ÉTALE COHOMOLOGY FOR NON-ARCHIMEDEAN ANALYTIC SPACES

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INTRODUCTION

The problem of constructing an étale cohomology theory for non-Archimedean analytic spaces has arisen from Drinfeld's work on elliptic modules [Dr1]. In his work, Drinfeld defined (among other things) the first cohomology group $H^1(X, \mu_n)$ as the set of pairs (L, φ), where L is an invertible sheaf on X, and φ is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} L^{\otimes n}$. He showed that for the one-dimensional p-adic upper half-plane Ω^2 this group gives rise to a certain infinite dimensional representation of $GL_2(k)$, where k is the ground local field. Afterwards, in [Dr2], Drinfeld constructed a certain family of equivariant coverings of the d-dimensional p-adic upper half-plane Ω^{d+1}, and suggested that all cuspidal representations of the group $GL_{d+1}(k)$ are realized in high dimensional étale cohomology groups of this family of coverings.

Since then, as far as I know, the only attempt to construct an étale cohomology theory for non-Archimedean analytic spaces was undertaken by O. Gabber. We understand that O. Gabber has made progress in the subject, but, unfortunately, he has written nothing on it. Besides that, in [FrPu] and [ScSt], definitions of an étale topology on a non-Archimedean analytic space were given, and in [ScSt] the cohomology of Ω^{d+1} is calculated for arbitrary d under the hypotheses that this cohomology satisfies certain reasonable properties. Finally, in [Car], a conjecture, which is an explicit form of Drinfeld's suggestion, is proposed. The conjecture predicts the decomposition of the representations of $GL_{d+1}(k)$ and the Galois group of k (k is a p-adic field) on the d-dimensional cohomology group of the equivariant system of coverings of Ω^{d+1} in terms of the Langlands correspondence.

The purpose of this work is to develop many basic results of étale cohomology for non-Archimedean analytic spaces. We define the étale cohomology and the étale cohomology with compact support and calculate the cohomological dimension of an analytic space. We prove a Comparison Theorem for Cohomology with Compact Support which states that, for a compactifiable morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ between schemes of locally finite type over the spectrum of a k-affinoid algebra and a torsion sheaf $\mathcal{F}$ on $\mathcal{Y}$, there is a canonical isomorphism $(R^q \varphi_! \mathcal{F})^{an} \xrightarrow{\sim} R^q \varphi_!^{an} \mathcal{F}^{an}$, $q \geqslant 0$. We also prove a Poincaré Duality Theorem, the acyclicity of the canonical projection $X \times D \to X$, where D is an open polydisc in the affine space, a Cohomological Purity Theorem, the invariance of the cohomology under algebraically closed extensions of the ground field, a Base Change Theorem for Cohomology with Compact Support and a Smooth Base Change Theorem (all the results are proved for torsion sheaves with torsion orders prime to the characteristics of the residue field of k). In particular, all the properties of the "abstract" cohomology theory from [ScSt] hold. Our main result is a Comparison Theorem which states that, for a morphism of finite type $\varphi : \mathcal{Y} \to \mathcal{X}$ between schemes of locally finite type over k and a constructible sheaf $\mathcal{F}$ on $\mathcal{Y}$ with torsion orders prime to the characteristics of the residue field of k, there is a canonical isomorphism $(R^q \varphi_* \mathcal{F})^{an} \xrightarrow{\sim} R^q \varphi_*^{an} \mathcal{F}^{an}$, $q \geq 0$. We note that the only previous proof of the Comparison Theorem in the classical situation over **C** uses Hironaka's Theorem on resolution of singularities. Our proof of the Comparison Theorem works over **C** as well and does not use Hironaka's Theorem.

Our approach to étale cohomology is completely based on the previous work [Ber]. In that work we introduced analytic spaces which are natural generalizations of the complex analytic spaces and have the advantage that they allow direct application of the geometrical intuition. One should say that although the analytic spaces from [Ber] were considered in a more general setting than that for rigid analytic geometry (for example, the valuation of the ground field is not assumed to be nontrivial), they don't give rise to all reasonable rigid spaces. And so our first purpose in this work is to extend the category of analytic spaces from [Ber] so that the new category gives rise to all reasonable rigid spaces, for example, to those that are associated with formal schemes of locally finite type over the ring of integers of k.

We now give a summary of the material which follows. Let k be a non-Archimedean field, and let $\widetilde{k}$ denote its residue field. As in [Ber], we don't assume that the valuation of k is nontrivial.

In § 1 we introduce a category of k-analytic spaces more general than those from [Ber]. These analytic spaces possess nice topological properties. For example, a basis of topology is formed by open locally compact paracompact arcwise connected sets. One does not need to use Grothendieck topology in the definition of the spaces, but they are naturally endowed with such a topology called the G-topology (§ 1.3). The latter is formed by analytic domains of an analytic space. The spaces from [Ber] (they are said to be " good ") are exactly those in which every point has an affinoid neighborhood. The G-topology on an analytic space is a natural framework for working with coherent sheaves. (If the space is good, then it is enough to work with the usual topology as in [Ber].) In § 1.4 we show that the category of analytic spaces introduced admits fibre products and the ground field extension functor, and we associate with every point x of an analytic space a non-Archimedean field $\mathcal{H}(x)$ so that any morphism $\varphi : Y \to X$ induces, for a point $y \in Y$, a canonical isometric embedding $\mathcal{H}(\varphi(y)) \hookrightarrow \mathcal{H}(y)$. In § 1.5 we define for a morphism $\varphi : Y \to X$ the relative interior $\mathrm{Int}(Y/X)$ (this is an open subset of Y), and we call the morphism closed if $\mathrm{Int}(Y/X) = Y$. In § 1.6 we construct a fully faithful functor from the category of Hausdorff (strictly) analytic spaces to the category of quasiseparated rigid spaces and show that it induces an equivalence between the category of paracompact analytic spaces and the category of quasiseparated rigid spaces that have an admissible affinoid covering of finite type.

In § 2 we establish properties of the local ring $\mathcal{O}_{X,x}$ and its residue field $\kappa(x)$, where x is a point of a k-affinoid space X. (The completion of $\kappa(x)$ is the field $\mathcal{H}(x)$.) First, we establish those properties which are mentioned without proof in [Ber], § 2.3.

Furthermore, we prove that the ring $\mathcal{O}_{X,x}$ is Henselian, and the canonical valuation on $\kappa(x)$ extends uniquely to any algebraic extension (fields with this property are said to be quasicomplete). The latter two facts are of crucial importance for the whole story. The quasicompleteness of $\kappa(x)$ implies, for example, an equivalence between the categories of finite separable extensions of $\kappa(x)$ and of $\mathcal{H}(x)$ and, in particular, an isomorphism of their Galois groups $G_{\kappa(x)} \xrightarrow{\sim} G_{\mathcal{H}(x)}$. In § 2.4 we establish properties of quasicomplete fields (whose proofs are borrowed from [BGR] and [ZaSa]), and in § 2.5 we show that the ℓ-cohomological dimension $\mathrm{cd}_\ell(\kappa(x))$ of the field $\kappa(x)$ (or, equivalently, of the field $\mathcal{H}(x)$) is a most $\mathrm{cd}_\ell(k) + \dim(X)$, where ℓ is a prime integer. In § 2.6 to every scheme $\mathcal{Y}$ of locally finite type over $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$, where $\mathcal{A}$ is an affinoid algebra, we associate an analytic space $\mathcal{Y}^{\mathrm{an}}$ over $X = \mathcal{M}(\mathcal{A})$. These objects are very important, in particular, for the proof of the Poincaré Duality Theorem. We establish some basic facts on the correspondence $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{an}}$, which are necessary for this work.

In § 3 we introduce and study the classes of étale and smooth morphisms. The first basic notion is that of a quasifinite morphism. A morphism $\varphi : Y \to X$ is said to be quasifinite if for any point $y \in Y$ there exist open neighborhoods $\mathcal{V}$ of y and $\mathcal{U}$ of $\varphi(y)$ such that φ induces a finite morphism $\mathcal{V} \to \mathcal{U}$. It turns out that a morphism is quasifinite if and only if it has discrete fibres and is closed (in the sense of § 1.5). Furthermore, we define étale morphisms. (By definition, they belong to the class of quasifinite morphisms.) For example, the canonical immersion of the closed unit disc in the affine line is not étale because it is not a closed morphism. In § 3.4 we introduce the notion of a germ of an analytic space and prove the very important fact that the category of germs finite and étale over the germ (X, x) of an analytic space X at a point x is equivalent to the category of schemes finite and étale over the field $\mathcal{H}(x)$. Furthermore, in § 3.5 we study smooth morphisms. A mosphism $\varphi : Y \to X$ is said to be smooth if locally it is a composition of an étale morphism to the affine space $\mathbf{A}^d_X = \mathbf{A}^d \times X$ and the canonical projection $\mathbf{A}^d_X \to X$. In particular, any smooth morphism is closed. The latter property of smooth morphisms is natural if we want to have for them Poincaré Duality. In § 3.6 and § 3.7 we describe the local structure of a smooth morphism. This description is very important for the sequel and is actually an analog of the trivial fact that locally any smooth morphism of complex analytic spaces is isomorphic to the projection $X \times D \to X$, where D is an open polydisc in the affine space.

In § 4 we define the étale topology on a k-analytic space X (the étale site $X_{\text{ét}}$) and establish first basic properties of étale cohomology. In § 4.1 we verify that certain reasonable presheaves are actually sheaves and give an interpretation of the first cohomology group with coefficients in a finite group. In § 4.2 we define the stalk F_x of a sheaf F at a point $x \in X$. It is a discrete $G_{\mathcal{H}(x)}$-set. It turns out that if π is the canonical morphism of sites $X_{\text{ét}} \to |X|$, where $|X|$ is the site generated by the usual topology of X, then for any abelian sheaf F on $X_{\text{ét}}$, there is an isomorphism $(R^q \pi_* F)_x \xrightarrow{\sim} H^q(G_{\mathcal{H}(x)}, F_x)$. It follows that the sheaf F is flabby if and only if, for any point $x \in X$, the fibre F_x is a flabby $G_{\mathcal{H}(x)}$-module and, for any étale morphism $U \to X$, the restriction of F to the usual topology of U is flabby. This fact reduces the verification of many properties of the étale cohomology established in § 4 and § 5 to the verification of certain properties of the cohomology of profinite groups and the usual cohomology with coefficients in sheaves. As first applications of these considerations we prove that if X is good then the étale cohomology of the sheaf induced by a coherent $\mathcal{O}_X$-module coincides with its usual cohomology and that the ℓ-cohomological dimension $cd_\ell(X)$ of a paracompact k-analytic space X is at most $cd_\ell(k) + 2\dim(X)$. The latter fact is easily obtained from the spectral sequence of the morphism of sites $\pi : X_{ét} \to |X|$, using the facts that the topological dimension of such a space is at most $\dim(X)$ and that $cd_\ell(\mathcal{H}(x)) \leqslant cd_\ell(k) + \dim(X)$.

In § 4.3 we study quasi-immersions of analytic spaces. A morphism $\varphi : Y \to X$ of analytic spaces over k is said to be a quasi-immersion if it induces a homeomorphism of Y with its image $\varphi(Y)$ in X and, for any point $y \in Y$, the field $\mathcal{H}(y)$ is a purely inseparable extension of $\mathcal{H}(\varphi(y))$. For example, the canonical embeddings of analytic domains in an analytic space and closed immersions are quasi-immersions. We prove that if $\varphi : Y \to X$ is a quasi-immersion such that the set $\varphi(Y)$ has a basis of paracompact neighborhoods, then for any abelian sheaf F on X one has $H^q(Y, F|_Y) = \varinjlim H^q(\mathcal{U}, F)$, where $\mathcal{U}$ runs through open neighborhoods of the set $\varphi(Y)$. Furthermore we construct a spectral sequence which relates the cohomology of a paracompact k-analytic space X to the cohomology of closed analytic domains from a locally finite coverings by such domains. We use it to show that the group $H^1(X, \mu_n)$ has the interpretation given to it by Drinfeld in [Dr1]. In § 4.4 we introduce and study quasiconstructible sheaves which play the role of constructible sheaves on schemes in the sense that any abelian torsion sheaf is a filtered inductive limit of quasiconstructible sheaves (the word " constructible " is reserved for a future development).

In § 5 we introduce and study the étale cohomology with compact support. All definitions and constructions are straightforward generalizations of the corresponding topological notions. In particular, the cohomology groups with compact support are defined as the right derived functors of the functor of sections with compact support. Theorem 5.3.1 gives, for an abelian sheaf F, a description of the stalks of the sheaves $R^q \varphi_! F$, where φ is a Hausdorff morphism of k-analytic spaces, in terms of the cohomology of the fibres of φ. As an application we show that if F is a torsion sheaf, then $R^q \varphi_! F = 0$ for all $q > 2d$, where d is the dimension of φ. In § 5.4 we construct for every separated flat quasifinite morphism $\varphi : Y \to X$ and for every abelian sheaf F on X a trace mapping $Tr_\varphi : \varphi_! \varphi^*(F) \to F$.

In § 6 we establish various facts on the cohomology of analytic curves. These facts are a basis for the induction used in the proof of the main theorems from § 7. In § 6.1 we prove the Comparison Theorem for Cohomology with Compact Support for curves. The proof of this theorem in the general case (§ 7.1) may be read immediately after § 6.1. In the rest of § 6 we assume that the ground field k is algebraically closed. In § 6.2 we construct, for every smooth separated analytic curve X, a trace mapping $\mathrm{Tr}_X : H_c^2(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$, where n is prime to $\mathrm{char}(k)$, and we show that it is an isomorphism if X is connected and n is prime to $\mathrm{char}(\widetilde{k})$. The central fact of § 6.3 (Theorem 6.3.2) states that any tame finite étale Galois covering of the one-dimensional closed disc is trivial (an étale morphism $\varphi : Y \to X$ is said to be tame if for any point $y \in Y$ the degree $[\mathscr{H}(y) : \mathscr{H}(\varphi(y))]$ is not divisible by $\mathrm{char}(\widetilde{k})$). We deduce from this, in particular, a Riemann Existence Theorem which states that, for an algebraic curve $\mathscr{X}$ of locally finite type over k, the functor $\mathscr{Y} \mapsto \mathscr{Y}^{an}$ defines an equivalence between the category of finite étale Galois coverings of $\mathscr{X}$ whose degree is prime to $\mathrm{char}(\widetilde{k})$ and the category of similar coverings of $\mathscr{X}^{an}$. We deduce also the Comparison Theorem for curves (the proof of this theorem in the general case (§ 7.5) does not use the particular case). In § 6.4 we prove that, for a one-dimensional k-affinoid space X and a positive integer n which is prime to $\mathrm{char}(\widetilde{k})$, the group $H^q(X, \mathbf{Z}/n\mathbf{Z})$ is finite for $q = 0, 1$ and equal to zero for $q \geq 2$. Furthermore, this group is preserved under algebraically closed extensions of the ground field.

In § 7 we obtain our main results. In § 7.1 we prove the Comparison Theorem for Cohomology with Compact Support. This result implies that the cohomology groups with compact support of a scheme of locally finite type over k (recall that k may have trivial valuation) can be defined as the right derived functors of the functor of sections with compact support over the associated k-analytic space. In § 7.2 we construct a trace mapping $\mathrm{Tr}_\varphi : R^{2d}\varphi_!(\mu_{n,X}^d) \to (\mathbf{Z}/n\mathbf{Z})_X$ for an arbitrary separated smooth morphism $\varphi : Y \to X$ of pure dimension d and for n prime to $\mathrm{char}(k)$, and we show that it is an isomorphism if the geometric fibres of φ are nonempty and connected and n is prime to $\mathrm{char}(\widetilde{k})$.

In the rest of § 7 all sheaves considered are torsion with torsion orders prime to $\mathrm{char}(\widetilde{k})$. In § 7.3 we prove the Poincaré Duality Theorem, which is actually a central result of this work. The main ingredients of the proof are our Theorem 3.7.2 and the Fundamental Lemma ([SGA4], Exp. XVIII, 2.14.2) from the proof of the Poincaré Duality Theorem for schemes. In § 7.4 we give first applications of Poincaré Duality. In particular, we prove the acyclicity of the canonical projections $X \times \mathbf{A}^d \to X$ and $X \times D \to X$ and the Cohomological Purity Theorem. In § 7.5 we prove the Comparison Theorem. The proof follows closely the proof of Deligne's "generic" theorem 1.9 from [SGA4½], Th. finitude, and uses it. (I am indebted to D. Kazhdan for suggesting that Deligne's "Th. finitude" could be useful for the proof of the Comparison Theorem.) The proof is actually a formal reasoning which works over the field of complex numbers $\mathbf{C}$ as well. In § 7.6 we prove that the cohomology groups $H^q(X, F)$ and $H_c^q(X, F)$ are preserved under algebraically closed extensions of the ground field. In § 7.7 we deduce from this and from Theorem 5.3.1 the Base Change Theorem for Cohomology with Compact Support. It implies, in particular, a Künneth Formula. In § 7.8 we prove the Smooth Base Change Theorem. The proof uses Poincaré Duality and the Base Change Theorem for Cohomology with Compact Support.

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In the first two versions only the analytic spaces from [Ber] were considered. I am very grateful to P. Deligne for his further remarks and comments that stimulated me to introduce more general analytic spaces and to extend to them all the results obtained.

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§ 1. Analytic spaces

1.1. Underlying topological spaces

In this subsection we introduce some structures on topological spaces that will be used in the sequel. We also fix general topology terminology.

All compact, locally compact and paracompact spaces are assumed to be Hausdorff. (A Hausdorff topological space is called paracompact if any open covering of it has a locally finite refinement.) Recall that a locally compact space is paracompact if and only if it is a disjoint union of open and closed subspaces countable at infinity ([Bou], Ch. I, § 10, n° 12, [En], 5.1.27). Recall also that if a Hausdorff topological space has a locally finite covering by paracompact closed subsets, then the space is paracompact ([En], 5.1.34). A topological space is said to be *locally Hausdorff* if each point of it has an open Hausdorff neighborhood.

Let X be a topological space, and let τ be a collection of subsets of X. (All subsets of X are provided with the induced topology.) For a subset $Y \subset X$ we set $\tau|_Y = \{ V \in \tau \mid V \subset Y \}$. We say that τ is *dense* if, for any $V \in \tau$, each point of V has a fundamental system of neighborhoods in V consisting of sets from $\tau|_V$. Furthermore, we say that τ is a *quasinet on* X if, for each point $x \in X$, there exist $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of x.

1.1.1. *Lemma. — Let τ be a quasinet on a topological space* X.

(i) *A subset $\mathcal{U} \subset X$ is open if and only if for each $V \in \tau$ the intersection $\mathcal{U} \cap V$ is open in* V.

(ii) *Suppose that τ consists of compact sets. Then* X *is Hausdorff if and only if for any pair* $U, V \in \tau$ *the intersection* $U \cap V$ *is compact.*

Proof. — The direct implication in both statements is trivial.

(i) Suppose that $\mathcal{U} \cap V$ is open in V for all $V \in \tau$. For a point $x \in \mathcal{U}$ we take $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \ldots \cap V_n$ and $V_1 \cup \ldots \cup V_n$ is a neighborhood of x in X. By hypothesis, there exist open sets $\mathcal{V}_i \subset X$ with $\mathcal{U} \cap V_i = \mathcal{V}_i \cap V_i$. Then the set $\mathcal{V} := \mathcal{V}_1 \cap \ldots \cap \mathcal{V}_n$ is an open neighborhood of x in X. It follows that the set $\mathcal{U} \cap (V_1 \cup \ldots \cup V_n)$ is a neighborhood of x because it contains the intersection $\mathcal{V} \cap (V_1 \cup \ldots \cup V_n)$ which is a neighborhood of x. Therefore $\mathcal{U}$ is open in X.

(ii) Suppose that $U \cap V$ are compact for all pairs $U, V \in \tau$. Since

$$\tau \times \tau := \{ U \times V \mid U, V \in \tau \}$$

is a quasinet on $X \times X$, then, by (1), it suffices to verify that the intersection of the diagonal with any $U \times V$ for $U, V \in \tau$ is closed in $U \times V$. But this intersection is homeomorphic to the compact set $U \cap V$, and therefore it is closed in $U \times V$. ∎

We remark that if X is Hausdorff, then to establish that a collection of compact subsets τ is a quasinet, it suffices to verify that each point of X has a neighborhood of the form $V_1 \cup \ldots \cup V_n$ with $V_i \in \tau$. We remark also that a Hausdorff space admitting a quasinet of compact subsets is locally compact.

Furthermore, we say that a collection τ of subsets of X is a *net on* X if it is a quasinet and, for any pair $U, V \in \tau$, $\tau|_{U \cap V}$ is a quasinet on $U \cap V$.

1.1.2. *Lemma. — Let* τ *be a net of compact sets on a topological space* X. *Then*

(i) *for any pair* $U, V \in \tau$, *the intersection* $U \cap V$ *is locally closed in* U *and* V;

(ii) *if* $V \subset V_1 \cup \ldots \cup V_n$ *for some* $V, V_1, \ldots, V_n \in \tau$, *then there exist* $U_1, \ldots, U_m \in \tau$ *such that* $V = U_1 \cup \ldots \cup U_m$ *and each* U_j *is contained in some* V_i.

Proof. — (i) It suffices to verify that $U \cap V$ is locally compact in the induced topology. But this is clear because $\tau|_{U \cap V}$ is a quasinet on $U \cap V$.

(ii) For each point $x \in V$ and for each i with $x \in V_i$, we take a neighborhood of x in $V \cap V_i$ of the form $V_{i1} \cup \ldots \cup V_{im_i}$, where $V_{ij} \in \tau$. Then the union of such neighborhoods over all i with $x \in V_i$ is a neighborhood of x in V of the form $U_1 \cup \ldots \cup U_m$ such that each U_j belongs to τ and is contained in some V_i. Since V is compact, we get the required fact. ∎

The underlying topological spaces of analytic spaces will be, by Definition 1.2.3 below, locally Hausdorff and provided with a net of compact subsets. It will follow from the definition (Remark 1.2.4 (iii)) that they admit a basis of open locally compact paracompact arcwise connected subsets (see also Proposition 1.2.18),

A continuous map of topological spaces $\varphi : Y \to X$ is said to be *Hausdorff* if for any pair of different points $y_1, y_2 \in Y$ with $\varphi(y_1) = \varphi(y_2)$ there exist open neighborhoods $\mathcal{V}_1$ of y_1 and $\mathcal{V}_2$ of y_2 with $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ (i.e., the image of Y in $Y \times_X Y$ is closed). We remark that if $\varphi : Y \to X$ is Hausdorff and X is Hausdorff, then Y is also Hausdorff. Furthermore, let X and Y be topological spaces and suppose that each point of X has a compact neighborhood. A continuous map $\varphi : Y \to X$ is said to be *compact* if the preimage of a compact subset of X is a compact subset of Y. It is clear that such a map is Hausdorff, it takes closed subsets of Y to closed subsets of X, and each point of Y has a compact neighborhood.

1.2. The category of analytic spaces

Throughout the paper we fix a non-Archimedean field k. (We don't assume that the valuation on k is nontrivial.) The category of k-affinoid spaces is, by definition, the category dual to the category of k-affinoid algebras (see [Ber], § 2.1). The k-affinoid space associated with a k-affinoid algebra $\mathcal{A}$ is denoted by X, where $X = \mathcal{M}(\mathcal{A})$. (For properties of k-affinoid spaces and their affinoid domains, see *loc. cit.*, § 2.) The notion of a k-analytic space we are going to introduce is based essentially on the following two fundamental facts. Let $\{V_i\}_{i \in I}$ be a finite affinoid covering of a k-affinoid space $X = \mathcal{M}(\mathcal{A})$.

Tate's Acyclicity Theorem. — *For any finite Banach $\mathcal{A}$-module* M, *the Čech complex*

$$0 \to M \to \prod_i M \otimes_{\mathcal{A}} \mathcal{A}_{V_i} \to \prod_{i,j} M \otimes_{\mathcal{A}} \mathcal{A}_{V_i \cap V_j} \to \dots$$

is exact and admissible. ∎

Kiehl's Theorem. — *Suppose we are given, for each $i \in I$, a finite $\mathcal{A}_{V_i}$-module M_i and, for each pair $i, j \in I$, an isomorphism of $\mathcal{A}_{V_i \cap V_j}$-modules $\alpha_{ij} : M_i \otimes_{\mathcal{A}_{V_i}} \mathcal{A}_{V_i \cap V_j} \xrightarrow{\sim} M_j \otimes_{\mathcal{A}_{V_j}} \mathcal{A}_{V_i \cap V_j}$ such that $\alpha_{il}|_W = \alpha_{jl}|_W \circ \alpha_{ij}|_W$, $W = V_i \cap V_j \cap V_l$, for all $i, j, l \in I$. Then there exists a finite $\mathcal{A}$-module M that gives rise to the $\mathcal{A}_{V_i}$-modules M_i and to the isomorphisms α_{ij}.* ∎

Both results are originally proved in the case when the valuation on k is nontrivial and all the spaces considered are strictly k-affinoid (see [BGR], 8.2.1/5 and 9.4.3/3). But the general case is reduced to this one by the standard argument from [Ber], § 2.1 (2.2.5 and 2.1.11). Tate's Acyclicity Theorem is sufficient to define the category of k-analytic spaces, and Kiehl's Theorem is used to establish their basic properties.

1.2.1. *Remarks.* — (i) Let V be a subset of a k-affinoid space $X = \mathcal{M}(\mathcal{A})$ which is a finite union of affinoid domains $\{V_i\}_{i \in I}$. From Tate's Acyclicity Theorem it follows that the commutative Banach k-algebra $\mathcal{A}_V = \mathrm{Ker}(\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i \cap V_j})$ does not depend (up to a canonical isomorphism) on the covering. Furthermore, V is an affinoid domain if and only if the Banach algebra $\mathcal{A}_V$ is k-affinoid and the canonical map $V \to \mathcal{M}(\mathcal{A}_V)$ is bijective. (In [Ber], 2.2.6 (iii), the latter condition was missed.)

(ii) From Tate's Acyclicity Theorem it follows that in the situation of Kiehl's Theorem the $\mathcal{A}$-module M is isomorphic to $\mathrm{Ker}(\prod_i M_i \to \prod_{i,j} M_i \otimes_{\mathcal{A}_{V_i}} \mathcal{A}_{V_i \cap V_j})$. (Recall that, by [Ber], 2.1.9, the category of finite Banach $\mathcal{A}$-modules is equivalent to the category of finite $\mathcal{A}$-modules.)

Our purpose is to introduce a category of Φ-analytic spaces associated with a system Φ of the following form. Suppose we are given for each non-Archimedean field K over k a class of K-affinoid spaces Φ_K so that the system $\Phi = \{\Phi_K\}$ satisfies the following conditions:

(1) $\mathcal{M}(K) \in \Phi_K$;

(2) Φ_K is stable under isomorphisms and direct products;

(3) if $\varphi : Y \to X$ is a finite morphism of K-affinoid spaces and $X \in \Phi_K$, then $Y \in \Phi_K$;

(4) if $\{V_i\}_{i\in I}$ is a finite affinoid covering of a K-affinoid space X such that $V_i \in \Phi_K$ for all $i \in I$, then $X \in \Phi_K$;

(5) if $K \hookrightarrow L$ is an isometric embedding of non-Archimedean fields over k, then for any $X \in \Phi_K$ one has $X \hat{\otimes}_K L \in \Phi_L$.

The class Φ_K is said to be *dense* if each point of each $X \in \Phi_K$ has a fundamental system of affinoid neighborhoods $V \in \Phi_K$. The system Φ is said to be *dense* if all Φ_K are dense.

The affinoid spaces from Φ_K (resp. Φ) and their algebras will be called Φ_K-(resp. Φ-) *affinoid*. From (2) and (3) it follows that Φ_K is stable under fibre products. In particular, if $\varphi : Y \to X$ is a morphism of Φ_K-affinoid spaces, then for any affinoid domain $V \subset X$ with $V \in \Phi_K$ one has $\varphi^{-1}(V) \in \Phi_K$.

1.2.2. *Remark.* — In fact we shall consider in this paper only analytic spaces for the system of all affinoid spaces. The more general setting is necessary for establishing connection with rigid analytic geometry in § 1.6 (see also Remark 1.2.16). For this one takes for Φ_K the class of strictly K-affinoid spaces. That this Φ_K satisfies (4) is shown as follows. Let $X = \mathcal{M}(\mathcal{A})$. By Tate's Acyclicity Theorem, the algebra $\mathcal{A}$ is a closed subalgebra of the direct product $\prod_i \mathcal{A}_{V_i}$. It follows that for the spectral radius $\rho(f)$ of an element $f \in \mathcal{A}$, one has $\rho(f) = \max_i \rho_{V_i}(f)$. Since $\rho_{V_i}(f) \in \sqrt{|k^*|} \cup \{0\}$, then $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$, and therefore $\mathcal{A}$ is strictly K-affinoid, by [Ber], 2.1.6. (Of course, in § 1.6 one assumes also that the valuation on k is nontrivial. In this case the system Φ is dense.) Here is one more example of Φ. Assume that the valuation on k is trivial. If the valuation on K is also trivial, then we take for Φ_K the class of K-affinoid spaces $X = \mathcal{M}(\mathcal{A})$ such that $\rho(f) \leqslant 1$ for all $f \in \mathcal{A}$. Otherwise we take for Φ_K the class of all K-affinoid spaces. The system $\Phi = \{\Phi_K\}$ satisfies the conditions (1)-(5) and it is dense.

