Uniform trigonometric polynomial B-spline curves

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Abstract This paper presents a new kind of uniform spline curve, named trigonometric polynomial B-splines, over space $\Omega = \text{span}\{\sin t, \cos t, t^{k-3}, t^{k-4}, \cdots, t, 1\}$ of which k is an arbitrary integer larger than or equal to 3. We show that trigonometric polynomial B-spline curves have many similar properties to traditional B-splines. Based on the explicit representation of the curve we have also presented the subdivision formulae for this new kind of curve. Since the new spline can include both polynomial curves and trigonometric curves as special cases without rational form, it can be used as an efficient new model for geometric design in the fields of CAD/CAM.

Keywords: C-curves, uniform B-splines, C-B-splines, trigonometric polynomial B-splines.

In most current CAD/CAM systems, NURBS (non-uniform rational B-splines) curves and surfaces have become the de facto standard primarily because they encompass under a unified mathematical model both freeform and some traditional analytical shapes, such as conics. However, there are several limitations of the NURBS model for shape design and analysis. For example:

• The differentiation of a rational polynomial of degree k is another rational curve of degree $2k$ generally. Rational curves and surfaces of high degree cannot always be dealt with in some CAD systems. Even if they can be dealt with, rational curves and surfaces can bring about uncertainties in numerical computation, difficulties for bounds estimation, etc.

• This model cannot encompass transcendental curves, such as the helix and the cycloid which have been used frequently in many CAD/CAM systems.

• Rational models need additional parameters, namely the weights for each control point, whose selection is not always clear.

For more limitations of NURBS model, please refer to refs. [1—4,11—14].

To overcome the shortcomings of NURBS model, several alternatives have been proposed for curves and surfaces modeling purpose recently. Among the known examples of such splines are certain non-polynomial counterparts to B-spline curves, for example exponential splines in tension[5] and C-B-splines[6,7]. All these curves belong to the class of Chebyshevian spline curves[5,8,9]. There is an extensive theory of T-spline functions[10].

In refs. [6,7], Zhang proposed C-B-spline curves based on the linear combination of $\{\sin t$,

$\cos t, t, 1\}$. C-B-spline curves are an extension of uniform B-spline curves, and they have many similar properties for shape modeling. On the other hand, the C-B-spline curve can express ellipse, circular arc exactly, which make it a potential tool for geometric design in CAD/CAM systems. However, the C-B-spline can only encompass linear polynomial curves, which restricts its application in CAD/CAM. In this paper we present an explicit construction of a kind of generalized curve over space $\Omega = \mathrm{span}\{\sin t, \cos t, t^{k-3}, t^{k-4}, \cdots, t, 1\}$. This kind of curve also has many similar properties to B-spline curves. It encompasses C-B-splines as a special case $k = 4$ and polynomial curves up to degree $k-3$.

Throughout this paper, we call the linear combination of $\{\sin t, \cos t, t^{k-3}, t^{k-4}, \cdots, t, 1\}$ trigonometric polynomial of order k. At the same time, we call a set of piecewise continuous trigonometric polynomial curves of order k trigonometric polynomial splines of order k. The rest of the paper is organized as follows. The definition of order-k trigonometric polynomial splines and its properties are given in section 1. In section 2 we show that trigonometric polynomial curves have many similar properties to B-splines. In section 3 we explain that trigonometric polynomial curve can be generated by subdivision algorithm, and consequently we obtain the V.D. and convex-preserving properties of this kind of curve. The conclusion is given in section 4.

1 The construction and properties of basis

Let $t_i = i\alpha$ $(i = 0, \pm 1, \pm 2, \cdots)$ (α is the interval length, $0 \leqslant \alpha \leqslant \pi$) be knots of a uniform partition of parameter-axis t and we denote the collection of all trigonometric polynomial of order k defined on $[t_i, t_{i+1}] (i = 0, \pm 1, \pm 2, \cdots)$ by $\Omega_{k,\alpha}$, where each function is $k-2$ times continuously differentiable at the knot $t_i (i = 0, \pm 1, \pm 2, \cdots)$. It can be easily observed that operations of addition and scale multiplication of functions over $\Omega_{k,\alpha}$ are closed, i.e. $\Omega_{k,\alpha}$ is a linear space. In this section we will show that, for k that is equal to or larger than 3, there exists a set of basis functions defined over $\Omega_{k,\alpha}$, for which the properties are similar to B-spline basis. The set of bases are then called trigonometric polynomial B-spline basis throughout this paper.

