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Gravitational Lensing by Stars in a Galaxy Halo: Theory of Combined Weak and Strong Scattering

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(*Invited article)*

Abstract. The theory of gravitational lensing of background quasars by stars in the halo of a galaxy is considered. In the limiting case of small 'optical depth', only one star is close enough to the beam to cause strong scattering, and the effect of all the other stars is treated as a perturbation with both systematic and random components. The perturbation coming from weak scattering can increase the number of images and the amplification in those cases where the amplification is already high; such events are preferentially selected in flux limited observations. The theory is applicable to the apparent association of background quasars with foreground galaxies. A comparison with earlier work on the same problem is given. The relevance of these results to gravitational lensing by galaxies as perturbed by random inhomogeneities surrounding the ray path is also briefly discussed.

Key words: gravitation lens—galaxy halos—quasars

1. Introduction

The possibility that the luminosity function of quasars derived from observations could be significantly influenced by gravitational lensing was originally proposed by Barnothy & Barnothy (1968) and more recently considered by Turner (1980). Avni (1981) and Peacock (1982) have subsequently examined this problem, including small effects due to the deamplification which is implied by flux conservation. The fluctuations in the intensity of the gravitationally lensed image of the quasar 0957 +561 due to 'mini-lensing' by stars close to the beam were considered by Chang & Refsdal (1979) and Gott (1981). The latter emphasised the unique opportunity this affords to detect low mass ($\sim 0.001\ M_{\odot}$) objects in galactic halos. Yet another aspect of lensing by stars in galactic halos was pointed out by Canizares (1981), *viz.*, that it could produce an apparent association between quasars and foreground galaxies. Vietri & Ostriker (1983) have reconsidered this problem, including not only the effects of individual stars but also that of the galaxy as a whole as well as flux conservation. They also introduced the very useful concept of an optical depth τ for significant amplification (defined below). All the problems described above centre around the distribution of amplifications produced by an encounter with a single galaxy. The basic concepts involved are: (1) lensing by individual stars; (2) superposition of the weak amplification caused by many distant encounters with strong amplification caused by a close encounter;

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(3) effect of the smoothed-out potential of the galaxy; (4) the deamplification required by flux conservation. The present analysis of the problem closely follows Vietri & Ostriker (1983, hereafter VO). They use a formalism similar to radiative transfer and treat the case $\tau \ll 1$. As discussed in the next section, we take an alternative view of the same problem which clarifies and extends their work. Most of the new results concern situations in which weak lensing effects are combined with a strong event. We find that such weak events do not simply superpose with the strong ones but can have a. significant, even surprisingly large, effect on the total amplification and on the number and geometry of the images. As Turner, Ostriker & Gott (1984; hereafter TOG) have shown, lensing events in a flux-limited sample are predominantly those with high amplification, and hence they are the ones where the effects discussed in this paper could be important.

The plan of this paper is as follows. In Section 2 we discuss two alternative points of view on gravitational lensing to clarify the issues of deamplification and flux conservation. Section 3 reviews the basic definitions, notation, and equations. Section 4 treats the superposition of weak and strong amplifications and Section 5 includes the smooth potential of the galaxy. Section 6 is a discussion and summary.

2. Filled and empty beams, flux conservation and negative amplification

There are two equivalent viewpoints in gravitational lensing which differ in the zero order description of the propagation of light. In the first, which we call the 'filled-beam' approach, the matter is first smoothed out to make the cosmology truly homogenous. One then considers under and over-densities which cause a beam of light to diverge or converge giving deamplification or amplification with respect to the flux calculated for a homogenous universe. In the alternative 'empty-beam' approach, the starting point is an evacuated tube in an otherwise homogenous universe. If the matter in the real universe is sufficiently lumpy so that we view at distant sources via empty regions, this may be a better first approximation. Zeldovich (1964) and Feynman (quoted by Gunn 1967) pointed out that this reduces the angular size and flux with respect to the filled-beam case. Gunn (1967) discussed the fluctuations produced by the discreteness of the matter outside the beam. In their classic paper on the lensing effects of a cosmological distribution of point masses, Press & Gunn (1973) used the empty beam approach. In this approach all density fluctuations are positive and act to increase the observed flux.

It is of course possible that a substantial fraction of the mass density in the universe is more smoothly distributed than the galaxies. One can then use a tube in which the density of galaxies alone has been removed. For reference, we give the deamplification as a function of redshift for an $\Omega = 1$ universe in which a fraction Ω_g of the density has been removed from a tube enclosing the beam. With the definitions

$$n_a \equiv [1-(1+24\Omega_g)^{1/2}]/2; \quad n_b \equiv [1+(1+24\Omega_g)^{1/2}]/2, \tag{1}$$

the effect of removing a fractional density Ω_g is to multiply the flux from point sources by the factor $A_g (A_g < 1)$ given below (Dashevskii & Slysh 1965)

$$A_g = \frac{(n_b - n_a)^2[(1+z_q)^{1/2}-1]^2}{[(1+z_q)^{n_b/2}-(1+z_q)^{n_a/2}]^2}. \tag{2}$$

For an empty tube ($\Omega_g = 1$), this takes the simpler form (Zeldovich 1964)

$$A_g = \frac{25[(1+z_q)^{1/2} - 1]^2}{[(1+z_q)^{3/2} - (1+z_q)^{-1}]^2}. \tag{3}$$

To take a somewhat extreme example, an empty tube in an $\Omega = 1$ universe produces deamplification by a factor 0.42 at a redshift of 3. So long as we are dealing with smaller redshifts, it is clear that deamplification effects amount to a fraction of a magnitude and cannot be more important than the comparable or greater uncertainties which are already present in the quasar luminosity function. The effect of this deamplification on the observed surface density of quasars also depends on the slope of the relation between number counts and apparent magnitude (VO). Typically, the fractional change in the number counts at a given magnitude due to deamplification is less than the fractional flux change. In the association problem considered by Canizares (1981), one compares the density of quasars in two fields one of which is close to a foreground galaxy. The deamplification is common to both the fields and hence does not affect the result.