Let X be a locally Hausdorff topological space, and let τ be a net of compact subsets on X.

1.2.3. *Definition.* — A *Φ_k-affinoid atlas $\mathcal{A}$ on X with the net τ* is a map which assigns, to each $V \in \tau$, a Φ_k-affinoid algebra $\mathcal{A}_V$ and a homeomorphism $V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$ and, to each pair $U, V \in \tau$ with $U \subset V$, a bounded homomorphism of k-affinoid algebras $\alpha_{V/U} : \mathcal{A}_V \to \mathcal{A}_U$ that identifies $(U, \mathcal{A}_U)$ with an affinoid domain in $(V, \mathcal{A}_V)$.

1.2.4. *Remarks.* — (i) It follows from the definition that, for any triple $U, V, W \in \tau$ with $U \subset V \subset W$, one has $\alpha_{W/U} = \alpha_{V/U} \circ \alpha_{W/V}$.

(ii) The family of Φ_k-affinoid atlases with the same net forms a category.

(iii) By [Ber], 2.2.8 and 3.2.1, each point of a k-affinoid space has a fundamental system of open arcwise connected subsets that are countable at infinity. It follows that a basis of topology of a Φ_k-analytic space is formed by open locally compact paracompact arcwise connected subsets.

A triple $(X, \mathcal{A}, \tau)$ of the above form is said to be a *Φ_k-analytic space*. To define morphisms between them, we need a preparatory work. First, we'll define a category Φ_k-$\widetilde{\mathcal{A}n}$ whose objects are the Φ_k-analytic spaces and whose morphisms will be called strong morphisms. After that the category of Φ_k-analytic spaces Φ_k-$\mathcal{A}n$ will be constructed as the category of fractions of Φ_k-$\widetilde{\mathcal{A}n}$ with respect to a certain system of strong morphisms that admits calculus of right fractions.

Let $(X, \mathcal{A}, \tau)$ be a Φ_k-analytic space.

1.2.5. *Lemma. — If* W *is a* Φ*-affinoid domain in some* $U \in \tau$, *then it is a* Φ*-affinoid domain in any* $V \in \tau$ *that contains* W.

Proof. — Since $\tau|_{U \cap V}$ is a net and W is compact, we can find $U_1, \ldots, U_n \in \tau|_{U \cap V}$ with $W \subset U_1 \cup \ldots \cup U_n$. Furthermore, since W and U_i are Φ-affinoid domains in U, then $W_i := W \cap U_i$ is a Φ-affinoid domain in U_i. It follows also that W_i and $W_i \cap W_j$ are Φ-affinoid domains in V. By Tate's Acyclicity Theorem, applied to the affinoid covering $\{ W_i \}$ of W the Banach algebra $\mathcal{A}_W = \mathrm{Ker}(\prod_i \mathcal{A}_{W_i} \to \prod_{i,j} \mathcal{A}_{W_i \cap W_j})$ is k-affinoid and $W \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_W)$. By Remark 1.2.1 (i), W is a Φ-affinoid domain in V. ∎

Let $\bar{\tau}$ denote the family of all W such that W is a Φ-affinoid domain in some $V \in \tau$. If Φ_k is dense, then $\bar{\tau}$ is dense.

1.2.6. *Proposition. — The family* $\bar{\tau}$ *is a net on* X, *and there exists a unique (up to a canonical isomorphism)* Φ_k*-affinoid atlas* $\bar{\mathcal{A}}$ *with the net* $\bar{\tau}$ *that extends* $\mathcal{A}$.

Proof. — Let $U, V \in \bar{\tau}$ and $x \in U \cap V$. Take $U', V' \in \tau$ with $U \subset U'$ and $V \subset V'$. We can find a neighborhood $W_1 \cup \ldots \cup W_n$ of x in $U' \cap V'$ with $W_i \in \tau$ and $x \in W_1 \cap \ldots \cap W_n$. Since U (resp. V) and W_i are Φ-affinoid domains in U' (resp. V'), then $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a Φ-affinoid domain in W_i, and therefore $U_i \cap V_i$ is a Φ-affinoid domain in W_i, i.e., $U_i \cap V_i \in \bar{\tau}|_{U \cap V}$. Since

$$\bigcup_i (U_i \cap V_i) = (U \cap V) \cap (\bigcup_i W_i),$$

then $\bigcup_i (U_i \cap V_i)$ is a neighborhood of x in $U \cap V$ with $x \in \bigcap_i (U_i \cap V_i)$. It follows that $\bar{\tau}$ is a net.

Furthermore, for each $V \in \tau$ we fix $V' \in \tau$ with $V \subset V'$ and assign to V the algebra $\mathcal{A}_V$ and the homeomorphism $V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$ arising from $(V', \mathcal{A}_{V'})$. We have to construct, for each pair $U, V \in \bar{\tau}$ with $U \subset V$, a canonical bounded homomorphism $\mathcal{A}_V \to \mathcal{A}_U$ that identifies $(U, \mathcal{A}_U)$ with an affinoid domain in $(V, \mathcal{A}_V)$. Consider first the case when $V \in \tau$. Since $\tau_{U' \cap V}$ is a quasinet, we can find sets $U_1, \ldots, U_n$ that are Φ-affinoid domains in U' and V and such that $U = U_1 \cup \ldots \cup U_n$. By Tate's Acyclicity Theorem, $\mathcal{A}_U = \mathrm{Ker}(\prod_i \mathcal{A}_{U_i} \to \prod_{i,j} \mathcal{A}_{U_i \cap U_j})$, and therefore the homomorphisms $\mathcal{A}_V \to \mathcal{A}_{U_i}$ and $\mathcal{A}_V \to \mathcal{A}_{U_i \cap U_j}$ induce a bounded homomorphism $\mathcal{A}_V \to \mathcal{A}_U$ that identifies $(U, \mathcal{A}_U)$ with an affinoid domain in $(V, \mathcal{A}_V)$. In particular, the homomorphism constructed does not depend on the choice of $U_1, \dots, U_n$. Assume now that V is arbitrary. Then $U \subset V'$ and, by the first case, there is a canonical bounded homomorphism $\mathscr{A}_{V'} \to \mathscr{A}_U$ that identifies $(U, \mathscr{A}_U)$ with an affinoid domain in $(V', \mathscr{A}_{V'})$. It follows that $(U, \mathscr{A}_U)$ is a Φ-affinoid domain in $(V, \mathscr{A}_V)$. ■

1.2.7. *Definition.* — A *strong morphism* of Φ_k-analytic spaces

$$\varphi : (X, \mathscr{A}, \tau) \to (X', \mathscr{A}', \tau')$$

is a pair which consists of a continuous map $\varphi : X \to X'$, such that for each $V \in \tau$ there exists $V' \in \tau'$ with $\varphi(V) \subset V'$, and of a system of compatible morphisms of k-affinoid spaces $\varphi_{V/V'} : (V, \mathscr{A}_V) \to (V', \mathscr{A}'_{V'})$ for all pairs $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$.

1.2.8. *Proposition.* — *Any strong morphism* $\varphi : (X, \mathscr{A}, \tau) \to (X', \mathscr{A}', \tau')$ *extends in a unique way to a strong morphism* $\bar{\varphi} : (X, \bar{\mathscr{A}}, \bar{\tau}) \to (X, \bar{\mathscr{A}}', \bar{\tau}')$.

Proof. — Let U and U′ be Φ-affinoid domains in $V \in \tau$ and $V' \in \tau'$, respectively, and suppose that $\varphi(U) \subset U'$. Take $W' \in \tau'$ with $\varphi(V) \subset W'$. Then $\varphi(U) \subset W_1 \cup \dots \cup W_n$ for some $W_1, \dots, W_n \in \tau'|_{V' \cap W'}$. The morphism of k-affinoid spaces $\varphi_{V/W'}$ induces a morphism $V_i := \varphi^{-1}_{V/W'}(W_i) \to W_i$ that induces, in its turn, a morphism $U_i := U \cap V_i \to U'_i := U' \cap W_i$ (the latter is a Φ-affinoid domain in V′). Thus, we have a system of morphisms of k-affinoid spaces $U_i \to U'_i \to U'$ that are compatible on intersections. It gives rise to a morphism $\bar{\varphi}_{U/U'} : (U, \mathscr{A}_U) \to (U', \mathscr{A}_{U'})$. It clear that the morphisms $\bar{\varphi}_{U/U'}$ are compatible. ■

We now define the composition χ of two strong morphisms

$$\varphi : (X, \mathscr{A}, \tau) \to (X', \mathscr{A}', \tau') \quad \text{and} \quad \psi : (X', \mathscr{A}', \tau') \to (X'', \mathscr{A}'', \tau'').$$

The map χ that is the composition of the maps φ and ψ satisfies the necessary condition of the Definition 1.2.7. Furthermore, by Proposition 1.2.8, we may assume that φ and ψ are extended to the morphisms $\bar{\varphi}$ and $\bar{\psi}$. Suppose now that we are given a pair $V \in \tau$ and $V'' \in \tau''$ with $\chi(V) \subset V''$. We have to define a morphism of k-affinoid spaces $\chi_{V/V''} : (V, \mathscr{A}_V) \to (V'', \mathscr{A}_{V''})$. For this we take $V' \in \tau'$ and $U'' \in \tau''$ with $\varphi(V) \subset V'$ and $\psi(V') \subset U''$. Since $\chi(V) \subset U'' \cap V''$ and V is compact, it follows that there exist $V''_1, \dots, V''_n \in \tau''|_{U'' \cap V''}$ with $\chi(V) \subset V''_1 \cup \dots \cup V''_n$. Then $V'_i := \psi^{-1}_{V'/U''}(V''_i)$ and $V_i := \varphi^{-1}_{V/V'}(V'_i)$ are Φ-affinoid domains in V′ and V, respectively, and $V = V_1 \cup \dots \cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ induce morphisms of k-affinoid spaces $V_i \to V''_i$, and since V''_i are Φ-affinoid domains in V″, they induce a system of morphisms $V_i \to V''$ that are compatible on intersections. It gives rise to the required morphism of k-affinoid spaces $\chi_{V/V''} : (V, \mathscr{A}_V) \to (V'', \mathscr{A}_{V''})$. It is easy to see that the morphisms $\chi_{V/V''}$ are compatible. Hence we get a morphism χ that is the composition of φ and ψ and is denoted by $\psi \circ \varphi$ (or simply by $\psi\varphi$). Thus, we get a category $\widetilde{\Phi_k\text{-}\mathscr{A}n}$.

1.2.9. *Definition.* — A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if φ induces a homeomorphism between X and X′ and, for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$, $\varphi_{V/V'}$ identifies V with an affinoid domain in V′.

It is easy to see that if φ is a quasi-isomorphism, then so is $\bar{\varphi}$.

1.2.10. *Proposition.* — *The system of quasi-isomorphisms in* Φ_k-$\widetilde{\mathcal{A}n}$ *admits calculus of right fractions.*

Proof. — We have to verify (see [GaZi], Ch. I, § 2, 2.2) that the system satisfies the following properties:

a) all identity morphisms are quasi-isomorphisms;

b) the composition of two quasi-isomorphisms is a quasi-isomorphism;

c) any diagram of the form $(X, \mathcal{A}, \tau) \xrightarrow{\varphi} (X', \mathcal{A}', \tau') \xleftarrow{g} (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$, where g is a quasi-isomorphism, can be complemented to a commutative square

$$\begin{array}{ccc} (X, \mathcal{A}, \tau) & \xrightarrow{\varphi} & (X', \mathcal{A}', \tau') \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} \\ (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}') \end{array}$$

where f is a quasi-isomorphism;

d) if for two strong morphisms $\varphi, \psi : (X, \mathcal{A}, \tau) \rightrightarrows (X', \mathcal{A}', \tau')$ and for a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$ one has $g\varphi = g\psi$, then there exists a quasi-isomorphism $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi f = \psi f$. (We'll show, in fact, that in this situation $\varphi = \psi$.)

The property *a)* is obviously valid. To verify *b)*, it suffices to apply the construction of the composition and Remark 1.2.1 (i). To verify *c)*, we need the following fact.

1.2.11. *Lemma.* — *Let* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ *be a strong morphism. Then for any pair* $V \in \bar{\tau}$ *and* $V' \in \bar{\tau}'$ *the intersection* $V \cap \varphi^{-1}(V')$ *is a finite union of* Φ*-affinoid domains in* V.

Proof. — Take $U' \in \bar{\tau}'$ with $\varphi(V) \subset U'$. Then we can find $U'_1, \ldots, U'_n \in \tau'|_{U' \cap V'}$ with $\varphi(V) \subset U'_1 \cup \ldots \cup U'_n$, and $V \cap \varphi^{-1}(V') = \bigcup_i \varphi^{-1}_{V/U'}(U'_i)$. ∎

Suppose that we have a diagram as in *c)*. We may assume that $\widetilde{X}' = X'$. Then $\widetilde{\tau}' \subset \tau'$. Let $\widetilde{\tau}$ denote the family of all $V \in \bar{\tau}$ for which there exists $\widetilde{V}' \in \widetilde{\tau}'$ with $\varphi(V) \subset \widetilde{V}'$. From Lemma 1.2.11 it follows that $\widetilde{\tau}$ is a net. The Φ_k-affinoid atlas $\overline{\mathcal{A}}$ defines a Φ_k-affinoid atlas $\widetilde{\mathcal{A}}$ with the net $\widetilde{\tau}$, and the strong morphism $\bar{\varphi}$ induces a strong morphism $\widetilde{\varphi} : (X, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X', \widetilde{\mathcal{A}}', \widetilde{\tau}')$. Then $\widetilde{\varphi}$ and the canonical quasi-isomorphism

$$f : (X, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$$

satisfy the required property *c)*.

Finally, we claim that in the situation *d)* the morphisms φ and ψ coincide. First of all, it is clear that they coincide as maps. Furthermore, let $V \in \tau$ and $V' \in \tau'$ be such that $\varphi(V) \subset V'$. Take $\widetilde{V}' \in \widetilde{\tau}'$ with $g(V') \subset \widetilde{V}'$. Then we have two morphisms of k-affinoid space $\varphi_{V/V'}, \psi_{V/V'} : V \rightrightarrows V'$ such that their compositions with $g_{V'/\widetilde{V}'}$ coincide. Since V' is an affinoid domain in $\widetilde{V}'$, it follows that $\varphi_{V/V'} = \psi_{V/V'}$. ∎

The category of Φ_k-analytic spaces Φ_k-$\mathcal{A}n$ is, by definition, the category of fractions of Φ_k-$\widetilde{\mathcal{A}n}$ with respect to the system of quasi-isomorphisms. By Proposition 1.2.10 morphisms in the category Φ_k-$\mathcal{A}n$ can be described as follows. Let $(X, \mathcal{A}, \tau)$ be a Φ_k-analytic space. If σ is a net on X, we write $\sigma \prec \tau$ if $\sigma \subset \bar{\tau}$. Then the Φ_k-affinoid atlas $\bar{\mathcal{A}}$ defines a Φ_k-affinoid atlas $\mathcal{A}_\sigma$ with the net σ, and there is a canonical quasi-isomorphism $(X, \mathcal{A}_\sigma, \sigma) \to (X, \mathcal{A}, \tau)$. The system of nets $\{\sigma\}$ with $\sigma \prec \tau$ is filtered and, for any Φ_k-analytic space $(X', \mathcal{A}', \tau')$, one has

$$\mathrm{Hom}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')) = \varinjlim_{\sigma \prec \tau} \mathrm{Hom}_{\widetilde{\mathcal{A}n}}((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')).$$

We remark that all the maps in the inductive system are injective.

We now want to construct a maximal Φ_k-affinoid atlas on a Φ_k-analytic space and to describe the set of morphisms between two Φ_k-analytic spaces in terms of their maximal atlases. (Kiehl's Theorem will be used here for the first time.)

Let $(X, \mathcal{A}, \tau)$ be a Φ_k-analytic space. We say that a subset $W \subset X$ is τ-*special* if it is compact and there exists a covering $W = W_1 \cup \ldots \cup W_n$ such that W_i, $W_i \cap W_j \in \tau$ and $\mathcal{A}_{W_i} \hat{\otimes} \mathcal{A}_{W_j} \to \mathcal{A}_{W_i \cap W_j}$ is an admissible epimorphism. A covering of W of the above type will be said to be a τ-*special covering* of W.

1.2.12. *Lemma. — Let* W *be a* τ-*special subset of* X. *If* $U, V \in \tau|_W$, *then* $U \cap V \in \bar{\tau}$ *and* $\mathcal{A}_U \hat{\otimes} \mathcal{A}_V \to \mathcal{A}_{U \cap V}$ *is an admissible epimorphism.*

Proof. — Since the sets $U \cap W_i$ and $V \cap W_j$ are compact, we can find finite coverings $\{U_{ik}\}_k$ of $U \cap W_i$ and $\{V_{jl}\}_l$ of $V \cap W_j$ by sets from τ. Furthermore, since $W_i \cap W_j \to W_i \times W_j$ are closed immersions, is follows that $U_{ik} \cap V_{jl} \in \bar{\tau}$ and $U_{ik} \cap V_{jl} \to U_{ik} \times V_{jl}$ is a closed immersion. Consider now the finite affinoid covering $\{U_{ik} \times V_{jl}\}_{i,j,k,l}$ of the k-affinoid space $U \times V$. For each quadruplet i, j, k, l, $\mathcal{A}_{U_{ik} \cap V_{jl}}$ is a finite $\mathcal{A}_{U_{ik} \times V_{jl}}$-algebra, and the system $\{\mathcal{A}_{U_{ik} \cap V_{jl}}\}$ satisfies the condition of Kiehl's Theorem. It follows that this system is defined by a finite $\mathcal{A}_{U \times V}$-algebra isomorphic to $\mathcal{A}_{U \cap V} := \mathrm{Ker}(\prod \mathcal{A}_{U_{ik} \cap V_{jl}} \to \prod \mathcal{A}_{U_{ik} \cap V_{jl} \cap U_{i'k'} \cap V_{j'l'}})$, and therefore the latter algebra is Φ_k-affinoid, $U \cap V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$, and $U \cap V \to U \times V$ is a closed immersion. By Remark 1.2.1 (i), $U \cap V$ is a Φ-affinoid domain in U and V, i.e., $U \cap V \in \bar{\tau}$. ∎

Let W be a τ-special subset of X. From Lemma 1.2.12 it follows that any finite covering of W by sets from τ is a τ-special covering. Furthermore, if $\{W_i\}$ is a τ-special covering, then from Tate's Acyclicity Theorem it follows that the commutative Banach k-algebra $\mathcal{A}_W := \mathrm{Ker}\,(\prod_i \mathcal{A}_{W_i} \to \prod_{ij} \mathcal{A}_{W_i \cap W_j})$ does not depend (up to a canonical isomorphism) on the covering, and a continuous map $W \to \mathcal{M}(\mathcal{A}_W)$ is well defined. Let $\hat{\tau}$ denote the collection of all $\bar{\tau}$-special subsets W such that the algebra $\mathcal{A}_W$ is k-affinoid, $W \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_W)$ and, for some $\bar{\tau}$-special covering $\{W_i\}$ of W, $(W_i, \mathcal{A}_{W_i})$ are affinoid domains in $(W, \mathcal{A}_W)$. We remark that from the condition (4) for the class Φ_k it follows that W belongs to Φ_k. Furthermore, the last property of W does not depend on the choice of the covering.

1.2.13. *Proposition.* — (i) *The collection* $\hat{\tau}$ *is a net, and for any net* $\sigma \prec \tau$ *one has* $\hat{\sigma} = \hat{\tau}$;

(ii) *there exists a unique (up to a canonical isomorphism)* Φ_k*-analytic atlas* $\hat{\mathcal{A}}$ *with the net* $\hat{\tau}$ *that extends the atlas* $\mathcal{A}$;

(iii) $\hat{\hat{\tau}} = \hat{\tau}$.

Proof. — (i) Let $U, V \in \hat{\tau}$. We take $\bar{\tau}$-special coverings $\{U_i\}$ of U and $\{V_j\}$ of V. Since $U \cap V = \bigcup_{i,j}(U_i \cap V_j)$ and $\bar{\tau}|_{U_i \cap V_j}$ are quasinets, it follows that $\hat{\tau}|_{U \cap V}$ is a quasinet. Furthermore, let σ is a net with $\sigma \prec \tau$. By Lemma 1.2.12, to verify the equality $\hat{\sigma} = \hat{\tau}$, it suffices to show that for any $V \in \bar{\tau}$ there exist $U_1, \dots, U_n \in \bar{\sigma}$ with $V = U_1 \cup \dots \cup U_n$. Since σ is a net on X, we can find $W_1, \dots, W_m \in \sigma$ with $V \subset W_1 \cup \dots \cup W_m$. Since $V, W_i \in \bar{\tau}$ and $\bar{\tau}$ is a net, then, by Lemma 1.1.2 (ii), we can find $U_1, \dots, U_n \in \bar{\tau}$ such that $V = U_1 \cup \dots \cup U_n$ and each U_i is contained in some W_j. Finally, since $W_j \in \sigma$, it follows that $U_i \in \bar{\sigma}$.

(ii) For each $V \in \hat{\tau}$ we fix a $\bar{\tau}$-special covering $\{V_i\}$ and assign to V the algebra $\mathcal{A}_V$ and the homeomorphism $V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$ arising from the covering. We have to construct for each pair $U, V \in \hat{\tau}$ with $U \subset V$ a canonical bounded homomorphism $\mathcal{A}_V \to \mathcal{A}_U$ that identifies $(U, \mathcal{A}_U)$ with an affinoid domain in $(V, \mathcal{A}_V)$. Consider first the case when $U \in \tau$. By Lemma 1.2.12, $U \cap V_i$ is an affinoid domain in V_i and therefore in V. It follows that U is an affinoid domain in V. If U is arbitrary, then by the first case each U_i from some $\bar{\tau}$-special covering of U is an affinoid domain in V. It follows that U is an affinoid domain in V.

(iii) From Lemma 1.2.12 it follows that $\bar{\hat{\tau}} = \hat{\tau}$. Let $\{V_i\}$ be a $\bar{\hat{\tau}}$-special covering of some $V \in \hat{\hat{\tau}}$. For each i we take a $\bar{\tau}$-special covering $\{V_{ij}\}_j$ of V_i. Then $\{V_{ij}\}_{i,j}$ is a $\bar{\tau}$-special covering of V, and therefore $V \in \hat{\tau}$. ∎

The sets from $\hat{\tau}$ are said to be Φ-*affinoid domains* in X. The $\hat{\tau}$-special sets are said to be Φ-*special domains* in X. They have a canonical Φ_k-analytic space structure. The following statement follows from Lemma 1.2.11 and Proposition 1.2.13 (i).

1.2.14. *Corollary.* — *If* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ *is a morphism of* Φ_k*-analytic spaces, then for any pair of* Φ*-affinoid domains* $V \subset X$ *and* $V' \subset X'$ *the intersection* $V \cap \varphi^{-1}(V')$ *is a* Φ*-special domain in* X. ∎

1.2.15. *Proposition.* — *Let* $(X, \mathcal{A}, \tau)$ *and* $(X', \mathcal{A}', \tau')$ *be* Φ_k*-analytic spaces.*

(i) *There is a one-to-one correspondence between the set* $\mathrm{Hom}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau'))$ *and the set of all pairs consisting of a continuous map* $\varphi : X \to X'$, *such that for each point* $x \in X$

there exist neighborhoods $V_1 \cup \ldots \cup V_n$ *of* x *and* $V'_1 \cup \ldots \cup V'_n$ *of* $\varphi(x)$ *with* $x \in V_1 \cap \ldots \cap V_n$ *and* $\varphi(V_i) \subset V'_i$, *where* $V_i \subset X$ *and* $V'_i \subset X'$ *are* Φ*-affinoid domains, and of a system of compatible morphisms of k-affinoid spaces* $\varphi_{V/V'} : (V, \mathcal{A}_V) \to (V', \mathcal{A}_{V'})$ *for all pairs of* Φ*-affinoid domains* $V \subset X$ *and* $V' \subset X'$ *with* $\varphi(V) \subset V'$.

(ii) *A morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ *is an isomorphism if and only if* φ *induces a homeomorphism between* X *and* X', $\varphi(\hat{\tau}) = \hat{\tau}'$ *and, for any* $V \in \hat{\tau}$, $(V, \mathcal{A}_V) \xrightarrow{\sim} (V', \mathcal{A}'_{V'})$, *where* $V' = \varphi(V)$.

Proof. — (i) Let σ be a net on X with $\sigma \prec \tau$, and let $\varphi : (X, \mathcal{A}_\sigma, \sigma) \to (X', \mathcal{A}', \tau')$ be a strong morphism. It is easy to extend the system of compatible morphisms of k-affinoid spaces $\varphi_{V/V'} : (V, \mathcal{A}_V) \to (V', \mathcal{A}_{V'})$ for all pairs $V \in \hat{\sigma}$ and $V' \in \hat{\tau}$ with $\varphi(V) \subset V'$. Since $\hat{\sigma} = \hat{\tau}$, we get a map (evidently injective) from the first set to the second one. Conversely, suppose that we have a pair of the above form. To verify that it comes from a morphism of Φ_k-analytic spaces, it suffices to show that the collection σ of all $V \in \hat{\tau}$ such that $\varphi(V) \subset V'$ for some $V' \in \hat{\tau}'$ is a net. For this we take a point $x \in X$ and neighborhoods $V_1 \cup \ldots \cup V_n$ of x and $V'_1 \cup \ldots V'_n$ of $\varphi(x)$ with $x \in V_1 \cap \ldots \cap V_n$ and $\varphi(V_i) \subset V'_i$, where $V_i \in \hat{\tau}$ and $V'_i \in \hat{\tau}'$. Then $V_i \in \sigma$, and we get the required fact.

(ii) follows from (i). ■

In practice we don't make a difference between $(X, \mathcal{A}, \tau)$ and the Φ_k-analytic spaces isomorphic to it. In particular, we shall denote it simply by X and assume that it is endowed with the maximal Φ_k-affinoid atlas. If it is necessary, we denote the underlying topological space by $|X|$. We remark that the functor that assigns to a Φ_k-affinoid space $X = \mathcal{M}(\mathcal{A})$ the Φ_k-analytic space $(X, \mathcal{A}, \{X\})$ is fully faithful. A Φ_k-analytic space isomorphic to such a space is called a Φ_k-affinoid space.

Furthermore, if Φ is the system of all affinoid spaces, then the category Φ_k-$\mathcal{A}n$ is denoted by k-$\mathcal{A}n$, and the corresponding spaces are called *k-analytic spaces*. In this case we withdraw the reference to Φ in the above and future definitions and notations. If Φ is the system of strictly affinoid spaces, then the category Φ_k-$\mathcal{A}n$ is denoted by *st-k-*$\mathcal{A}n$, and the corresponding spaces are called *strictly k-analytic spaces*. Similarly, instead of referring to Φ, we use the word " strictly " (strictly affinoid domains and so on).

1.2.16. *Remark.* — For an arbitrary Φ there is an evident functor Φ_k-$\mathcal{A}n \to k$-$\mathcal{A}n$. From Proposition 1.2.15 it follows that this functor is faithful. But we don't know whether it is fully faithful. (This was another reason for introducing the category of Φ_k-analytic spaces.) The only fact in this connection is Proposition 1.2.17.

We say that a Φ_k-analytic space is *good* if each point of it has a Φ-affinoid neighborhood.

1.2.17. *Proposition.* — *Let* X *and* Y *be* Φ_k*-analytic spaces, and assume that the class* Φ_k *is dense and* X *is good. Then any morphism of k-analytic spaces* $\varphi : Y \to X$ *is a morphism of* Φ_k*-analytic spaces.*

Proof. — It suffices to show that the family τ of all Φ-affinoid domains $V \subset Y$, for which there exists a Φ-affinoid domain $U \subset X$ with $\varphi(V) \subset U$, is a net. For an arbitrary point $y \in Y$ we take a Φ-affinoid neighborhood U of $\varphi(y)$. Since Φ_k is dense, we can find a neighborhood of y in $\varphi^{-1}(U)$ of the form $V_1 \cup \ldots \cup V_n$, where V_i are Φ-affinoid domains in Y and $y \in V_1 \cap \ldots \cap V_n$. We get $V_i \in \tau$, and therefore τ is a net. ∎

The *dimension* $\dim(X)$ of a Φ_k-analytic space X is the supremum of the dimensions of its Φ-affinoid domains. (The dimension of a k-affinoid space is defined in [Ber], p. 34.) We remark that the supremum can be taken over Φ-affinoid domains from some net, and, in particular, the dimension of X is the same whether the space is considered as an object of Φ_k-$\mathcal{A}n$ or of k-$\mathcal{A}n$.

1.2.18. *Proposition.* — *The topological dimension of a paracompact Φ_k-analytic space is at most the dimension of the space. If the space is strictly k-analytic, both numbers are equal.*

Proof. — Suppose first that the space $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. If X is strictly k-affinoid, the statement is proved in [Ber], 3.2.6. If X is arbitrary, we take a non-Archimedean field K of the form $K_{r_1, \ldots, r_n}$ (see [Ber], § 2.1) such that the algebra $\mathcal{A}' = \mathcal{A} \widehat{\otimes} K$ is strictly k-affinoid, and consider the map $\sigma : X \to X' = \mathcal{M}(\mathcal{A}')$ which takes a point $x \in X$ to the point $x' \in X'$ that corresponds to the multiplicative seminorm $\sum_\nu a_\nu T^\nu \mapsto \max_\nu | a_\nu(x) | r^\nu$. The map σ induces a homeomorphism of X with a closed subset of X'. Therefore the topological dimension of X is at most $\dim(X') = \dim(X)$.

