J. Astrophys. Astr. (1982) **3**, 193–206

Structure and Stability of Rotating Fluid Disks around Massive Objects. II. General Relativistic Formulation

D. K. Chakraborty* and A. R. Prasanna *physical Research Laboratory, Navrangpura, Ahmedabad 380009*

Received 1982 January 4; accepted 1982 May 24

Abstract. In this paper we have considered the structure of a thick perfect fluid disk of constant density rotating around a Schwarzschild black hole and its stability under axisymmetric perturbation. The inner edge of such disk cannot lie within $4m$. The critical γ_c for neutral stability is found to be much less than 4/3 indicating that the disks are generally stable.

Key words: fluid disks—general relativity—stability

1. Introduction

Recent developments in the study of high energy emission from cosmic sources has emphasised quite frequently the importance of the study of structure and stability of accretion disks around compact objects. After the early analysis of Pringle and Rees (1972), Novikov and Thorne (1973), Shakura and Sunyaev (1976), the subject has been treated in a more detailed way by many authors with the analysis of both thin and thick disks. Subsequent to the review of Lightman, Shapiro and Rees (1978), the Polish school has considered several aspects of accretion disk models, a reference to which may be seen in Paczynski (1980). It is now well known that if the accretion rate exceeds the critical limit the inner regions of the disk render a thick structure as first pointed out by Shakura and Sunyaev (1973), and thus it is very relevant to consider the detailed analysis of the structure and stability of thick disks in the same spirit as has been done earlier for thin disks. However almost all these analyses restricted themselves to the study of disks under equilibria with respect to the gravitational, centrifugal and pressure gradient forces only. Prasanna and Chakraborty (1981; hereafter referred to as Paper I) emphasised the necessity of considering the analysis including the self-generated electromagnetic fields also and they showed that

*Permanent Address: Government College, Jagdalpur 494001

pressureless thin disks of charged perfect fluid are indeed generally stable under radial pulsations. However an analysis of the thick disks under the action of all the four forces (gravitational, electromagnetic, centrifugal and pressure gradient) has been quite formidable and as such we consider the detailed structure and stability of thick disks around Schwarzschild black hole without the electromagnetic field. The analysis of such disks in Newtonian formulation showed that both under radial as well as axisymmetric perturbation there are large regions of stability (Chakraborty and Prasanna 1981, hereafter referred to as Paper II). We present in this paper the analysis of similar disks with a fully relativistic treatment.

2. Steady state solutions

The general set of equations governing the dynamics of a non-self-gravitating perfect fluid disk can be obtained from the general momentum equations (Paper I, Equations 2.15 – 2.17) and are given by

$$\left(\rho + \frac{p}{c^2}\right)\left\{\frac{DV^{(r)}}{Dt} + \frac{mc^2}{r^2}\left(1 - \frac{V^{(r)2}}{c^2}\right) - \left(1 - \frac{2m}{r}\right)\left(\frac{V^{(\theta)2} + V^{(\phi)2}}{r}\right)\right\}$$
$$= -\left(1 - \frac{V^2}{c^2}\right)\left\{\left(1 - \frac{2m}{r}\right)\frac{\partial p}{\partial r} + \frac{V^{(r)}}{c^2}\frac{\partial p}{\partial t}\right\}, \tag{2.1}$$

$$\left(\rho + \frac{p}{c^2}\right)\left\{\frac{DV^{(\theta)}}{Dt} + \left(1 - \frac{3m}{r}\right)\frac{V^{(r)}\,V^{(\theta)}}{r} - \left(1 - \frac{2m}{r}\right)^{1/2}\frac{\cot\theta\, V^{(\phi)2}}{r}\right\}$$
$$= -\left(1 - \frac{V^2}{c^2}\right)\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{V^{(\theta)}}{c^2}\frac{\partial p}{\partial r}\right\}, \tag{2.2}$$

$$\left(\rho + \frac{p}{c^2}\right)\left\{\frac{DV^{(\phi)}}{Dt} + \left(1 - \frac{3m}{r}\right)\frac{V^{(r)}\,V^{(\theta)}}{r} + \left(1 - \frac{2m}{r}\right)^{1/2}\frac{\cot\theta\, V^{(\theta)}\,V^{(\phi)}}{r}\right\}$$
$$= -\left(1 - \frac{V^2}{c^2}\right)\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \frac{V^{(\phi)}}{c^2}\frac{\partial p}{\partial t}\right\}, \tag{2.3}$$

the continuity equation (Paper I, Equation 2.6) is given by

$$\frac{D}{Dt}\left(\rho - \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right)\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,V^{(r)})\right.\right.$$
$$\left.\left.+ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, V^{(\theta)}) + \frac{\partial V^{(\phi)}}{\partial\phi}\right)\right]\right\} + \frac{1}{c^2}\left(1 - \frac{V^2}{c^2}\right)\frac{\partial p}{\partial t} = 0, \tag{2.4}$$

the equation of baryon conservation $(nu^i)_{;\,i} = 0$, as given by

$$n\left(1 - \frac{2m}{r}\right)^{1/2}\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,V^{(r)}) + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, V^{(\theta)}) + \frac{\partial V^{(\phi)}}{\partial\phi}\right)\right\}$$
$$+ \frac{Dn}{Dt} - \frac{n}{c^2}\left(\rho + \frac{p}{c^2}\right)^{-1}\left\{\frac{Dp}{Dt} + \left(1 - \frac{V^2}{c^2}\right)\frac{\partial p}{\partial t}\right\} = 0 \tag{2.5}$$

and the equation for the adiabatic flow

$$\frac{D}{Dt}(p\, n^{-\gamma}) = 0 \tag{2.6}$$

where p, ρ, n and $V^{(\alpha)}$ are the pressure, density, baryon number density and the components of 3-velocity in the local Lorentz frame and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(1 - \frac{2m}{r}\right)^{1/2} \left\{\left(1 - \frac{2m}{r}\right)^{1/2} V^{(r)} \frac{\partial}{\partial r} + \frac{1}{r} V^{(\theta)} \frac{\partial}{\partial \theta} + \frac{1}{r \sin\theta} V^{(\phi)} \frac{\partial}{\partial \phi}\right\},$$

$$V^2 = V^{(r)2} + V^{(\theta)2} + V^{(\phi)2},$$

$$m = MG/c^2. \tag{2.7}$$

Restricting ourselves to the case of an axisymmetric disk in pure rotational flow as expressed by $V_0^{(r)} = 0$, $V_0^{(\theta)} = 0$, $V_0^{(\phi)} = V_0$, the equations governing the steady state reduce to