The same empty and filled beam approaches also apply to lensing by stars in galaxies. VO use the filled beam approach and represent the galaxy by a smooth mass distribution. Individual stars are then represented by point masses with surrounding compensating negative mass spheres to avoid double counting. In the present paper, we use the empty beam approach and thus deal only with overdensities and amplification.

3. Review of lensing definitions and equations

We briefly review the basic equations governing lensing, following the treatment of Young (1981). The geometry of a gravitational lensing event is shown in Fig. 1. The

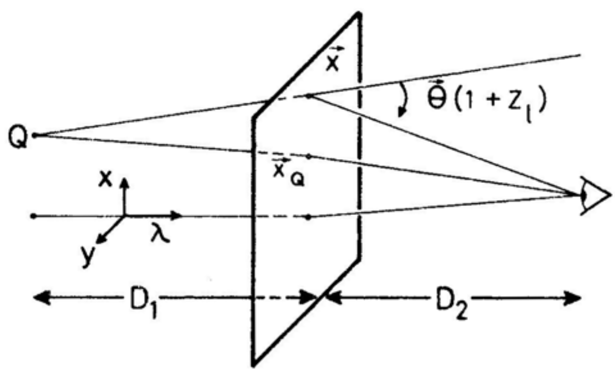

Figure 1. Geometry of a gravitational lensing event. Rays from the background source (quasar) at Q at redshift z_q undergo a proper vector deflection θ at a redshift z_1 and propagate to the observer. x and y are transverse proper distances. x_Q is the intercept on the deflector plane of the unperturbed ray from Q to the observer. D_1 and D_2 are affine distances.

source (quasar) is located at a red shift z_q and the lensing object (also called the deflector) at a red shift z_l We use proper distances in the deflector plane as Cartesian coordinates x, y. The unperturbed ray connecting the quasar and the observer in the absence of gravitational lensing cuts the deflector plane at x_q, y_q. The deflection at a point x, y in the deflector plane is a vector θ whose components θ_x and θ_y are expressed in terms of the Newtonian gravitational potential ϕ (x, y, z) in the weak field limit (Bourassa & Kantowski 1975).

$$\begin{aligned} \theta_x &= -\frac{2}{c^2}\frac{\partial}{\partial x}\int \phi(x, y, z)\,\mathrm{d}z \equiv -\frac{2}{c^2}\frac{\partial}{\partial x}\Phi(x, y), \\ \theta_y &= -\frac{2}{c^2}\frac{\partial}{\partial y}\int \phi(x, y, z)\,\mathrm{d}z \equiv -\frac{2}{c^2}\frac{\partial}{\partial y}\Phi(x, y). \end{aligned} \tag{4}$$

We have denoted the integral of the gravitational potential along the line of sight (z direction) by $\Phi(x, y)$ in (2). This is related to the surface density Σ (x, y) by Poisson's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 4\pi G\Sigma(x, y). \tag{5}$$

Strictly speaking, the potential due to a two-dimensional mass distribution suffers from a logarithmic divergence if it is required to vanish as $x, y \to \infty$. We can cure this by choosing any (finite) point as the zero of potential, without affecting the gradient of the potential which determines the deflection of a ray. This 'two-dimensional' view of lensing is valid so long as the extent of the deflector along the line of sight is much smaller than its distance from either the source or observer. Turner, Ostriker & Gott (1984) have shown how a uniform sheet of matter at a different red shift from a given galaxy can be replaced by an equivalent sheet at the same red shift so that the two-dimensional picture can then be used.

In the 'empty-beam' approach, the rays are regarded as travelling in an evacuated tube in the universe, with deflections by galaxies and stars put in through Equation (4). Press & Gunn (1973) have introduced a particularly convenient coordinate system (shown in Fig. 1) in which distances along the line of sight are measured in terms of the affine parameter* λ. In the transverse direction, the proper lengths x and y are used. Of course, it is assumed that x and y are much smaller than characteristic distances along the ray. In this coordinate system, light rays close to the axis are straight lines so long as there is circular symmetry around the beam and no matter. In a Friedman cosmology with a scale factor $a(t)$, we have $\mathrm{d}\lambda = ca(t)\mathrm{d}t$. The affine length normalized in this way agrees with the proper distance at the present epoch, and is less than it by the factor $1 + z$ at earlier epochs. The angles in Fig. 1 are thus $1 + z$ times proper angles. From the geometry of Fig. 1, we have the following relation between x, the image position, and x_q, the unperturbed quasar position.

$$(\mathbf{x} - \mathbf{x}_q)(1/D_1 + 1/D_2) = -\boldsymbol{\theta}(1 + z_l). \tag{6}$$

It is convenient to define the effective distance D by

$$(1 + z_l)/D \equiv 1/D_1 + 1/D_2. \tag{7}$$

* The properties of the affine parameter are treated in the texts by Misner, Thorne & Wheeler (1972) and Hawking & Ellis (1973).