If X is an arbitrary paracompact k-analytic (resp. strictly k-analytic) space, then it has a locally finite covering by affinoid (resp. strictly affinoid) domains, and therefore the statement follows from [En], 7.2.3. ∎

1.3. Analytic domains and G-topology on an analytic space

1.3.1. *Definition.* — A subset Y of a Φ_k-analytic space X is said to be a *Φ-analytic domain* if, for any point $y \in Y$, there exist Φ-affinoid domains $V_1, \ldots, V_n$ that are contained in Y and such that $y \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of y in Y (i.e., the restriction of the net of Φ-affinoid domains on Y is a net on Y).

We remark that the intersection of two Φ-analytic domains is a Φ-analytic domain, and the preimage of a Φ-analytic domain with respect to a morphism of a Φ_k-analytic spaces is a Φ-analytic domain. Furthermore, the family of Φ-affinoid domains that are contained in a Φ-analytic domain $Y \subset X$ defines a Φ_k-affinoid atlas on Y, and there is a canonical morphism of Φ_k-analytic spaces $\nu : Y \to X$. For any morphism $\varphi : Z \to X$ with $\varphi(Z) \subset Y$ there exists a unique morphism $\psi : Z \to Y$ with $\varphi = \nu\psi$. It is clear that a Φ-analytic domain that is isomorphic to a k-affinoid space is a Φ-affinoid domain. A morphism $\varphi : Y \to X$ that induces an isomorphism of Y with an open Φ-analytic domain in X is said to be an *open immersion*. If the class Φ_k is dense, then all open subsets of X are Φ-analytic domains.

1.3.2. *Proposition. — Let* $\{Y_i\}_{i\in I}$ *be a covering of a* Φ_k*-analytic space* X *by* Φ*-analytic domains such that each point of* X *has a neighborhood of the form* $Y_{i_1} \cup \ldots \cap Y_{i_n}$ *with* $y \in Y_{i_1} \cap \ldots \cap Y_{i_n}$ *(i.e.,* $\{Y_i\}_{i\in I}$ *is a quasinet on* X*). Then for any* Φ_k*-analytic space* X′ *the following sequence of sets is exact*

$$\mathrm{Hom}(X, X') \to \prod_i \mathrm{Hom}(Y_i, X') \rightrightarrows \prod_{i,j} \mathrm{Hom}(Y_i \cap Y_j, X').$$

Proof. — Let $\varphi_i : Y_i \to X'$ be a family of morphisms such that, for all pairs $i, j \in I$, $\varphi_i|_{Y_i \cap Y_j} = \varphi_j|_{Y_i \cap Y_j}$. Then these φ_i define a map $X \to X'$ which is continuous, by Lemma 1.1.1 (i). Furthermore, let τ be the collection of Φ-affinoid domains $V \subset X$ such that there exist $i \in I$ and a Φ-affinoid domain $V' \subset X'$ with $V \subset Y_i$ and $\varphi_i(V) \subset V'$. It is easy to see that τ is a net on X, and therefore there is a morphism $\varphi : X \to X'$ that gives rise to all the morphisms φ_i. ∎

We now consider a process of gluing of analytic spaces. Let $\{X_i\}_{i\in I}$ be a family of Φ_k-analytic spaces, and suppose that, for each pair $i, j \in I$, we are given a Φ-analytic domain $X_{ij} \subset X_i$ and an isomorphism of Φ_k-analytic spaces $\nu_{ij} : X_{ij} \stackrel{\sim}{\to} X_{ji}$ so that $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $X_{ij} \cap X_{il}$. We are looking for a Φ_k-analytic space X with a family of morphisms $\mu_i : X_i \to X$ such that:

(1) μ_i is an isomorphism of X_i with a Φ-analytic domain in X;
(2) all $\mu_i(X_i)$ cover X;
(3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$;
(4) $\mu_i = \mu_j \circ \nu_{ij}$ on X_{ij}.

If such X exists, we say that it is obtained by *gluing of* X_i *along* X_{ij}.

1.3.3. *Proposition. — The space* X *obtained by gluing of* X_i *along* X_{ij} *exists and is unique (up to a canonical isomorphism) in each of the following cases:*

a) all X_{ij} *are open in* X_i*;*

b) for any $i \in I$*, all* X_{ij} *are closed in* X_i *and the number of* $j \in I$ *with* $X_{ij} \neq \emptyset$ *is finite. Furthermore, in the case a) all* $\mu_i(X_i)$ *are open in* X*. In the case b) all* $\mu_i(X_i)$ *are closed in* X *and, if all* X_i *are Hausdorff (resp. paracompact), then* X *is Hausdorff (resp. paracompact).*

Proof. — Let $\tilde{X}$ be the disjoint union $\coprod_i X_i$. The system $\{\nu_{ij}\}$ defines an equivalence relation R on $\tilde{X}$. We denote by X the quotient space $\tilde{X}/R$ and by μ_i the induced maps $X_i \to X$. In the case *a)*, the equivalence relation R is open (see [Bou], Ch. I, § 9, nº 6), and therefore all $\mu_i(X_i)$ are open in X. In the case *b)*, the equivalence relation R is closed (see *loc. cit.*, nº 7), and therefore all $\mu_i(X_i)$ are closed in X and μ_i induces a homeomorphism $X_i \stackrel{\sim}{\to} \mu_i(X_i)$. Moreover, if all X_i are Hausdorff, then X is Hausdorff, by *loc. cit.*, exerc. 6. If all X_i are paracompact, then X is paracompact because it has a locally finite covering by closed paracompact subsets ([En], 5.1.34).

Furthermore, let τ denote the collection of all subsets $V \subset X$ for which there exists $i \in I$ such that $V \subset \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is a Φ-affinoid domain in X_i (in this case $\mu_j^{-1}(V)$ is a Φ-affinoid domain in X_j for any j with $V \subset \mu_j(X_j)$). It is easy to see that τ is a net, and there is an evident Φ_k-affinoid atlas $\mathscr{A}$ with the net X. In this way we get a Φ_k-analytic space $(X, \mathscr{A}, \tau)$ that satisfies the properties (1)-(4). That X is unique up to canonical isomorphism follows from Proposition 1.3.2. ∎

Let X be a Φ_k-analytic space. The family of its Φ-analytic domains can be considered as a category, and it gives rise to a Grothendieck topology generated by the pretopology for which the set of coverings of an analytic domain $Y \subset X$ is formed by the families $\{ Y_i \}_{i \in I}$ of analytic domains in Y that are quasinets on Y. For brevity, the above Grothendieck topology is called the *G-topology on* X, and the corresponding site is denoted by X_G. From Proposition 1.3.2 it follows that any representable presheaf on X_G is a sheaf. The G-topology on X is a natural framework for working with coherent sheaves.

Recall ([Ber], 1.5) that the n-dimensional affine space $\mathbf{A}^n$ is the set of all multiplicative semi-norm on the ring of polynomials $k[T_1, \dots, T_n]$ that extend the valuation on k endowed with the evident topology. The family of closed polydiscs with center at zero $E(0; r_1, \dots, r_n) = \{ x \in \mathbf{A}^n \mid |T_i(x)| \leqslant r_i, 1 \leqslant i \leqslant n \}$ defines a k-affinoid atlas on $\mathbf{A}^n$. (We remark that $\mathbf{A}^n$ is a good k-analytic space.) We remark that the affine line $\mathbf{A}^1$ is a ring object of the category k-$\mathscr{A}n$. If $X = \mathscr{M}(\mathscr{A})$ is a k-affinoid space, then $\mathrm{Hom}(X, \mathbf{A}^1) = \mathscr{A}$.

We return to Φ-analytic spaces. Applying Proposition 1.3.2 to $X' = \mathbf{A}^1$ and the category k-$\mathscr{A}n$, we get a *structural sheaf* $\mathcal{O}_{X_G}$ *on* X_G (this is a sheaf of rings). The category of $\mathcal{O}_{X_G}$-modules is denoted by $\mathrm{Mod}(X_G)$. An $\mathcal{O}_{X_G}$-module is said to be *coherent* if there exists a quasinet τ of Φ-affinoid domains in X such that, for each $V \in \tau$, $\mathcal{O}_{X_G}|_{V_G}$ is isomorphic to the cokernel of a homomorphism of free $\mathcal{O}_{V_G}$-modules of finite rank. For example, suppose that $X = \mathscr{M}(\mathscr{A})$ is a Φ_k-affinoid space. Then a finite $\mathscr{A}$-module M defines a coherent $\mathcal{O}_{X_G}(M)$ by $V \mapsto M \otimes_{\mathscr{A}} \mathscr{A}_V$, and Kiehl's Theorem tells that any coherent $\mathcal{O}_{X_G}$-module is isomorphic to $\mathcal{O}_{X_G}(M)$ for some M. The latter fact enables one to define for a coherent $\mathcal{O}_{X_G}$-module F the *support* Supp(F) of F. Namely, if $X = \mathscr{M}(\mathscr{A})$ is k-affinoid and $F = \mathcal{O}_{X_G}(M)$, then Supp(F) is the support of the annihilator of M. If X is arbitrary, then Supp(F) is the set of point $x \in X$ such that for some (and therefore for any) affinoid domain V that contains x the support of $F|_{V_G}$ contains x. Let $\mathrm{Coh}(X_G)$ denote the category of coherent $\mathcal{O}_{X_G}$-modules, and let $\mathrm{Pic}(X_G)$ denote the *Picard group* of invertible $\mathcal{O}_{X_G}$-modules. (One has $\mathrm{Pic}(X_G) = H^1(X_G, \mathcal{O}^*_{X_G})$.) From Kiehl's Theorem it follows that $\mathrm{Coh}(X_G)$ and $\mathrm{Pic}(X_G)$ are the same whether X is considered as an object of Φ_k-$\mathscr{A}n$ or of k-$\mathscr{A}n$.

We now consider connection of the above objects with their analogs in the usual topology of X. For this *we assume that the class* Φ_k *is dense*. Then all open subsets of X are Φ-analytic domains, and there is a morphism of G-topological spaces $\pi : X_G \to X$ which induces a morphism of the corresponding topoi $(\pi_*, \pi^*) : X_G^\sim \to X^\sim$. The direct image functor π_* is simply the restriction functor. In particular, we have the *structural sheaf* $\mathcal{O}_X := \pi_* \mathcal{O}_{X_G}$ *on* X. The functor π_* is not fully faithful (see Remark 1.3.8). The inverse image functor π^* is as follows. For a sheaf F on X and a Φ-affinoid (or Φ-special) domain, one has

$$\pi^* F(V) = \varinjlim_{\mathcal{U} \supset V} F(\mathcal{U}),$$

where $\mathcal{U}$ runs through open neighborhoods of V. It is easy to see that $F \xrightarrow{\sim} \pi_* \pi^* F$. In particular, the functor π^* is fully faithful.

Let Mod(X) denote the category of $\mathcal{O}_X$-modules. The functor π_* defines an evident functor $\mathrm{Mod}(X_G) \to \mathrm{Mod}(X)$. The natural functor in the inverse direction is as follows:

$$\mathrm{Mod}(X) \to \mathrm{Mod}(X_G) : F \mapsto F_G = \pi^* F \otimes_{\pi^* \mathcal{O}_X} \mathcal{O}_{X_G}.$$

An $\mathcal{O}_X$-module is said to be *coherent* if locally (in the usual topology of X) it is isomorphic to the cokernel of a homomorphism of free modules of finite rank. (For example, if $X = \mathcal{M}(\mathcal{A})$ is Φ_k-affinoid, then $\mathcal{O}_X(M) := \pi_* \mathcal{O}_{X_G}(M)$ is a coherent $\mathcal{O}_X$-module.) The *Picard group* Pic(X) is the group of invertible $\mathcal{O}_X$-modules. One has $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$.

1.3.4. *Proposition. — If* X *is a good* Φ_k*-analytic space, then*

(i) *for any* $\mathcal{O}_X$*-module* F *one has* $F \xrightarrow{\sim} \pi_* F_G$; *in particular, the functor*

$$\mathrm{Mod}(X) \to \mathrm{Mod}(X_G) : F \mapsto F_G$$

is fully faithful;

(ii) *the functor* $F \mapsto F_G$ *induces an equivalence of categories* $\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X_G)$;

(iii) *a coherent* $\mathcal{O}_X$*-module* F *is locally free if and only if* F_G *is locally free.*

Proof. — (i) It suffices to verify that for any point $x \in X$ there is an isomorphism of stalks $F_x \xrightarrow{\sim} (\pi_* F_G)_x$. But this easily follows from the definitions because x has an affinoid neighborhood.

(ii) By (i), it suffices to verify that for a coherent $\mathcal{O}_{X_G}$-module $\mathcal{F}$ the $\mathcal{O}_X$-module $F = \pi_* \mathcal{F}$ is coherent and $F_G \xrightarrow{\sim} \mathcal{F}$. This also follows easily from the definitions.

(iii) We may assume that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. It suffices to show that a finite $\mathcal{A}$-module M is projective if and only if the $\mathcal{O}_{X_G}$-module $\mathcal{O}_{X_G}(M)$ is locally free. The direct implication is simple. Conversely, suppose that for some finite affinoid covering $\{V_i\}_{i \in I}$ of X the finite $\mathcal{A}_{V_i}$-modules $M \otimes_{\mathcal{A}} \mathcal{A}_{V_i}$ are free. It suffices to verify that M is flat over $\mathcal{A}$. For this we take an injective homomorphism of finite $\mathcal{A}$-modules $P \to Q$. Then the homomorphisms $(M \otimes_{\mathcal{A}} P) \otimes_{\mathcal{A}} \mathcal{A}_{V_i} \to (M \otimes_{\mathcal{A}} Q) \otimes_{\mathcal{A}} \mathcal{A}_{V_i}$ are also injective. Applying Tate's Acyclicity Theorem to the finite $\mathcal{A}$-modules $M \otimes_{\mathcal{A}} P$ and $M \otimes_{\mathcal{A}} Q$, we obtain the injectivity of the homomorphism $M \otimes_{\mathcal{A}} P \to M \otimes_{\mathcal{A}} Q$. ■

1.3.5. *Corollary. — If* X *is a good* Φ_k*-analytic space, then there is an isomorphism* $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X_G)$. ■

The structural sheaf $\mathcal{O}_X$ will be used only for good spaces X. The group $\mathrm{Pic}(X_G)$ will appear in Corollary 4.3.8. We now compare the cohomology groups in both topologies.

1.3.6. *Proposition.* — (i) *For any abelian sheaf* F *on* X, *one has* $H^q(X, F) \xrightarrow{\sim} H^q(X_G, \pi^* F)$, $q \geqslant 0$.

(ii) *If* X *is good, then* $H^q(X, F) \xrightarrow{\sim} H^q(X_G, F_G)$, $q \geqslant 0$, *for any coherent* $\mathcal{O}_X$*-module* F.

(iii) *If* X *is paracompact, then* $\check{H}^1(X, F) \xrightarrow{\sim} \check{H}^1(X_G, \pi^* F)$ *for any sheaf of groups* F *on* X.

Proof. — (i) An open covering of X is a covering in the usual and the G-topology, and therefore it generates two Leray spectral sequences that are convergent to the groups $H^q(X, F)$ and $H^q(X_G, \pi^* F)$, respectively. Comparing them, we see that it suffices to verify the statement for sufficiently small X. In particular, we may assume that X is paracompact. It suffices to verify that if F is injective, then $H^q(X_G, \pi^* F) = 0$ for $q \geqslant 1$. Since X is paracompact, it suffices to verify that the Čech cohomology groups of $\pi^* F$ with respect to a locally finite covering by compact analytic domains are trivial. But this is clear because they are also the Čech cohomology groups of F with respect to the same covering.

(ii) The same reasoning reduces the situation to the case when X is an open paracompact subset of a k-affinoid space. (In particular, the intersection of two affinoid domains is an affinoid domain.) In this case $H^q(X, F)$ is an inductive limit of the q-th cohomology groups of the Čech complexes associated with locally finite open coverings $\{ \mathcal{U}_i \}_{i \in I}$ of X. On the other hand, since the cohomology groups of a coherent sheaf on a G-ringed k-affinoid space are trivial, then $H^q(X_G, F_G)$ is the q-th cohomology group of the Čech complex associated with an arbitrary locally finite affinoid covering $\{ V_j \}_{j \in J}$ of X. It remains to remark that for any $\{ \mathcal{U}_i \}_{i \in I}$ we can find $\{ V_j \}_{j \in J}$ such that each V_j is contained in some $\mathcal{U}_i$ and $\bigcup_{j \in J} \mathrm{Int}(V_j/X) = X$.

(iii) is trivial. ∎

We remark that a morphism of Φ_k-analytic spaces $\varphi : Y \to X$ induces a morphism of G-ringed topological spaces $\varphi_G : Y_G \to X_G$. If the spaces X and Y are good, then for any coherent $\mathcal{O}_X$-module F there is a canonical isomorphism of coherent $\mathcal{O}_{Y_G}$-modules $(\varphi^* F)_G \xrightarrow{\sim} \varphi_G^* F_G$.

We finish this subsection by introducing several classes of morphisms.

1.3.7. *Lemma.* — *The following properties of a morphism of* Φ_k*-analytic spaces* $\varphi : Y \to X$ *are equivalent:*

a) for any point $x \in X$ *there exist* Φ*-affinoid domains* $V_1, \dots, V_n \subset X$ *such that* $x \in V_1 \cap \dots \cap V_n$ *and* $\varphi^{-1}(V_i) \to V_i$ *are finite morphisms (resp. closed immersions) of* k*-affinoid spaces;*

b) for any Φ*-affinoid domain* $V \subset X$, $\varphi^{-1}(V) \to V$ *is a finite morphism (resp. a closed immersion) of* k*-affinoid spaces.*

Proof. — Suppose that *a)* is true. Then the collection τ of all Φ-affinoid domains $V \subset X$ such that $\varphi^{-1}(V) \to V$ is a finite morphism (resp. a closed immersion) of k-affinoid spaces is a net. Let V be an arbitrary Φ-affinoid domain. Then $V \subset V_1 \cup \dots \cup V_n$ for some

$V \in \tau$. By Lemma 1.1.2 (ii), we can find Φ-affinoid domains $U_1, \ldots, U_m \subset X$ such that $V = U_1 \cup \ldots \cup U_m$ and each U_j is contained in some V_i. Then $U_j \in \tau$. It remains to apply Kiehl's Theorem. ■

A morphism $\varphi : Y \to X$ satisfying the equivalent properties of Lemma 1.3.7 is said to be *finite* (resp. a *closed immersion*). It is clear that this property of φ is the same whether we consider it in the category Φ_k-$\mathscr{A}n$ or in k-$\mathscr{A}n$. A finite morphism $\varphi : Y \to X$ induces a compact map with finite fibres $|Y| \to |X|$, and $\varphi_{G_*}(\mathcal{O}_{Y_G})$ is a coherent $\mathcal{O}_{X_G}$-module. If φ is a closed immersion, then it induces a homeomorphism of $|Y|$ with its image in $|X|$, and the homomorphism $\mathcal{O}_{X_G} \to \varphi_{G_*}(\mathcal{O}_{Y_G})$ is surjective. Its kernel is a coherent sheaf of ideals in $\mathcal{O}_{X_G}$. Furthermore, we say that a subset $\Sigma \subset X$ is *Zariski closed* if, for any Φ-affinoid domain $V \subset X$, the intersection $\Sigma \cap V$ is Zariski closed in V. The complement to a Zariski closed subset is called *Zariski open*. For example, the support of a coherent $\mathcal{O}_{X_G}$-module is Zariski closed in X. If $\varphi : Y \to X$ is a closed immersion, then the image of Y is Zariski closed in X. Conversely, if Σ is Zariski closed in X, then there is a closed immersion $Y \to X$ that identifies $|Y|$ with Σ.

Furthermore, a morphism of Φ_k-analytic spaces $\varphi : Y \to X$ is said to be a G-*locally* (resp. *locally*) *closed immersion* if there exist a quasinet τ of Φ-analytic (resp. open Φ-analytic) domains in Y and, for each $V \in \tau$, a Φ-analytic (resp. an open Φ-analytic) domain $U \subset X$ such that φ induces a closed immersion $V \to U$. (It is clear that this property of φ is the same whether we consider it in the category Φ_k-$\mathscr{A}n$ or in k-$\mathscr{A}n$.) Of course, a locally closed immersion is a G-locally closed immersion. If the both spaces are good, then the converse is also true.

Let now $\varphi : Y \to X$ be a G-locally closed immersion, and let V and U be as above. If $\mathcal{J}$ is the sheaf of ideals in $\mathcal{O}_{U_G}$ that corresponds to V, then $\mathcal{J}/\mathcal{J}^2$ can be considered as an $\mathcal{O}_{V_G}$-module. All these sheaves are compatible on intersections, and so they define a coherent $\mathcal{O}_{Y_G}$-module that is said to be the *conormal sheaf of* φ and is denoted by $\mathcal{N}_{Y_G/X_G}$. If both spaces are good, then one can also define a similar $\mathcal{O}_X$-module $\mathcal{N}_{Y/X}$, and one has $(\mathcal{N}_{Y/X})_G \xrightarrow{\sim} \mathcal{N}_{Y_G/X_G}$.

1.3.8. *Remark.* — Here is an example showing that the direct image functor $\pi_* : X_G^\sim \to X^\sim$ is not fully faithful. Let X be the closed unit disc

$$E(0, 1) = \{ x \in \mathbf{A}^1 \mid |T(x)| \leqslant 1 \},$$

and let x_0 be the maximal point of X (it corresponds to the norm of the algebra $k\{T\}$). We construct two sheaves F and F′ on X_G as follows. Let Y be an analytic domain in X. Then $F(Y) = \mathbf{Z}$ if $x_0 \in Y$, and $F(Y) = 0$ otherwise. Furthermore, $F'(Y) = \mathbf{Z}$ if $\{ x \in X \mid r < |T(x)| < 1 \} \cup \{ x_0 \} \subset Y$ for some $0 < r < 1$, and $F'(Y) = 0$ otherwise. The sheaves F and F′ are not isomorphic, but $\pi_* F = \pi_* F' = i_* \mathbf{Z}$, where i is the embedding $\{ x_0 \} \to X$.

1.4. Fibre products and the ground field extension functor

1.4.1. *Proposition.* — *The category Φ_k-$\mathscr{A}n$ admits fibre products.*

Proof. — First we shall show the existence of fibre products in the category k-$\mathscr{A}n$, and after that we'll use this fact to show that the same is true for the category Φ_k-$\mathscr{A}n$. Let $\varphi : Y \to X$ and $f : X' \to X$ be morphisms of k-analytic spaces.

Consider first the case when all three spaces are paracompact. In this case we may assume that φ and f are represented by strong morphisms $(Y, \mathscr{B}, \sigma) \to (X, \mathscr{A}, \tau)$ and $(X', \mathscr{A}', \tau') \to (X, \mathscr{A}, \tau)$, where τ, σ and τ' are *locally finite* nets. Let S denote the family of all triples (V, U, U'), where $V \in \sigma$, $U \in \tau$, $U' \in \tau'$ and $\varphi(V), f(U') \subset U$. For $\alpha = (V, U, U') \in S$ we denote by W_α the k-affinoid space $V \times_U U'$ and by Σ_α the topological space $|V| \times_{|U|} |U'|$. The latter is a compact subset of the topological space $\Sigma := |Y| \times_{|X|} |X'|$, and the canonical map $W_\alpha \to \Sigma_\alpha$ induces a map $\pi_\alpha : W_\alpha \to \Sigma$. We claim that, for any pair $\alpha, \beta \in S$, the set $W_{\alpha\beta} := \pi_\alpha^{-1}(\Sigma_\alpha \cap \Sigma_\beta)$ is a special domain in W_α, and there is a canonical isomorphism of k-analytic spaces $\nu_{\alpha\beta} : W_{\alpha\beta} \xrightarrow{\sim} W_{\beta\alpha}$. Indeed, let $\beta = (\overline{V}, \overline{U}, \overline{U}')$. Then $U \cap \overline{U} = U_1 \cap \ldots \cap U_n$ for $U_i \in \tau$. Furthermore, for each $1 \leqslant i \leqslant n$, one has $\varphi_{V/U}^{-1}(U_i) \cap \varphi_{\overline{V}/\overline{U}}^{-1}(U_i) = \bigcup_{j=1}^{p_i} V_{ij}$ and

$$f_{U'/U}^{-1}(U_i) \cap f_{\overline{U}'/\overline{U}}^{-1}(U_i) = \bigcup_{l=1}^{q_i} U'_{il}$$

for some $V_{ij} \in \sigma$ and $U'_{il} \in \tau'$. One has $W_{\alpha\beta} = \bigcup_{i,j,l} V_{ij} \times_{U_i} U'_{il}$. The right hand side of the latter equality can also be considered as a subset of $W_{\beta\alpha}$. It follows that $W_{\alpha\beta}$ and $W_{\beta\alpha}$ are special domains in W_α and W_β, respectively, and we get an isomorphism $\nu_{\alpha\beta} : W_{\alpha\beta} \xrightarrow{\sim} W_{\beta\alpha}$ that does not depend on the choice of the above coverings. It is clear that $W_{\alpha\alpha} = W_\alpha$, $\nu_{\alpha\beta}(W_{\alpha\beta} \cap W_{\alpha\gamma}) = W_{\beta\alpha} \cap W_{\beta\gamma}$ and $\nu_{\alpha\gamma} = \nu_{\beta\gamma} \circ \nu_{\alpha\beta}$ on $W_{\alpha\beta} \cap W_{\alpha\gamma}$. By Proposition 1.3.3, we can glue all W_α along $W_{\alpha\beta}$ and get a k-analytic space $(Y', \mathscr{B}', \sigma')$ which is a fibre product of $(Y, \mathscr{B}, \sigma)$ and $(X', \mathscr{A}', \tau')$ over $(X, \mathscr{A}, \tau)$.

Consider now the case when only the space X is paracompact. In this case we take coverings $\{Y_i\}_{i \in I}$ of Y and $\{X'_j\}_{j \in J}$ of X' by open paracompact subsets, and we glue all the spaces $Y_i \times_X X'_j$ along the open subspaces $(Y_i \cap Y_k) \times_X (X'_j \cap X'_l)$. We get a locally Hausdorff space Y'. The collection σ' of sets of the forms $V \times_U U'$, where $V \subset Y_i$, $U \subset X$ and $U' \subset X'_j$ are affinoid domains, is a net of compact subsets on Y', and there is an evident k-affinoid atlas $\mathscr{B}'$ with the net σ'. The triple $(Y', \mathscr{B}', \sigma')$ is a fibre product of Y and X' over X.

Finally, in the case when all three spaces are arbitrary we take a covering $\{X_i\}_{i \in I}$ of X by open paracompact subsets and construct a fibre product Y' by gluing the spaces $\varphi^{-1}(X_i) \times_{X_i} f^{-1}(X_i)$ along the open subspaces $\varphi^{-1}(X_i \cap X_j) \times_{X_i \cap X_j} f^{-1}(X_i \cap X_j)$.

We remark that the above construction gives also a compact map

$$\pi : Y' \to |Y| \times_{|X|} |X'|.$$

In particular, if $V \subset Y$, $U \subset X$ and $U' \subset X'$ are affinoid domains with $\varphi(V), f(U') \subset U$, then the set $\pi^{-1}(|V| \times_{|U|} |U'|)$ is compact. It follows that the canonical morphism $V' := V \times_U U' \to Y'$ identifies V' with an affinoid domain in Y'.

Suppose now that $\varphi : Y \to X$ and $f : X' \to X$ are morphisms of Φ_k-analytic spaces. Then the collection σ' of all affinoid domains of the form $V \times_U U'$, where $V \subset Y$, $U \subset X$ and $U' \subset X'$ are Φ-affinoid domains with $\varphi(V), f(U') \subset U$, is a net on Y', and there is an evident Φ_k-affinoid atlas with the net σ'. It defines a Φ_k-analytic space structure on Y'. It is easy to see that the canonical projections $Y' \to Y$ and $Y' \to X'$ are morphisms of Φ_k-analytic spaces and that Y' is a fibre product of Y and X' over X in the category Φ_k-$\mathscr{A}n$. ∎

Similarly, one constructs, for a non-Archimedean field K over k, a *ground field extension functor* $\Phi_k\text{-}\mathscr{A}n \to \Phi_K\text{-}\mathscr{A}n : X \mapsto X \hat{\otimes} K$ and a compact map $X \hat{\otimes} K \to X$. A *Φ-analytic space* is a pair (K, X), where K is a non-Archimedean field K over k and $X \in \Phi_K\text{-}\mathscr{A}n$. A morphism $(L, Y) \to (K, X)$ is a pair consisting of an isometric embedding $K \hookrightarrow L$ over k and a morphism of Φ_L-analytic spaces $Y \to X \hat{\otimes}_K L$. The category of Φ-analytic spaces is denoted by $\Phi\text{-}\mathscr{A}n_k$. If Φ is the family of all affinoid spaces, then the category is denoted by $\mathscr{A}n_k$ and its objects are called *analytic spaces over k*. For brevity we denote the analytic space (K, X) by X.