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\frac{mc^2}{r^2} - \left(1 - \frac{2m}{r}\right)\frac{V_0^2}{r}\right] = -\left(1 - \frac{2m}{r}\right)\left(1 - \frac{V_0^2}{c^2}\right)\frac{\partial p_0}{\partial r}, \tag{2.8}$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\cot\theta\, V_0^2 = \left(1 - \frac{V_0^2}{c^2}\right)\frac{\partial p_0}{\partial \theta}, \tag{2.9}$$

the remaining equations being identically satisfied. The above two equations can be solved exactly for the special case, ρ_0 = constant. Using this in Equations (2.8) and (2.9), we obtain

$$\cot\theta \frac{\partial}{\partial r} V_0^2 - \frac{1}{r}\frac{\partial}{\partial \theta} V_0^2 + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} \frac{\partial}{\partial \theta} V_0^2 = 0 \tag{2.10}$$

whose solution is given by

$$V_0^2 = Ac^2\,(1 - 2m/r)\,/\,r^2 \sin^2\theta, \tag{2.11}$$

A being a constant. Substituting this in Equations (2.8) and (2.9), we get

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\frac{mc^2}{r^2} - Ac^2\left(1 - \frac{2m}{r}\right)^2 \Big/ r^3 \sin^2\theta\right]$$

$$= -\left[1 - A\left(1 - \frac{2m}{r}\right)\Big/ r^2 \sin^2\theta\right]\left(1 - \frac{2m}{r}\right)\frac{\partial p_0}{\partial r}, \tag{2.12}$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right) Ac^2 \cot\theta \left(1 - \frac{2m}{r}\right)\Big/ r^2 \sin^2\theta = \left[1 - A\left(1 - \frac{2m}{r}\right)\Big/ r^2 \sin^2\theta\right]\frac{\partial p_0}{\partial \theta} \tag{2.13}$$

whose solution may be obtained as

$$p_0/c^2 = B\left[\left(1 - \frac{2m}{r}\right)^{-1} - A/r^2 \sin^2\theta\right]^{1/2} - \rho_0, \tag{2.14}$$

where B is another constant. Using the boundary condition $p_0 = 0$ at r_a and r_b, the inner and outer edges at the plane $\theta = \pi/2$ we obtain the solutions of steady state as

$$\rho_0 = \text{constant}, \tag{2.15}$$

$$\frac{V_0^2}{c^2} = A\left(1 - \frac{2}{R}\right)\Big/ R^2 \sin^2\theta, \tag{2.16}$$

$$\frac{p_0}{c^2} = \rho_0\left[B\left\{\left(1 - \frac{2}{R}\right)^{-1} - \frac{A}{R^2\sin^2\theta}\right\}^{1/2} - 1\right], \tag{2.17}$$

wherein

$$A = 2\,a^2b^2\,/\,(a+b)\,(a-2)\,(b-2),$$

$$B = \left[\frac{(b^2 - a^2)\,(b-2)\,(a-2)}{b^3\,(a-2) - a^3\,(b-2)}\right]^{1/2},$$

$$R = \frac{r}{m}, \quad a = \frac{r_a}{m}, \quad b = \frac{r_b}{m}. \tag{2.18}$$

The solutions obtained above are physically acceptable if $p_0 > 0$ throughout the interior of the disk and it goes over to zero at the boundary. The former condition leads us to the constraint that the inner edge cannot lie within $4m$ and further

$$b > 2a/(a-4), \quad \text{if} \quad 4 < a < 6. \tag{2.19}$$

There is no restriction on outer edge if $a \geqslant 6$. The latter condition $(p_0)_b = 0$ gives the edge of the disk θ_e (and $\pi - \theta_e$) on the meridional plane as given by

$$\sin^2\theta_e = AB^2\,/\,R^2\,[B^2\,(1 - 2/R)^{-1} - 1]. \tag{2.20}$$

Fig. 1 shows the profiles of velocity and pressure as the functions of equatorial distance while Fig. 2 shows the meridional section of the disk.

3. Stability analysis

We consider the axisymmetric perturbations of the disk as described above and use the normal mode analysis restricting the perturbations to linear terms only; the general procedure of the analysis remains the same as used in Paper II which is based on the technique developed by Chandrasekhar and Friedman (1972a, b). The set of equations governing the perturbations are obtained from Paper I, Equations (3.1)–(3.4) as given by