Theorem 1.1. There exists no trigonometric polynomial B-spline basis over $\Omega_{2,\alpha}$.

Proof. Assume there exist trigonometric polynomial B-spline bases over $\Omega_{2,\alpha}$. Then each basis should be a linear combination of $\{\sin t, \cos t\}$ on each interval, and all bases should sum up to one. This implies that 1 is also a linear combination of $\{\sin t, \cos t\}$, which contradicts the fact that $\{\sin t, \cos t, 1\}$ are linear independent. This proves the proposition.

To construct the bases in space $\Omega_{k,\alpha}$ for $k \geqslant 3$, we first define a set of functions over $\Omega_{2,\alpha}$

$$N_{0,2}(t) = \begin{cases} -\dfrac{\alpha}{2(\cos\alpha - 1)} \sin t, & 0 \leqslant t \leqslant \alpha, \\ -\dfrac{\alpha}{2(\cos\alpha - 1)} \sin(2\alpha - t), & \alpha \leqslant t \leqslant 2\alpha, \\ 0, & \text{elsewhere}, \end{cases} \tag{1}$$

and

$$N_{i,2}(t) = N_{0,2}(t - i\alpha) \quad (i = 0, \pm 1, \pm 2, \cdots). \tag{2}$$

And for $k \geqslant 3$ let

$$N_{i,k}(t) = \frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,k-1}(x)\mathrm{d}x \quad (i = 0, \pm 1, \pm 2, \cdots). \tag{3}$$

It can be easily derived that $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \cdots)$ possess the following properties, from which we conclude that $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \cdots)$ constitute a set of bases in $\Omega_{k,\alpha}$ for $k \geqslant 3$. Then, we call $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \cdots)$ a trigonometric polynomial B-spline bases of order k. Trigonometric polynomial B-spline bases of order 3—8 are illustrated in fig. 1.

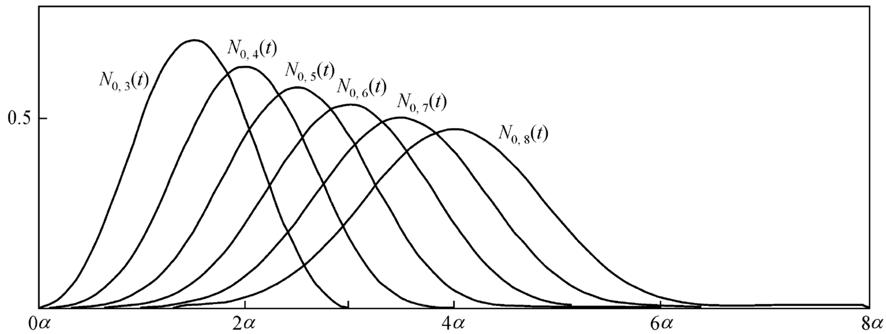

Fig. 1. Trigonometric polynomial B-spline basis of order 3—8 ($\alpha = \pi/2$).

Some basic properties of trigonometric polynomial B-spline basis of order k are listed as follows, and these properties can be easily derived from formulae (1)—(3):

Property 1.1. Non-negative: $N_{i,k} \geqslant 0, t \in (-\infty, +\infty)$.

Property 1.2. Local support: $N_{i,k} \begin{cases} > 0, & t \in (i\alpha, (i+k)\alpha), \\ = 0, & \text{elsewhere.} \end{cases}$

That is to say, the local support of $N_{i,k}(t)$ is k intervals. This is one reason why we say its order is k.

Property 1.3. Partition of unity:

$$\sum_i N_{i,k}(t) \equiv 1. \tag{4}$$

Property 1.4. Linear independence: $N_{i,k}(t)|_{-\infty}^{+\infty}$ are linear independent on $(-\infty, +\infty)$. Specially, $N_{i,k}(t), N_{i+1,k}(t), \cdots, N_{i+n,k}(t) (n \geqslant k)$ are linear independent on interval $[(i + k - 1)\alpha, (i + n + 1)\alpha]$.

Property 1.5. Derivative formula: $N'_{i,k}(t) = \frac{1}{\alpha}(N_{i,k-1}(t) - N_{i+1,k-1}(t))$.

Property 1.6. Symmetry: $N_{i,k}(i\alpha + k\alpha - t) = N_{i,k}(i\alpha + t)$ $t \in [0, k\alpha]$.