We remark that with each point $x \in X \in \Phi_k\text{-}\mathscr{A}n$ one can associate a non-Archimedean field $\mathscr{H}(x)$ over x so that, for any Φ-affinoid domain $V \subset X$ that contains x, there is a canonical bounded character $\mathscr{A}_V \to \mathscr{H}(x)$ that identifies $\mathscr{H}(x)$ with the corresponding field of the point x with respect to V (see [Ber], 1.2.2 (i)). A morphism $\varphi : Y \to X$ induces, for each point $y \in Y$, an isometric embedding $\mathscr{H}(\varphi(y)) \hookrightarrow \mathscr{H}(y)$. Furthermore, let x be a point of X. If V is a Φ-affinoid domain that contains x, then the caracter $\mathscr{A}_V \hat{\otimes} \mathscr{H}(x) \to \mathscr{H}(x) : f \otimes \lambda \mapsto \lambda f(x)$ defines an $\mathscr{H}(x)$-point $x'_V \in V \hat{\otimes} \mathscr{H}(x)$. It is clear that the image x' of x'_V in $X \hat{\otimes} \mathscr{H}(x)$ does not depend on the choice of V. If now $\varphi : Y \to X$ is a morphism of Φ_k-analytic spaces, then the $\mathscr{H}(x)$-analytic space $(Y \hat{\otimes} \mathscr{H}(x)) \hat{\otimes}_{X \hat{\otimes} \mathscr{H}(x)} \mathscr{M}(\mathscr{H}(x))$, where the morphism $\mathscr{M}(\mathscr{H}(x)) \to X \hat{\otimes} \mathscr{H}(x)$ corresponds to the point x', is denoted by Y_x and is said to be the *fibre of φ at the point x*. The canonical morphism $Y_x \to Y$ induces a homeomorphism $Y_x \xrightarrow{\sim} \varphi^{-1}(x)$. The *dimension of* φ, $\dim(\varphi)$, is the supremum of the dimensions $\dim(Y_x)$ over all $x \in X$.

Let $\varphi : Y \to X$ be a morphism of Φ_k-analytic spaces, and consider the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$. The collection τ of Φ-affinoid domains $V \subset Y$ for which there exists a Φ-affinoid domain $U \subset X$ with $\varphi(V) \subset U$ is a net, and, for such V and U, $V \times_U V$ is a Φ-affinoid domain in Y and $\Delta_{Y/X}$ induces a closed immersion $V \to V \times_U V$. Thus, $\Delta_{Y/X}$ is a G-locally closed immersion. The conormal sheaf of $\Delta_{Y/X}$ is said to be the *sheaf of differentials of* φ and is denoted by Ω_{Y_G/X_G}. If both spaces are good, then one can also define a similar coherent $\mathscr{O}_Y$-module $\Omega_{Y/X}$, and one has $(\Omega_{Y/X})_G \xrightarrow{\sim} \Omega_{Y_G/X_G}$. The sheaves Ω_{Y_G/X_G} and $\Omega_{Y/X}$ will be studied in § 3.3.

A morphism of Φ_k-analytic spaces $\varphi : Y \to X$ is said to be *separated* (resp. *locally separated*) if the diagonal morphism $\Delta_{Y/X}$ is a closed (resp. a locally closed) immersion.

If the canonical morphism $X \to \mathcal{M}(k)$ is separated (resp. locally separated), then X is said to be *separated* (resp. *locally separated*). For example good Φ_k-analytic spaces and morphisms between them are locally separated. If a morphism $\varphi : Y \to X$ is separated, then $|Y|$ is closed in $|Y \times_X Y|$. Since the map $\pi : |Y \times_X Y| \to |Y| \times_{|X|} |Y|$ is compact, then $|Y|$ is closed also in $|Y| \times_{|X|} |Y|$, and therefore the map $|Y| \to |X|$ is Hausdorff. In particular, if Y is separated, then its underlying topological space $|Y|$ is Hausdorff.

1.4.2. *Proposition. — A locally separated morphism of Φ_k-analytic spaces $\varphi : Y \to X$ is separated if and only if the induced map $|Y| \to |X|$ is Hausdorff.*

Proof. — Suppose that the map $|Y| \to |X|$ is Hausdorff. Then the complement $\mathcal{W}$ of Y in $|Y| \times_{|X|} |Y|$ is open. Since the diagonal morphism $\Delta = \Delta_{Y/X}$ is a composition of a closed immersion with an open immersion, it suffices to show that $\Delta(Y)$ is closed in $|Y \times_X Y|$. For this we consider the compact map $\pi : |Y \times_X Y| \to |Y| \times_{|X|} |Y|$. Let $z \in (Y \times_X Y) \backslash \Delta(Y)$, and let $\pi(z) = (y_1, y_2)$. If $y_1 \neq y_2$, then $\pi^{-1}(\mathcal{W})$ is an open neighborhood of z that does not meet $\Delta(Y)$. If $y_1 = y_2$, then we take an open neighborhood $\mathcal{V}$ of $y_1 = y_2$ such that $\Delta_{\mathcal{V}} : \mathcal{V} \to \mathcal{V} \times_X \mathcal{V}$ is a closed immersion. Since $\mathcal{V} \times_X \mathcal{V}$ is an open subset of $Y \times_X Y$ and $z \notin \Delta(Y)$, then we can find an open neighborhood of z that does not meet $\Delta(Y)$. The required fact follows. ∎

It is clear that the classes of closed and locally closed immersions, finite, separated and locally separated morphisms are preserved under composition, under any base change functor and under extensions of the ground field.

1.4.3. *Remarks.* — (i) The converse implication of Proposition 1.4.2 is not true in general. For example, the space obtained by gluing two copies of the unit one-dimensional disc along the closed annulus of radius one is Hausdorff but is not separated.

(ii) We conjecture that every point of a separated k-analytic space has an open neighborhood which is isomorphic to an analytic domain in a k-affinoid space.

1.5. Analytic spaces from [Ber]

In this subsection we recall the notion of a k-analytic space from [Ber] (with the necessary details that were omitted in [Ber]), and we show that the category of k-analytic spaces from [Ber] is equivalent to the category of good k-analytic spaces from the previous subsection.

First of all, recall that a *k-quasiaffinoid space* is a pair $(\mathcal{U}, \nu)$ consisting of a locally ringed space $\mathcal{U}$ and an open immersion ν of $\mathcal{U}$ in a k-affinoid space X. We remark that the immersion ν induces a net τ of all $V \subset \mathcal{U}$ for which $\nu(V)$ is an affinoid domain in X and a k-affinoid atlas $\mathcal{A}$ with the net τ for which $\mathcal{A}_V = \mathcal{A}_{\nu(V)}$, and therefore we get a

k-analytic space $(\mathcal{U}, \mathcal{A}, \tau)$ from k-$\mathcal{A}n$. We remark also that if V is an affinoid domain in $\mathcal{U}$, then for any pair of open subsets $\mathcal{V}, \mathcal{W} \subset \mathcal{U}$ with $\mathcal{V} \subset V \subset \mathcal{W}$ there are canonical homomorphisms $\mathcal{O}(\mathcal{W}) \to \mathcal{A}_V \to \mathcal{O}(\mathcal{V})$.

Furthermore, a *morphism of k-quasiaffinoid spaces* $(\mathcal{U}, \nu) \to (\mathcal{U}', \nu')$ is a morphism of locally ringed spaces $\varphi : \mathcal{U} \to \mathcal{U}'$ such that for any pair of affinoid domains $V \subset \mathcal{U}$ and $V' \subset \mathcal{U}'$ with $\varphi(V) \subset \mathcal{V} = \mathrm{Int}(V'/\mathcal{U}')$ (the topological interior of V' in $\mathcal{U}'$), the induced homomorphism $\mathcal{A}'_{V'} \to \mathcal{O}(\mathcal{V}) \to \mathcal{O}(\varphi^{-1}(\mathcal{V})) \to \mathcal{A}_V$ is bounded. We remark that from the definition it follows that for any pair of affinoid domains $U \subset V$ and $U' \subset V'$ with $\varphi(U) \subset \mathrm{Int}(U'/\mathcal{U}')$ the homomorphisms $\mathcal{A}'_{U'} \to \mathcal{A}_U$ and $\mathcal{A}'_{V'} \to \mathcal{A}_V$ are compatible.

1.5.1. *Lemma. — The system of homomorphism $\mathcal{A}'_{V'} \to \mathcal{A}_V$ extends canonically to the family of all pairs of affinoid domains $V \subset \mathcal{U}$ and $V' \subset \mathcal{U}'$ with $\varphi(V) \subset V'$ so that one gets a well-defined morphism $(\mathcal{U}, \mathcal{A}, \tau) \to (\mathcal{U}', \mathcal{A}', \tau')$.*

Proof. — Let V, V′ be such a pair. Assume first that $\varphi(V) \subset \mathrm{Int}(V'/\mathcal{U}')$. We claim that the two maps from V to V′ induced by φ and by the homomorphism $\mathcal{A}'_{V'} \to \mathcal{A}_V$ coincide. Let ψ denote the second map, and let $x \in V$. Take affinoid neighborhoods U of x in $\mathcal{U}$ and U′ of $\varphi(x)$ in $\mathcal{U}'$ such that $\varphi(U) \subset \mathrm{Int}(U'/\mathcal{U}')$. Then $\varphi(U \cap V) \subset \mathrm{Int}(U' \cap V'/\mathcal{U}')$. The homomorphisms $\mathcal{A}'_{V'} \to \mathcal{A}_V$ and $\mathcal{A}'_{U' \cap V'} \to \mathcal{A}_{U \cap V}$ are compatible, and therefore $\psi(U \cap V) \subset U' \cap V'$. Since U and U′ can be taken sufficiently small, then $\varphi(x) = \psi(x)$, and our claim follows. It follows that one can construct in a canonical way bounded homomorphisms $\mathcal{A}'_{U'} \to \mathcal{A}_U$ for every pair of affinoid domains $U \subset V$ and $U' \subset V'$ with $\varphi(U) \subset U'$, and the two maps from U to U′ induced by φ and by the homomorphism $\mathcal{A}'_{U'} \to \mathcal{A}_U$ coincide.

Assume now that V and V′ are arbitrary. Then we can find affinoid domains $V_1, \ldots, V_n \subset \mathcal{U}$ and $V'_1, \ldots, V'_n \subset \mathcal{U}'$ such that $V \subset V_1 \cup \ldots \cup V_n$, $V' \subset V'_1 \cup \ldots \cup V'_n$ and $\varphi(V_i) \subset \mathrm{Int}(V'_i/\mathcal{U}')$. By the first case, there are canonical bounded homomorphisms $\mathcal{A}'_{V' \cap V'_i} \to \mathcal{A}_{V \cap V_i}$ and $\mathcal{A}'_{V' \cap V'_i \cap V'_j} \to \mathcal{A}_{V \cap V_i \cap V_j}$ that induce the maps

$$\varphi : V \cap V_i \to V' \cap V'_i \quad \text{and} \quad V \cap V_i \cap V_j \to V' \cap V'_i \cap V'_j.$$

Applying Tate's Acyclicity Theorem to the coverings $\{ V \cap V_i \}$ of V and $\{ V' \cap V'_i \}$ of V′, we get a bounded homomorphism $\mathcal{A}'_{V'} \to \mathcal{A}_V$ that is compatible with the homomorphisms $\mathcal{A}'_{V' \cap V'_i} \to \mathcal{A}_{V \cap V_i}$ and such that the maps from V to V′ induced by φ and by the homomorphism $\mathcal{A}'_{V'} \to \mathcal{A}_V$ coincide. Thus, we get the required morphism $(\mathcal{U}, \mathcal{A}, \tau) \to (\mathcal{U}', \mathcal{A}', \tau')$. ∎

We remark that any morphism $(\mathcal{U}, \mathcal{A}, \tau) \to (\mathcal{U}', \mathcal{A}', \tau')$ comes from a unique morphism $(\mathcal{U}, \nu) \to (\mathcal{U}', \nu')$. Thus, k-quasiaffinoid spaces form a category which is equivalent to a full subcategory of k-$\mathcal{A}n$. The latter consists of all k-analytic spaces that admit an open immersion in a k-affinoid space.

1.5.2. *Corollary. — Let* $(\mathcal{U}, \nu)$ *and* $(\mathcal{U}', \nu')$ *be k-quasiaffinoid spaces, and let* $\varphi : \mathcal{U} \to \mathcal{U}'$ *be a morphism (resp. an isomorphism) of locally ringed spaces. Then the following are equivalent:*

a) φ *induces a morphism (resp. an isomorphism) of k-quasiaffinoid spaces* $(\mathcal{U}, \nu) \to (\mathcal{U}', \nu')$;

b) there exist open coverings $\{\mathcal{U}_i\}_{i\in I}$ *of* $\mathcal{U}$ *and* $\{\mathcal{U}'_j\}_{j\in J}$ *of* $\mathcal{U}'$ *such that, for each pair* i, j, φ *induces a morphism (resp. an isomorphism) of k-quasiaffinoid spaces*

$$(\mathcal{U}_i \cap \varphi^{-1}(\mathcal{U}'_j), \nu) \to (\mathcal{U}'_j, \nu')$$

(resp. $(\mathcal{U}_i \cap \varphi^{-1}(\mathcal{U}'_j), \nu) \xrightarrow{\sim} (\varphi(\mathcal{U}_i) \cap \mathcal{U}'_j, \nu')$*);*

c) property b) is true for arbitrary open coverings of $\mathcal{U}$ *and* $\mathcal{U}'$. ∎

Let X be a locally ringed space. An (open) *k-analytic atlas* on X is a collection of k-quasiaffinoid spaces $\{(\mathcal{U}_i, \nu_i)\}_{i\in I}$ called *charts* of the atlas such that $\{\mathcal{U}_i\}_{i\in I}$ is an open covering of X (each $\mathcal{U}_i$ is provided with the locally ringed structure induced from X) and, for each pair $i, j \in I$, the identity morphism induces an isomorphism of k-quasiaffinoid spaces $(\mathcal{U}_i \cap \mathcal{U}_j, \nu_i) \xrightarrow{\sim} (\mathcal{U}_i \cap \mathcal{U}_j, \nu_j)$. Furthermore, suppose that we are given an open subset $\mathcal{U} \subset X$ and an open immersion ν of $\mathcal{U}$ in a k-affinoid space. Then $(\mathcal{U}, \nu)$ is *compatible* with the atlas $\{(\mathcal{U}_i, \nu_i)\}_{i\in I}$ if, for each $i \in I$, the identity morphism induces an isomorphism of k-quasiaffinoid spaces $(\mathcal{U} \cap \mathcal{U}_i, \nu) \xrightarrow{\sim} (\mathcal{U} \cap \mathcal{U}_i, \nu_i)$. Two atlases are said to be *compatible* if every chart of one atlas is compatible with the other atlas. From Corollary 1.5.2 it follows that the compatibility of atlases is an equivalence relation. A k-analytic space from [Ber] is a locally ringed space X provided with an equivalence class of k-analytic atlases.

Let X, X′ be two k-analytic spaces defined in the above way, and let $\varphi : X \to X'$ be a morphism of locally ringed spaces. Then φ is called a morphism of k-analytic spaces if there exists an atlas $\{(\mathcal{U}_i, \nu_i)\}_{i\in I}$ of X and an atlas $\{(\mathcal{U}'_j, \nu'_j)\}_{j\in J}$ of X′ such that, for each pair i, j, φ induces a morphism of k-quasiaffinoid spaces $(\mathcal{U}_i \cap \varphi^{-1}(\mathcal{U}'_j), \nu_i) \to (\mathcal{U}'_j, \nu'_j)$. From Corollary 1.5.2 it follows that the same condition holds for any choice of atlases on X and X′ defining the same k-analytic structure, and that one can compose morphisms. Thus, one gets a category. This is the category introduced in [Ber] (and denoted there by k-$\mathcal{A}n$).

We now construct a functor from the category of k-analytic spaces from [Ber] to k-$\mathcal{A}n$. For each k-analytic space X from [Ber] we fix an open k-analytic atlas $\{(\mathcal{U}_i, \nu_i)\}_{i\in I}$. Let τ be the family of the subsets $V \subset X$ for which there exists $i \in I$ such that V is an affinoid domain in $\mathcal{U}_i$ (in this case V is an affinoid domain in any $\mathcal{U}_j$ that contains V). Then τ is a net on X, and there is an evident k-affinoid atlas $\mathcal{A}$ with the net τ. The k-analytic spaces $(X, \mathcal{A}, \tau)$ obtained in this way is evidently good. Let now $\varphi : X \to X'$ be a morphism of k-analytic spaces from [Ber]. We denote by σ the family of all $V \in \tau$ for which there exists $V' \in \tau'$ with $\varphi(V) \subset V'$. It is clear that σ is a net with $\sigma \prec \tau$, and the morphism φ gives rise to a strong morphism $(X, \mathcal{A}_\sigma, \sigma) \to (X', \mathcal{A}', \tau')$. Therefore we have the required functor, and it is easy to see that it is fully faithful. Let now X be a good k-analytic space from k-$\mathcal{A}n$. For an affinoid domain $V \subset X$ we denote by $\mathcal{U}_V$ the topological interior of V in X and by ν_V the canonical open immersion of locally ringed spaces $\mathcal{U}_V \to V$. Then $\{(\mathcal{U}_V, \nu_V)\}$ is an open k-analytic atlas on X, and the k-analytic space from [Ber] obtained in this way gives rise to a k-analytic space from k-$\mathcal{A}n$ isomorphic to X. Thus, the correspondence $X \mapsto (X, \mathcal{A}_\sigma, \sigma)$ is an equivalence of the category of k-analytic spaces from [Ber] and the category of good k-analytic spaces.

We now extend to the category k-$\mathcal{A}n$ several classes of morphisms that were introduced in [Ber] for good k-analytic spaces. Let P be a class or morphisms of good k-analytic spaces which is preserved under compositions, under any base change and under extensions of the ground field. We say that a morphism $\varphi : Y \to X$ in k-$\mathcal{A}n$ is of class $\widetilde{P}$ if for any morphism $X' \to X$ from a good analytic space over k the space $Y \times_X X'$ is good and the induced morphism $Y \times_X X' \to X'$ is of class P. It follows from the definition that the class $\widetilde{P}$ is also preserved under the same operations. Furthermore, if P contains locally closed immersions, then $\widetilde{P}$ processes the following property: if $Y \to X$ is a locally separated morphism, then any morphism $Z \to Y$, for which the composition $Z \to X$ is of class $\widetilde{P}$, is of class $\widetilde{P}$.

1.5.3. *Examples.* — (i) If P is the class of all morphisms of good analytic spaces, then the morphisms from $\widetilde{P}$ are said to be *good*. For example, finite morphisms and locally closed immersions are good morphisms.

(ii) If P is the class of closed morphisms of good analytic spaces ([Ber], p. 49), then the morphisms from $\widetilde{P}$ are said to be *closed*. For example, finite morphisms and locally closed immersions are closed morphisms.

(iii) If P is the class of proper morphisms of good analytic spaces ([Ber], p. 50), then the morphisms from $\widetilde{P}$ are said to be *proper*. It follows from the definitions that a morphism is proper if and only if it is compact and closed. For example, finite morphisms are closed. Conversely, if a proper morphism has discrete fibres, then it is finite ([Ber], 3.3.8).

1.5.4. *Definition.* — The *relative interior* of a morphism $\varphi : Y \to X$ is the set Int(Y/X) of all points $y \in Y$ for which there exists an open neighborhood $\mathcal{V}$ of y such that the induced morphism $\mathcal{V} \to X$ is closed. The complement of Int(Y/X) is called the *relative boundary* of φ and is denoted by $\partial(Y/X)$. If $X = \mathcal{M}(k)$, these sets are denoted by Int(Y) and $\partial(Y)$ and are called the *interior* and the *boundary* of Y, respectively.

It follows from the definition that $\partial(Y/X) = \emptyset$ if and only if the morphism φ is closed. The following properties of the relative interior are easily deduced from the definition and [Ber], 3.1.3.

1.5.5. *Proposition.* — (i) *If* Y *is an analytic domain in* X, *then* Int(Y/X) *coincides with the topological interior of* Y *in* X.

(ii) *For a sequence of morphisms* $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$, *one has*

$$\mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Z/X).$$

If φ *is locally separated (resp. and good) then*

$$\mathrm{Int}(Z/X) \subset \mathrm{Int}(Y/X) \quad (\textit{resp.}\ \mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))).$$

(iii) *For a morphism* $f: X' \to X$, *one has* $f'^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y'/X')$, *where* f' *is* $Y' = Y \times_X X' \to Y$.

(iv) *For a non-Archimedean field* K *over* k, *one has* $\pi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y \hat{\otimes} K/X \hat{\otimes} K)$, *where* π *is* $Y \hat{\otimes} K \to Y$. ∎

1.5.6. *Remark.* — The notion of a strictly k-analytic space introduced in [Ber], p. 48, is not consistent with that introduced in the previous subsection. First of all, if the valuation on k is trivial, the two notions are completely different. (For example, the affine line $\mathbf{A}^1$ is strictly k-analytic in the sense of [Ber] but is not such a space in the sense of § 2.2.) Assume now that the valuation on k is nontrivial. In this case the difference is that in [Ber] strictly k-analytic spaces were considered as objects of the whole category of k-analytic spaces, but here we consider them as objects of their own category $st\text{-}k\text{-}\mathcal{A}n$ because we do not know whether the faithful functor $st\text{-}k\text{-}\mathcal{A}n \to k\text{-}\mathcal{A}n$ is fully faithful.

1.6. Connection with rigid analytic geometry

We work here with the category of rigid k-analytic spaces which is defined in [BGR], § 9.

Assume that the valuation on k is nontrivial, and let X be a Hausdorff strictly k-analytic space. The corresponding rigid k-analytic structure will be defined on the set $X_0 = \{ x \in X \mid [\mathcal{H}(x) : k] < \infty \}$. (We remark that from [Ber], 2.1.15, it follows that the set X_0 is everywhere dense in X.) First of all, if $X = \mathcal{M}(\mathcal{A})$ is strictly k-affinoid, then the maximal spectrum $X_0 = \mathrm{Max}(\mathcal{A})$ is endowed with a rigid k-analytic space structure as in [BGR], § 9.3.1. Suppose that X is arbitrary. We say that a subset $\mathcal{U} \subset X_0$ is *admissible open* if, for any strictly affinoid domain $V \subset X$, the intersection $\mathcal{U} \cap V_0$ is an admissible open set in the rigid k-affinoid space V_0. Furthermore, a covering $\{ \mathcal{U}_i \}_{i \in I}$ of an admissible open subset $\mathcal{U} \subset X_0$ by admissible open subsets is *admissible* if, for any strictly affinoid domain $V \subset X$, $\{ \mathcal{U}_i \cap V_0 \}_{i \in I}$ is an admissible open covering of $\mathcal{U} \cap V_0$. In this way we get a G-topology on the set X_0. The sheaves of rings $\mathcal{O}_{V_0}$, where V runs through the strictly affinoid domains in X, are compatible on intersections, and to they glue together to form a sheaf of rings $\mathcal{O}_{X_0}$ on the G-topological space X_0. The locally G-ringed space $(X_0, \mathcal{O}_{X_0})$ satisfies the conditions of Definition 9.3.1/4 from [BGR], and so we get a rigid k-analytic space. We remark that the rigid k-analytic space constructed is quasiseparated. (A rigid k-analytic space is called *quasiseparated* if the intersection of two open affinoid domains is a finite union of open affinoid domains.)

1.6.1. *Theorem.* — *The correspondence* $X \mapsto X_0$ *is a fully faithful functor from the category of Hausdorff strictly* k*-analytic spaces to the category of quasiseparated rigid* k*-analytic* *spaces. Furthermore, this functor induces an equivalence between the category of paracompact strictly k-analytic spaces and the category of quasiseparated rigid k-analytic spaces that have an admissible affinoid covering of finite type.*

A collection of subsets of a set is said to be of finite type if each subset of the collection meets only a finite number of other subsets of the collection.

Proof. — Let X be a Hausdorff strictly k-analytic space. First of all we establish the following fact.

1.6.2. *Lemma.* — (i) *Any open affinoid domain in the rigid k-analytic space* X_0 *is of the form* V_0, *where* V *is a strictly affinoid domain in* X.

(ii) *Let* $\{V_i\}_{i\in I}$ *be a system of strictly affinoid domains in* X. *Then* $\{V_{i,0}\}_{i\in I}$ *is an admissible covering of* X_0 *if and only if each point of* X *has a neighborhood of the form* $V_{i_1}\cup\ldots\cup V_{i_n}$ *(i.e.,* $\{V_i\}_{i\in I}$ *is a quasinet on* X*).*

Proof. — (i) An open affinoid domain in X_0 is an open immersion of rigid k-analytic spaces $f: U_0\to X_0$, where U is a strictly k-affinoid space. In particular, for any strictly affinoid domain $V\subset X$, $f^{-1}(V_0)$ is a finite union of affinoid domains in U_0, and $\{f^{-1}(V_0)\}$, where V runs through strictly affinoid domains in X, is an affinoid covering. It follows that we can find strictly affinoid domains $U_1,\ldots,U_n\subset U$ and $V_1,\ldots,V_n\subset X$ such that $U=U_1\cup\ldots\cup U_n$ and $f|_{U_{i,0}}$ comes from a morphism of strictly affinoid spaces $\varphi_i: U_i\to V_i$ that identifies U_i with an affinoid domain in V_i. Moreover, all φ_i are compatible on intersections. Therefore, we get a morphism of strictly k-analytic spaces $\varphi: U\to X$. Since φ, as a map of topological spaces, is compact and induces an injection on the everywhere dense subset $U_0\subset U$, it follows that φ induces a homeomorphism of U with its image in X. Finally, φ identifies U_i with a strictly affinoid domain in V_i, and therefore φ identifies U with a strictly analytic domain in X. It is clear that this is a strictly affinoid domain.

(ii) Suppose first that $\{V_{i,0}\}_{i\in I}$ is an admissible covering of X_0. This means that, for any strictly affinoid domain $V\subset X$, $\{V_{i,0}\cap V_0\}_{i\in I}$ is an admissible covering of V_0. It follows that V is contained in a finite union $V_{i_1}\cup\ldots\cup V_{i_n}$, and therefore each point of X has a neighborhood of the required form. Conversely, assume that the latter property is true. Then any strictly affinoid domain is contained in a finite union $V_{i_1}\cup\ldots\cup V_{i_n}$, and therefore $\{V_{i,0}\cap V_0\}_{i\in I}$ is an admissible covering of V_0. ∎

Let $\varphi: Y\to X$ be a morphism of strictly k-analytic spaces. First of all we claim that the induced map $\varphi_0: Y_0\to X_0$ is continuous with respect to the G-topologies on X_0 and Y_0. Let $\mathcal{U}\subset X_0$ be an admissible open subset, and let $V\subset Y$ be a strictly affinoid domain. By [BGR], 9.1.4/2, and Lemma 1.6.2 (i), the set $\mathcal{U}$ has an admissible covering $\{U_{i,0}\}_{i\in I}$, where U_i are strictly affinoid domains in X. By Corollary 1.2.14, for each $i\in I$ one has $\varphi^{-1}(U_i)\cap V=\bigcup_{j\in J_i}V_{ij}$, where V_{ij} are strictly affinoid domains in V and J_i is finite. We get a covering $\{V_{ij,0}\}$ of the set $\varphi_0^{-1}(\mathcal{U})\cap V_0$. To verify that the latter set is admissible open in V_0, it suffices to show that for any morphism of strictly k-affinoid spaces $\psi : W \to V$ with $\psi_0(W_0) \subset \varphi_0^{-1}(\mathscr{U}) \cap V_0$, the covering $\{ \varphi_0^{-1}(V_{ij,0})\}$ has a finite affinoid covering that refines it. But this follows from the fact that the latter condition is satisfied by the covering $\{(\varphi\psi)_0^{-1}(U_{i,0}\}_{i\in I}$. Thus, the set $\varphi_0^{-1}(\mathscr{U})$ is admissible open in Y_0. In the same way one shows that the preimage of an admissible covering of an admissible open set is an admissible open covering. Hence the map of G-topological spaces $\varphi_0 : Y_0 \to X_0$ is continuous. That φ induces a morphism of locally G-ringed spaces easily follows from this.

Let now $f: Y_0 \to X_0$ be a morphism between the above rigid k-analytic spaces. We have to show that it comes from a unique morphism of strictly k-analytic spaces $\varphi : Y \to X$. First of all, from Proposition 1.3.2 it follows that it suffices to verify the required fact only in the case when Y is strictly k-affinoid. For this we remark that the system $\{ U_0 \}$, where U runs through strictly affinoid domains in X, is an admissible covering of X_0. Therefore $\{f^{-1}(U_0)\}$ is an admissible covering of Y_0 by admissible open subsets. By [BGR], 9.1.4/2, the latter covering has a finite affinoid covering that refines it. In this way we get strictly affinoid domains $V_i, \dots, V_n \subset Y$ and $U_1, \dots, U_n \subset X$ such that $Y_0 = V_{1,0} \cup \dots \cup V_{n,0}$ (and therefore $Y = V_1 \cup \dots \cup V_n$) and $f(V_{i,0}) \subset U_{i,0}$. The induced morphisms of strictly affinoid spaces $V_i \to U_i$ are obviously compatible on intersections, and therefore we get a morphism of strictly k-analytic spaces $\varphi : Y \to X$. It is easy to see that $\varphi_0 = f$ and that φ is a unique morphism satisfying this property.