For a given quasar position x_q one can solve for the values of x, each corresponding to an image of the source. Further, the Jacobian of x with respect to x_q gives the amplification A. Young (1981) has used the concepts of shear κ and convergence γ of an infinitesimal bundle of rays (defined below) to write the amplification A in a simple form.

$$\gamma_1 \equiv -\frac{D}{2}\left(\frac{\partial\theta_x}{\partial x} - \frac{\partial\theta_y}{\partial y}\right); \quad \gamma_2 \equiv -\frac{D}{2}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right); \tag{8}$$

$$\gamma^2 = \gamma_1^2 + \gamma_2^2; \quad \kappa \equiv -\frac{D}{2}\left(\frac{\partial\theta_x}{\partial x} + \frac{\partial\theta_y}{\partial y}\right); \quad A = [(1-\kappa)^2 - \gamma^2]^{-1}. \tag{9}$$

We can substitute the expression (4) for the deflection into the definition (8) of the convergence, which turns out to be proportional to the two-dimensional Laplacian (5) of the potential, *i.e.*, to the surface density Σ (x, y). The convergence thus vanishes when there is no matter in the beam. The shear γ represents tidal deformation produced by matter not in the beam. Note that κ and γ^2 are scalars under rotation by an angle α in the x-y plane while γ_1 and γ_2 transform as components of a second-rank symmetric tensor, *viz.*

$$\gamma_1' = \gamma_1 \cos 2\alpha + \gamma_2 \sin 2\alpha, \quad \gamma_2' = -\gamma_1 \sin 2\alpha + \gamma_2 \cos 2\alpha. \tag{10}$$

It is also clear from (9) that for weak scattering ($\kappa \ll 1$, $\gamma \ll 1$), the convergence makes a first-order contribution to the amplification while the shear appears only in the second order, since it involves expansion in one direction and contraction in the perpendicular one.

We now review briefly the lensing properties of a point mass (VO give more details). There is a critical radius r_0 in the deflector plane given in terms of the Schwarzschild radius r_s and the effective distance D by

$$r_0^2 = 2r_s D. \tag{11}$$

When the unperturbed ray approaches within r_0 of the point mass, the amplification is significant (greater than 1.34) and the two images become of comparable intensity (the intensity ratio is less than 6.9). Thus πr_0^2 is a natural cross section for a strong lensing event. VO introduce the optical depth τ in terms of the density n of stars by

$$d\tau = \pi r_0^2 n \, dl.$$

When all the lensing matter lies at the same red shift, r_0 is effectively constant and the optical depth can then be expressed in terms of the surface density n_s of stars

$$\tau \equiv n_s \sigma_0 = n_s \pi r_0^2. \tag{12}$$

For small optical depth ($\tau \ll 1$), circles of radius r_0 around the projected position of each star in the x-y plane do not overlap and are in fact separated typically by $\sim \tau^{-1/2} r_0$ which is greater than r_0. For a $\tau \ll 1$, strong lensing event is due to the beam passing within $\sim r_0$ from a single star. The effect of the other stars can be treated as a perturbation (see Section 4 below).

The finite angular size of the source and the breakdown of geometrical optics can both limit the amplification produced by lensing. This was discussed briefly by Gott (1981) and VO; we go into some more details here. For high amplification A, the closest approach r of the unperturbed ray is less than the critical radius r_0 and we have $A = r_0/r$.

To ensure that the amplification does not vary appreciably over an extended source for which r varies by Δr we must have $\Delta r < fr$ where f is a safety factor less than 1. This defines a critical linear size in the deflector plane, below which we obtain the same amplification as for a point source.

$$\Delta r < fr_0 / A. \tag{13}$$

Gott (1981) and VO used the weaker condition

$$\Delta r < fr_0/A^{1\,2},$$

which implicitly assumes that the amplification is isotropic. For a point mass the amplification is entirely in the tangential direction and our condition (11) applies. This correction raises by a factor A, the minimum mass of a star which can act as a 'mini-lens' for a quasar of a given angular size. The interpolation formula given by Peacock (1982) for the amplification of extended sources is consistent with the criterion (13) since the maximum amplification scales inversely as the source size.

We now consider the possible breakdown of geometrical optics which involves two distinct effects. A ray connecting the source and the observer represents a path of stationary phase. We can draw the so-called Fresnel zone (*cf.* Born & Wolf 1975) in the deflector plane with a radius r_F around each image. Paths passing within this zone differ by less than half a wavelength and hence contribute significantly to the observed intensity. In terms of the effective distance D and the observed wavelength W_0, we have

$$r_F = [Dw_0/(1+z_l)]^{1/2}. \tag{14}$$

The geometry of point-mass lensing shows that an event with amplification A involves rays which pass within $r_0/2A$ of the critical radius r_0. To ensure that the amplification does not vary significantly over r_F, we need

$$fr_0/2A > r_F$$

or equivalently

$$w_0/(1+z_l) < f^2 r_0^2/4A^2 D = f^2 r_S/2A^2. \tag{15}$$

VO give a similar criterion for a typical lensing event ($A - 1$ of the order of unity). It is interesting that apart from red shift and amplification-dependent factors, the scale of wavelengths is set by the Schwarzschild radius of the lensing mass. We note, as a consequence of (15), that geometrical optics breaks down more easily, *i.e.*, at shorter wavelengths, for high amplification events. We note that even for high amplification ($A \sim 10$) and small masses ($10^{-2}\ M_\odot$) the criterion (15) is comfortably fulfilled upto decametre wavelengths and geometrical optics is valid.