Proof. Since $N_{i,k}(i\alpha + t) = N_{0,k}(t)$, $t \in [0, k\alpha]$, we only need to show $N_{0,k}(k\alpha - t) = N_{0,k}(t)$. For $k = 3$

$$\begin{aligned} N_{0,3}(3\alpha - t) =& \frac{1}{\alpha}\int_{2\alpha-t}^{3\alpha+t} N_{0,2}(x)\mathrm{d}x = \frac{1}{\alpha}\int_{2\alpha-t}^{3\alpha+t} N_{0,2}(2\alpha - x)\mathrm{d}x \\ =& \frac{1}{\alpha}\int_{t-\alpha}^{t} N_{0,2}(y)\mathrm{d}y = N_{0,3}(t), \quad t \in [0, 3\alpha]. \end{aligned}$$

Now assume the property holds for $k = l - 1$. For $k = l$ we have

$$\begin{aligned} N_{0,l}(t) =& \frac{1}{\alpha}\int_{t-\alpha}^{t} N_{0,l-1}(x)\mathrm{d}x = \frac{1}{\alpha}\int_{t-\alpha}^{t} N_{0,l-1}((l-1)\alpha - x)\mathrm{d}x \\ =& \frac{1}{\alpha}\int_{-t-\alpha}^{-t} N_{0,l-1}(l\alpha - (\alpha + x))\mathrm{d}x \\ =& N_{0,l}(l\alpha - t). \end{aligned}$$

This completed the proof.

Property 1.7. Continuity: $N_{i,k}(t)$ is $k-2$-differentiable on the whole parameter space.

2 Trigonometric polynomial B-spline curves

With the basis defined in the above section, we can define trigonometric polynomial B-spline curves over the whole parameter space. However, in practical geometric modeling applications, the span of parameter t is always restricted to a finite interval such as $[a, b]$ $(a < b)$.

We denote the space of trigonometric polynomial B-splines of order k defined over $[a, b]$ as $\Omega_{k,\alpha}[a, b]$. If $a = k\alpha, b = (n+1)\alpha$, then $N_{1,k}(t), N_{2,k}(t), \cdots, N_{n,k}(t)(n \geqslant k)$ are the bases of space $\Omega_{k,\alpha}[a, b]$ (cf. fig. 2). Therefore, we can define spline curves in $\Omega_{k,\alpha}[a, b]$, with bases $N_{i,k}(i = 1, 2, \cdots, n)$ as:

$$p_k(t) = \sum_{i=1}^{n} P_i N_{i,k}(t), \quad k\alpha \leqslant t \leqslant (n+1)\alpha \quad (n \geqslant k), \tag{5}$$

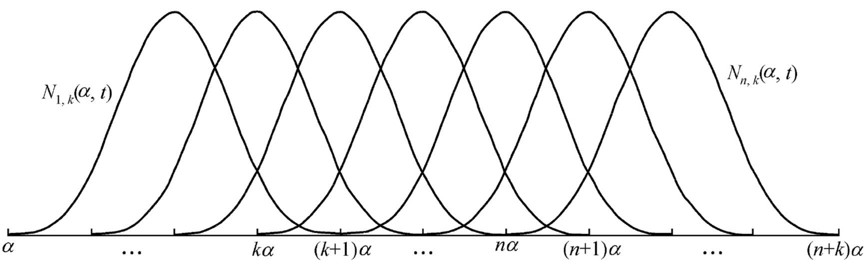

Fig. 2. $N_{l,k}(\alpha, t), \cdots, N_{n,k}(\alpha, t)$ are basis of the space $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$.

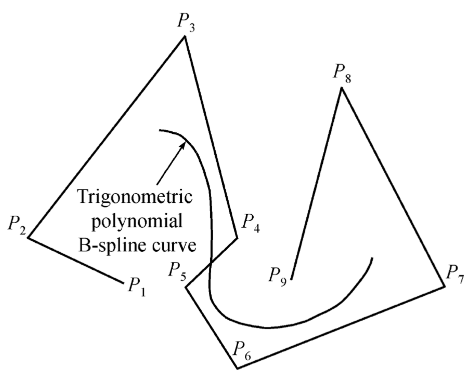

Fig. 3. Trigonometric polynomial B-spline curve of order 6 ($\alpha = \pi/2$).

where $P_i(i = 1, 2, \cdots, n)$ are the control points, and $P = [P_1, P_2, \cdots, P_n]$ stands for the control polygon. Fig. 3 illustrates an example of trigonometric polynomial B-spline curve of order 6.