If X is a paracompact strictly k-analytic space, then it has a strictly k-affinoid atlas with a locally finite net, and therefore the rigid k-analytic space X_0 has an admissible affinoid covering of finite type. It is also evident that it is quasiseparated. Conversely, let $\mathscr{X}$ be a quasiseparated rigid k-analytic space that has an admissible affinoid covering $\{ \mathscr{U}_i \}_{i\in I}$ of finite type. First of all, let $\mathscr{U}_i = U_{i,0}$, where the U_i are strictly k-affinoid spaces. Since $\mathscr{X}$ is quasiseparated, for any pair $i, j \in I$ the intersection $\mathscr{U}_i \cap \mathscr{U}_j$ is a finite union of open affinoid domains in $\mathscr{X}$. Thus, there are strictly special domains $U_{ij} \subset U_i$ and $U_{ji} \subset U_j$ that correspond to $\mathscr{U}_i \cap \mathscr{U}_j$ under the identifications $\mathscr{U}_i = U_{i,0}$ and $\mathscr{U}_j = U_{j,0}$. Let ν_{ij} denote the induced isomorphism $U_{ij} \xrightarrow{\sim} U_{ji}$. It is clear that $U_{ii} = U_i$, $\nu_{ij}(U_{ij} \cap U_{il}) = U_{ji} \cap U_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $U_{ij} \cap U_{il}$. By Proposition 1.3.3, we can glue all U_i along U_{ij} and get a paracompact strictly k-analytic space X. It is easy to see that X_0 is isomorphic to $\mathscr{X}$. ∎

Let X be a Hausdorff strictly k-analytic space. From Lemma 1.6.2 it follows that there is an isomorphism of topoi $X_G^\sim \xrightarrow{\sim} X_0^\sim$. In particular, there is a morphism of topoi $(\pi_*, \pi^*) : X_0^\sim \to X^\sim$ such that the functor π^* is fully faithful (and π_* is not). Furthermore, from Proposition 1.3.6 it follows that if F is an abelian sheaf on X, then $H^q(X, F) \xrightarrow{\sim} H^q(X_0, \pi^* F)$, $q \geq 0$, and if X is good and F is a coherent $\mathcal{O}_X$ module, then $H^q(X, F) \xrightarrow{\sim} H^q(X_0, F_0)$, $q \geq 0$, where $F_0 = \pi^* F \otimes_{\pi^* \mathcal{O}_X} \mathcal{O}_{X_0}$. Finally, there are equivalences of categories $\mathrm{Mod}(X_G) \xrightarrow{\sim} \mathrm{Mod}(X_0)$ and $\mathrm{Coh}(X_G) \xrightarrow{\sim} \mathrm{Coh}(X_0)$ and an isomorphism of groups $\mathrm{Pic}(X_G) \xrightarrow{\sim} \mathrm{Pic}(X_0)$.

§ 2. Local rings and residue fields of points of affinoid spaces

2.1. The local rings $\mathcal{O}_{X,x}$

Throughout the section we consider a k-affinoid space $X = \mathcal{M}(\mathcal{A})$. The stalk $\mathcal{O}_{X,x}$ of the structural sheaf $\mathcal{O}_X$ at a point $x \in X$ is a local ring. Its maximal ideal is denoted by $\mathbf{m}_x$, and its residue field $\mathcal{O}_{X,x}/\mathbf{m}_x$ is denoted by $\kappa(x)$. The field $\kappa(x)$ has a canonical valuation. The completion of $\kappa(x)$ is the field $\mathcal{H}(x)$. Furthermore, let $\mathcal{X}$ denote the affine scheme $\mathrm{Spec}(\mathcal{A})$. There is a morphism of locally ringed spaces $\pi : X \to \mathcal{X}$. For a point $x \in X$ we denote by $\mathbf{x}$ its image in $\mathcal{X}$, by $\wp_{\mathbf{x}}$ the corresponding prime ideal of $\mathcal{A}$ (it is the kernel of the seminorm on $\mathcal{A}$ which corresponds to the point x) and by $k(\mathbf{x})$ the fraction field of $\mathcal{A}/\wp_{\mathbf{x}}$.

2.1.1. *Proposition.* — *The map* $\pi : X \to \mathcal{X}$ *is surjective.*

Proof. — Suppose that the algebra is strictly k-affinoid. It suffices to show that if $\mathcal{A}$ has no zero divisors, then there exists a point $x \in X$ with $\wp_{\mathbf{x}} = 0$. By Noether Normalization Lemma, there exists a finite injective homomorphism $\mathcal{B} = k\{T_1, \ldots, T_n\} \to \mathcal{A}$. By [Ber], 2.1.16, the map $\mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is surjective. So it suffices to consider the algebra $k\{T_1, \ldots, T_n\}$. In this case $\wp_{\mathbf{x}} = 0$ for the point x corresponding to the norm of $k\{T_1, \ldots, T_n\}$.

2.1.2. *Lemma.* — *For a k-affinoid algebra $\mathcal{A}$ and a non-Archimedean field* K *over k the algebra $\mathcal{A}' = \mathcal{A} \hat{\otimes} K$ is faithfully flat over $\mathcal{A}$.*

Proof. — First of all we recall that for Banach spaces B and M over k the canonical map $M \otimes B \to M \hat{\otimes} B$ is injective and, if $0 \to M \to N \to P \to 0$ is an exact admissible sequence of Banach spaces over k, then the sequence $0 \to M \hat{\otimes} B \to N \hat{\otimes} B \to P \hat{\otimes} B \to 0$ is also exact and admissible (see [Gru]). (One assumed in [Gru] that the valuation on k is nontrivial, but in the case of trivial valuation one obtains the same fact by tensoring with the field K_r for some $0 < r < 1$.) Let now M be a finite $\mathcal{A}$-module. It can be regarded as a finite Banach $\mathcal{A}$-module and, in particular, as a Banach space over k (see [Ber], 2.1.9). We have $M \otimes_{\mathcal{A}} \mathcal{A}' = M \hat{\otimes}_{\mathcal{A}} \mathcal{A}' = M \hat{\otimes} K$. By the above fact, if $M \neq 0$, then $M \otimes_{\mathcal{A}} \mathcal{A}' \neq 0$. Furthermore, if a homomorphism $M \to N$ of finite $\mathcal{A}$-modules is injective, the homomorphism of Banach K-spaces $M \hat{\otimes} K \to N \hat{\otimes} K$ is injective. This implies that the homomorphism $M \otimes_{\mathcal{A}} \mathcal{A}' \to N \otimes_{\mathcal{A}} \mathcal{A}'$ is injective. Hence, $\mathcal{A}'$ is a faithfully flat $\mathcal{A}$-algebra. ■

2.1.3. *Corollary.* — *In the situation of Lemma* 2.1.2 *for any pair of points* $x \in X$ $x' \in X' = \mathcal{M}(\mathcal{A}')$ *with* $\varphi(x') = x$, *where* φ *is the canonical map* $X' \to X$, $\mathcal{O}_{X',x'}$ *is a faithfully flat* $\mathcal{O}_{X,x}$*-algebra.*

Proof. — It suffices to show that for any pair of affinoid subdomains $V \subset X'$ and $U \subset X$ with $\varphi(V) \subset U$, $\mathcal{A}'_V$ is a flat $\mathcal{A}_U$-algebra. By Lemma 2.1.2, $\mathcal{A}'_{\varphi^{-1}(U)} = \mathcal{A}_U \hat{\otimes} K$ is a flat $\mathcal{A}_U$-algebra. Since $\mathcal{A}'_V$ is a flat $\mathcal{A}'_{\varphi^{-1}(U)}$-algebra ([Ber], 2.2.4 (ii)), then $\mathcal{A}'_V$ is a flat $\mathcal{A}_U$-algebra. ∎

We now consider an arbitrary k-affinoid algebra $\mathcal{A}$. We can take a non-Archimedean field K over k such that the algebra $\mathcal{A}' = \mathcal{A} \hat{\otimes} K$ is strictly K-affinoid. By the previous case, the map $X' = \mathcal{M}(\mathcal{A}') \to \mathcal{X}' = \mathrm{Spec}(\mathcal{A}')$ is surjective, and, by Lemma 2.1.2, the map $\mathcal{X}' \to \mathcal{X}$ is surjective. It follows that the map π is surjective. ∎

2.1.4. *Theorem. — The ring $\mathcal{O}_{X,x}$ is a Noetherian ring faithfully flat over $\mathcal{O}_{\mathcal{X},\mathbf{x}} = \mathcal{A}_{\wp_\mathbf{x}}$.*

Proof. — Suppose first that the valuation of k is nontrivial, the algebra $\mathcal{A}$ is strictly k-affinoid and $x \in \mathrm{Max}(\mathcal{A})$. In this case $\mathcal{O}_{X,x}$ coincides with the algebra of germs of affinoid functions on $\mathrm{Max}(\mathcal{A})$ considered in [BGR], 7.3.2. By [BGR], 7.3.2/7, the ring $\mathcal{O}_{X,x}$ is Noetherian, and, by [BGR], 7.3.2/3, there is an isomorphism $\hat{\mathcal{A}} \xrightarrow{\sim} \widehat{\mathcal{A}_{\wp_\mathbf{x}}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x}}$ between $\wp_\mathbf{x}$, $\wp_\mathbf{x} \mathcal{A}_{\wp_\mathbf{x}}$, $\mathbf{m}_x$-adic completions of the rings $\mathcal{A}$, $\mathcal{A}_{\wp_\mathbf{x}}$, $\mathcal{O}_{X,x}$, respectively. By [Mat], 8.14, the ring $\widehat{\mathcal{A}_{\wp_\mathbf{x}}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x}}$ is faithfully flat over $\mathcal{A}_{\wp_\mathbf{x}}$ and $\mathcal{O}_{X,x}$.

We now consider the general case. We can find a non-Archimedean field K over k with nontrivial valuation such that the algebra $\mathcal{A}' = \mathcal{A} \hat{\otimes} K$ is strictly K-affinoid, and there exists a point $x' \in \mathrm{Max}(\mathcal{A}')$ which goes to x under the canonical map $X' = \mathcal{M}(\mathcal{A}') \to X$. By Lemma 2.1.2 (resp. Corollay 2.1.3), the algebra $\mathcal{A}'_{\wp_{\mathbf{x}'}}$ (resp. $\mathcal{O}_{X',x'}$) is faithfully flat over $\mathcal{A}_{\wp_\mathbf{x}}$ (resp. $\mathcal{O}_{X,x}$). Since $\mathcal{O}_{X',x'}$ is faithfully flat over $\mathcal{A}'_{\wp_{\mathbf{x}'}}$, then $\mathcal{O}_{X,x}$ is faithfully flat over $\mathcal{A}_{\wp_\mathbf{x}}$. Furthermore, Corollary 2.1.3 implies that $\mathbf{a} = \mathbf{a}\mathcal{O}_{X',x'} \cap \mathcal{O}_{X,x}$ for any finitely generated ideal $\mathbf{a} \in \mathcal{O}_{X,x}$. From this it follows easily that $\mathcal{O}_{X,x}$ is a Noetherian ring. ∎

2.1.5. *Theorem. — The ring $\mathcal{O}_{X,x}$ is Henselian.*

Proof. — We use the following criterion for a local ring A to be Henselian (see [Ray], I.1.5). A is Henselian if and only if any finite free A-algebra B is a direct product of local rings.

Let B be a finite free $\mathcal{O}_{X,x}$-algebra. We claim that there exist an affinoid neighborhood U of the point x and a finite free $\mathcal{A}_U$-algebra $\mathcal{B}$ such that $B = \mathcal{B} \otimes_{\mathcal{A}_U} \mathcal{O}_{X,x}$. Indeed, let $b_1, \ldots, b_n$ be free generators of the $\mathcal{O}_{X,x}$-module B and set $1 = \sum_{i=1}^n a_i b_i$ and $b_i b_j = \sum_{l=1}^n a_{ijl} b_l$, where $a_i, a_{ijl} \in \mathcal{O}_{X,x}$. The fact that B is an associative and commutative ring with identity is equivalent to certain identities between the coefficients a_i and a_{ijl}. Take a sufficiently small affinoid neighborhood U of the point x such that all the a_i, a_{ijl} come from $\mathcal{A}_U$ and all the identities are true in $\mathcal{A}_U$. Consider the free $\mathcal{A}_U$-module $\mathcal{B} = \mathcal{A}_U b_1 + \ldots + \mathcal{A}_U b_n$ and endow it with the multiplication $b_i b_j = \sum_{l=1}^n a_{ijl} b_l$. Then $\mathcal{B}$ is a finite free $\mathcal{A}_U$-algebra, and, by construction, $B = \mathcal{B} \otimes_{\mathcal{A}_U} \mathcal{O}_{X,x}$.

Furthermore, we may assume that $U = X$ and consider $\mathcal{B}$ as a finite Banach $\mathcal{A}$-algebra (see [Ber], 2.1.12). Then we have a finite morphism of k-affinoid spaces $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X$, and the theorem follows from the following lemma.

2.1.6. *Lemma.* — *Let $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X = \mathcal{M}(\mathcal{A})$ be a finite morphism of k-affinoid spaces. Then for any point $x \in X$ there is an isomorphism of rings $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{O}_{X,x} = \prod_{i=1}^{d} \mathcal{O}_{Y,y_i}$ where $\varphi^{-1}(x) = \{y_1, \dots, y_d\}$.*

Proof. — Since φ is a map of compact spaces, one has $\varphi^{-1}(U) = \bigcup_{i=1}^{d} V_i$, for any sufficiently small affinoid neighborhood U of x, where V_i are affinoid neighborhoods of the points y_i such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Moreover, the domains V_i form a basis of affinoid neighborhoods of y_i. We have

$$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{O}_{X,x} = \mathcal{B}_{\varphi^{-1}(U)} \otimes_{\mathcal{A}_U} \mathcal{O}_{X,x} = \prod_{i=1}^{d} \mathcal{B}_{V_i} \otimes_{\mathcal{A}_U} \mathcal{O}_{X,x}.$$

Here we used the equality $\mathcal{B}_{\varphi^{-1}(U)} = \mathcal{B} \hat{\otimes}_{\mathcal{A}} \mathcal{A}_U = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}_U$ which follows from the fact that $\mathcal{B}$ is a finite Banach $\mathcal{A}$-module. Therefore, $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{O}_{X,x} = \prod_{i=1}^{d} \mathcal{O}_{Y,y_i}$. ∎

2.2. Comparison of properties of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{\mathcal{X},x}$

Let P be a property of local rings which is preserved under localizations with respect to the complements to prime ideals. A commutative ring A is said to possess the property P (or A is a P-ring) if all of the local rings $A_{\wp}$, where $\wp$ runs through prime ideals of A, possess the property P. More generally, let Y be a locally ringed space. The set of points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is a P-ring is denoted by P(Y). If P(Y) = Y, then Y is said to possess the property P.

2.2.1. *Theorem.* — *Let* P *be the property of being* Red *(reduced),* Nor *(normal),* Reg *(regular),* CI *(complete intersection),* Gor *(Gorenstein),* CM *(Cohen-Macauley). Then* P(X) *is Zariski open in* X *and* $P(X) = \pi^{-1}(P(\mathcal{X}))$.

For the definition of these properties and the verification of the fact that P is preserved under localizations see Matsumura's book [Mat]. We shall deduce Theorem 2.2.1 from known results which are formulated in the following lemmas.

2.2.2. *Lemma.* — *Let $(A, \mathbf{m}) \to (B, \mathbf{n})$ be a faithfully flat homomorphism of local Noetherian rings.*

(i) *If* B *is a* P-*ring, then so is* A.

(ii) *If* A *is a* P-*ring, where* P $\neq$ Red, Nor, *and* $\mathbf{n} = \mathbf{m}B$, *then* B *is a* P-*ring.* ∎

2.2.3. *Lemma.* — *Strictly k-affinoid algebras are excellent rings.* ∎

2.2.4. *Lemma.* — *Let* A *be an excellent ring.*

(i) *If* $\wp$ *is a prime ideal of* A *such that* $A_\wp$ *is a* P-*ring, where* P = Red *or* Nor, *then the completion* $\widehat{\mathcal{A}_\wp}$ *is a* P-*ring.*

(ii) *The set of prime ideals* $\wp \subset A$ *such that* $A_\wp$ *is a* P-*ring is open in* Spec(A). ∎

For Lemma 2.2.2 and the assertion (ii) of Lemma 2.2.4 in the cases P = CI or Gor see [Mat], § 23-24. Lemma 2.2.3 is proved in [Kie]. The assertions (i) and (ii) (for P ≠ CI, Gor) of Lemma 2.2.4 are proved in [EGAIV], 7.8.3.

Proof of Theorem **2.2.1.** — From Theorem 2.1.4 and Lemma 2.2.2 (i) it follows that $P(X) \subset \pi^{-1}(P(\mathcal{X}))$. Let x be a point of X such that $\mathcal{A}_{\wp_x}$ is a P-ring. We have to show that $\mathcal{O}_{X,x}$ is a P-ring.

Suppose that the valuation of k is nontrivial and the algebra $\mathcal{A}$ is strictly k-affinoid. From Lemmas 2.2.3 and 2.2.4 it follows that the set $P(\mathcal{X})$ is open in $\mathcal{X}$. Furthermore, if $x \in \mathrm{Max}(\mathcal{A})$, then $\widehat{\mathcal{A}_{\wp_x}} = \widehat{\mathcal{O}_{X,x}}$, and therefore $\mathcal{O}_{X,x}$ is a P-ring. Let x be an arbitrary point. Since $P(\mathcal{X})$ is open in $\mathcal{X}$, then $V \subset \pi^{-1}(P(\mathcal{X}))$ for any sufficiently small strictly affinoid neighborhood V of x. If $y \in \mathrm{Max}(\mathcal{A}_V) \subset \mathrm{Max}(\mathcal{A})$, then $\mathcal{O}_{\mathcal{X},\pi(y)}$ is a P-ring, and therefore $\mathcal{O}_{X,y}$ is a P-ring. Since $\mathcal{O}_{X,y} = \mathcal{O}_{V,y}$, then $\mathcal{O}_{\mathcal{V},\pi(y)}$ is a P-ring, where $\mathcal{V} = \mathrm{Spec}(\mathcal{A}_V)$. It follows that $P(\mathcal{V}) = \mathcal{V}$ because $P(\mathcal{V})$ is open in $\mathcal{V}$. Thus, the algebras $\mathcal{A}_V$ are P-rings for all sufficiently small strictly affinoid neighborhoods V of x. Since $\mathcal{O}_{X,x} = \varinjlim \mathcal{A}_V$, then $\mathcal{O}_{X,x}$ is a P-ring. Indeed, this is evident if P = Red or Nor. If P ≠ Red, Nor, we remark that $\mathbf{m}_x = \wp_{x,V}\, \mathcal{O}_{X,x}$ for a sufficiently small strictly affinoid neighborhood V of x, where $\wp_{x,V}$ is the prime ideal of $\mathcal{A}_V$ corresponding to the point x. From Lemma 2.2.2 (ii) it follows that $\mathcal{O}_{X,x}$ is a P-ring.

We remark that (under the same assumptions) the fact already verified implies that the subsheaf of ideals $\mathcal{I}_X \subset \mathcal{O}_X$ consisting of nilpotent elements is coherent and, in fact, is generated by the nilradical $\mathrm{rad}(\mathcal{A})$ of $\mathcal{A}$. Furthermore, if X is reduced, then the subsheaf $\mathcal{O}_X^{\mathrm{norm}}$ of the sheaf $\mathcal{M}_X$ of meromorphic functions consisting of elements, whose images in all stalks $\mathcal{M}_{X,x}$ are integral over $\mathcal{O}_{X,x}$, is coherent, and there exists $a \in \mathcal{A}$, which is not a zero divisor, such that $a\mathcal{O}_X^{\mathrm{norm}} \subset \mathcal{O}_X$.

We now consider the general case.

2.2.5. *Lemma.* — *Let* K *be a field of the form* $K_{r_1,\dots,r_n}$ *(see* [Ber], § 2.1*) and* $\mathcal{A}' = \mathcal{A} \widehat{\otimes} K$. *Consider the map* $\sigma : X \to X' = \mathcal{M}(\mathcal{A}')$ *which sends a point* $x \in X$ *to the point* $x' \in X'$ *corresponding to the multiplicative semi-norm* $\sum_\nu a_\nu T^\nu \mapsto \max_\nu |a_\nu(x)| r^\nu$. *Then* $\wp_{x'} = \wp_x \mathcal{A}'$. *Furthermore, if* Y' *is a Zariski closed subset of* X', *then* $\sigma^{-1}(Y')$ *is a Zariski closed subset of* X.

Proof. — Let $f_1, \dots, f_n$ be generators of $\wp_x$. Since the canonical epimorphism $\mathcal{A}^n \to \wp_x : (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i f_i$ is admissible (see [Ber], 2.1.9), there exists a constant $C > 0$ such that any element $a \in \wp_x$ can be represented in the form $\sum_{i=1}^n a_i f_i$ with $\|a_i\| \leqslant C \|a\|$, $1 \leqslant i \leqslant n$. Let $a' = \sum_\nu a_\nu T^\nu \in \wp_{x'}$. By construction, all the a_ν

belong to $\wp_x$. For every ν we take a representation $a_\nu = \sum_{i=1}^n a_{\nu,i} f_i$ as above. Then $b_i = \sum_\nu a_{\nu,i} T^\nu$ are well defined elements of $\mathscr{A}'$, and we have $a' = \sum_{i=1}^n b_i f_i \in \wp_x \mathscr{A}'$. Thus $\wp_{x'} = \wp_x \mathscr{A}'$.

Let Y' be defined by an ideal $\mathbf{a}' \subset \mathscr{A}'$. Denote by $\mathbf{a}$ the ideal of $\mathscr{A}$ generated by all of the coefficients a_ν from the representations $a' = \sum_\nu a_\nu T^\nu$ of elements $a' \in \mathbf{a}'$. We claim that $\sigma^{-1}(Y')$ is the closed k-analytic subset of X defined by the ideal $\mathbf{a}$. Indeed, let $x \in X$ and $x' = \sigma(x)$. Then $x \in \sigma^{-1}(Y') \Leftrightarrow x' \in Y' \Leftrightarrow \mathbf{a}' \subset \wp_{x'} \Leftrightarrow \mathbf{a} \subset \wp_x$. ∎

Take a field K of the form $K_{r_1,\ldots,r_n}$, $n \geqslant 1$, such that the algebra $\mathscr{A}' = \mathscr{A} \hat{\otimes} K$ is strictly K-affinoid (the valuation on K is nontrivial since $n \geqslant 1$). First we consider the case when $P \neq$ Red, Nor. Since $\wp_{x'} = \wp_x \mathscr{A}'$, where $x' = \sigma(x)$, it follows from Lemma 2.2.2 (ii) that $\mathscr{A}'_{\wp_{x'}}$ is a P-ring. By the strictly affinoid case, $\mathcal{O}_{X',x'}$ is a P-ring. Therefore $\mathcal{O}_{X,x}$ is a P-ring, by Lemma 2.2.2 (i). We have

$$P(X) = \pi^{-1}(P(\mathscr{X})) = \sigma^{-1}(P(X')).$$

From Lemma 2.2.5 it follows that P(X) is Zariski open in X.

Let P = Red. It suffices to verify that the subsheaf of ideals $\mathscr{I}_X \subset \mathcal{O}_X$ consisting of nilpotent elements is generated by $\mathrm{rad}(\mathscr{A})$. We may assume that $\mathrm{rad}(\mathscr{A}) = 0$. Then $\mathrm{rad}(\mathscr{A}') = 0$. Hence $\mathscr{I}_{X'} = 0$. Since $\mathcal{O}_{X,x}$ is embedded to $\mathcal{O}_{X',x'}$, where $x' = \sigma(x)$, we have $\mathscr{I}_X = 0$.

Let P = Nor. By the previous case, we may assume that X is reduced. We want to verify that there exists $a \in \mathscr{A}$ which is not a zero-divisor such that $a\mathcal{O}_X^{\mathrm{norm}} \subset \mathcal{O}_X$. Since this is true for the subsheaf of $\mathscr{M}_X$ generated by the normalization of $\mathscr{A}$ (see [Ber], 2.1.14 (i)), we may assume that $\mathscr{A}$ is a (normal) integral domain. By the strictly affinoid case, there exists a non-zero element $a' = \sum_\nu a_\nu T^\nu \in \mathscr{A}'$ with $a' \mathcal{O}_{X'}^{\mathrm{norm}} \subset \mathcal{O}_{X'}$. It suffices to show that $a_\nu \mathcal{O}_X^{\mathrm{norm}} \subset \mathcal{O}_X$ for any ν. Let $\mathscr{U}$ be an open subset of X, and let f be an element from the full ring of fractions of $\mathcal{O}_X(\mathscr{U})$. Then $a'f \in \mathcal{O}_{X'}(\varphi^{-1}(\mathscr{U}))$, where φ denotes the canonical map $X' \to X$. But $\mathcal{O}_{X'}(\varphi^{-1}(\mathscr{U}))$ consists of the series $\sum_\nu f_\nu T^\nu$ such that $f_\nu \in \mathcal{O}_X(\mathscr{U})$, and, for any affinoid subdomain $V \subset \mathscr{U}$, $\|f_\nu\|_V r^\nu \to 0$ as $\nu \to \infty$. It follows that $a_\nu f \in \mathcal{O}_X(\mathscr{U})$ for any ν. ∎

2.2.6. *Corollary. — Let* P *be one of the properties in Theorem* 2.2.1. *Then for any good k-analytic space* Y *the set* P(Y) *is Zariski open in* Y. *Furthermore, if* Y *is reduced, then the complement to* Reg(Y) *is nowhere dense in* Y. ∎

2.2.7. *Corollary. — Let* P *be one of the properties in Theorem* 2.2.1. *Let* $\mathscr{Y}$ *be a scheme of locally finite type over k, and let π be the canonical map* $\mathscr{Y}^{\mathrm{an}} \to \mathscr{Y}$. *Then* $P(\mathscr{Y}^{\mathrm{an}}) = \pi^{-1}(P(\mathscr{Y}))$.

Proof. — We may assume that $\mathscr{Y} = \mathrm{Spec}(B)$ is an affine scheme, and B is a finitely generated k-algebra. Suppose first that the valuation on k is trivial. If $f_1, \ldots, f_n$ generate B over k, then $\mathscr{Y}^{\mathrm{an}}$ is a union of affinoid subdomains of the type

$$V = \{y \in \mathscr{Y}^{\mathrm{an}} \mid |f_i(x)| \leqslant r, 1 \leqslant i \leqslant n\}.$$

But if $r \geqslant 1$, then $\mathscr{B}_V = B$. Therefore the required statement follows from Theorem 2.2.1.

Suppose now that the valuation on k is nontrivial, and let $y \in \mathcal{Y}^{\mathrm{an}}$. By [Ber], 3.4.1, $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}},y}$ is a faithfully flat $\mathcal{O}_{\mathcal{Y},y}$-algebra, and if $y \in \mathrm{Max}(\mathrm{B})$, then $\widehat{\mathcal{O}_{\mathcal{Y}^{\mathrm{an}},y}} = \widehat{\mathcal{O}_{\mathcal{Y},y}}$. Since A is an excellent ring, we get the required statement, by the reasoning from the proof of Theorem 2.2.1. ∎

2.2.8. *Corollary.* — *For any affinoid subdomain* $\mathrm{V} \subset \mathrm{X}$ *one has* $\mathrm{Reg}(\mathrm{V}) = \mathrm{Reg}(\mathrm{X}) \cap \mathrm{V}$.

Proofs. — If X and V are strictly k-affinoid, this is clear because both sets are determined by their intersections with X_0. The general case is reduced to the strictly affinoid one by the reasoning from the proof of Theorem 2.2.1. ∎

2.2.9. *Remark.* — The maximal ideal $\mathbf{m}_x$ of the local ring $\mathcal{O}_{\mathrm{X},x}$ can be strictly larger than $\wp_x \mathcal{O}_{\mathrm{X},x}$. This is related to the fact that the Zariski topology on an affinoid subdomain $\mathrm{V} \subset \mathrm{X}$ can be strictly stronger than that induced by the Zariski topology on X. Here is an example. Let $\widetilde{f}(\mathrm{T}) = \Sigma_{i=1}^{\infty} \widetilde{a}_i \mathrm{T}^i$ be a formal power series in one variable over the residue field $\widetilde{k}$, which is algebraically independent of T, and set $f(\mathrm{T}) = \Sigma_{i=1}^{\infty} a_i \mathrm{T}^i$, where a_i are representatives of $\widetilde{a}_i$ in k^0. We claim that for any non-zero $g(\mathrm{T}_1, \mathrm{T}_2) \in k\{\mathrm{T}_1, \mathrm{T}_2\}$ one has $g(\mathrm{T}, f(\mathrm{T})) \neq 0$. Indeed, let $g(\mathrm{T}_1, \mathrm{T}_2) = \Sigma_{i,j=0}^{\infty} a_{i,j} \mathrm{T}_1^i \mathrm{T}_2^j$. Multiplying g by a constant, we may assume that $\| g \| = \max_{i,j} | a_{i,j} | = 1$. Let $\mathrm{S} = \{(i,j) \mid |a_{i,j}| = 1\}$ (it is a finite set) and set

$$\mathrm{P}(\mathrm{T}_1, \mathrm{T}_2) = \Sigma_{(i,j) \in \mathrm{S}} a_{i,j} \mathrm{T}_1^i \mathrm{T}_2^j$$

and

$$h(\mathrm{T}_1, \mathrm{T}_2) = g(\mathrm{T}_1, \mathrm{T}_2) - \mathrm{P}(\mathrm{T}_1, \mathrm{T}_2).$$

Since $\| h \| < 1$, we have $\| h(\mathrm{T}, f(\mathrm{T})) \| < 1$. If $g(\mathrm{T}, f(\mathrm{T})) = 0$, then $\| \mathrm{P}(\mathrm{T}, f(\mathrm{T})) \| < 1$. It follows that $\widetilde{\mathrm{P}}(\mathrm{T}, \widetilde{f}(\mathrm{T})) = 0$. This is impossible because $\widetilde{f}(\mathrm{T})$ is algebraically independent of T over $\widetilde{k}$. Thus, the Zariski closed subset of the two-dimensional disc V of radius $r < 1$, which is defined by the equation $\mathrm{T}_2 - f(\mathrm{T}_1) = 0$, does not extend to a Zariski closed subset of the two-dimensional unit disc X. If now x is the point of X, which corresponds to the multiplicative seminorm on $k\{\mathrm{T}_1, \mathrm{T}_2\} : g(\mathrm{T}_1, \mathrm{T}_2) \mapsto | g(\mathrm{T}, f(\mathrm{T})) |$, then $\mathbf{m}_x \neq 0$ because $\mathrm{T}_2 - f(\mathrm{T}_1) \in \mathbf{m}_x$, but $\wp_x = 0$.

