = \sigma F^i_k U^k \tag{4.1}$$

alongwith the set of dynamical Equations (2.3) to (2.8) then it is clear that the electric and magnetic fields are coupled through the velocity field and it would not be consistent to choose certain component of V^α to be zero *a priori* This would mean a more complex set of coupled nonlinear equations for which the existence of an equilibrium solution may not be always guaranteed.

In conclusion one finds that for a situation wherein the current contribution of the plasma on the existing electromagnetic field of the compact object is neglected a fully relativistic Keplerian angular momentum distribution without radial or meridional velocity does admit a reasonable pressure profile in equilibrium with the electromagnetic field for an incompressible fluid.

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