Even after fulfilling the condition (15) which guarantees the validity of geometrical optics for the intensity of each ray, we can get interference effects between the two images for a source of small enough angular size. From Fig. 1, the two rays part at the angle $2r_0/D_1\,(1+z_q)$ at the source. They will be incoherent only for wavelengths Satisfying

$$\frac{w_0}{1+z_q} < \frac{2r_q \cdot 2r_0}{D_1(1+z_q)}, \tag{16}$$

where $2r_q$ is the linear size of the source. This is basically the same as Gott's (1981) condition that the gravitational lens of size $2r_0$ should be able to resolve the source of angular size $2r_0/D_1$. For smaller source or lens sizes, one can obtain fringes—but subject to one more condition. Given a path difference between the two rays of n wavelengths, one needs a fractional bandwidth of less than $1/n$. The path difference is directly related to the time delay between the two images which has been calculated by Refsdal (1964). Using this result we find

$$n = (1+z_1)\, r_s / A w_0. \tag{17}$$

Comparing (17) and (15), we see that the fractional bandwidth has to be less than a critical value to get two beam interference.

$$\Delta w_0/w_0 < f_2/2A.$$

4. Superposition of weak and strong amplifications

Gott (1981) has shown that rays from background objects which are not doubly imaged by an isothermal galaxy encounter an optical depth due to individual stars that is less than 1/4. The condition $\tau \ll 1$ applies to all the cases of interest in this paper. We thus have two possibilities: (1) rays which pass at a distance significantly greater than r_0 from the projected position of all stars, (2) rays which pass one star at a distance $\sim r_0$ and hence typically at $> r_0$ from all the others. In case (1) we are dealing with the superposition of weak amplifications. VO (in their Appendix A) show that it is correct, on the average, to multiply the amplifications which individual stars would have produced acting on their own. In case (2), we have to combine a weak and a strong amplification. Although VO have used the superposition principle in this case as well, the result proved in their Appendix A actually applies to the superposition, on the average, of the shears F_1 and F_2 produced by two stars. Since the relation (9) between shear and amplification is nonlinear, we do not expect the superposition principle to hold for the amplification in case (2) as shown below.

Let the image considered lie at the origin and let $(r_i\ \theta_i)$ be the polar coordinates of stars of mass m. The shear components γ_1, γ_2 defined in (8) can easily be evaluated

$$\gamma_1 = -\sum_i \left(\frac{r_0}{r_i}\right)^2 \cos 2\theta_i; \quad \gamma_2 = -\sum_i \left(\frac{r_0}{r_i}\right)^2 \sin 2\theta_i. \tag{18}$$

For example, we consider the case when a close encounter with one star produces a shear F_1 of magnitude close to one (strong amplification) and is at a polar angle zero, while a second star has a shear $F_2 \ll 1$ at polar angle θ. The amplification A is given by

$$\begin{aligned} A^{-1} &= 1 - (F_1 + F_2\cos 2\theta)^2 - (F_2 \sin 2\theta)^2 \\ &= 1 - F_1^2 - F_2^2 - 2F_1F_2\cos 2\theta. \end{aligned} \tag{19}$$

The average amplification $\bar{A}$ is given by averaging over θ:

$$\bar{A} = [(1 - F_1^2 - F_2^2) - 4F_1^2F_2^2]^{1/2}. \tag{20}$$

We can compare this to the result given by the superposition principle.

$$\bar{A}_{\mathrm{Su}} = (1 - F_1^2)^{-1}(1 - F_2^2)^{-1}. \tag{21}$$

We see from (20) and (21) that even for $F_2 \ll 1$, if we have $F_2 \sim 1 - F_1^2$, the true amplification can be significantly greater than that given by superposition. In the limiting case $A_1^{-1} = 1 - F_1^2 \ll 1, F_2 \ll 1$, we find

$$\bar{A}/\bar{A}_{\mathrm{Su}} = |1 - 4F_2^2 A_1^2|^{-1/2}. \tag{22}$$

This shows that a weak shear F_2 is boosted by a factor of the order of A_1 the amplification due to the strong scattering alone.

In the rest of this section, we treat the problem of gravitational lensing by a point mass, perturbed by a weak shear $\gamma \ll 1$ produced by neighbouring masses. As suggested by (22), the imaging is strongly perturbed for $F_2 \sim 1 - F_1^2$, *i.e.*, for amplifications of the order of γ^{-1}. As shown below, one can even obtain four images instead of two. For a surface density n_s, typical nearest neighbour distances are given by $r \sim n_s^{-1/2}$. From (18), the shear γ is typically given by

$$\gamma \sim r_0^2 n_s \sim \tau. \tag{23}$$

We evaluate the probability distribution of the shear γ and average the probability distribution of amplifications over it. This approach to the problem does full justice to the tensorial nature of the shear, the deviations from the superposition principle, and the random distribution of the projected star positions.

As just noted, a weak shear γ significantly affects high amplification events with $A \sim \gamma^{-1}$. The equations governing lensing by a point mass in the presence of shear were given by Chang & Refsdal (1979) in the context of the *B* image of the quasar 0957 + 561 (Walsh, Carswell & Weymann 1979). Since $\gamma \sim 1$ in this case, the problem had to be solved numerically. We need the case $\gamma \ll 1$. Defining dimensionless distances in the deflector plane by

$$\mathrm{X} \equiv (\mathrm{X}_1, \mathrm{X}_2) \equiv \mathrm{x}/r_0; \qquad \mathrm{X_q} \equiv (\mathrm{X_{q1}}, \mathrm{X_{q2}}) \equiv \mathrm{x_q}/r_0, \tag{24}$$

the basic lensing Equation (3) reads

$$X_1 - X_{q1} = \frac{X_1}{X_1^2 + X_2^2} + \gamma X_1; \quad X_2 - X_{q2} = \frac{X_2}{X_1^2 + X_2^2} - \gamma X_2. \tag{25}$$