2.3. The residue fields $\kappa(x)$

2.3.1. *Definition.* — A field K with valuation is said to be *quasicomplete* if the valuation extends uniquely to any algebraic extension of K.

For example, if K is complete with respect to its valuation (i.e. K is a valuation field in the terminology of [Ber], § 1.1), then it is quasicomplete. The following lemma easily follows from [BGR], § 3.2.

2.3.2. *Lemma.* — *The following properties of a field* K *with valuation are equivalent*:

(i) K *is quasicomplete*;

(ii) *for any irreductible polynomial* $T^n + a_1 T^{n-1} + \ldots + a_n \in K[T]$, *one has* $|a_i|^{1/i} \leqslant |a_n|^{1/n}$, $1 \leqslant i \leqslant n$;

(iii) *the spectral norm of any finite extension of* K *is a valuation.* ∎

2.3.3. *Theorem.* — *The residue field* $\kappa(x)$ *of a point* $x \in X$ *is quasicomplete.*

Proof. — Note that the field $\kappa(x)$ does not change if we replace X by a smaller affinoid neighborhood of x or by a closed k-analytic subset which contains x. Since the ring $\mathcal{O}_{X,x}$ is Noetherian, $\mathbf{m}_x = \wp_{x,V}\,\mathcal{O}_{X,x}$ for some affinoid neighborhood V of x, where $\wp_{x,V}$ is the prime ideal of $\mathcal{A}_V$ which corresponds to x. So we can replace X by $\mathcal{M}(\mathcal{A}_V/\wp_{x,V})$ and assume that $\mathbf{m}_x = 0$. Furthermore, since X is regular at x, we can decrease X and assume that the algebra $\mathcal{A}$ is regular and has no zero divisors.

Let L be a finite extension of $K = \kappa(x)$. It suffices to show that the spectral norm $|\ |_{sp}$ of L is a valuation. Recall (see [BGR], 3.2.1/1) that for an element $g \in L$ one has $|g|_{sp} = \max_{1 \leqslant i \leqslant n} |f_i|^{1/i}$, where $T^n + f_1 T^{n-1} + \ldots + f_n$ is the minimal polynomial of g and $|\ |$ is the valuation of K (if f comes from $\mathcal{A}$, then $|f| = |f(x)|$). We may assume that L is separable over K and, in particular, that L is generated by one element α. Let $T^m + a_1 T^{m-1} + \ldots + a_m$ be the minimal polynomial of α over K. Decreasing X, we may assume that all the a_i and f_j belong to $\mathcal{A}$. Let $\mathcal{K}$ be the fraction field of $\mathcal{A}$. Consider the finite extension $\mathcal{L}$ of $\mathcal{K}$ which corresponds to the minimal polynomial of α. We may assume that $\alpha, g \in \mathcal{L}$ and $L = K\mathcal{L}$. Let $\mathcal{B}$ be the integral closure of $\mathcal{A}$ in $\mathcal{L}$, and let φ denote the morphism of k-affinoid spaces $Y = \mathcal{M}(\mathcal{B}) \to X$. By construction, $\varphi^{-1}(x) = \{y\}$ and $\kappa(y) = L$. It suffices to show that $|g|_{sp} = |g(y)|$. One has

$$|g(y)| = \inf_V \rho_V(g),$$

where V runs through a basis of affinoid neighborhoods of y, and $\rho_V(g)$ is the spectral norm of g in the Banach algebra $\mathcal{A}_V$. Since $\varphi^{-1}(x) = \{y\}$, we have

$$|g(y)| = \inf_U \rho_{\varphi^{-1}(U)}(g),$$

where U runs through a basis of affinoid neighborhoods of x.

2.3.4. *Lemma.* — *Let* $\mathcal{A} \to \mathcal{B}$ *be a finite injective homomorphism of regular k-affinoid algebras, and suppose that* $\mathcal{B}$ *has no zero divisors. Then for any element* $g \in \mathcal{B}$ *one has*

$$\rho(g) = \max_{1 \leqslant i \leqslant n} \rho(f_i)^{1/i},$$

where $T^n + f_1 T^{n-1} + \ldots + f_n$ *is the minimal polynomial of* g *over* $\mathcal{A}$.

Proof. — Suppose that the valuation of k is nontrivial and the algebra $\mathscr{A}$ is strictly k-affinoid. Then for an element $f \in \mathscr{A}$ (resp. $g \in \mathscr{B}$) the spectral norm $\rho(f)$ (resp. $\rho(g)$) is equal to the supremum semi-norm $|f|_{\sup}$ (resp. $|g|_{\sup}$) on the maximal spectrum $\mathrm{Max}(\mathscr{A})$ (resp. $\mathrm{Max}(\mathscr{B})$). Hence the required fact follows from [BGR], 3.8.1/7.

In the general case we take a field k' of the form $K_{r_1,\dots,r_n}$, $n \geqslant 1$, such that the algebra $\mathscr{A}' = \mathscr{A} \hat{\otimes} k'$ is strictly k'-affinoid; then the rings $\mathscr{A}'$ and $\mathscr{B}' = \mathscr{B} \hat{\otimes} k'$ are regular. From Lemma 2.2.5 it follows that $\mathscr{B}'$ has no zero divisors. Since the canonical homomorphisms $\mathscr{A} \to \mathscr{A}'$ and $\mathscr{B} \to \mathscr{B}'$ are isometric, it suffices to show that the minimal polynomial of g over $\mathscr{A}$ remains to be irreducible over $\mathscr{A}'$. But this is clear because $\mathscr{B}'$ has no zero divisors. ∎

Using Lemma 2.3.4, we have

$$\begin{aligned} |g(y)| &= \inf_{\mathrm{U}} \rho_{\varphi^{-1}(\mathrm{U})}(g) = \inf_{\mathrm{U}} \max_{1 \leqslant i \leqslant n} \rho_{\mathrm{U}}(f_i)^{1/i} \\ &= \max_{1 \leqslant i \leqslant n} \inf_{\mathrm{U}} \rho_{\mathrm{U}}(f_i)^{1/i} = \max_{1 \leqslant i \leqslant n} |f_i(x)|^{1/i} = |g|_{\mathrm{sp}}. \end{aligned}$$

Theorem 2.3.3 is proved. ∎

2.4. Quasicomplete fields

In this subsection we establish properties of quasicomplete fields which will be very useful in the sequel. The Galois group of a normal extension L/K will be denoted by G(L/K). The Galois group $G(K^s/K)$ of the separable closure K^s of K will be denoted by G_K. If K is a quasicomplete field, then the valuation on K uniquely extends to its algebraical closure K^a. The same, of course, is true for the completion $\hat{K}$ of K.

2.4.1. *Proposition.* — *Let* K *be a quasicomplete field. Then for any finite separable extension* L/K *one has* $\hat{L} \xrightarrow{\sim} L \otimes_K \hat{K}$, *and the correspondence* $L \mapsto \hat{L}$ *induces an equivalence between the categories of finite separable extensions of* K *and of* $\hat{K}$. *In particular, there is an isomorphism* $G_{\hat{K}} \xrightarrow{\sim} G_K$.

Proof. — Since the valuation of L coincides with the spectral norm, L is weakly K-cartesian ([BGR], 3.5.1/3). By [BGR], 2.3.3/6, one has $[\hat{L} : \hat{K}] = [L : K]$, and therefore $\hat{L} \xrightarrow{\sim} L \otimes_K \hat{K}$. Our assertion now follows from Krasner's Lemma (see [BGR], 3.4.2). ∎

2.4.2. *Corollary.* — *Let* K *be a quasicomplete field, and let* K′ *be a bigger quasicomplete field whose valuation extends the valuation of* K. *Suppose that the maximal purely inseparable extension of* $\hat{K}$ *in* $\hat{K}'$ *is dense in* $\hat{K}'$. *Then the correspondence* $L \mapsto L \otimes_K K'$ *induces an equivalence between the categories of finite separable extensions of* K *and of* K′, *and there is an isomorphism* $G_{K'} \xrightarrow{\sim} G_K$. ∎

2.4.3. *Proposition. — The following properties of a field* K *with valuation are equivalent:*

a) K *is quasicomplete;*

b) the local ring $K^0 = \{ \alpha \in K \mid |\alpha| \leqslant 1 \}$ *is Henselian.*

Proof. — *a)* ⇒ *b)* (see [BGR], 3.3.4). Let F be a polynomial in $K^0[T]$ with $|F| = 1$ (the Gauss norm). Suppose that $\widetilde{F}$ is a product of two coprime polynomials $g, h \in \widetilde{K}[T]$. Take a decomposition $F = F_1, \ldots, F_n$ of F into irreducible polynomials. We may assume that $|F_i| = 1$ for all i and that the polynomials $F_1, \ldots, F_m$ are monic and the leading coefficients of $F_{m+1}, \ldots, F_n$ have the norm < 1. Then $\widetilde{F}_i = f_i^{l_i}$ for some irreducible polynomials $f_i \in \widetilde{K}[T]$, $1 \leqslant i \leqslant m$ (this follows from Proposition 2.4.4 (ii)). From Lemma 2.3.2 it follows that $\widetilde{F}_{m+1}, \ldots, \widetilde{F}_n$ are elements of $\widetilde{K}^{\bullet}$. Since g and h are coprime in $\widetilde{K}[T]$, we may assume that $g = \widetilde{a} f_1^{l_1}, \ldots, f_r^{l_r}$ for some $a \in K$ with $|a| = 1$ and $r \leqslant m$. Then for the polynomials $G = aF_1, \ldots, F_r$ and $H = a^{-1} F_{r+1}, \ldots, F_n$ one has $\widetilde{G} = g$, $\widetilde{H} = h$ and $F = GH$.

b) ⇒ *a)*. Let L be a finite extension of K, and let B the integral closure of $A = K^0$ in L. We claim that B is a local ring. Indeed, it suffices to show that the set $\mathbf{b} = B \backslash B^{\bullet}$ is an ideal in B. Suppose that for some elements $f, g \in \mathbf{b}$ one has $f + g \notin \mathbf{b}$. Consider the A-subalgebra C of B generated by the elements f, g and $(f+g)^{-1}$. Then C is a finite A-algebra. But any finite algebra over a local Henselian ring is a product of local rings. Since C is an integral domain, it follows that C is a local ring. We get that the elements f and g belong to the maximal ideal of C but their sum $f + g$ is invertible in C. Thus, B is a local ring.

Furthermore, from the definition of the spectral norm it follows that

$$B = \{ f \in L \mid |f|_{sp} \leqslant 1 \}.$$

Let $|\ |_1, \ldots, |\ |_n$ be the valuations on L which extend the valuation on K. One has $|f|_{sp} = \max_{1 \leqslant i \leqslant n} |f|_i$ (see [BGR], § 3.3). Suppose that $n > 1$. By the Artin-Waples Lemma, one can find for each $1 \leqslant i \leqslant n$ an element $f_i \in L$ such that $|f_i|_i = 1$ and $|f_i|_j < 1$ for $j \neq i$. The elements $f_1, \ldots, f_n$ are not invertible in B, and therefore belong to the maximal ideal of B. But for the element $f = f_1 + \ldots + f_n$ one has $|f|_i = 1$, $1 \leqslant i \leqslant n$. It follows that $|f^{-1}|_i = 1$, $1 \leqslant i \leqslant n$, and therefore $|f^{-1}|_{sp} = 1$, i.e., f is invertible in B. Hence, $n = 1$, i.e., the spectral norm on L is a valuation. ■

Let K be a quasicomplete field, let L be a Galois extension of K (finite or infinite). We set $I(L/K) = \{ \sigma \in G(L/K) \mid \sigma \text{ acts trivially on } \widetilde{L} \}$ ($\widetilde{L}$ is the residue field of L) and $W(L/K) = \{ \sigma \in G(L/K) \mid |{}^{\sigma}\alpha - \alpha| < |\alpha| \text{ for all } \alpha \in L^{\bullet} \}$. We set $p = \mathrm{char}(\widetilde{K})$.

2.4.4. *Proposition.* — (i) I(L/K) *and* W(L/K) *are normal divisors of* G(L/K) *and* $W(L/K) \subset I(L/K)$;

(ii) *the extension* $\widetilde{L}/\widetilde{K}$ *is normal and* $G(L/K)/I(L/K) \xrightarrow{\sim} G(\widetilde{L}/\widetilde{K})$;

(iii) *there is a canonical isomorphism* $I(L/K)/W(L/K) \xrightarrow{\sim} \mathrm{Hom}(|L^{\bullet}|/|K^{\bullet}|, \widetilde{L}^{\bullet})$;

(iv) W(L/K) *is a pro-p-group.*

Proof. — (i) is trivial.

(ii) Let $\tilde{\alpha} \in \tilde{L}$. We take a representative α of $\tilde{\alpha}$ in L^0 and the minimal polynomial $P(T) = T^n + a_1 T^{n-1} + \ldots + a_n$ of α over K. Since the valuation of L coincides with the spectral norm,

$$|\alpha| = \max_{1 \leqslant i \leqslant n} |a_i|^{1/i} \leqslant 1,$$

and therefore all a_i belong to K^0. The element $\tilde{\alpha}$ is a root of the polynomial $\tilde{P}(T) = T^n + \tilde{a}_1 T^{n-1} + \ldots + \tilde{a}_n \in \tilde{K}[T]$. Hence all the elements of $(\tilde{K})^a$ conjugated to $\tilde{\alpha}$ are also roots of $\tilde{P}(T)$. Since the group $G(L/K)$ acts transitively on the set of roots of $P(T)$, its image in the automorphism group of $\tilde{L}$ over $\tilde{K}$ acts transitively on the set of roots of $\tilde{P}(T)$. It follows that $\tilde{L}$ is normal over $\tilde{K}$, and the canonical map $G(L/K) \to G(\tilde{L}/\tilde{K})$ is surjective.

(iii) For $\sigma \in I(L/K)$ and $\alpha \in L^*$ we denote by $\psi(\sigma, \alpha)$ the image of the element ${}^{\sigma}\alpha/\alpha$ in $\tilde{L}^*$. The map $\psi : I(L/K) \times L^* \to \tilde{L}^*$ is bilinear because for $\sigma, \tau \in I(L/K)$ one has

$$\left| \frac{{}^{\sigma\tau}\alpha}{\alpha} - \frac{{}^{\sigma}\alpha}{\alpha}\frac{{}^{\tau}\alpha}{\alpha} \right| = \left| \frac{{}^{\sigma}\alpha}{\alpha}\left(\frac{{}^{\sigma\tau}\alpha}{{}^{\sigma}\alpha} - \frac{{}^{\tau}\alpha}{\alpha}\right) \right| = \left| {}^{\sigma}\!\left(\frac{{}^{\tau}\alpha}{\alpha}\right) - \frac{{}^{\tau}\alpha}{\alpha} \right| < 1.$$

Furthermore, if $|\alpha| = |\beta|$, then

$$\left| \frac{{}^{\sigma}\alpha}{\alpha} - \frac{{}^{\sigma}\beta}{\beta} \right| = \left| \frac{{}^{\sigma}\beta}{\alpha}\left(\frac{{}^{\sigma}\alpha}{{}^{\sigma}\beta} - \frac{\alpha}{\beta}\right) \right| = \left| {}^{\sigma}\!\left(\frac{\alpha}{\beta}\right) - \frac{\alpha}{\beta} \right| < 1,$$

i.e., $\psi(\sigma, \alpha)$ depends only on $|\alpha|$. If $\alpha \in K^*$, then $\psi(\sigma, \alpha) = 1$. Therefore ψ induces an embedding

$$M(L/K) = I(L/K)/W(L/K) \hookrightarrow \operatorname{Hom}(|L^*|/|K^*|, \tilde{L}^*).$$

2.4.5. *Lemma.* — *If* K′ *is a Galois extension of* K *with* $K \subset K' \subset L$, *then there are exact sequences*

$$0 \to I(L/K') \to I(L/K) \to I(K'/K) \to 0,$$

$$0 \to W(L/K') \to W(L/K) \to W(K'/K) \to 0.$$

Proof. — The only nontrivial fact is the surjectivity of the maps $I(L/K) \to I(K'/K)$ and $W(L/K) \to W(K'/K)$. The surjectivity of the first map is equivalent to the surjectivity of the map $G(L/K) \to G(\tilde{L}/\tilde{K})$ which is proved in (ii). The surjectivity of the second map is equivalent to the injectivity of the map $M(L/K') \to M(L/K)$. The latter fact follows from the part of (iii) which is already verified. ∎

Suppose that L is finite over K and set $K' = L^{W(L/K)}$. By Lemma 2.4.5, $W(K'/K) = 0$. We claim that $\tilde{K}'$ is the maximal subfield $\tilde{L}_s \subset \tilde{L}$ separable over $\tilde{K}$, and $[I(K'/K) : 1] = [\operatorname{Hom}(|K'^*|/|K^*|, \tilde{K}'^*) : 1] = [|K'^*| : |K^*|]$. Indeed, since

$$G(\tilde{K}'/\tilde{K}) = G(\tilde{L}/\tilde{K}) = G(\tilde{L}_s/\tilde{K}),$$

we have $\widetilde{L}_s \subset \widetilde{K}'$. Furthermore,

$$\begin{aligned}[K' : K] &= [G(\widetilde{K}'/\widetilde{K}) : 1]\,[I(K'/K) : 1] \\ &\leqslant [\widetilde{K}' : \widetilde{K}]\,[\mathrm{Hom}(|K'^*|/|K^*|, \widetilde{K}'^*) : 1] \\ &\leqslant [\widetilde{K}' : \widetilde{K}]\,[|K'^*| : |K^*|] \leqslant [K' : K].\end{aligned}$$

Therefore all the inequalities are actually equalities, and our claim follows. The case of an arbitrary L is deduced from this, using Lemma 2.4.5.

(iv) As above, it suffices to assume that L is finite over K. Suppose that W contains an element σ of order l which is prime to p. Let K′ be the subfield of L, which consists of elements fixed by σ. Then L is a cyclic extension of K′ of degree l. Take an element $\alpha \in L$ with $L = K'(\alpha)$. If $a = \mathrm{Tr}_{L/K'}(\alpha)$, then replacing α by $\alpha - a/l$, we may assume that $\mathrm{Tr}_{L/K'}(\alpha) = 0$. On the other hand, since $\sigma \in W$ then ${}^{\sigma^i}\alpha = \alpha + \beta_i$, where $|\beta_i| < |\alpha|$. We have

$$0 = \alpha + {}^{\sigma}\alpha + \dots + {}^{\sigma^{l-1}}\alpha = l\alpha + \sum_{i=0}^{l-1} \beta_i.$$

This is impossible because $|\sum_{i=0}^{l-1} \beta_i| < |\alpha| = |l\alpha|$. ■

The group I(L/K) is said to be the *inertia group*, and the group

$$M(L/K) = I(L/K)/W(L/K)$$

(resp. W(L/K)) is said to be the *moderate* (resp. *wild*) *ramification group* of the Galois extension L/K. Furthermore, applying Proposition 2.4.3 to the separable closure K^s of K, one gets the *maximal unramified* (resp. *moderately ramified*) *extension* K^{nr} (resp. K^{mr}) of K. We set $G_K^{mr} = G(K^{mr}/K)$, $G_K^{nr} = G(K^{nr}/K)$, $M_K = G(K^{mr}/K^{nr})$ and $W_K = G(K^s/K^{mr})$.

2.4.6. *Corollary.* — (i) $\widetilde{K^{nr}}$ *is the separable closure* $\widetilde{K}^s$ *of* $\widetilde{K}$ *and* $|K^{nr}| = |K|$;
(ii) $G_K^{nr} = G_{\widetilde{K}}$;
(iii) $M_K \xrightarrow{\sim} \mathrm{Hom}\left(\sqrt{|K^*|}/|K^*|, (\widetilde{K}^s)^*\right)$;
(iv) W_K *is a pro-p-group.* ■

We say that an algebraic extension L/K is *unramified* (resp. *moderately ramified*) if $L \subset K^{nr}$ (resp. $L \subset K^{mr}$).

2.4.7. *Proposition.* — *A finite separable extension* L/K *is unramified (resp. moderately ramified) if and only if it satisfies the following conditions*:

a) L *is* K*-cartesian, i.e.,* $[L : K] = [\widetilde{L} : \widetilde{K}]\,[|L^*| : |K^*|]$;
b) $\widetilde{L}$ *is separable over* $\widetilde{K}$;
c) $|L^*| = |K^*|$ *(resp.* $p \nmid [|L^*| : |K^*|]$*).*

Proof. — The direct implication follows from Proposition 2.4.4 and the fact that a subfield of a K-cartesian field is K-cartesian.

2.4.8. *Lemma.* — *Any finite extension* L/K *with* $p \nmid [L:K]$ *is moderately ramified.*

Proof. — The assertion follows from the fact that the wild ramification group W(L′/K) is an invariant p-subgroup of the Galois group G(L′/K), where L′ is the minimal Galois extension of K which contains L. ■

Suppose that L satisfies the conditions *a)-c)*. Let L′ be a finite Galois extension of K which contains L, and let K_0 be the maximal subfield of L′ which is unramified over K. We take the field $K' \subset K_0$ for which $\widetilde{K}' = \widetilde{L}$ and claim that $K' \subset L$. Indeed, let α be an element of K'^0 with $\widetilde{K}' = \widetilde{K}(\widetilde{\alpha})$. Since $[K':K] = [\widetilde{K}':\widetilde{K}]$, we have $K' = K(\alpha)$. Let P(T) be the (monic) minimal polynomial of α over K. It is clear that $P(T) \in K^0[T]$. The polynomial $\widetilde{P}(T)$ is separable and has a root in $\widetilde{L}$. Since L^0 is Henselian, there exists a root β of P(T) in L with $\widetilde{\beta} = \widetilde{\alpha}$. From this it follows that $\beta = \alpha$ because that polynomial $\widetilde{P}(T)$ is separable.

We have $[L:K'] = [|L^*|:|K^*|]$. If $|L^*| = |K^*|$, then $L = K'$. If p does not divide $[|L^*|:|K^*|]$, then $p \nmid [L:K']$, and, by Lemma 2.4.8, L is moderately ramified over K′. Since K′ is unramified over K, L is moderately ramified over K. ■

2.5. The cohomological dimension of the fields $\kappa(x)$

Recall that the l-cohomological dimension $cd_l(G)$ of a profinite group G is the minimal integer n (or ∞) such that $H^i(G, A) = 0$ for all $i > n$ and all l-torsion G-modules A (l is a prime integer). The l-cohomological dimension $cd_l(K)$ of a field K is, by definition, the l-cohomological dimension $cd_l(G_K)$. Recall also that if $l = \mathrm{char}(K)$, then $cd_l(K) \leqslant 1$ ([Ser], Ch. II, § 2.2).

2.5.1. *Theorem.* — *For a point* $x \in X$, *one has* $cd_l(\kappa(x)) \leqslant cd_l(k) + \dim(X)$.

Proof. — First of all we remark that the statement is evidently true if $\dim(X) = 0$ and that, by Proposition 2.4.1, one has $cd_l(\kappa(x)) = cd_l(\mathcal{H}(x))$. Consider first the case when X is a closed disc in $\mathbf{A}^1$. If $[\kappa(x):k] < \infty$, then $cd_l(\kappa(x)) \leqslant cd_l(k)$. Assume therefore that $[\kappa(x):k] = \infty$. Then the field of the rational functions in one variable $k(T)$ is embedded in $\kappa(x)$ and everywhere dense in it. Fix an embedding $k(T)^s \hookrightarrow \kappa(x)^s$ over the canonical embedding $k(T) \hookrightarrow \kappa(x)$. Since the field $\kappa(x)$ is quasicomplete, it follows that $\kappa(x)^s = k(T)^s \kappa(x)$. In particular, the Galois group $G_{\kappa(x)}$ can be identified with a closed subgroup of $G_{k(T)}$, and therefore one has $cd_l(\kappa(x)) \leqslant cd_l(k(T))$ (*loc. cit.*, Ch. I, § 3.3). By Tsen's Theorem (*loc. cit.*, Ch. II, § 4.2), one has $cd_l(k(T)) \leqslant cd_l(k) + 1$, and hence $cd_l(\kappa(x)) \leqslant cd_l(k) + 1$.

Suppose now that $\dim(X) \geqslant 1$ and that the theorem is true for affinoid spaces whose dimension is at most $\dim(X) - 1$. Take an analytic function f on W which is nonconstant at any irreducible component of X and consider the induced morphism $f: X \to \mathbf{A}^1$. The morphism f can be considered as a morphism to a closed disc of a sufficiently big radius Y. Let $y = f(x)$. The point x is also a point of the $\mathscr{H}(y)$-affinoid space X_y, whose dimension is at most $\dim(X) - 1$. By induction,

$$\mathrm{cd}_l(\mathscr{H}(x)) \leqslant \mathrm{cd}_l(\mathscr{H}(y)) + \dim(X) - 1,$$

and, by the first case, $\mathrm{cd}_l(\mathscr{H}(y)) \leqslant \mathrm{cd}_l(k) + 1$. The required inequality follows. ■

If l is not equal to the characteristic of the residue field of k, then one can get as follows a more strong inequality for $\mathrm{cd}_l(\kappa(x))$ (the result will not be used in the sequel). For this we recall some definitions and facts from [Ber], § 9.1.

Let K be an extension of k with a valuation which extends the valuation of k. We denote by $s(K/k)$ the transcendence degree of $\widetilde{K}$ over $\widetilde{k}$ and by $t(K/k)$ the dimension of the $\mathbf{Q}$-vector space $\sqrt{|K^*|}/\sqrt{|k^*|}$, and we set $d(K/k) = s(K/k) + t(K/k)$ (the *dimension of* K *over* k). It is clear that $d(K/k) = d(\widehat{K}/k)$.

2.5.2. *Lemma.* — *For a point* $x \in X$, *one has* $d(\kappa(x)/k) \leqslant \dim(X)$. *Moreover, the equality is achieved for some point of* X.

Proof. — If X is strictly k-affinoid, the assertion is proved in [Ber], 9.1.3. The proof also shows that $d(\kappa(x)/k) = \dim(X)$ for some point x from the Shilov boundary of $\mathscr{A}$ (see [Ber], § 2.4). In the general case we take a field K of the form $K_{r_1, \dots, r_n}$ for which the algebra $\mathscr{A}' = \mathscr{A} \widehat{\otimes} K$ is strictly K-affinoid. Let φ denote the canonical map $X' = \mathscr{M}(\mathscr{A}') \to X$, and let σ denote the map $X \to X'$ from Lemma 2.2.5. Since the fiber of φ at x coincides with $\mathscr{M}(\mathscr{H}(x) \widehat{\otimes} K)$, it follows that $d(\kappa(\sigma(x))/\kappa(x)) = n$. By the strictly affinoid case, $d(\kappa(\sigma(x))/K) \leqslant \dim(X)$. Let now x' be a point from the Shilov boundary of $\mathscr{A}'$ for which $d(\kappa(x')/K) = \dim(X)$. It is easy to see that $x' = \sigma(x)$ where $x = \varphi(x')$. Therefore $d(\kappa(x)/k) = d(\kappa(x')/K) = \dim(X)$. ■

2.5.3. *Theorem.* — *Suppose that* $l \neq \mathrm{char}(\widetilde{k})$. *Then for a point* $x \in X$, *one has* $\mathrm{cd}_l(\kappa(x)) \leqslant \mathrm{cd}_l(k) + d(\kappa(x)/k)$.

2.5.4. *Lemma.* — *Let* K *be a quasicomplete field, and let* ℓ *be a prime integer different from* $\mathrm{char}(\widetilde{K})$. *Suppose that the numbers* $\mathrm{cd}_l(\widetilde{K})$ *and* $s_l(K) = \dim_{\mathbf{F}_l}(|K^*|/|K^*|^l)$ *are finite. Then* $\mathrm{cd}_l(K) \leqslant \mathrm{cd}_l(\widetilde{K}) + s_l(K)$.