In a similar way to B-spline curves, trigonometric polynomial B-spline curves have the following properties:

Property 2.1. Convex hull property: A curve defined on $[i\alpha, (i+1)\alpha]$, namely $p_k(t)(i\alpha \leqslant t \leqslant (i+1)\alpha, i = k, \cdots, n)$, must lie inside the convex hull H_i of control points $P_{i-k+1}, \cdots, P_i$, and the entire curve (5) lies inside $H = \bigcup_{i=k}^{n} H_i$, which is union of H_i.

This follows, since the trigonometric polynomial B-spline basis is nonnegative. They sum to one as shown in (4).

Property 2.2. Geometric invariance: Because $p_k(t)$ is affine combination of the control points $P_i (i = 1, \cdots, n)$, the shape of trigonometric polynomial B-spline curves is independent of the choice of coordinate.

Property 2.3. Local control property: Change of one control vertex will alter at most k segments of trigonometric polynomial B-spline curve of order k, then, local adjustment can be made without disturbing the rest of the curve.

Property 2.4. Symmetry: Just like the curve in fig. 3, it is clear that the control points can be labeled $P_1, P_2, \cdots, P_n$ or $P_n, P_{n-1}, \cdots, P_1$ without changing the shape of the curve. They differ only in the direction in which they are traversed. If we do not consider the direction of a curve, we have

$$\sum_{i=1}^{n} P_i N_{i,k}(i\alpha + t) = \sum_{i=1}^{n} P_{n-i} N_{i,k}(i\alpha + k\alpha - t) \qquad t \in [0, k\alpha].$$

Property 2.5. The derivative of a trigonometric polynomial B-spline curve:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = \frac{1}{\alpha} \sum_{i=2}^{n} N_{i,k-1}(t) \Delta P_i \quad (k\alpha \leqslant t \leqslant (n+1)\alpha), \tag{6}$$

where $\Delta P_i = P_i - P_{i-1}$.

Corollary. The rth derivative of a trigonometric polynomial B-spline curve:

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} p_k(t) = \frac{1}{\alpha^r} \sum_{i=r+1}^{n} N_{i,k-r}(t) \varDelta^r P_i, \quad r = 0, 1, \cdots, k-2, \tag{7}$$

where $\varDelta^r P_i = \varDelta^{r-1} P_i - \varDelta^{r-1} P_{i-1}$.

The proof of (7) is by repeated application of (6).

Property 2.6. Continuity: $p_k(t) \in C^{k-2}[k\alpha, (n+1)\alpha]$.

Proof. $p_k(t)$ is defined as (5), i.e. $p_k(t) = \sum_{i=1}^{n} P_i N_{i,k}(t)$ $(k\alpha \leqslant t \leqslant (n+1)\alpha)$, in other words $p_k(t)$ is a linear combination of $N_{i,k}(t)$. And from Property 1.7, we have $N_{i,k}(t)$ is $k - \alpha$-differentiable on the parameter space, hence $p_k(t) \in C^{k-2}[k\alpha, (n+1)\alpha]$.

3 Subdivision formulae for trigonometric polynomial B-splines

In this section, we will discuss the subdivision formulae for trigonometric polynomial B-spline curves. Let $N_{i,k}(\alpha, t)(i = 0, \pm 1, \pm 2, \cdots)$ denote the basis of order k based on a set of uniform knots $t_i = i\alpha$ $(i = 0, \pm 1, \pm 2, \cdots)$ as defined in (1)—(3). If we partition the parameter-axis t with unit interval length $\frac{\alpha}{2}$, that is take $t'_i = \frac{i\alpha}{2} (i = 0, \pm 1, \pm 2, \cdots)$ as knots, we have a new set of bases with this new set of knots. Consequently, we denote these new bases of order k as $N_{i,k}(\alpha/2, t)$ $(i = 0, \pm 1, \pm 2, \cdots)$ (see fig. 4).