The first term on the right sides of (24) and (25) represents the deflection produced by a point mass and the second that due to the shear γ.The amplification (9) now reads

$$A^{-1} = 1 - \left[\frac{X_1^2 - X_2^2}{(X_1^2 + X_2^2)^2} - \gamma\right]^2 - \left[\frac{2X_1 X_2}{(X_1^2 + X_2^2)^2}\right]^2. \tag{26}$$

In terms of polar coordinates defined by

$$X_1 = R\cos\theta, \quad X_2 = R\sin\theta; \quad X_{q1} = R_q \cos\theta_q, \quad X_{q2} = R_q \sin\theta_q,$$

we can rewrite (24) and (25) as radial and tangential equations

$$R - R_q \cos(\theta - \theta_q) = 1/R + \gamma R \cos 2\theta, \tag{27}$$

$$-R_q \sin(\theta - \theta_q) = \gamma R \sin 2\theta, \tag{28}$$

with the amplification give by

$$A^{-1} = 1 - \left(\frac{\cos 2\theta}{R^2} - \gamma\right)^2 - \left(\frac{\sin 2\theta}{R^2}\right)^2. \tag{29}$$

Further, in the high amplification limit we can set

$$R = 1 + \delta R; \quad \delta R \ll 1.$$

As is clear from (29), we have $\delta R \sim \gamma$ for $A \sim \gamma^{-1}$. Retaining terms of order γ, Equations (27) and (29) simplify to

$$2\delta R = R_q \cos(\theta - \theta_q) + \gamma \cos 2\theta, \tag{30}$$

$$-R_q \sin(\theta - \theta_q) = \gamma \sin 2\theta, \tag{31}$$

$$A^{-1} = 4\delta R + 2\gamma \cos 2\theta. \tag{32}$$

Equations (30)–(32) describe the high-amplification limit of lensing by a point mass with weak external shear. Equation (31) gives the angular position of the image, giving four solutions for $R \ll \gamma$ and two for $R_q \gg \gamma$. For each value of θ satisfying (31), one can compute the radial position of the image from (30) and the amplification from (32). It is clear from (30)–(32) that R_q R and A all scale with the shear γ and we only need to solve for $A\gamma$ in terms of R_q/γ and θ_q.

Figure 2 shows the probability distribution of amplifications obtained by numerical solution of (30)–(32). At low amplifications, we have $P(A) \propto dA/A^3$ as for isolated point masses (VO). As the amplification approaches γ^{-1}, the probability falls below its unperturbed value. This is consistent with Equation (22) which predicts that shear shifts events to higher amplification. Then, at $A = \gamma^{-1}$ there is a sharp rise in $P(A)$, due to the onset of events with four images instead of two. At still higher values of A, the curve returns to its A^{-3} form. The net effect of the shear is thus to shift a certain fraction of the events with $A \sim \gamma^{-1}$ to amplifications higher by a factor of two or more.

Just as we define a cross-section $\sigma_0 = \pi r_0^2$ for strong lensing events, one can define a cross-section σ_4 for events in which the shear is typically strong enough to produce four images. This is comparable to the cross-section for producing amplifications higher than τ^{-1}. We have $\sigma_4 \simeq \sigma_0 \tau^2$, for small optical depth. As τ approaches 1, it is more and more probable that more than two images form.

5. Effect of the smooth potential of the galaxy

So far, we have discussed lensing by individual stars, perturbed by the random shear produced by their neighbours close to the ray, as if they made up a homogeneous sheet. However, there is also a systematic potential acting on a light ray produced by the overall mass distribution of the galaxy. For clarity, we first look at the case when this mass is dominated by stars which act as point masses, rather than something like neutrinos which act as a continuous distribution. There are two basic effects: (1) a systematic shear γ_g and (2) an increase, by a factor of $1 + 2\tau$, of observed solid angles over those in the absence of lensing.

Imaging by a galaxy has been discussed by Gott & Gunn (1974) and reviewed more recently by VO and TOG. The smoothed-out potential of the galaxy idealised as a singular isothermal sphere, produces a constant deflection angle θ_g given by

$$\theta_g = 4\pi\sigma^2 / c^2, \tag{33}$$

where σ is the one-dimensional velocity dispersion of the matter making up the isothermal. The critical radius r_g defined by

$$r_g = D\theta_g \tag{34}$$

plays a role similar to that of r_0 in the case of a point mass with the cross-section for