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\frac{\partial}{\partial t}\delta V^{(r)} - \frac{2}{r}\left(1 - \frac{2m}{r}\right) V_0 \delta V^{(\phi)}\right] + \left(\delta\rho + \frac{\delta p}{c^2}\right)\left[\frac{mc^2}{r^2} - \frac{1}{r}\left(1 - \frac{2m}{r}\right)V_0^2\right]$$

$$= -\left(1 - \frac{2m}{r}\right)\left[\left(1 - \frac{V_0^2}{c^2}\right)\frac{\partial}{\partial r}\delta p - \frac{2V_0 \delta V^{(\phi)}}{c^2}\frac{\partial p_0}{\partial r}\right], \tag{3.1}$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\frac{\partial}{\partial t}\delta V^{(\theta)} - \frac{2}{r}\left(1 - \frac{2m}{r}\right)^{1/2} V_0\, \delta V^{(\phi)} \cot\theta\right] - \left(\delta\rho + \frac{\delta p}{c^2}\right)\frac{V_0^2}{r}$$

$$\times \left(1 - \frac{2m}{r}\right)^{1/2} \cot\theta = -\left(1 - \frac{2m}{r}\right)^{1/2}\left[\left(1 - \frac{V_0^2}{c^2}\right)\frac{1}{r}\frac{\partial}{\partial\theta}\delta p - \frac{2V_0\,\delta V^{(\phi)}}{rc^2}\frac{\partial p_0}{\partial\theta}\right] \tag{3.2}$$

and

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\frac{\partial}{\partial t}\delta V^{(\phi)} + \frac{1}{r}\left(1 - \frac{2m}{r}\right)^{1/2}\left\{\frac{\partial V_0}{\partial\theta} + V_0 \cot\theta\right\}\delta V^{(\theta)}\right.$$

$$\left. + \left\{\left(1 - \frac{2m}{r}\right)\frac{\partial V_0}{\partial r} + \frac{1}{r}\left(1 - \frac{3m}{r}\right)V_0\right\}\delta V^{(r)}\right] = -\left(1 - \frac{V_0^2}{c^2}\right)\frac{V_0}{c^2}\frac{\partial}{\partial t}\delta p \tag{3.3}$$

while Equations (2.4) – (2.6) yield

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left(1 - \frac{2m}{r}\right)^{1/2}\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\delta V^{(r)}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\;\delta V^{(\theta)})\right\}$$

$$+ \frac{\partial}{\partial t}\left(\delta\rho - \frac{\delta p}{c^2}\right) + \left(1 - \frac{2m}{r}\right)^{1/2}\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\delta V^{(r)}\frac{\partial}{\partial r} + \frac{\delta V^{(\theta)}}{r}\frac{\partial}{\partial\theta}\right\}\left(\rho_0 - \frac{p_0}{c^2}\right)$$

$$+ \frac{1}{c^2}\left(1 - \frac{V_0^2}{c^2}\right)\frac{\partial}{\partial t}\delta p = 0, \tag{3.4}$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right) n_0 \left(1 - \frac{2m}{r}\right)^{1/2}\left\{\left(1 - \frac{2m}{r}\right)^{1/2}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\delta V^{(r)}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\;\delta V^{(\theta)})\right\}$$

$$+ \left(\rho_0 + \frac{p_0}{c^2}\right)\left\{\frac{\partial}{\partial t}\delta n + \left(1 - \frac{2m}{r}\right)^{1/2}\left[\left(1 - \frac{2m}{r}\right)^{1/2}\delta V^{(r)}\frac{\partial}{\partial r} + \frac{\delta V^{(\theta)}}{r}\frac{\partial}{\partial\theta}\right]n_0\right\}$$

$$- \frac{n_0}{c^2}\left\{\frac{\partial}{\partial t}\delta p + \left(1 - \frac{2m}{r}\right)^{1/2}\left[\left(1 - \frac{2m}{r}\right)^{1/2}\delta V^{(r)}\frac{\partial}{\partial r} + \frac{\delta V^{(\theta)}}{r}\frac{\partial}{\partial\theta}\right](p_0)\right.$$

$$\left. - \left(1 - \frac{V_0^2}{c^2}\right)\frac{\partial}{\partial t}\delta p\right\} = 0, \tag{3.5}$$

$$\frac{\partial}{\partial t}\,(n_0^{-\gamma}\,\delta p - \gamma\, p_0\, n_0^{-\gamma-1}\,\delta n)$$

$$+\left(1-\frac{2m}{r}\right)^{1/2}\left\{\left(1-\frac{2m}{r}\right)^{1/2}\delta V^{(r)}\,\frac{\partial}{\partial r}+\frac{\delta V^{(\theta)}}{r}\,\frac{\partial}{\partial\theta}\right\}(p_0\, n_0^{-\gamma})=0. \tag{3.6}$$

Defining the Lagrangian displacement ξ^{α}, ($\alpha = r,\ \theta$) through

$$\delta V^{a}=\frac{\partial\xi^{a}}{\partial t},\qquad \xi^{a}\,(r,\,\theta,\,t)=\xi^{a}\,(r,\,\theta)\exp\,(i\sigma t), \tag{3.7}$$

we obtain from Equations (3.1) – (3.6)

$$\left(\rho_0+\frac{p_0}{c^2}\right)\delta V^{(\phi)} = -\,S_2 V_0\,\delta p/c^2, \tag{3.8}$$

$$\delta\rho = -\left(\rho_0+\frac{p_0}{c^2}\right)\sqrt{S_1}\left[\frac{\sqrt{S_1}}{r^2}\,\frac{\partial}{\partial r}(r^2\,\xi^r)+\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\,\xi^{\theta})\right]$$

$$+\frac{V_0^2}{c^4}\delta p+\sqrt{S_1}\left\{\sqrt{S_1}\,\xi^r\frac{\partial}{\partial r}+\frac{\xi^{\theta}}{r}\frac{\partial}{\partial\theta}\right\}(p_0/c^2), \tag{3.9}$$

$$\delta n = -\,n_0\,\sqrt{S_1}\left[\frac{\sqrt{S_1}}{r^2}\,\frac{\partial}{\partial r}(r^2\,\xi^r)+\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\,\xi^{\theta})\right]$$