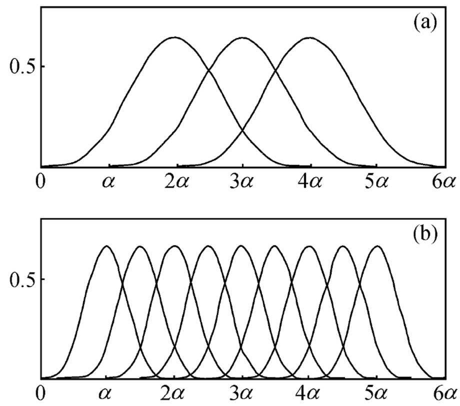

Fig. 4. Trigonometric polynomial B-spline basis of order-4. (a) Interval length is α; (b) interval length is $\alpha/2$.

From the above sections we know that $\Omega_{k,\alpha}[k\alpha,(n+1)\alpha]$ is a space consisting of trigonometric polynomial B-spline curves of order k which are $k-2$ times continuously differentiable at knots $t_i = i\alpha (i = 0, \pm 1, \pm 2, \cdots)$. Similarly, $\Omega_{k,\alpha/2}[k\alpha,(n+1)\alpha]$ is another space consisting of trigonometric polynomial B-spline curves of order k defined with knots $t'_i = \frac{i\alpha}{2}$ on interval $[k\alpha,(n+1)\alpha]$. It is clear that $\Omega_{k,\alpha}[k\alpha,(n+1)\alpha]$ is a subspace of $\Omega_{k,\alpha/2}[k\alpha,(n+1)\alpha]$, and then curves in $\Omega_{k,\alpha}[k\alpha,(n+1)\alpha]$ can be expressed by $N_{i,k}(\alpha/2,t)$. When a curve in $\Omega_{k,\alpha}[k\alpha,(n+1)\alpha]$ has been expressed by $N_{i,k}(\alpha,t)$ and $N_{i,k}(\alpha/2,t)$, respectively, the relationship between their control points is based on the following theorem (see fig. 5 and fig. 6).

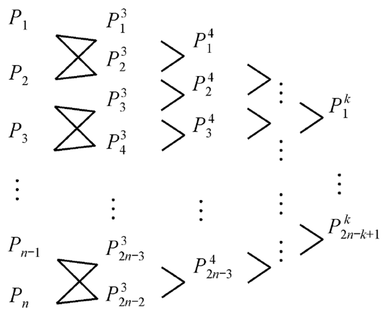

Fig. 5. The recursion of the control points on subdivision.

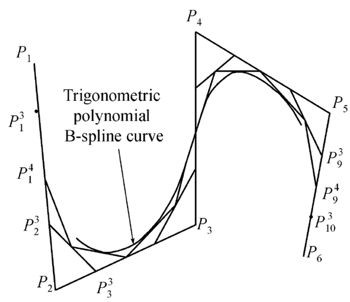

Fig. 6. Subdivision of trigonometric polynomial B-spline curve ($\alpha = \pi/2$).

Theorem 3.1 (Subdivision). Order-k trigonometric polynomial B-spline curve $p_k(t)$ can be expressed by $N_{i,k}(\alpha,t)$ and $N_{i,k}(\alpha/2,t)$ respectively as follows:

$$\begin{aligned} p_k(t) &= \sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) \\ &= \sum_{i=1}^{2n-k+1} P_i^k N_{i,k}(\alpha/2,t), \quad t \in [k\alpha,(n+1)\alpha], \end{aligned}$$

where

$$P_i^3 = \begin{cases} ((1+2\cos(\alpha/2))P_{(i+1)/2} + P_{(i+3)/2})/(2(1+\cos(\alpha/2))), & i \text{ odd}, \\ (P_{i/2} + (1+2\cos(\alpha/2))P_{i/2+1})/(2(1+\cos(\alpha/2))), & i \text{ even}, \end{cases} \tag{8}$$

$$i = 0, 1, 2, \cdots, 2n-2,$$

$$P_i^k = (P_i^{k-1} + P_{i+1}^{k-1})/2, \quad i = 1, 2, \cdots, 2n-k+1, \quad k > 3. \tag{9}$$