Proof. — Since W_K is a p-group, then $\mathrm{cd}_l(K) = \mathrm{cd}_l(G_K^{\mathrm{mr}})$. Furthermore, since the group M_K is abelian, $\mathrm{cd}_l(M_K)$ coincides with the l-cohomological dimension of the l-component of M_K. The latter group is isomorphic to $\mathbf{Z}_l^{s_l(K)}$, and therefore $\mathrm{cd}_l(M_K) = s_l(K)$. Since $G_K^{\mathrm{nr}} = G_{\widetilde{K}}$, the required fact follows from the spectral sequence $H^p(G_{\widetilde{K}}, H^q(M_K, A)) \Rightarrow H^{p+q}(G_K^{\mathrm{mr}}, A)$. ■

Proof of Theorem **2.5.3.** — As in the proof of Theorem 2.5.1 consider first the case when X is a closed disc in $\mathbf{A}^1$. Let x' be a point of $X' = X \widehat{\otimes} \widehat{k}^a$ over x. Since the field $\kappa(x)\, k^s$ is everywhere dense in $\kappa(x')$ and the both fields are quasicomplete, from Corollary 2.4.2 it follows that $G_{\kappa(x')} \xrightarrow{\sim} G_{\kappa(x)\, k^s}$. Therefore there is an exact sequence

$$0 \to G_{\kappa(x')} \to G_{\kappa(x)} \to G(\kappa(x)\, k^s/\kappa(x)) \to 0.$$

The latter group is a closed subgroup of G_k, and hence its cohomological dimension is at most $\mathrm{cd}_l(k)$. It follows that $\mathrm{cd}_l(\kappa(x)) \leqslant \mathrm{cd}_l(k) + \mathrm{cd}_l(\kappa(x'))$. Since $d(\kappa(x')/\widehat{k}^a) = d(\kappa(x)/k)$, Lemma 2.5.4 implies that $\mathrm{cd}_l(\kappa(x')) = d(\kappa(x)/k)$.

Suppose now that $\dim(X) \geqslant 1$ and that the theorem is true for affinoid spaces whose dimension is at most $\dim(X) - 1$. As in the proof of Theorem 2.5.1 we can find a morphism $f: X \to Y$, where Y is a closed disc in $\mathbf{A}^1$, such that all of the fibres of X has dimension at most $\dim(X) - 1$. Let $y = f(x)$. By induction,

$$\mathrm{cd}_l(\mathscr{H}(x)) \leqslant \mathrm{cd}_l(\mathscr{H}(y)) + d(\mathscr{H}(x)/\mathscr{H}(y)),$$

and, by the first case, $\mathrm{cd}_l(\mathscr{H}(y)) \leqslant \mathrm{cd}_l(k) + d(\mathscr{H}(y)/k)$. Since

$$d(\mathscr{H}(x)/\mathscr{H}(y)) + d(\mathscr{H}(y)/k) = d(\mathscr{H}(x)/k),$$

the required inequality follows. ∎

2.6. GAGA over an affinoid space

Let $\mathscr{Y}$ be a scheme of locally finite type over $\mathscr{X}$, and let F be the functor from the category of morphisms $Z \to X$, where Z is a good analytic space over k, to the category of sets which associates with $Z \to X$ the set of morphisms of locally ringed spaces over $\mathscr{X}$, $\mathrm{Hom}_{\mathscr{X}}(Z, \mathscr{Y})$.

2.6.1. *Proposition. — The functor* F *is representable by a closed morphism of k-analytic spaces* $\mathscr{Y}^{\mathrm{an}} \to X$ *and a morphism of locally ringed spaces* $\pi: \mathscr{Y}^{\mathrm{an}} \to \mathscr{Y}$. *The correspondence* $\mathscr{Y} \mapsto \mathscr{Y}^{\mathrm{an}}$ *is a functor which commutes with extensions of the ground field and with fibred products.*

Proof. — The k-analytic space $\mathscr{Y}^{\mathrm{an}}$ is constructed in the same way as in the case when $X = \mathscr{M}(k)$ (see [Ber], 3.4.1). Namely, one shows that if $\mathscr{Y}$ is the affine space over $\mathscr{X}$, $A^d_{\mathscr{X}}$, then $\mathscr{Y}^{\mathrm{an}} = \mathbf{A}^d_X = X \times \mathbf{A}^d$. After that one shows that if $\mathscr{Y}^{\mathrm{an}}$ exists for $\mathscr{Y}$, then $\mathscr{Z}^{\mathrm{an}}$ exists for any subscheme $\mathscr{Z} \subset \mathscr{Y}$. In particular, $\mathscr{Y}^{\mathrm{an}}$ exists for any affine scheme of finite type over $\mathscr{X}$ and for any its open subscheme. Finally, if $\mathscr{Y}$ is an arbitrary scheme of locally finite type over $\mathscr{X}$, then one takes an open covering $\{\mathscr{Y}_i\}_{i \in I}$ of $\mathscr{Y}$ by affine subschemes of finite type over $\mathscr{X}$. One glues together all of the $\mathscr{Y}_i^{\mathrm{an}}$'s and obtains the k-analytic space $\mathscr{Y}^{\mathrm{an}}$ associated with $\mathscr{Y}$. That the correspondence $\mathscr{Y} \mapsto \mathscr{Y}^{\mathrm{an}}$ is a functor possessing the necessary properties follows from the universal property of $\mathscr{Y}^{\mathrm{an}}$. ∎

2.6.2. *Proposition. — The map* $\pi: Y = \mathscr{Y}^{\mathrm{an}} \to \mathscr{Y}$ *is surjective, and for any point* $y \in Y$ *the ring* $\mathcal{O}_{Y,y}$ *is flat over* $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$, *where* $\mathbf{y} = \pi(y)$ *(i.e.,* π *is a faithfully flat morphism).*

Proof. — By Proposition 2.1.1, the map $X \to \mathscr{X}$ is surjective. Therefore to show that the map $\pi : Y \to \mathscr{Y}$ is surjective, it suffices to verify that for any point $x \in X$ the map $Y_x \to \mathscr{Y}_{\mathbf{x}}$ is surjective. One has

$$Y_x = (\mathscr{Y}_{\mathbf{x}} \otimes_{k(\mathbf{x})} \mathscr{H}(x))^{\mathrm{an}}.$$

Since the morphism of schemes $\mathscr{Y}_{\mathbf{x}} \otimes_{k(\mathbf{x})} \mathscr{H}(x) \to \mathscr{Y}_{\mathbf{x}}$ is faithfully flat, the situation is reduced to the case when $X = \mathscr{M}(k)$. In this case it suffices to verify that if $\mathscr{Y}$ is an irreducible affine scheme of finite type over k and $\mathbf{y}$ is its generic point, then there exists a point $y \in Y$ whose image in $\mathscr{Y}$ is $\mathbf{y}$. The Noether Normalization Lemma reduces the problem to the case when $\mathscr{Y}$ is the affine space A^d. In this case, for any point $y \in Y = \mathbf{A}^d$ associated with a closed polydisc, $\pi(y)$ is the generic point of $\mathscr{Y} = \mathrm{A}^d$.

2.6.3. *Lemma. — The map $\pi : Y = \mathscr{Y}^{\mathrm{an}} \to \mathscr{Y}$ induces a bijection*

$$Y_0 \xrightarrow{\sim} \mathscr{Y}_0 = \{\, \mathbf{y} \in \mathscr{Y} \mid [k(\mathbf{y}) : k] < \infty \,\}.$$

If $y \in Y_0$, then there is an isomorphism of completions $\widehat{\mathcal{O}_{\mathscr{Y},\mathbf{y}}} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y,y}}$.

Proof. — Let $\mathbf{y} \in \mathscr{Y}_0$. For $n \geqslant 1$ we set $\mathscr{Z} = \mathrm{Spec}(\mathcal{O}_{\mathscr{Y},\mathbf{y}}/\mathbf{m}_{\mathbf{y}}^n)$. The scheme $\mathscr{Z}$ consists of one point $\mathbf{z}$ and is finite over k. Therefore $Z = \mathscr{Z}^{\mathrm{an}}$ consists of one point z, and one has $\mathcal{O}_{\mathscr{Z},\mathbf{z}} = \mathcal{O}_{\mathscr{Y},\mathbf{y}}/\mathbf{m}_{\mathbf{y}}^n \xrightarrow{\sim} \mathcal{O}_{Z,z}$. Furthermore, there is a canonical closed immersion $\mathscr{Z} \to \mathscr{Y}$ which takes $\mathbf{z}$ to $\mathbf{y}$. Therefore $Z \to Y$ is also a closed immersion, and the point y, which is the image of z in Y, is the only preimage of $\mathbf{y}$ in Y. (In particular, $Y_0 \xrightarrow{\sim} \mathscr{Y}_0$.) Moreover, one has

$$\mathcal{O}_{\mathscr{Y},\mathbf{y}}/\mathbf{m}_{\mathbf{y}}^n \xrightarrow{\sim} \mathcal{O}_{Z,z} = \mathcal{O}_{Y,y}/\mathbf{m}_{\mathbf{y}}^n\, \mathcal{O}_{Y,y}.$$

If $n = 1$, we get $\mathbf{m}_y = \mathbf{m}_{\mathbf{y}}\, \mathcal{O}_{Y,y}$ and $k(\mathbf{y}) = \kappa(y)$. Hence $\widehat{\mathcal{O}_{\mathscr{Y},\mathbf{y}}} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y,y}}$. ∎

From Lemma 2.6.3 it follows that $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$ at least in the case when $y \in Y_0$. In the general case we take a sufficiently big non-Archimedean field K over k such that there exists a K-point $y' \in Y' = Y \hat{\otimes} K$ over y. We set $X' = \mathscr{M}(\mathscr{A} \hat{\otimes} K)$ and $\mathscr{X}' = \mathrm{Spec}(\mathscr{A} \hat{\otimes} K)$. One has $Y' = \mathscr{Y}'^{\mathrm{an}}$, where $\mathscr{Y}' = \mathscr{Y} \times_{\mathscr{X}} \mathscr{X}'$. Let $\mathbf{y}'$ be the image of the point y' in $\mathscr{Y}'$. We know that $\mathcal{O}_{Y',y'}$ is flat over $\mathcal{O}_{\mathscr{Y}',\mathbf{y}'}$. Since $\mathscr{X}'$ is faithfully flat over $\mathscr{X}$ (Lemma 2.1.2), it follows that $\mathcal{O}_{\mathscr{Y}',\mathbf{y}'}$ is flat over $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$, and therefore $\mathcal{O}_{Y',y'}$ is flat over $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$. Finally, from Corollary 2.1.3 it follows that $\mathcal{O}_{Y',y'}$ is faithfully flat over $\mathcal{O}_{Y,y}$. Hence $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$. ∎

2.6.4. *Proposition. — Let T be a constructible subset of $\mathscr{Y}$. Then $\pi^{-1}(\overline{T}) = \overline{\pi^{-1}(T)}$.*

2.6.5. *Lemma. — Suppose that $\mathscr{Y}$ is affine of finite type over $\mathscr{X}$. Let $\mathbf{y}$, $\mathbf{z}$ be points of $\mathscr{Y}$ with $\mathbf{z} \in \overline{\mathbf{y}}$, and let z be a point of $\mathscr{Y}^{\mathrm{an}}$ with $\pi(z) = \mathbf{z}$. Then any open neighborhood of z contains a point y with $\pi(y) = \mathbf{y}$.*

Proof. — For a non-Archimedean field K over k, we set $\mathscr{A}' = A \hat{\otimes} K$, $X' = \mathscr{M}(\mathscr{A}')$, $\mathscr{X}' = \mathrm{Spec}(\mathscr{A}')$, $\mathscr{Y}' = \mathscr{Y} \times_{\mathscr{X}} \mathscr{X}'$. Since $\mathscr{Y}'$ is faithfully flat over $\mathscr{Y}$, there exists points

$\mathbf{y}', \mathbf{z}' \in \mathscr{Y}'$ over $\mathbf{y}$ and $\mathbf{z}$, respectively, with $\mathbf{z}' \in \overline{\mathbf{y}}'$. Since the tensor product $\kappa(z) \otimes_{k(\mathbf{z})} k(\mathbf{z}')$ is nontrivial, the valuation on $\kappa(z)$ extends to a multiplicative seminorm on it. It follows that there exists a point $z' \in \mathscr{Y}'^{\mathrm{an}}$ whose image in $\mathscr{Y}'$ is $\mathbf{z}'$ and in $\mathscr{Y}^{\mathrm{an}}$ is z. Thus, we can extend the field k, and, in particular, we may assume that the valuation on k is nontrivial and $\mathscr{A}$ is strictly k-affinoid. Let $\mathscr{Y} = \mathrm{Spec}(\mathrm{B})$. Replacing B by $\mathrm{B}/\wp_{\mathbf{y}}$, we may assume that $\mathscr{Y}$ is reduced and irreducible and that $\mathbf{y}$ is its generic point. Since $\mathscr{Y}_0^{\mathrm{an}}$ is everywhere dense in $\mathscr{Y}^{\mathrm{an}}$, we may assume that $z \in \mathscr{Y}_0^{\mathrm{an}}$. Furthermore, since the ring B is excellent (it is finitely generated over the excellent ring $\mathscr{A}$), $\mathrm{Reg}(\mathscr{Y})$ is an everywhere dense Zariski open subset of $\mathscr{Y}$. Shrinking $\mathscr{Y}$, we may assume that $\mathscr{Y}$ is regular. In this case the homomorphism $\mathrm{B} \to \mathcal{O}_{\mathscr{Y},\mathbf{z}}$ is injective. From Lemma 2.6.3 it follows that the homomorphism $\mathrm{B} \to \mathcal{O}_{\mathscr{Y}^{\mathrm{an}},z}$ is injective. Take a sufficiently small connected strictly affinoid neighborhood $\mathrm{V} = \mathscr{M}(\mathscr{B}_{\mathrm{V}})$ of the point z. The latter homomorphism goes through $\mathscr{B}_{\mathrm{V}}$, therefore the homomorphism $\mathrm{B} \to \mathscr{B}_{\mathrm{V}}$ is injective. Since $\mathscr{Y}^{\mathrm{an}}$ is regular and V is connected, the ring $\mathscr{B}_{\mathrm{V}}$ is an integral domain. By Proposition 2.1.1, there exists a point $y \in \mathrm{V}$ for which the character $\mathscr{B}_{\mathrm{V}} \to \mathscr{H}(y)$ is injective. It follows that the character $\mathscr{B} \to \mathscr{H}(y)$ is injective, and therefore $\pi(y)$ is the generic point of $\mathscr{Y}$. ∎

Proof of Proposition **2.6.4.** — It is clear that $\overline{\pi^{-1}(\mathrm{T})} \subset \pi^{-1}(\overline{\mathrm{T}})$. To verify the inverse inequality, we may assume that $\overline{\mathrm{T}} = \mathscr{Y}$. Since T is constructible, it contains an everywhere dense Zariski open subset of $\mathscr{Y}$ and, in particular, all of the generic points of $\mathscr{Y}$. From Lemma 2.6.5 it follows that $\overline{\pi^{-1}(\mathrm{T})} = \mathscr{Y}^{\mathrm{an}}$. ∎

2.6.6. *Corollary.* — *A constructible set* T *is open (resp. closed, resp. everywhere dense) in* $\mathscr{Y}$ *if and only if* $\pi^{-1}(\mathrm{T})$ *is open (resp. closed, resp. everywhere dense) in* $\mathscr{Y}^{\mathrm{an}}$. ∎

2.6.7. *Corollary.* — *A morphism* $\varphi : \mathscr{Z} \to \mathscr{Y}$ *between schemes of locally finite type over* $\mathscr{X}$ *is separated if and only if* $\varphi^{\mathrm{an}} : \mathscr{Z}^{\mathrm{an}} \to \mathscr{Y}^{\mathrm{an}}$ *is separated.*

Proof. — Let $\Delta : \mathscr{Z} \to \mathscr{Z} \times_{\mathscr{Y}} \mathscr{Z}$ be the diagonal morphism. If φ is separated, then Δ is a closed immersion, and it follows that so is Δ^{an}. Assume that φ^{an} is separated. It suffices to verify that the set $\Delta(\mathscr{Z})$ is closed in $\mathscr{Z} \times_{\mathscr{Y}} \mathscr{Z}$. This follows from Corollary 2.6.6 because $\Delta(\mathscr{Z})$ is a constructible set and its preimage in $\mathscr{Z}^{\mathrm{an}} \times_{\mathscr{Y}^{\mathrm{an}}} \mathscr{Z}^{\mathrm{an}}$ is closed. ∎

2.6.8. *Proposition.* — *For a morphism* $\varphi : \mathscr{Z} \to \mathscr{Y}$ *between schemes of locally finite type over* $\mathscr{X}$, *one has* $\pi^{-1}(\varphi(\mathscr{Z})) = \varphi^{\mathrm{an}}(\mathscr{Z}^{\mathrm{an}})$.

Proof. — From the construction of the analytification it follows that the statement is true when $\mathscr{Z}$ is an open subscheme of $\mathscr{Y}$ and, if $\{ \mathscr{Y}_i \}_{i \in \mathrm{I}}$ is an open covering of $\mathscr{Y}$, then $\{ \mathscr{Y}_i^{\mathrm{an}} \}_{i \in \mathrm{I}}$ is an open covering of $\mathscr{Y}^{\mathrm{an}}$. Hence the situation is reduced to the case when $\mathscr{Y} = \mathrm{Spec}(\mathrm{B})$ and $\mathscr{Z} = \mathrm{Spec}(\mathrm{C})$ are affine schemes of finite type over $\mathscr{X}$. The inclusion $\varphi^{\mathrm{an}}(\mathscr{Z}^{\mathrm{an}}) \subset \pi^{-1}(\varphi(\mathscr{Z}))$ is evident. To verify the converse inclusion, it suffices to show that if $\mathbf{q}$ is a prime ideal of C, then any multiplicative norm on $B/\mathbf{p}$, where $\mathbf{p}$ is the preimage of $\mathbf{q}$ in B, extends to a multiplicative norm on $C/\mathbf{q}$. But this follows from the well known fact that a valuation on a field can be extended to a valuation on any bigger field. ∎

The following statement is proved in the same way as its particular case when $\mathscr{X} = \mathcal{M}(k)$ (see [Ber], 3.4.7).

2.6.9. *Proposition.* — *Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite type between schemes of locally finite type over $\mathscr{X}$. Then φ is proper (resp. finite, resp. a closed immersion) if and only if φ^{an} possesses the same property.* ∎

2.6.10. *Proposition.* — *Let $\varphi : \mathcal{X} = \mathrm{Spec}(C) \to \mathcal{Y} = \mathrm{Spec}(B)$ be a finite morphism between affine schemes of finite type over $\mathscr{X}$. Let $\mathbf{z} \in \mathcal{X}$ and $y \in \mathcal{Y}^{\mathrm{an}}$ be points with $\varphi(\mathbf{z}) = \pi(y) = \mathbf{y}$, and let $\varphi^{\mathrm{an}^{-1}}(y) = \{z_1, \ldots, z_n\}$ and $\varphi^{\mathrm{an}^{-1}}(y) \cap \pi^{-1}(\mathbf{z}) = \{z_1, \ldots, z_m\}$, $m \leqslant n$. Then there is an isomorphism of rings*

$$\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y} \otimes_{\mathcal{O}_{\mathcal{Y}, \mathbf{y}}} \mathcal{O}_{\mathcal{X}, \mathbf{z}} \xrightarrow{\sim} \prod_{i=1}^{m} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, z_i} \times \prod_{i=m+1}^{n} (\mathcal{O}_{\mathcal{X}^{\mathrm{an}}, z_i})_{\wp_{\mathbf{z}}},$$

where $(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}, z_i})_{\wp_{\mathbf{z}}}$ is the localization with respect to the complement of the prime ideal of C *corresponding to the point* $\mathbf{z}$.

Proof. — If V is an affinoid domain in $\mathcal{Y}^{\mathrm{an}}$, then for its preimage W in $\mathcal{X}^{\mathrm{an}}$ one has $\mathscr{C}_W = \mathscr{B}_V \otimes_B C$. Since φ^{an} is finite, from Lemma 2.1.6 it follows that $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y} \otimes_B C \xrightarrow{\sim} \prod_{i=1}^{n} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}, z_i}$. The ring $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y} \otimes_{\mathcal{O}_{\mathcal{Y}, \mathbf{y}}} \mathcal{O}_{\mathcal{X}, \mathbf{z}}$ is the localization of $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y} \otimes_B C$ with respect to the complement of $\wp_{\mathbf{z}}$. If $\pi(z_i) = \mathbf{z}$, then the ring $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}, z_i}$ does not change under this localization. The required statement follows. ∎

§ 3. Etale and smooth morphisms

3.1. Quasifinite morphisms

3.1.1. *Definition.* — A morphism of k-analytic spaces $\varphi : Y \to X$ is said to be *finite at a point $y \in Y$* if there exist open neighborhoods $\mathcal{V}$ of y and $\mathcal{U}$ of $\varphi(y)$ such that φ induces a finite morphism $\mathcal{V} \to \mathcal{U}$; φ is said to be *quasifinite* if it is finite at any point $y \in Y$.

It follows from the definition that quasifinite morphisms are locally separated and closed.

3.1.2. *Lemma.* — *If a morphism $\varphi : Y \to X$ is finite at a point $y \in Y$, then the neighborhoods $\mathcal{V}$ and $\mathcal{U}$ from the Definition* 3.1.1 *can be found arbitrary small.*

Proof. — We may assume that the morphism φ is finite. Let $x = \varphi(y)$ and let $\varphi^{-1}(x) = \{y_1 = y, y_2, \ldots, y_n\}$. Since the map $|Y| \to |X|$ is compact, we can find a sufficiently small open neighborhood $\mathcal{U}$ of x such that $\varphi^{-1}(\mathcal{U}) = \coprod_{i=1}^{n} \mathcal{V}_i$, where $y_i \in \mathcal{V}_i$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i \neq j$. It follows that all of the morphisms $\mathcal{V}_i \to \mathcal{U}$ are finite, and the neighborhood $\mathcal{V}_1$ of y is sufficiently small. ■

3.1.3. *Corollary.* — *Quasifinite morphisms are preserved under compositions, under any base change functor, and under any ground field extension functor.* ■

We remark that if $\varphi : Y \to X$ is a quasifinite morphism and the space X is good, then Y is also good. Quasifinite morphisms between good k-analytic spaces can be characterized as follows.

3.1.4. *Proposition.* — *Let $\varphi : Y \to X$ be a morphism of good k-analytic spaces, and let $y \in Y$ and $x = \varphi(y)$. The following are equivalent:*

a) φ is finite at y;

b) there exist sufficiently small affinoid neighborhoods V *of y and* U *of x such that φ induces a finite morphism* $V \to U$;

c) the point y is isolated in the fibre $\varphi^{-1}(x)$ and $y \in \mathrm{Int}(Y/X)$.

Proof. — The implications *a)* $\Rightarrow$ *c)* and *b)* $\Rightarrow$ *c)* are trivial. The implication *b)* $\Rightarrow$ *a)* follows from the following Lemma.

3.1.5. *Lemma.* — *Let $\varphi : Y \to X$ be a morphism, and let $V \subset Y$ and $U \subset X$ be affinoid subdomains such that φ induces a finite morphism $\psi : V \to U$. Then* $\mathrm{Int}(V/Y) = \psi^{-1}(\mathrm{Int}(U/X))$. *In particular, φ induces a finite morphism* $\mathrm{Int}(V/Y) \to \mathrm{Int}(U/X)$.

Proof. — From [Ber], 3.1.3, it follows that

$$\mathrm{Int}(V/X) = \mathrm{Int}(V/Y) \cap \mathrm{Int}(Y/X) = \mathrm{Int}(V/Y).$$

On the other hand, $\mathrm{Int}(V/X) = \mathrm{Int}(V/U) \cap \psi^{-1}(\mathrm{Int}(U/X)) = \psi^{-1}(\mathrm{Int}(U/X))$. Therefore $\mathrm{Int}(V/Y) = \psi^{-1}(\mathrm{Int}(U/X))$. ■

To verify the implication *c)* $\Rightarrow$ *b)*, it suffices to assume that $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ are k-affinoid. Furthermore, since X and Y are compact, we can decrease them and assume that $\varphi^{-1}(x) = \{y\}$. Since $y \in \mathrm{Int}(Y/X)$, there exists an admissible epimorphism

$$\pi : \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$$

such that $|\pi(T_i)(y)| < r_i$, $1 \leqslant i \leqslant n$. For any affinoid neighborhood U of x, π induces an admissible epimorphism

$$\pi_U : \mathcal{A}_U\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}_{\varphi^{-1}(U)}.$$

If U is sufficiently small, then $\varphi^{-1}(U)$ is a sufficiently small affinoid neighborhood of the point y. Therefore we can find sufficiently small U such that $|\pi(T_i)(y')| < r$

for all points $y' \in \varphi^{-1}(U)$. This means that the induced morphism of k-affinoid spaces $\varphi^{-1}(U) \to U$ is closed. By [Ber], 2.5.13, the latter morphism is finite. ■

3.1.6. *Corollary.* — *If a morphism of good k-analytic spaces* $\varphi : Y \to X$ *is finite at a point* $y \in Y$, *then* $\mathcal{O}_{Y,y}$ *is a finite* $\mathcal{O}_{X,\varphi(y)}$*-algebra.* ■

Recall that a morphism of schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is called quasifinite if any point $\mathbf{y} \in \mathcal{Y}$ is isolated in the fibre $\varphi^{-1}(\mathbf{x}) = \mathcal{Y}_{\mathbf{x}}$, where $\mathbf{x} = \varphi(\mathbf{y})$ (see [SGA1], I.2).

3.1.7. *Corollary.* — *A morphism* $\varphi : \mathcal{Z} \to \mathcal{Y}$ *between schemes of locally finite type over* $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$, *where* $\mathcal{A}$ *is a k-affinoid algebra, is quasifinite if and only if the corresponding morphism* $\varphi^{an} : \mathcal{Z}^{an} \to \mathcal{Y}^{an}$ *is quasifinite.*

Proof. — For any point $y \in \mathcal{Y}^{an}$, there is an isomorphism of $\mathcal{H}(y)$-analytic spaces

$$(\mathcal{Z}_{\mathbf{y}} \otimes_{k(\mathbf{y})} \mathcal{H}(y))^{an} \xrightarrow{\sim} \mathcal{Z}^{an}_{y}.$$

Therefore our assertion follows from the fact that the morphism φ^{an} is closed (Proposition 2.6.1). ■

3.1.8. *Proposition.* — *Let* $\varphi : Y \to X$ *be a morphism of k-analytic spaces, and let* $y \in Y$ *and* $x = \varphi(y)$. *Then the following are equivalent:*

a) φ *is finite at* y;

b) there exist analytic domains $X_1, \ldots, X_n \subset X$ *such that* $x \in X_1 \cap \ldots \cap X_n$, $X_1 \cup \ldots \cup X_n$ *is a neighborhood of* x *and the morphisms* $\varphi^{-1}(X_i) \to X_i$ *are finite at* y;

c) there exist affinoid domains $V_1, \ldots, V_n \subset Y$ *and* $U_1, \ldots, U_n \subset X$ *such that* $y \in V_1 \cap \ldots \cap V_n$, $V_1 \cup \ldots \cup V_n$ *and* $U_1 \cup \ldots \cup U_n$ *are neighborhoods of* y *and* x, *respectively,* $\varphi(V_i) \subset U_i$, *and the induced morphisms* $\varphi_i : V_i \to U_i$ *and* $\varphi_{ij} : V_i \cap V_j \to U_i \cap U_j$ *are finite at* y.

We remark that if the spaces X and Y are separated, then the finiteness at y of all the morphisms φ_i from *c)* implies the same property for the morphisms φ_{ij}. Indeed, in this case $U_i \cap U_j$ and $V_i \cap V_j$ are affinoid domains and, by Proposition 3.1.4, it suffices to verify that $y \in \mathrm{Int}(V_i \cap V_j / U_i \cap U_j)$. If $V_{ij} := \varphi_i^{-1}(U_i \cap U_j)$, then $y \in \mathrm{Int}(V_{ij}/U_i \cap U_j)$, and therefore $y \in \mathrm{Int}(V_{ij} \cap V_{ji}/U_i \cap U_j)$. It remains to note that $V_{ij} \cap V_{ji} = V_i \cap V_j$.

Proof. — The implication *a)* ⇒ *b)* is trivial. Furthermore, we remark that all three properties remain true if we replace X and Y by sufficiently small open neighborhoods of the point x and y, respectively. In particular, we may assume that X and Y are Hausdorff.

b) ⇒ *c)*. We may assume that $X_i = U'_i$ are affinoid domains. Then we can find affinoid neighborhoods V'_i of y in $\varphi^{-1}(U'_i)$ such that the induced morphisms

$\varphi'_i : V'_i \to U'_i$ are finite at y. Furthermore, shrinking X and Y, we may assume that the morphisms φ'_i are finite and $\varphi'^{-1}_i(x) = \{y\}$. Thus we get a morphism $\psi : V \to U$, where $V = V'_1 \cup \ldots \cup V'_n$ and $U = U'_1 \cup \ldots \cup U'_n$ are compact analytic neighborhoods of y and x, respectively. It follows that we can find on open neighborhood $\mathcal{U}$ of x in X with $\mathcal{U} \subset U$ such that $\mathcal{V} := \psi^{-1}(\mathcal{U})$ is an open neighborhood of y in Y and, for every $1 \leqslant i \leqslant n$, $\psi^{-1}(U'_i \cap \mathcal{U}) \subset V'_i$. If now U_i is an affinoid neighborhood of x in $U'_i \cap \mathcal{U}$, then $V_i := \psi^{-1}(U_i)$ is an affinoid neighborhood of y in $V'_i \cap \mathcal{V}$, and the induced morphism $\varphi_i : V_i \to U_i$ is finite. Since $V_i \cap V_j = \varphi_i^{-1}(U_i \cap U_j)$, the induced morphisms $\varphi_{ij} : V_i \cap V_j \to U_i \cap U_j$ are finite. Hence, φ satisfies *c)*.