$$-\sqrt{S_1}\left[\sqrt{S_1}\,\xi^r\frac{\partial n_0}{\partial r}+\frac{\xi^{\theta}}{r}\,\frac{\partial n_0}{\partial\theta}\right]+\left(\rho_0+\frac{p_0}{c^2}\right)^{-1}\frac{n_0}{c^2}\left[\delta p+\sqrt{S_1}\left\{\sqrt{S_1}\,\xi^r\frac{\partial p_0}{\partial r}\right.\right.$$

$$\left.\left.+\frac{\xi^{\theta}}{r}\,\frac{\partial p_0}{\partial\theta}\right\}-S_2\,\delta p\right], \tag{3.10}$$

$$\delta p=\frac{\gamma p_0}{n_0}\delta n-\sqrt{S_1}\left\{\sqrt{S_1}\,\xi^r\frac{\partial p_0}{\partial r}+\frac{\xi^{\theta}}{r}\,\frac{\partial p_0}{\partial\theta}\right\}+\frac{\gamma p_0}{n_0}\sqrt{S_1}\left\{\sqrt{S_1}\,\xi^r\,\frac{\partial n_0}{\partial r}+\frac{\xi^{\theta}}{r}\frac{\partial n_0}{\partial\theta}\right\}, \tag{3.11}$$

$$-\left(\rho_0+\frac{p_0}{c^2}\right)\sigma^2\,\xi^r=\left(\rho_0+\frac{p_0}{c^2}\right)\frac{2}{r}\,S_1\,V_0\,\delta V^{(\phi)}-S_1\,S_2\,\frac{\partial}{\partial r}\delta p$$

$$-\left(\delta\rho+\frac{\delta p}{c^2}\right)\left[\frac{mc^2}{r^2}-\frac{S_1}{r}\,V_0^2\right]+2S_1\frac{V_0}{c^2}\frac{\partial p_0}{\partial r}\,\delta V^{(\phi)}, \tag{3.12}$$

$$-\left(\rho_0+\frac{p_0}{c^2}\right)\sigma^2\,\xi^{\theta}=\left(\rho_0+\frac{p_0}{c^2}\right)\frac{2}{r}\sqrt{S_1}\cot\theta\;V_0\,\delta V^{(\phi)}-\sqrt{S_1}\,S_2\,\frac{1}{r}\frac{\partial}{\partial\theta}\delta p$$

$$+\left(\delta\rho+\frac{\delta p}{c^2}\right)\frac{V_0^2}{r}\sqrt{S_1}\cot\theta+2\sqrt{S_1}\frac{V_0}{c^2}\frac{1}{r}\frac{\partial p_0}{\partial\theta}\delta V^{(\phi)}, \tag{3.13}$$

wherein

$$S_1 = \left(1 - \frac{2m}{r}\right), \quad S_2 = \left(1 - \frac{V_0^2}{c^2}\right) \tag{3.14}$$

and all the perturbed variables represent only the spatial parts. Equations (3.8)–(3.11) are the initial value equations while (3.12) and (3.13) are the pulsation equations. In the above treatment, we have dropped out those terms which become zero because of the steady-state solutions and also we have integrated the initial-value equations with respect to time. Equation (3.9) together with (3.10) yields

$$\frac{\Delta\rho}{(\rho_0 + p_0/c^2)} = \frac{\Delta n}{n_0}, \tag{3.15}$$

while Equation (3.11) can be rewritten as

$$\frac{\Delta p}{p_0} = \gamma \frac{\Delta n}{n_0} \tag{3.16}$$

in terms of Lagrangian perturbations. From Equations (3.9)–(3.11), we obtain

$$\begin{aligned} \delta p &\left[1 - (\rho + p_0/c^2)^{-1} \frac{\gamma p_0}{c^2} \frac{V_0^2}{c^2}\right] \\ &= - \left[1 - \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1} \frac{\gamma p_0}{c^2}\right] \left(S_1 \, \xi^r \frac{\partial p_0}{\partial r} + \sqrt{S_1} \, \frac{\xi^\theta}{r} \frac{\partial p_0}{\partial \theta}\right) \\ &\quad - \gamma p_0 \left[\frac{S_1}{r^2} \frac{\partial}{\partial r} (r^2 \, \xi^r) + \frac{\sqrt{S_1}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \; \xi^\theta)\right], \end{aligned} \tag{3.17}$$

$$\begin{aligned} \delta \rho &\left[1 - \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1} \frac{\gamma p_0}{c^2} \frac{V_0^2}{c^2}\right] \\ &= - \left(\rho_0 + \frac{p_0}{c^2}\right) \left[\frac{S_1}{r^2} \frac{\partial}{\partial r} (r^3 \, \xi^r) + \frac{\sqrt{S_1}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \; \xi^\theta)\right] \\ &\quad + \frac{S_2}{c^2} \left(S_1 \, \xi^r \frac{\partial p_0}{\partial r} + \sqrt{S_1} \, \frac{\xi^\theta}{r} \frac{\partial p_0}{\delta \theta}\right). \end{aligned} \tag{3.18}$$

The problem is then to solve Equations (3.12) and (3.13) as the eigen-value equations, consistently with the initial value equation (3.17) and (3.18) and the appropriate boundary conditions.