Proof. (By induction on k) Let $k=3$,

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,3}(\alpha,t) &= \sum_{i=1}^{n} P_i \cdot \left(\frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,2}(\alpha,x)\mathrm{d}x\right)\\ &= \sum_{i=1}^{n} P_i \Bigg(\frac{1}{\alpha}\int_{t-\alpha}^{t}\Bigg(\frac{1}{1+\cos(\alpha/2)}N_{2i-3,2}(\alpha/2,x)+\frac{2\cos(\alpha/2)}{1+\cos(\alpha/2)}N_{2i-2,2}(\alpha/2,x)\\ &\quad+\frac{1}{1+\cos(\alpha/2)}N_{2i-1,2}(\alpha/2,x)\Bigg)\mathrm{d}x\Bigg)\\ &= \sum_{i=1}^{n} P_i \Bigg(\frac{1}{2(1+\cos(\alpha/2))}N_{2i-3,3}(\alpha/2,t)+\frac{1+2\cos(\alpha/2)}{2(1+\cos(\alpha/2))}N_{2i-2,3}(\alpha/2,t)\\ &\quad+\frac{1+2\cos(\alpha/2)}{2(1+\cos(\alpha/2))}N_{2i-1,3}(\alpha/2,t)+\frac{1}{2(1+\cos(\alpha/2))}N_{2i,3}(\alpha/2,t)\Bigg).\end{aligned}$$

Since $N_{-1,3}(\alpha/2,t)=N_{0,3}(\alpha/2,t)=N_{2n-1,3}(\alpha/2,t)=N_{2n,3}(\alpha/2,t)=0$ for $t\in[3\alpha,(n+1)\alpha]$, we have

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,3}(\alpha,t) &= \sum_{i=1}^{n-1}\Bigg(\frac{(1+2\cos(\alpha/2))P_i+P_{i+1}}{2(1+\cos(\alpha/2))}N_{2i-1,3}(\alpha/2,t)\\ &\quad+\frac{P_i+(1+2\cos(\alpha/2))P_{i+1}}{2(1+\cos(\alpha/2))}N_{2i,3}(\alpha/2,t)\Bigg)\\ &= \sum_{i=1}^{2n-2} P_i^3 N_{i,3}(\alpha/2,t).\end{aligned}$$

Thus, (8) holds. Now assume (9) holds for all l, $3\leqslant l<k$. For $l=k$ we have

$$\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t)=\sum_{i=1}^{n} P_i\left(\frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,k-1}(\alpha,x)\mathrm{d}x\right)=\frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\sum_{i=1}^{n} P_i N_{i,k-1}(\alpha,x)\mathrm{d}x\right) \tag{10}$$

for $t\in[k\alpha,(n+1)\alpha]$. Now by our induction hypothesis, (10) reduces algebraically to

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) &= \frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i,k-1}(\alpha/2,x)\right)\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{1}{\alpha/2}\left(\int_{t-\alpha/2}^{t}+\int_{t-\alpha}^{t-\alpha/2}\right)\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i,k-1}(\alpha/2,x)\mathrm{d}x\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i,k}(\alpha/2,t)+\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i,k}(\alpha/2,t-\alpha/2)\right).\end{aligned} \tag{11}$$

Since $N_{2n-k+2,k}(\alpha/2,t)=N_{1,k}(\alpha/2,t-\alpha/2)=0$ for $t\in[k\alpha,(n+1)\alpha]$, and after dropping terms concerning these basis functions and rearranging the remaining term in (11), we have

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) &= \frac{1}{2}\left(\sum_{i=1}^{2n-k+1} P_i^{k-1} N_{i,k}(\alpha/2,t)+\sum_{i=2}^{2n-k+2} P_i^{k-1} N_{i,k}(\alpha/2,t-\alpha/2)\right)\\ &= \sum_{i=1}^{2n-k+1}\left(P_i^{k-1}+P_{i+1}^{k-1}\right)\Big/ 2\cdot N_{i,k}(\alpha/2,t)\\ &= \sum_{i=1}^{2n-k+1} P_i^{k}\cdot N_{i,k}(\alpha/2,t).\end{aligned}$$

This proves the theorem.

This theorem indicates that new control polygon can be generated by the old polygon after one subdivision. We can repeatedly apply subdivision to the consequent control polygon, and generate a series of control polygons.

For convenience of description, we use $B_k(P;1,n)(t)$ to denote k-order trigonometric polynomial B-spline curve $p_k(t)$, where $P = \Phi^0[P] = [P_1, P_2, \cdots, P_n]$ is control polygon. Define $\Phi^1[P] = [P_1^{k,1}, P_2^{k,1}, \cdots, P_n^{k,1}]$, where $P_i^{k,1} = P_i^k$ is given by (8), (9) and $\Phi^l[P] = \Phi[\Phi^{l-1}P] = [P_1^{k,l}, P_2^{k,l}, \cdots, P_{r(l,k)}^{k,l}]$ for $r(l,k) = 2^{l+1}(n-k+1)+k-1$, $l \geqslant 1$, where $\Phi^l[P]$ denotes the control polygon after l times of subdivision. In fact, we can show that when the subdivision time increases, the control polygon series will converge to the spline curve.