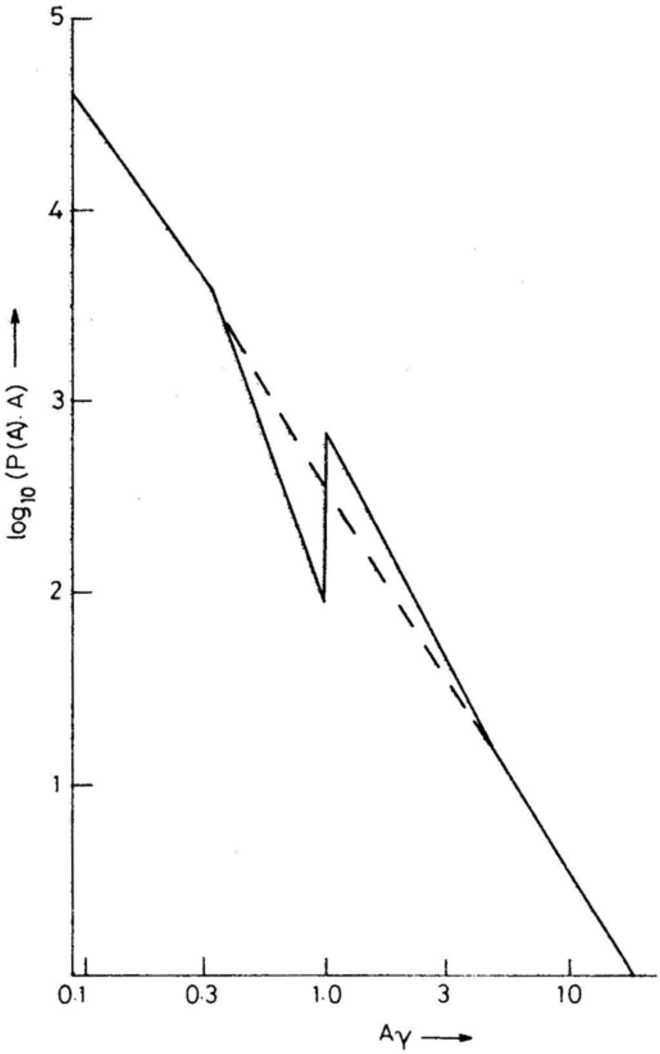

Figure 2, Probability distribution $AP(A)$ (in arbitrary units) of finding amplification A in a given logarithmic interval. The amplification along the x-axis is measured in units of γ^{-1} where γ is a shear (see Equations 30–32).

galaxy lensing given by πr_g^2. There is significant amplification accompanied by formation of two images with comparable intensity when the impact parameter is less than r_g. For impact parameters greater than r_g, only one image is formed. Gott (1981) has shown that the optical depth τ for scattering by the stars making up the galaxy seen by rays passing at a distance r from the centre is given by

$$\tau(r) = r_g/2r \tag{35}$$

This result is independent of the stellar mass so long as geometrical optics holds, *i.e.*, condition (15) is fulfilled.

The convergence κ and shear γ produced by the smooth potential of an isothermal galaxy can be calculated from Equations (8), (33) and (35), the result being

$$\kappa_g(r) = -\gamma_g(r) = r_g/2r = \tau(r). \tag{36}$$

We now show that the rays which propagate in between the stars experience the shear γ but not the convergence κ. The argument is similar to that used in discussions of the Lorentz local field in dielectrics (*cf.* Kittel 1966). We use the fact that the spacing between the stars (projected onto the deflector plane) is five or six orders of magnitude smaller than length-scales like r_g and r associated with the lensing galaxy. It is therefore possible to choose an area element at r with radius r_d with the following properties: (1) The surface density n_s does not vary appreciably within r_d. (2) The discreteness of the mass distribution outside the circle r_d can be neglected in calculating the shear and convergence of rays near the centre. The distribution of the matter outside r_d is thus that of a smooth galaxy with a uniform disc of radius r_d removed. The convergence κ is entirely produced by the local surface density (as noted following Equations 8 and 9) and hence vanishes when the disc is removed. The removal of a circularly symmetric disc leaves the shear γ_g unaffected. We are still left with the contribution of the stars within r_d. A single star at a distance l produces a shear proportional to $1/l^2$ (Equation 18). The area element $l\mathrm{d}l$ translates into a γ^{-2} dg probability for shear γ. We show in the appendix that the distribution of the random as well as the total shear is in fact dominated by nearest neighbour contributions and has a γ^{-2} tail.

We now discuss the effect of lensing on solid angles in the sky. As just shown, the magnification produced by a smooth galaxy does not apply to a beam (coming from a sufficiently small source) which slips in between the stars. Nevertheless, this magnification does apply to extended sources and hence to the angles between quasars on the sky. There is no paradox involved here, just a difference of scales. Fig. 3 shows rays from two point sources which undergo a relative deflection $\sim \theta_g$ as computed from the smooth galaxy potential. If either or both underwent a close encounter with a star, this would contribute extra deflections of the order of r_0/D. The correction coming from the discreteness of the mass distribution is thus of the order

$$r_0/D\theta_g \sim (m_{\text{star}}/m_{\text{galaxy}})^{-1/2} \sim N_{star}^{-1/2} \tag{37}$$

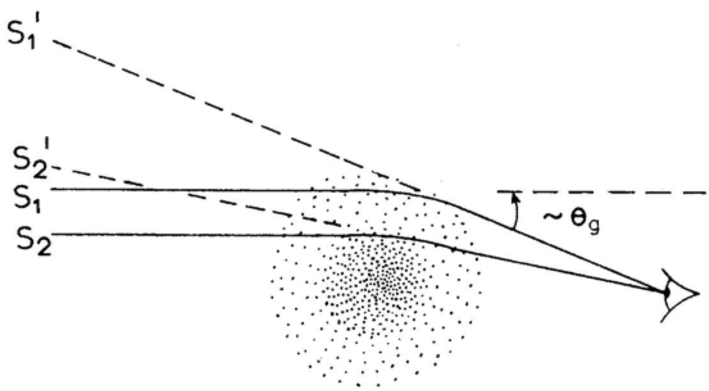

Figure 3. Magnification of the angle between two point sources S_1 and S_2 by a galaxy. S_1' and S_2' are their images. As argued in Section 5, the relative deflection of the two rays is dominated by the smooth potential of the galaxy.

and hence small. Basically, the deflection produced by a point mass falls off slowly (as $1/r$). The relative deflection of two rays separated by many stars is thus not dominated by nearest neighbours and the continuum picture applies with small corrections. Solid angles at a distance r from the centre of an isothermal galaxy are scaled up by a factor

$$A \simeq [1 - \kappa_{\rm g}(r)]^{-2} \simeq 1 + 2\tau(r), \tag{38}$$

where terms of order τ^2 have been neglected. The number of quasars per unit solid angle goes down by the same factor.