As we did in Paper II, we define ‘ trial displacements’ $\overline{\xi^r}$ and $\overline{\xi^\theta}$, multiply Equation (3.12) by $\overline{\xi^r}$ and Equation (3.13) by $\overline{\xi^\theta}$, add and integrate over the range of r and θ. In order to bring the resultant expression in a symmetrical form in barred and unbarred variables, we limit ourselves to the class of perturbations such that $\delta p = 0$ at the boundary of the disk. This in turn requires that both ξ^r and ξ^θ be zero at the boundary. Performing several integrations by parts and using Equation (3.8) and the steady state equations we finally obtain

$$\sigma^2 \iint \frac{1}{S_2}\left(\rho_0 + \frac{p_0}{c^2}\right)(\bar{\xi}^r \xi^r + \bar{\xi}^\theta \xi^\theta)\, r^2 \sin\theta \, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \iint \Bigg[S_1\left(\bar{\xi}^r \frac{\partial}{\partial r}\delta p + \xi^r \frac{\partial}{\partial r}\overline{\delta p}\right) + \frac{\sqrt{S_1}}{r}\left(\bar{\xi}^\theta \frac{\partial}{\partial\theta}\delta p + \xi^\theta \frac{\partial}{\partial\theta}\overline{\delta p}\right)$$

$$+ \frac{2m}{r^2}(\bar{\xi}^r \delta p + \xi^r \overline{\delta p})$$

$$+ \frac{V_0^2}{c^4}\left(\rho_0 + \frac{p_0}{c^2}\right)^{-2}\left(S_1 \xi^r \frac{\partial p_0}{\partial r} + \sqrt{S_1}\frac{\xi^\theta}{r}\frac{\partial p_0}{\partial\theta}\right)\left((S_1 \bar{\xi}^r \frac{\partial p_0}{\partial r} + \sqrt{S_1}\frac{\bar{\xi}^\theta}{r}\frac{\partial p_0}{\partial\theta}\right)$$

$$+ \frac{\gamma p_0 S_2}{c^2}\left(\rho_0 + \frac{p_0}{c^2}\right)^{-2}\Bigg\{\delta\rho\left(S_1 \bar{\xi}^r \frac{\partial p_0}{\partial r} + \frac{\sqrt{S_1}\,\bar{\xi}^\theta}{r}\frac{\partial p_0}{\partial\theta}\right)$$

$$+ \overline{\delta\rho}\left(S_1 \xi^r \frac{\partial p_0}{\partial r} + \frac{\sqrt{S_1}\,\xi^\theta}{r}\frac{\partial p_0}{\partial\theta}\right)\Bigg\}$$

$$- \frac{\gamma p_0}{(\rho_0 + p_0/c^2)^2}\left\{1 - \frac{\gamma p_0/c^2}{(\rho_0 + p_0/c^2)}\frac{V_0^2}{c^2}\right\}\delta\rho\,\overline{\delta\rho}\Bigg] r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta, \tag{3.19}$$

where $\overline{\delta\rho}$ and $\overline{\delta p}$ are variations in perturbed density and pressure obtained by using the trial displacements in initial-value equations. As it was shown in Paper II the symmetrical expression of σ^2 implies a variational principle: identifying barred variables with the unbarred ones in Equation (3.19), we write the expression for σ^2 and calculate σ^2 by using two trial displacements ξ^α and $\xi^\alpha + \delta\, \xi^\alpha$. If we now demand that the resultant variation $\delta\sigma^2$ in σ^2 is zero, then it amounts to solving the original eigenvalue equations (3.12) and (3.13). To calculate the critical value of adiabatic index for neutral stability we limit ourselves to the situations where $(\gamma p_0/c^2)(V_0^2/c^2) \times (\rho_0 + p_0/c^2)^{-1}$ is very small compared to unity so that we can write Equations (3.17) and (3.18) in the form

$$\delta p = -\Bigg[\left\{1 - \frac{\gamma p_0/c^2}{(\rho_0 + p_0/c^2)}\right\}\left\{S_1 \xi^r \frac{\partial p_0}{\partial r} + \sqrt{S_1}\frac{\xi^\theta}{r}\frac{\partial p_0}{\partial\theta}\right\}$$

$$+ \gamma p_0 \left\{\frac{S_1}{r^2}\frac{\partial}{\partial r}(r^2 \xi^r) + \frac{\sqrt{S_1}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, \xi^\theta)\right\}\Bigg]\left[1 + \frac{\gamma p_0/c^2}{(\rho_0 + p_0/c^2)}\frac{V_0^2}{c^2}\right] \tag{3.20}$$

and

$$\delta\rho = -\left[\left(\rho_0 + \frac{p_0}{c^2}\right)\left\{\frac{S_1}{r^2}\frac{\partial}{\partial r}(r^2\,\xi^r) + \frac{\sqrt{S_1}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)\right\}\right.$$
$$\left. -\frac{S_2}{c^2}\left(S_1\,\xi^r\,\frac{\partial p_0}{\partial r} + \sqrt{S_1}\,\frac{\xi^\theta}{r}\,\frac{\partial p_0}{\partial\theta}\right)\right]\left[1 + \frac{\gamma p_0/c^2}{(\rho_0 + p_0/c^2)}\,\frac{V_0^2}{c^2}\right], \tag{3.21}$$