Theorem 3.2. Let $B_k(P;1,n)(t)$ and $\Phi^l[P]$ be defined as above. Then

$$\lim_{l\to\infty} \Phi^l[P] = B_k(P;1,n)(t).$$

Proof. From Theorem 3.1 and simple induction on l we have

$$B_k(\Phi^l[P];1,n)(t) = B_k(P;1,n)(t).$$

Now let $M = \max_i |P_{i+1} - P_i|$. It can be easily derived that

$$|P_{i+1}^k - P_i^k| \leqslant \frac{1}{1+\cos(\alpha/2)} M,$$

and therefore

$$|P_{i+1}^{k,l} - P_i^{k,l}| \leqslant \frac{1}{(1+\cos(\alpha/2))\cdots(1+\cos(\alpha/2^l))} M.$$

Since $0 < \alpha < \pi, 0 < \alpha/2 < \pi/2$, we have

$$|P_{i+1}^{k,l} - P_i^{k,l}| \leqslant \frac{1}{(1+\cos(\alpha/2))^l} M.$$

That is, $\lim_{l\to\infty} |P_{i+1}^{k,l} - P_i^{k,l}| = 0$, and then

$$\lim_{l\to\infty} |P_{i+j}^{k,l} - P_i^{k,l}| = 0 \tag{12}$$

for $\forall i \in \{1, \cdots, r(l,k)\}, j = 1, \cdots, k, i+j \leqslant r(l,k)$.

From the convex hull property we know $B_k(P;1,n)(t_0), \forall t_0 \in [k\alpha, (n+1)\alpha]$ lies within the convex hull of $P_i^{k,l}, P_{i+1}^{k,l}, \cdots, P_{i+k}^{k,l}$ for some i. But with (12) we can then conclude

$$\lim_{l\to\infty} \Phi^l[P] = B_k(P;1,n)(t).$$

Theorem 3.2 indicates that subdivision of control polygon can lead to its corresponding trigonometric polygon B-spline curve. And the following theorem describes such a curve possesses variation diminishing property (V.D. property), which is critical for work in CAD since it prevents the curve from wiggling too much.

Theorem 3.3 (V.D. Property). No plane has more intersections with the curve than with the control polygon in a trigonometric polynomial B-spline.

Proof. For arbitrary selected plane P, the intersections between the plane P and control polygons will not increase after subdivision. And because the series of control polygons converge to the trigonometric polynomial B-spline after repeated subdivisions in the end, the V.D. property comes true.

Theorem 3.4 (Convexity preserving). If the control polygon is convex, the corresponding trigonometric polynomial B-spline curve is convex.

Proof. The convex control polygon preserves convex after every subdivision, so from the converge property the corresponding trigonometric polynomial B-spline curve is also convex.

4 Conclusion

In this paper we have obtained a general piecewise trigonometric polynomial B-spline basis of order $k(k \geqslant 3)$ over the space $\Omega = \mathrm{span}\{\sin t, \cos t, t^{k-3}, t^{k-4}, \cdots, t, 1\}$, and trigonometric polynomial B-spline curve models are presented simultaneously. Trigonometric polynomial B-spline curves not only inherit the advantage of the polynomial curve, but have the property of the trigonometric curve. In addition to the ellipse, they can be used to represent remarkable transcendental curves, such as the cycloid, graph of sinusoidal functions[4]. Trigonometric polynomial B-spline curves have nearly all the same properties as uniform B-splines. The subdivision formulae of this kind of curve are given, and therefore the curve can be derived from repeated subdivisions.

In fact, we can construct tensor product trigonometric polynomial B-surfaces just like B-spline surfaces. Therefore, trigonometric polynomial B-spline model is a new powerful tool for constructing free-form curves and surfaces in CAGD. Because trigonometric polynomial B-spline curves are defined over space Ω mixed by trigonometric function $\sin t, \cos t$ and polynomial, Ω is closed for the integral operation, that is to say, for $\forall f(t) \in \Omega$, we have $\int f(t) \in \Omega$.

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