It is now straightforward to deal with the case when there is a significant amount of smoothly distributed matter in the galaxy, in addition to the stars. The convergence $\kappa_{\rm s}$ due to the surface density of this matter should be included in (9) while calculating the amplification. As before, the stars do not contribute to κ.

6. Discussion and summary

The main purpose of this paper has been to discuss the problem of imaging background sources such as quasars by point masses (such as low-mass stars) distributed in an isothermal halo around a galaxy, stressing the compound effect of weak and strong lensing. We have tried to include in a consistent way, the lensing by a single mass as modified by its neighbours and the potential of the whole galaxy. The importance of this modification is measured by the optical depth τ and the calculations presented in this paper are valid for $\tau < 1$. In practice, this includes all lines of sight which pass more than 10 kpc from the centre of a typical massive galaxy. Two new points are (1) weak shear can strongly perturb high amplification encounters multiplying the number of images, and (2) the probability distribution of the random shear coming from neighbours of a given point mass has a tail extending to many times the typical shear. Combining these, the net effect is that an appreciable fraction of events with amplifications of the order of $\frac{1}{2}\tau^{-1}$ are further brightened by approximately one magnitude. This should be important for the QSO–galaxy association problem (Canizares 1981, VO).

The general idea that weak shear can appreciably perturb high-amplification events is applicable to the cosmological situation where quasars are lensed by galaxies along the line of sight. TOG have pointed out that there is a strong bias towards high amplification events in flux limited observations of gravitationally lensed quasars. They estimate that the average amplification of a lensed quasar could be in the range 10–40 or even higher. The basic reason for this bias is the steep fall in the luminosity function of quasars at the bright end. At a given observed flux, this favours higher amplification of intrinsically fainter sources. Such events will be sensitive to the random cosmological shear coming from the lumpiness of matter in the universe which is always present (Gunn 1967). This random shear is of the same order as the optical depth for lensing by galaxies which TOG estimate to be 0.05–0.1. We therefore have a situation where the distributed cosmologically-induced shear can play a significant role in determining image geometry and amplification. In addition, the lensing galaxy can be in a cluster as in the case of the QSO 0957 + 561 (Young *et al.* 1981). While TOG emphasize the role of the convergence produced by the cluster potential in increasing image splittings, there is also a significant shear produced by the cluster, especially when it is not centred on the beam. We hope to return to these applications in detail in a future publication.

Acknowledgements

We would like to thank J. E. Gunn for his illuminating remarks on the relation between lensing by point masses and by a continuous sheet. One of us (RN) would like to express his thanks to the Astrophysical Sciences Department, Princeton University, for support and hospitality while this work was being done, and to IAU Commission 38 for travel funds under their exchange of astronomers programme.

Appendix

Probability Distribution of the Shear by Randomly Distributed Stars

The tidal force (that is, shear) produced at the origin by a star at (r_i, θ_i) falls as r_i^{-2}. The law of superposition of shears, expressed by (18), shows that we can regard the total shear as the result of a random walk with steps of length (r_0^2/r_i^2) and direction $2\theta_i$. Chandrasekhar (1943) has reviewed the Holtsmark-Markoff method of calculating the distribution of the net displacement in such random walks and we follow it here.

It is convenient to use the following scaled variables to present the coordinates of the stars and the shear components occuring in (18),

$$S_1 \equiv \gamma_1/n_s r_0^2 \equiv \gamma_1 \pi/\tau; \quad S_2 \equiv \gamma_2 \frac{\pi}{\tau}; \qquad X_i \equiv r_i \cos 2\theta_i/n_s^{-1/2}$$
$$Y_i \equiv r_i \sin 2\theta_i/n_s^{-1/2}; \quad S^2 \equiv S_1^2 + S_2^2; \quad R_i^2 \equiv X_i^2 + Y_i^2. \tag{A1}$$

Equation (18) for the shear produced by a random collection of stars then reads

$$S_1 = -\sum_i \frac{X_i}{R_i^3}; \quad S_2 = \sum_i \frac{Y_i}{R_i^3}. \tag{A2}$$

The probability distribution $P(S_1, S_2)$ of the scaled shear components is easier to compute in terms of its Fourier transform, the so-called characteristic function $Q(t_1, t_2)$ defined by

$$Q(t_1, t_2) \equiv \langle e^{i(t_1 S_1 + t_2 S_2)} \rangle \equiv \iint \mathrm{d}S_1\, \mathrm{d}S_2\, P(S_1, S_2) e^{i(t_1 S_1 + t_2 S_2)}. \tag{A3}$$

Let all the stars which lie at distances less than r_d be included in (A2). In terms of scaled variables, we have

$$R_i < R_d = r_d/n_s^{-1/2}. \tag{A4}$$

The probability distribution of R_i is uniform over the disc $R_i < R_d$.

$$P(R_i)\,\mathrm{d}R_i = 2R_i\,\mathrm{d}R_i/R_d^2. \tag{A5}$$

The total number of points is given by

$$N = \pi r_d^2 n_s = \pi R_d^2 \tag{A6}$$

with small fluctuations since r_d is chosen to enclose many points.