and using these we obtain

$$\frac{m^2\,\sigma^2}{c^2}\iint \frac{1}{S_2}(\rho_0 + p_0/c^2)\,(\xi^{r2} + \xi^{\theta 2})\,R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta$$
$$= \iint\left[\frac{V_0^2/c^2}{(\rho_0 + p_0/c^2)}\,T_2^2 - 2S_1\,\xi^r\,\frac{\partial T_2}{\partial R} - \frac{2\sqrt{S_1}}{R}\,\xi^\theta\,\frac{\partial T_2}{\partial\theta} - \frac{4}{R^2}\,\xi^r\,T_2\right]R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta$$
$$+ \gamma\iint\left[\left(\frac{2\sqrt{S_1}}{R}\right)\left(R\sqrt{S_1}\,\xi^r\,\frac{\partial}{\partial R} + \xi^\theta\,\frac{\partial}{\partial\theta}\right)\left(-T_1\frac{p_0}{c^2} + S_2\,S_3\,T_2\right)\right.$$
$$- \frac{4}{R^2}\,\xi^r\left(\frac{p_0}{c^2}T_1 - S_2\,S_3\,T_2\right) - 2S_2\,S_3\,[T_1\,T_2 - S_2\,T_2^2\,(\rho_0 + p_0/c^2)^{-1}]$$
$$\left. - S_3\left\{\left(\rho_0 + \frac{p_0}{c^2}\right)T_1^2 + S_2^2\,T_2^2\,(\rho_0 + p_0/c^2)^{-1} - 2S_2\,T_1\,T_2\right\}\right]R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta$$
$$+ \gamma^2\iint\left[\left[-2S_1\xi^r\frac{\partial}{\partial R}(S_5T_1\,p_0/c^2) + 2S_1\xi^r\,\frac{\partial}{\partial R}(S_4^2T_2)\right.\right.$$
$$- \frac{2\sqrt{S_1}\xi^\theta}{R}\,\frac{\partial}{\partial\theta}(S_5\,T_1\,p_0/c^2) + \frac{2\sqrt{S_1}\,\xi^\theta}{R}\,\frac{\partial}{\partial\theta}(S_4^2\,T_2) + \frac{4}{R^2}\,\xi^r\left[\left(S_4^2\,T_2 - \frac{p_0}{c^2}\,T_1\,S_5\right)\right]$$
$$- 2\,S_1\,S_3\left\{S_5\,T_1\,T_2 - S_2\,S_5\,T_2^2\left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}\right\}$$
$$+ S_3\,S_5\,\left\{\left(\rho_0 + \frac{p_0}{c^2}\right)T_1^2 + \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}S_2^2\,T_2^2 - 2\,S_2\,T_1\,T_2\right\}$$
$$\left. - S_3\left\{2\left(\rho_0 + \frac{p_0}{c^2}\right)S_5\,T_1^2 + \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}2\,S_2^2\,S_5\,T_2^2 - 4\,S_5\,S_2\,T_1\,T_2\right\}\right]$$
$$+ \gamma^3\left[-S_3\,\left\{\left(\rho_0 + \frac{p_0}{c^2}\right)S_5^2\,T_1^2 + \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}S_5^2\,S_2^2\,T_2^2 - 2\,S_5^2\,T_1\,T_2\,S_2\right\}\right.$$
$$\left. + S_3\,S_5\,\left\{2\left(\rho_0 + \frac{p_0}{c^2}\right)S_5\,T_1^2 + \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}2\,S_2^2\,S_5\,T_2^2 - 4\,S_5\,S_2\,T_1\,T_2\right\}\right]$$
$$\left.+ \gamma^4\left[S_3\,S_5\,\left\{\left(\rho_0 + \frac{p_0}{c^2}\right)S_5^2\,T_1^2 + \left(\rho_0 + \frac{p_0}{c^2}\right)^{-1}S_5^2\,S_2^2\,T_2^2 - 2\,S_2\,S_5^2\,T_1\,T_2\right\}\right]\right]$$
$$\times\, R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta, \tag{3.22}$$

wherein

$$S_3 = \frac{p_0}{c^2} \Big/ \left(\rho_0 + \frac{p_0}{c^2}\right), \quad S_5 = \frac{V_0}{c} S_4 = \frac{V_0^2}{c^2} S_3,$$

$$T_1 = \frac{S_1}{R^2} \frac{\partial}{\partial R} (R^2 \xi^r) + \frac{\sqrt{S_1}}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \ \xi^\theta),$$

$$T_2 = S_1 \xi^r \frac{\partial p_0}{\partial R} + \sqrt{S_1} \frac{\xi^\theta}{R} \frac{\partial}{\partial \theta} \left(\frac{p_0}{c^2}\right). \tag{3.23}$$

We chosse a function q as

$$q = \frac{1}{\sin^2 \theta} - \frac{R^2 [B^2 (1 - 2/R)^{-1} - 1]}{AB^2} \tag{3.24}$$

Which is zero at the boundary of the disk and the take

$$\xi^r = q + \alpha q^2, \quad \xi^\theta = q + \beta q^2 , \tag{3.25}$$

wherein α and β are constants determined by extremising σ^2 as calculated by using these trial displacements in Equation (3.22). For such choice of ξ^r and ξ^θ we calculate critical value γ_c of the adiabatic index for the neutral stability.

Table 1 shows the values of γ_c for the onset of instability ($\gamma < \gamma_c$ for instability) for different values of 'a' and 'b' for general-relativistic as well as for the Newtonian case. It turns out that the coefficients of γ^2, γ^3 and γ^4 on the right-hand side of Equation (3.22) are very small as compared to the first two terms and therefore in the calculation of γ_c we can drop them out. The critical g for the Newtonian case is calculated by taking the limit $c \to \infty$, of Equations (3.17), (3.18) and (3.19). In this case we obtain

$$m^2\sigma^2 \iint \rho_0 [(\xi^r)^2 + (\xi^\theta)^2] R^2 \sin \theta \, dR \, d\theta$$
$$= - \iint \left\{ 2 \left(\xi^r \frac{\partial T_2}{\partial R} + \frac{\xi^\theta}{R} \frac{\partial T_2}{\partial \theta} \right) + \gamma \left[2 \left(\xi^r \frac{\partial}{\partial R} + \frac{\xi^\theta}{R} \frac{\partial}{\partial \theta} \right) \left(\frac{p_0 T_1}{c^2} \right) + \frac{p_0}{c_2} T_1^2 \right] \right\} R^2 \sin \theta \, dR \, d\theta, \tag{3.26}$$

Table 1. Values of γ_c and $\theta_{e\,(\min)}$ for different choices of a and b.