The characteristic function Q in (A3)is the expectation of a product in N independent random variables and hence factors

$$Q(t_1, t_2) = \left[\left\langle \exp\left(i\frac{t_1 X}{R^3} + i\frac{t_2 Y}{R^3} \right) \right\rangle \right]^N \equiv [q(t_1, t_2)]^N. \tag{A7}$$

It is convenient to introduce polar coordinates related to t_1, t_2 by

$$t_1 = t\cos\phi, \qquad t_2 = t\sin\phi \tag{A8}$$

Using (A5), one of the factors in (A7) reads

$$q(t_1, t_2) = \int \frac{d\theta}{2\pi} \exp\left\{ \frac{it\cos(2\theta - \phi)}{R^2} \right\} \frac{2R\,dR}{R_d^2}$$
$$= \frac{2}{R_d^2} \int_0^{R_d} J_0(t/R^2) R\,dR. \tag{A9}$$

J_0 is the usual zero-order Bessel function. Clearly, q is very close to 1, but raised to a very high power in (A7). Substituting (A9) into (A7), we have

$$Q(t, \phi) = \left[1 - \frac{2}{R_d^2} \int_0^{R_d} [1 - J_0(t/R^2)] R\,dR \right]^{\pi R_d^2}. \tag{A10}$$

Making the substitution

$$u = t/R^2$$

and using the integral

$$\int_0^\infty \left(\frac{1 - J_0(u)}{u^2} \right) du = 1$$

we can take the limit of large R_d in (A10) which simplifies to

$$Q(t, \phi) = e^{-\pi t}. \tag{A11}$$

As expected for circular symmetry, there is no ϕ dependence. The probability distribution of S_1 and S_2 is given by inverting the Fourier transform in (A3)

$$P(S_1, S_2) = \iint \frac{dt_1\,dt_2}{(2\pi)^2} e^{-i(t_1 S_1 + t_2 S_2)} e^{-\pi(t_1^2 + t_2^2)^{1/2}}. \tag{A12}$$

The probability is a function of the magnitude S of the shear and can be written

$$P(S)\,dS = 2\pi S\,dS \int \frac{t\,dt}{2\pi} J_0(St) e^{-\pi t}. \tag{A13}$$

Fig. 4 shows the function $P(S)$.

We now superpose a systematic shear S_0, taken for convenience to be along the x-axis. The new probability distribution $P'(S_1, S_2)$ is obtained from the old one by a shift along the S_1 direction.

$$P'(S_1, S_2) = P(S_1 - S_0, S_2). \tag{A14}$$

The corresponding characteristic function $Q'(t_1, t_2)$ is given

$$Q'(t_1, t_2) = Q(t_1, t_2) e^{iS_0 t_1} = e^{-\pi t} e^{iS_0 t\cos\phi}. \tag{A15}$$

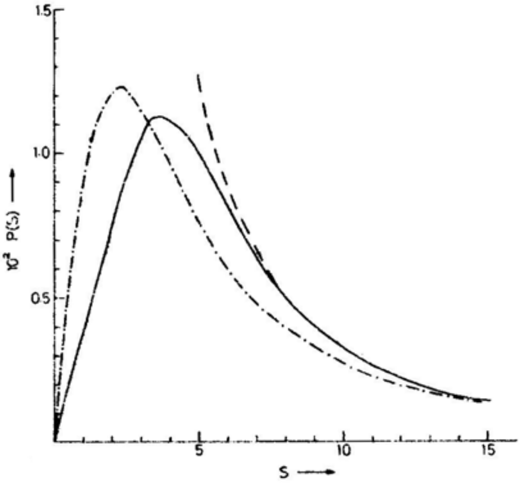

Figure 4. Probability distribution of the normalised shear S defined in (Al). The dot dashed line is the probability distribution $P(S)$ defined in (A 13) which includes only the random part of the shear. The solid line gives the probability distribution $P''(S)$ defined in (A l7) which includes the systematic shear produced by the smoothed-out galaxy potential. The dashed line gives the asymptotic form S^{-2} dominated by the contribution of one star very close to the beam.

Equations (A14) and (A15) show that the problem is no longer circularly symmetric about $S_1, S_2 = 0$. However, we are only interested in the magnitude S of the shear and can therefore average (A14) over θ, which is equivalent to averaging (A l5) over ϕ. The resulting characteristic function $Q''(t)$ and probability distribution $P''(S)$ are given by

$$Q''(t) = \int \frac{d\phi}{2\pi} e^{-\pi t} e^{itS_0 \cos\phi} = e^{-\pi t} J_0(S_0 t) \tag{A16}$$

$$P''(S) = \int \frac{t\,dt}{2\pi} e^{-\pi t} J_0(S_0 t) J_0(St) \cdot 2\pi S. \tag{A17}$$

In the case when the galaxy is entirely made out of stars with no smooth component, we can use Equation (36) to evaluate the scaled systematic shear S_0.

$$S_0 = \gamma_g(r) \cdot \pi / \tau_g(r) = \pi.$$

The probability (A17) is also plotted in Fig. 4 for this case. Also shown in the figure is the function πS^{-2}. Simple arguments show that P and P'' approach this form asymptotically. At large S, we have one star at a distance $R \ll 1$, with the shear given by

$$S = R^{-2}.$$

Since the surface density of stars has been scaled to one in (Al), the probability distribution of S is given by

$$2\pi R\,dR = |2\pi S^{-1/2}(-\tfrac{1}{2}) S^{-3/2}\,dS| = \pi S^{-2}\,dS.$$

This asymptotic form, dominated by the nearest neighbour contribution, also follows from the πt cusp at the origin in the two-dimensional Fourier transform (A11) of the probability distribution.

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Note added in proof

K. Chang & S. Refsdal (1984, *Astr. Astrophys.*, **132,** 168) have recently published a detailed numerical study of Lensing by a star in a galaxy which provides additional shear and convergence. M. Vietri (private communication) has pointed out that many of the claimed cases of quasar-galaxy associations correspond to large angular separations where the probability of Lensing would be very small. We thank M. Vietri for drawing our attention to the work of Chang & Refsdal and also for his close and critical reading of the manuscript.