		General relativistic		Newtonian	
a	b	γ_c	$\theta_{e\,(\min)}$	γ_c	$\theta_{e\,(\min)}$
8·1	100	0·7488	0·63	0·7107	0·55
7·1	100	0·7758	0·61	0·7420	0·52
6·1	100	0·8007	0·56	0·7732	0·48
5·1	100	0·8214	0·56	0·8043	0·44
4·1	100	0·8320	0·55	0·8356	0·40
4·1	140	0·8850	0·47	0·8726	0·33
4·1	180	0·9155	0·42	0·8936	0·30
4·05	180	0·9155	0·42	0·8948	0·30

where

$$T_1 = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\,\xi^r) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)$$

$$T_2 = \left(\xi^r\frac{\partial}{\partial R} + \frac{\xi^\theta}{R}\frac{\partial}{\partial\theta}\right)(p_0/c^2), \tag{3.27}$$

with the steady state solutions as

$$\frac{V_0^2}{c^2} = \frac{A}{R^2\sin^2\theta} \qquad ; \quad A = \frac{2ab}{(a+b)},$$

$$\frac{p_0}{c^2} = \rho_0\left[\frac{1}{R} - \frac{A}{2R^2\sin^2\theta} + B\right]; \quad B = -\frac{1}{(a+b)}. \tag{3.28}$$

We note that the σ^2 equation obtained here for the Newtonian case has a different form than that reported in Paper I. This is because of the different boundary conditions used for ξ^r and ξ^θ in the two sets of calculations.

As a special case we find that in case $p_0 = 0$, the disk collapses to $\theta = \frac{\pi}{2}$ plane, rotating with velocity

$$V_0 = \left[\frac{mc^2}{r}\left(1 - \frac{2m}{r}\right)^{-1}\right]^{1/2} \tag{3.29}$$

as may be seen from Equations (2.8) and (2.9) . Considering further the radial oscillations of such disk $\xi^\theta = 0$, $\xi^r \neq 0$ with $\delta p = 0$, we have

$$\delta V^{(\phi)} = -\left[\left(1 - \frac{2m}{r}\right)\frac{\partial V_0}{\partial r} + \frac{1}{r}\left(1 - \frac{3m}{r}\right)V_0\right]\xi^r \tag{3.30}$$

$$\sigma^2\,\xi^r = \frac{2}{r}\left(1 - \frac{2m}{r}\right)V_0\,\delta V^{(\phi)} \tag{3.31}$$

as the equations governing the radial perturbations appropriate to this case. Combining these, we get

$$\sigma^2 = \frac{mc^2}{r^4}(r - 6m) \tag{3.32}$$

which shows that such disks are stable for $r > 6m$.

4. Results and discussion

The steady-state parameters of velocity and pressure as a function of radial distance along the equatorial plane for a constant-density thick disk rotating around a Schwarzschild black hole is presented in Fig. 1 while Fig. 2 shows the meridional

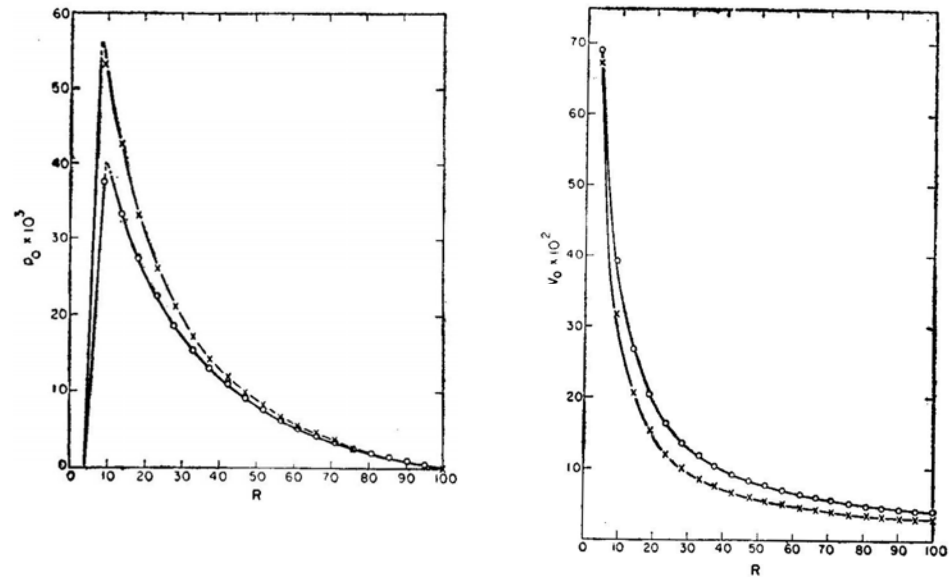

Figure 1. Profiles of pressure (left) and velocity (right) for relativistic (circles) and for Newtonian (crosses) disks along the equatorial plane α=4·1, b=100.

section of such a disk. The corresponding plots in the Newtonian formulation for the same values of inner and outer radii a and b (and for a constant density mode) are also shown in these figures. We find that for the same a and b the Newtonian disk occupies more volume than the relativistic one. It seems that the relativistic disks show a formation of cusp at the inner edge specially when it is near 4. For a Newtonian disk the pressure at any point is higher while the velocity at any point is lower at the equatorial plane, than in the case of a relativistic disk.

For the case of a relativistic disk we find a constraint that the inner edge cannot be less than 4*. Further, if $4 < a < 6$, then $b = 2a/(a-4)$. For $a \geqslant 6$, any $b > a$ gives rise to a plausible disk. No such restriction appeared in the Newtonian formulation indicating a pure general-relativistic origin of the present constraint.

From the values of γ_c as tabulated in Table 1 we find that the disks considered here represent stable configurations ($\gamma_c < 4/3$). In calculating γ_c through Equation (3.22) we have used the approximation that $(V_0^2/c^2)\,(p_0/c^2)\,(\rho_0 + p_0/c^2)^{-1} \ll 1$, which is quite justified from the values of V_0/c and p_0/c as we obtained. There is a qualitative agreement between the γ_c calculated for relativistic and the corresponding Newtonian disks. In these two cases, although the inner and outer radii a and b are the same, the regions occupied by the disk in the two cases are not the same. In general, Newtonian disks are thicker [minimum angular elevation$= \pi - 2\theta_e$ (min)]. We also find from the numbers that γ_c depends upon the size of the disk. In the calculation of γ_c, the effects due to general relativistic convections and that due to the difference in sizes contribute simultaneously and therefore the agreement between the general relativistic γ_c and the Newtonian γ_c is no better than a qualitative one.

* The fact that the inner edge r_α of the disk should always be greater than $4m$ is consistent with the general proof given by Kozlowsky, Jaroszynski and Abramowicz (1978) that $R_i \geqslant R_{mb}$ for the marginally bound orbit.

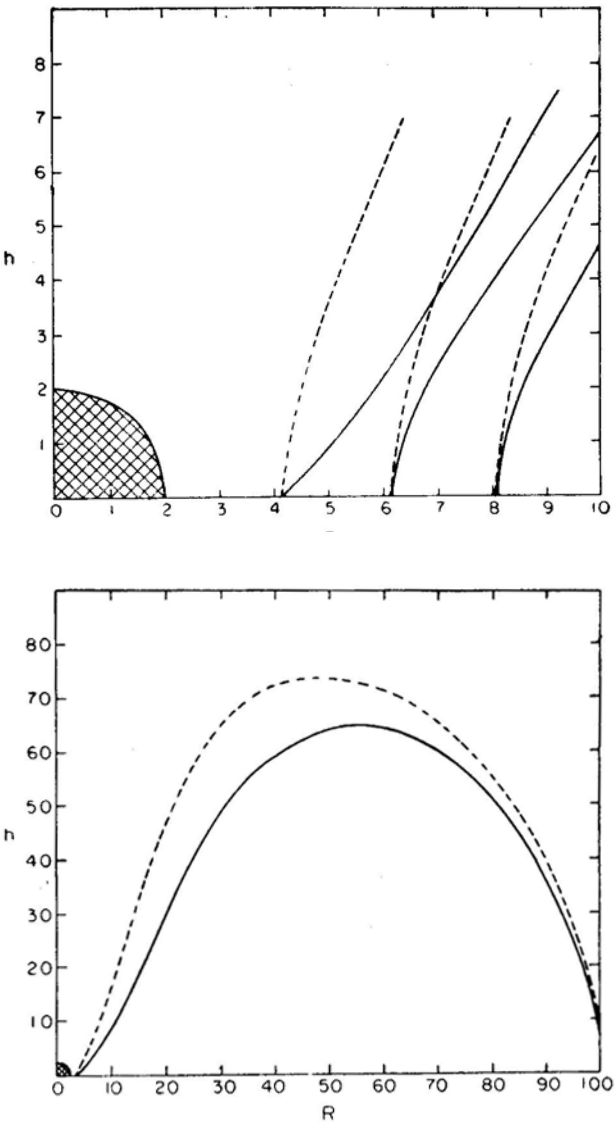

Figure 2. Meridional section of the disk in general relativistic (solid curves) and of corresponding Newtonian case (dashed curves). Top: Various choices of a, b = 100; inner portion of the disk Bottom: $a = 4{\cdot}1$, $b = 100$.

It does not seem to be possible to identify separately the contributions from general relativity, in the present formulation.

Starting from the general equations, if we take $p_0 = 0$, we found that the disk collapses on to $\theta = \pi/2$ plane and is stable under radial perturbation only if the inner edge is beyond $6m$ as its local frequency is $[mc^2\,(r - 6m)/r^4]^{1/2}$. Now, since a pressureless fluid is essentially an aggregate of non-interacting particles, the above conclusion can be regarded as an alternative derivation of the well-known result that the last stable circular orbit for Schwarzschild geometry is at $6m$.

The general conclusion that the perfect-fluid thick disk rotating around Schwarzschild black hole is generally stable under axisymmetric perturbation, at least when the density is constant (or a function of r when considered in Newtonian framework) may have important significance in the study of the models of accretion disks developed for explaining the radiation from high-energy sources.

Acknowledgement

One of us (DKC) would like to thank UGC for the award of Teacher Fellowship and Director, PRL, for the hospitality during his stay at Ahmedabad.

References

Chakraborty, D. K., Prasanna, A. R. 1981, *J. Astrophys. Astr.*, **2,** 421 (Paper II).
Chandrasekhar, S., Friedman, J. L. 1972a, *Astrophys. J.*, **175,** 379.
Chandrasekhar, S., Friedman, J. L. 1972b, *Astrophys. J.*, **176,** 745.
Kozlowsky, M., Jaroszynski, M., Abramowicz, M.A. 1978, *Astr. Astrophys.*, **63,** 209.
Lightman, A. P., Shapiro, S. L., Rees, M. J. 1978, in *Physics and Astrophysics of Neutron Stars and Black Holes*, Eds R. Giacconi and R. Ruffini, North Holland, Amsterdam, p. 786.
Novikov, I. D., Thorne, K. S. 1973, in *Les Houches: Black Holes*, Eds C. DeWitt and B. S. DeWitt, Gordon and Breach, New York.
Paczynski, B. 1980, *Acta Astr.*, **30.4,** 347.
Prasanna, A. R., Chakraborty, D. K. 1981, *J. Astrophys. Astr.*, **2,** 1 (Paper I).
Pringle, J. E., Rees, M. J. 1972, *Astr. Astrophys.*, **21,** 1.
Shakura, N. I., Sunyaev, R. A. 1973, *Astr. Astrophys.*, **24,** 337.
Shakura, N. I., Sunyaev, R. A. 1976, *Mon. Not. R. astr. Soc.*, **175,** 613.