3.1.9. *Lemma. — If* Y *is an analytic domain in* X *and the canonical morphism* $Y \to X$ *is quasifinite, then* Y *is open in* X.

Proof. — Since quasifinite morphisms are closed, $\mathrm{Int}(Y/X) = Y$. By Proposition 1.5.5 (ii), $\mathrm{Int}(Y/X)$ coincides with the topological interior of Y in X. It follows that Y is open in X. ■

c) ⇒ *a)*. We can shrink all the affinoid domains V_i and assume that $\varphi_i^{-1}(x) = \{y\}$. Then for sufficiently small affinoid neighborhoods U'_i of x in U_i, $\varphi_i^{-1}(U'_i)$ and $\varphi_{ij}^{-1}(U'_i \cap U'_j) = \varphi_i^{-1}(U'_i) \cap \varphi_j^{-1}(U'_j)$ are sufficiently small neighborhoods of y in V and $V_i \cap V_j$, respectively. Thus, we can shrink all the affinoid domains U_i and assume that all of the morphisms φ_i and φ_{ij} are finite.

Furthermore, the morphism φ_i induces finite morphisms $\varphi_{ij} : V_i \cap V_j \to U_i \cap U_j$ and $V_{ij} := \varphi_i^{-1}(U_i \cap U_j) \to U_i \cap U_j$, and therefore the canonical embedding of special domains $V_i \cap V_j \to V_{ij}$ is finite. From Lemma 3.1.9 it follows that $V_i \cap V_j$ is open in V_{ij}, and therefore $V_{ij} = (V_i \cap V_j) \amalg W_{ij}$ for some special domain $W_{ij} \subset V_i$. We get $\varphi^{-1}(U_i) = V_i \amalg W_i$, where $W_i = \bigcup_{j \neq i} W_{ji}$. Since $y \in V_1 \cap \ldots \cap V_n$, we have $x \notin \varphi(W_i)$, and therefore we can find, for each $1 \leqslant i \leqslant n$, an affinoid neighborhood U'_i of x in U_i such that $U'_i \cap \varphi(W_i) = \emptyset$. The latter implies that $\varphi^{-1}(U'_i) = V'_i := \varphi_i^{-1}(U'_i)$. Hence, the morphism $V'_1 \cup \ldots \cup V'_n \to U'_1 \cup \ldots \cup U'_n$ is finite, and the required statement follows. ■

3.1.10. *Corollary. — A morphism of k-analytic spaces* $\varphi : Y \to X$ *is finite at a point* $y \in Y$ *if and only if the point* y *is isolated in the fibre* $\varphi^{-1}(\varphi(y))$ *and* $y \in \mathrm{Int}(Y/X)$.

Proof. — The direct implication is clear. Suppose that y is isolated in $\varphi^{-1}(\varphi(y))$ and $y \in \mathrm{Int}(Y/X)$. We can shrink Y and assume that φ is closed. Let $U_1, \ldots, U_n$ be affinoid domains in X such that $\varphi(y) \in U_1 \cap \ldots \cap U_n$ and $U_1 \cup \ldots \cup U_n$ is a neighborhood of $\varphi(y)$. From Proposition 3.1.4 is follows that $\varphi^{-1}(U_i) \to U_i$ are closed morphisms of good k-analytic spaces. In particular, the property *b)* of Proposition 3.1.8 holds. It follows that φ is finite at y. ■

3.2. Flat quasifinite morphisms

A morphism of good k-analytic spaces $\varphi : Y \to X$ is said to be *flat at a point* $y \in Y$ if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,\varphi(y)}$-algebra. φ is said to be *flat* if it is flat at all points $y \in Y$.

3.2.1. *Proposition. — A finite morphism of k-affinoid spaces $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X = \mathcal{M}(\mathcal{A})$ is flat at a point $y \in Y$ if and only if $\mathcal{B}_{\wp_y}$ is a flat $\mathcal{A}_{\wp_x}$-algebra, where $x = \varphi(y)$. In particular, φ is flat if and only if $\mathcal{B}$ is a flat $\mathcal{A}$-algebra.*

Proof. — If $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$, then it is also flat over $\mathcal{A}_{\wp_x}$, by Theorem 2.1.4. Thus, the ring $\mathcal{O}_{Y,y}$ is faithfully flat over $\mathcal{B}_{\wp_y}$ and is flat over $\mathcal{A}_{\wp_x}$. It follows that $\mathcal{B}_{\wp_y}$ is flat over $\mathcal{A}_{\wp_x}$. Conversely, suppose that $\mathcal{B}_{\wp_y}$ is a flat $\mathcal{A}_{\wp_x}$-algebra. Then the ring $\mathcal{O}_{X,x} \otimes_{\mathcal{A}_{\wp_x}} \mathcal{B}_{\wp_y}$ is flat over $\mathcal{O}_{X,x}$. If $\varphi^{-1}(x) = \{y_1 = y, y_2, \ldots, y_n\}$, then $\mathcal{O}_{X,x} \otimes_{\mathcal{A}} \mathcal{B} = \prod_{i=1}^{n} \mathcal{O}_{Y,y_i}$ (Lemma 2.1.6). But the ring $\mathcal{O}_{X,x} \otimes_{\mathcal{A}_{\wp_x}} \mathcal{B}_{\wp_y}$ is the localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{A}} \mathcal{B}$ with respect to the complement of $\wp_y$, and therefore, is a direct product of the same localizations of the rings $\mathcal{O}_{Y,y_i}$. Since the ring $\mathcal{O}_{Y,y}$ does not change under this localization, it is a direct summand of the flat $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x} \otimes_{\mathcal{A}\wp_x} \mathcal{B}_{\wp_y}$. It follows that $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-algebra. ∎

3.2.2. *Corollary. — A morphism of good k-analytic spaces $\varphi : Y \to X$ is flat quasifinite if and only if for any point $y \in Y$ there exist affinoid neighborhoods V of y and U of $x = \varphi(y)$ such that $\varphi(V) \subset U$ and $\mathcal{B}_V$ is a flat finite $\mathcal{A}_U$-algebra.*

Proof. — The converse implication follows from Proposition 3.2.1. Suppose that φ is flat quasifinite. By Corollary 3.1.6, $\mathcal{O}_{Y,y}$ is a finite $\mathcal{O}_{X,x}$-algebra. Therefore there is an isomorphism $\mathcal{O}_{X,x}^n \xrightarrow{\sim} \mathcal{O}_{Y,y}$. It is clear that it comes from a homomorphism $\mathcal{A}_U^n \to \mathcal{B}_V$ for some affinoid neighborhoods V of y and U of x such that φ induces a finite morphism $\psi : V \to U$. The homomorphism considered is related to a homomorphism of sheaves $\mathcal{O}_U^n \to \psi_*(\mathcal{O}_V)$. The supports of the kernel and cokernel of the latter homomorphism are Zariski closed in U and do not contain the point x. It follows that one can decrease U and V such that $\mathcal{O}_U^n \xrightarrow{\sim} \psi_*(\mathcal{O}_V)$, i.e., $\mathcal{A}_U^n \xrightarrow{\sim} \mathcal{B}_V$. Hence $\psi : V \to U$ is a flat finite morphism. ∎

3.2.3. *Proposition. — Let $\varphi : Y \to X$ be a finite morphism of k-analytic spaces, and let $y \in Y$ and $x = \varphi(y)$. Then the following are equivalent:*

a) there exist affinoid domains $V_1, \ldots, V_n \subset X$ such that $x \in V_1 \cap \ldots \cap V_n$, $V_1 \cup \ldots \cup V_n$ is a neighborhood of x and $\varphi^{-1}(V_i) \to V_i$ are flat at y;

b) for any affinoid domain $x \in V \subset X$, $\varphi^{-1}(V) \to V$ is flat at y.

Proof. — The implication *b)* ⇒ *a)* is trivial. Suppose that *a)* is true. Then *b)* is true for any V that is contained in some V_i. Assume that V is arbitrary. Replacing V by a small affinoid neighborhood of x in V, we may assume that $V \subset V_1 \cup \ldots \cup V_n$. By

Lemma 1.1.2 (ii), there exist affinoid domains $U_1, \ldots, U_m$ such that $V = U_1 \cup \ldots \cup U_m$ and each U_j is contained in some V_i. By the first remark, if $x \in U_j$, then the finite morphism $\varphi^{-1}(U_j) \to U_j$ is flat at y. Therefore the required fact follows from the following lemma.

3.2.4. *Lemma.* — *A finite morphism of k-affinoid spaces $\varphi : Y \to X$ is flat at a point $y \in Y$ if and only if there exists a finite affinoid covering $\{V_i\}_{1 \leq i \leq n}$ of X such that, for each i with $\varphi(y) \in V_i$, the induced finite morphism $\varphi^{-1}(V_i) \to V_i$ is flat at y.*

Proof. — Shrinking X, we may assume that $\varphi(y) \in V_1 \cap \ldots \cap V_n$. Furthermore, from Proposition 3.2.1 it follows that we can shrink X and Y and assume that all of the morphisms $\varphi^{-1}(V_i) \to V_i$ are flat. It suffices to show that φ is flat after an extension of the ground field. Therefore, we may assume that the valuation on k is nontrivial, and all V_i are strictly k-affinoid. (Then X and Y are also strictly k-affinoid.) If $x \in X_0$, then $\mathcal{O}_{X,x} = \mathcal{O}_{V_i,x}$ for any V_i that contains x. It follows that φ is flat at all points of Y, i.e., φ is flat. ∎

Let $\varphi : Y \to X$ be a quasifinite morphism of k-analytic spaces.

3.2.5. *Definition.* — The morphism φ is said to be *flat at a point* $y \in Y$ if there exist open neighborhoods $\mathcal{V}$ of y and $\mathcal{U}$ of $\varphi(y)$ such that φ induces a finite morphism $\mathcal{V} \to \mathcal{U}$ that possesses the equivalent properties of Proposition 3.2.3; φ is said to be *flat* if it is flat at all points of Y.

From Proposition 3.2.3 it follows that if a quasifinite morphism $\varphi : Y \to X$ is flat at a point $y \in Y$, then the neighborhoods $\mathcal{V}$ and $\mathcal{U}$ can be found sufficiently small.

3.2.6. *Corollary.* — *Flat quasifinite morphisms are preserved under compositions, under any base change functor, and under any ground field extension functor.* ∎

3.2.7. *Proposition.* — *A flat quasifinite morphism $\varphi : Y \to X$ is an open map.*

Proof. — We may assume that X and Y are k-affinoid. Let $y \in Y$ and $x = \varphi(y)$. By the proof of Corollary 3.2.2, there exist affinoid neighborhoods V of y and U of x for which there is an isomorphism $\mathcal{B}_V \xrightarrow{\sim} \mathcal{A}_U^n$. In particular, the canonical homomorphism $\mathcal{A}_U \to \mathcal{B}_V$ is injective and finite. By [Ber], 2.1.16, the map

$$\psi : V = \mathcal{M}(\mathcal{B}_V) \to U = \mathcal{M}(\mathcal{A}_U)$$

is surjective. By Lemma 3.1.5, $\mathrm{Int}(V/Y) = \psi^{-1}(\mathrm{Int}(U/X))$, and therefore

$$\psi(\mathrm{Int}(V/Y)) = \mathrm{Int}(U/X). \quad ∎$$

3.2.8. *Proposition.* — *Let $\varphi : Y \to X$ be a quasifinite morphism. Then the set of points $y \in Y$ such that φ is not flat at y is Zariski closed.*

Proof. — We may assume that φ is a finite morphism of k-affinoid spaces. In this case Proposition 3.2.1 reduces the statement to the corresponding fact for morphisms of affine schemes. ■

3.2.9. *Proposition.* — *Let* $\varphi : Y \to X$ *be a closed morphism of pure one-dimensional good k-analytic spaces, and suppose that* X *is regular and* Y *is reduced. Then* φ *is flat quasifinite if and only if* φ *is nonconstant at any irreducible component of* Y.

Proof. — The direct implication is trivial. Suppose that φ is nonconstant at any irreducible component of Y. Then φ has discrete fibres, and therefore is quasifinite, by Proposition 3.1.4. It follows that for any point $y \in Y$ we can find connected affinoid neighborhoods V of y and U of $\varphi(y)$ such that φ induces a finite morphism of k-affinoid spaces $V \to U$. Then $\mathscr{A}_U$ is a one-dimensional regular integral domain, the ring $\mathscr{B}_V$ is reduced, and the canonical homomorphism $\mathscr{A}_U \to \mathscr{B}_V$ is injective. By [Ha2], III.9.7, $\mathscr{B}_V$ is a flat $\mathscr{A}_U$-algebra. ■

3.2.10. *Proposition.* — *A morphism* $\varphi : \mathscr{Z} \to \mathscr{Y}$ *between schemes of locally finite type over* $\mathscr{X} = \mathrm{Spec}(\mathscr{A})$, *where* $\mathscr{A}$ *is a k-affinoid algebra, is flat quasifinite if and only if the corresponding morphism* $\varphi^{an} : \mathscr{Z}^{an} \to \mathscr{Y}^{an}$ *is flat quasifinite.*

Proof. — Since $\mathscr{Y}^{an}$ and $\mathscr{Z}^{an}$ are faithfully flat over $\mathscr{Y}$ and $\mathscr{Z}$, respectively, the converse implication follows. Assume that φ is flat quasifinite. Then φ^{an} is quasifinite, by Corollary 3.1.7. Let $z \in \mathscr{Z}^{an}$, $y = \varphi^{an}(z)$, $\mathbf{z} = \pi(z)$, and $\mathbf{y} = \pi(y)$. We may replace $\mathscr{Y}$ and $\mathscr{Z}$ by open affine subschemes of finite type over $\mathscr{X}$. By Zariski's Main Theorem, there is an open immersion of $\mathscr{Z}$ in an affine scheme $\overline{\mathscr{Z}}$ finite over $\mathscr{Y}$. Then $\mathscr{Z}^{an} \to \overline{\mathscr{Z}}^{an}$ is also an open immersion, and $\overline{\mathscr{Z}}^{an}$ is finite over $\mathscr{Y}^{an}$. By hypothesis, $\mathcal{O}_{\mathscr{Z},\mathbf{z}}$ is flat over $\mathcal{O}_{\mathscr{Y},\mathbf{y}}$. Therefore $\mathcal{O}_{\mathscr{Y}^{an},y} \otimes_{\mathcal{O}_{\mathscr{Y},\mathbf{y}}} \mathcal{O}_{\mathscr{Z},\mathbf{z}}$ is flat over $\mathcal{O}_{\mathscr{Y}^{an},y}$. By Proposition 2.6.10, $\mathcal{O}_{\mathscr{Z}^{an},z}$ is a direct factor of the above tensor product. It follows that $\mathcal{O}_{\mathscr{Z}^{an},z}$ is flat over $\mathcal{O}_{\mathscr{Y}^{an},y}$. ■

3.3. Étale morphisms

We start this subsection with establishing basic properties of the sheaves of differentials that were introduced in § 1.4. Of course, the essential case is that of k-affinoid spaces.

Let $\varphi : Y = \mathscr{M}(\mathscr{B}) \to X = \mathscr{M}(\mathscr{A})$ be a morphism of k-affinoid spaces. In this case the sheaf $\Omega_{Y/X}$ is associated with the finite Banach $\mathscr{B}$-module $\Omega_{\mathscr{B}/\mathscr{A}} = J/J^2$, where J is the kernel of the multiplication $\mu : \mathscr{B} \hat{\otimes}_{\mathscr{A}} \mathscr{B} \to \mathscr{B}$. Furthermore, let M be a Banach $\mathscr{B}$-module. An $\mathscr{A}$-*derivation* from $\mathscr{B}$ to M is a bounded map $D : \mathscr{B} \to M$ such that $D(x+y) = Dx + Dy$, $D(xy) = x\,Dy + y\,Dx$ and $D(\mathscr{A}) = 0$. The set of all $\mathscr{A}$-derivations from $\mathscr{B}$ to M is a Banach $\mathscr{B}$-module with respect to the evident norm. It is denoted by $\mathrm{Der}_{\mathscr{A}}(\mathscr{B}, M)$. For example, the mapping $\mathscr{B} \to J : x \mapsto 1 \otimes x - x \otimes 1$ induces an $\mathscr{A}$-derivation $d : \mathscr{B} \to \Omega_{\mathscr{B}/\mathscr{A}}$.

3.3.1. *Proposition.* — (i) *The finite* $\mathcal{B}$*-module* $\Omega_{\mathcal{B}/\mathcal{A}}$ *is generated by the elements* dx, $x \in \mathcal{B}$.

(ii) *For any Banach* $\mathcal{B}$*-module* M *there is a canonical isomorphism of Banach* $\mathcal{B}$*-modules*

$$\mathrm{Hom}_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}}, \mathrm{M}) \xrightarrow{\sim} \mathrm{Der}_{\mathcal{A}}(\mathcal{B}, \mathrm{M})$$

(the left hand side is the set of all bounded $\mathcal{B}$*-homomorphisms).*

Proof. — (i) Let T be the $\mathcal{B}$-submodule of $\Omega_{\mathcal{B}/\mathcal{A}}$ generated by the elements dx, $x \in \mathcal{B}$. Recall that the ring $\mathcal{B}$ is Noetherian, and all its ideals are closed. Since $\Omega_{\mathcal{B}/\mathcal{A}}$ is a finite Banach $\mathcal{B}$-module, T is closed in it. We claim that for any $w \in \Omega_{\mathcal{B}/\mathcal{A}}$ and any $\varepsilon > 0$ there exists an element $t \in \mathrm{T}$ with $\| w - t \| < \varepsilon$. Indeed, let v be an inverse image of w in J. There exists an element $\sum_{i=1}^n x_i \otimes y_i \in \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ such that $\| v - \sum_{i=1}^n x_i \otimes y_i \| < \varepsilon$. Since $\mu(v) = 0$, $\| \sum_{i=1}^n x_i y_i \| < \varepsilon$. We have

$$\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1) + \sum_{i=1}^n x_i y_i \otimes 1.$$

Therefore

$$\| v - \sum_{i=1}^n (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1) \| \leqslant \max(\| v - \sum_{i=1}^n x_i \otimes y_i \|, \| \sum_{i=1}^n x_i y_i \otimes 1 \|) < \varepsilon.$$

The required claim follows.

(ii) It is clear that the homomorphism considered is bounded. From (i) it follows that it is injective. Therefore it suffices to construct an inverse bounded homomorphism. Consider the following Banach $\mathcal{B}$-algebra $\mathcal{B} * \mathrm{M}$. As a Banach $\mathcal{B}$-module it is the direct sum of $\mathcal{B}$ and M. Its multiplication is defined as follows: $(x, m)(y, n) = (xy, xn + ym)$. Let now $\mathrm{D} : \mathcal{B} \to \mathrm{M}$ be an $\mathcal{A}$-derivation. The bounded $\mathcal{A}$-bilinear mapping

$$\mathcal{B} \times \mathcal{B} \to \mathcal{B} * \mathrm{M} : (x, y) \mapsto (xy, x\,\mathrm{D}y)$$

induces a bounded homomorphism of Banach $\mathcal{A}$-algebras $\varphi : \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B} \to \mathcal{B} * \mathrm{M}$. The reasoning from (i) shows that $\varphi(\mathrm{J}) \subset \mathrm{M}$. Since $\mathrm{M}^2 = 0$, the homomorphism φ induces a bounded homomorphism of Banach $\mathcal{B}$-modules $f : \Omega_{\mathcal{B}/\mathcal{A}} = \mathrm{J}/\mathrm{J}^2 \to \mathrm{M}$. We have $\mathrm{D}x = f(dx)$ for all $x \in \mathcal{B}$. That the correspondence $\mathrm{D} \mapsto f$ is bounded follows from the construction. ∎

3.3.2. *Proposition.* — *Suppose we are given a commutative diagram of morphisms of k-analytic spaces*

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\varphi} & \mathrm{X} \\ & \searrow \quad \swarrow_{f} & \\ & \mathrm{S} & \end{array}$$

(i) *There is an exact sequence*

$$\varphi_G^*(\Omega_{X_G/S_G}) \to \Omega_{Y_G/S_G} \to \Omega_{Y_G/X_G} \to 0.$$

(ii) *If* φ *is a closed immersion and* $\mathcal{I}_G$ *is the subsheaf of ideals in* $\mathcal{O}_{X_G}$ *that corresponds to* Y, *then there is an exact sequence*

$$\mathcal{I}_G/\mathcal{I}_G^2 \to \varphi_G^*(\Omega_{X_G/S_G}) \to \Omega_{Y_G/S_G} \to 0.$$

Proof. — We may assume that $X = \mathcal{M}(\mathcal{A})$, $Y = \mathcal{M}(\mathcal{B})$ and $S = \mathcal{M}(\mathcal{C})$ are k-affinoid.

(i) We have to show that the sequence of finite Banach $\mathcal{B}$-modules

$$\Omega_{\mathcal{A}/\mathcal{C}} \otimes_{\mathcal{A}} \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{C}} \to \Omega_{\mathcal{B}/\mathcal{A}} \to 0$$

is exact (note that $\Omega_{\mathcal{A}/\mathcal{C}} \otimes_{\mathcal{A}} \mathcal{B} = \Omega_{\mathcal{A}/\mathcal{C}} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}$). For this it suffices to show that for any finite Banach $\mathcal{B}$-module M the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}}, M) \to \mathrm{Hom}_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{C}}, M) \to \mathrm{Hom}_{\mathcal{C}}(\Omega_{\mathcal{A}/\mathcal{C}}, M)$$

is exact. But, by Proposition 3.3.1, the latter sequence coincides with the sequence $0 \to \mathrm{Der}_{\mathcal{A}}(\mathcal{B}, M) \to \mathrm{Der}_{\mathcal{C}}(\mathcal{B}, M) \to \mathrm{Der}_{\mathcal{C}}(\mathcal{A}, M)$ which is exact for trivial reasons.

(ii) Let J be the ideal of $\mathcal{A}$ corresponding to $\mathcal{I}$. We have to show that the sequence of finite Banach $\mathcal{B}$-modules $J/J^2 \xrightarrow{\delta} \Omega_{\mathcal{A}/\mathcal{C}} \otimes_{\mathcal{A}} \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{C}} \to 0$ is exact, where $\delta(x) = dx \otimes 1$. As above, it suffices to show that for any finite Banach $\mathcal{B}$-module M the sequence $0 \to \mathrm{Der}_{\mathcal{C}}(\mathcal{B}, M) \to \mathrm{Der}_{\mathcal{C}}(\mathcal{A}, M) \to \mathrm{Hom}_{\mathcal{B}}(J/J^2, M)$ is exact, but this is evident. ■

3.3.3. *Proposition.* — *Let* $\varphi : Y \to X$ *be a morphism of k-analytic spaces. Then:*

(i) *for any morphism of k-analytic spaces* $f: X' \to X$, *one has* $\Omega_{Y'_G/X'_G} = f_G'^*(\Omega_{Y_G/X_G})$, *where* f' *is the induced morphism* $Y' = Y \times_X X' \to Y$;

(ii) *for any non-Archimedean field* K *over* k, *one has* $\Omega_{Y'_G/X'_G} = f_G'^*(\Omega_{Y_G/X_G})$, *where* f' *is the induced morphism* $Y' = Y \widehat{\otimes} K \to Y$.

Proof. — We may assume that $Y = \mathcal{M}(\mathcal{B})$, $X = \mathcal{M}(\mathcal{A})$ and $X' = \mathcal{M}(\mathcal{A}')$ from (i) are k-affinoid. In this case $\Omega_{Y/X}$ is defined by the finite Banach $\mathcal{B}$-module J/J^2, where J is the kernel of the multiplication $\mathcal{C} = \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$. Note that the exact admissible sequence $0 \to J \to \mathcal{C} \to \mathcal{B} \to 0$ is split.

(i) If $\mathcal{B}' = \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{A}'$, then $\Omega_{Y'/X'}$ is defined by the finite Banach $\mathcal{B}'$-module J'/J'^2, where J' is the kernel of the multiplication $\mathcal{C}' = \mathcal{B}' \widehat{\otimes}_{\mathcal{A}'} \mathcal{B}' \to \mathcal{B}'$. We have to show that $J/J^2 \otimes_{\mathcal{B}} \mathcal{B}' \xrightarrow{\sim} J'/J'^2$ (since J/J^2 is a finite Banach $\mathcal{B}$-module, $J/J^2 \otimes_{\mathcal{B}} \mathcal{B}' = J/J^2 \widehat{\otimes}_{\mathcal{B}} \mathcal{B}'$).

The exact sequence $0 \to J' \to \mathcal{C}' \to \mathcal{B}' \to 0$ is obtained from the above exact sequence by tensoring with $\mathcal{A}'$ over $\mathcal{A}$. It follows that $J' = J \widehat{\otimes}_{\mathcal{A}} \mathcal{A}' = J \widehat{\otimes}_{\mathcal{C}} \mathcal{C}' = J\mathcal{C}'$ because J is a finite Banach $\mathcal{C}$-module. Tensoring the exact sequence

$$0 \to J^2 \to J \to J/J^2 \to 0$$

with $\mathscr{C}'$ over $\mathscr{C}$, we get an exact sequence $J^2 \otimes_{\mathscr{C}} \mathscr{C}' \to J' \to J/J^2 \otimes_{\mathscr{C}} \mathscr{C}' \to 0$. Since $J/J^2 \otimes_{\mathscr{C}} \mathscr{C}' = J/J^2 \otimes_{\mathscr{B}} \mathscr{B}'$ and the image of $J^2 \otimes_{\mathscr{C}} \mathscr{C}'$ in $J' = J\mathscr{C}$ coincides with $J'^2 = J^2 \mathscr{C}$, we get the required isomorphism.

(ii) If $\mathscr{A}' = \mathscr{A} \hat{\otimes} K$ and $\mathscr{B}' = \mathscr{B} \hat{\otimes} K$, then $\Omega_{Y'/X'}$ is defined by J'/J'^2, where J' is the kernel of the multiplication $\mathscr{C}' = \mathscr{B}' \hat{\otimes}_{\mathscr{A}'} \mathscr{B}' \to \mathscr{B}'$. As above we get that $J' = J \hat{\otimes} K = J\mathscr{C}'$ and $J'^2 = J^2 \hat{\otimes} K = J^2 \mathscr{C}'$. Therefore $J/J^2 \hat{\otimes} K = J'/J'^2$. It remains to note that $J/J^2 \hat{\otimes} K = J/J^2 \otimes_{\mathscr{B}} \mathscr{B}'$. ∎

Let $\varphi : Y \to X$ be a quasifinite morphism.

3.3.4. *Definition.* — The morphism φ is said to be *unramified* if $\Omega_{Y_G/X_G} = 0$. It is said to be *étale* if it is unramified and flat. It is said to be *unramified* (resp. *étale*) *at a point* $y \in Y$ if there exists an open neighborhood $\mathscr{V}$ of y such that the induced morphism $\mathscr{V} \to X$ is unramified (resp. étale).

For example, if φ is a *local isomorphism at a point* $y \in Y$ (i.e., there exist open neighborhoods $\mathscr{V}$ of y and $\mathscr{U}$ of $\varphi(x)$ such that φ induces an isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{U}$), then φ is étale at y. Therefore if φ is a *local isomorphism* (i.e., φ is a local isomorphism at every point $y \in Y$), then φ is étale.

3.3.5. *Lemma.* — *If $\varphi : Y \to X$ is a quasifinite morphism of good k-analytic spaces, then the stalk of $\Omega_{Y/X}$ at a point $y \in Y$ coincides with the module of differentials $\Omega_{B/A}$, where $B = \mathcal{O}_{Y,y}$, $A = \mathcal{O}_{X,x}$ and $x = \varphi(y)$.*

Proof. — We may assume that φ is a finite morphism of k-affinoid spaces $Y = \mathscr{M}(\mathscr{B}) \to X = \mathscr{M}(\mathscr{A})$. In this case $\mathscr{B} \hat{\otimes}_{\mathscr{A}} \mathscr{B} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}$, and therefore $\Omega_{Y/X}$ is defined by the module of differentials $\Omega_{\mathscr{B}/\mathscr{A}}$ (regarded as a finite Banach $\mathscr{B}$-module). The required statement easily follows from this. ∎

3.3.6. *Corollary.* — *A quasifinite morphism of good k-analytic spaces $\varphi : Y \to X$ is unramified (resp. étale) at a point $y \in Y$ if and only if $\mathcal{O}_{Y,y}/\mathbf{m}_x \mathcal{O}_{Y,y}$ is a finite separable extension of the field $\kappa(x)$ (resp. and $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$), where $x = \varphi(y)$.* ∎

The following statement is straightforward from the definitions.

3.3.7. *Proposition.* — *Let $\varphi : Y \to X$ be a quasifinite morphism and $y \in Y$. Then the following are equivalent:*

a) φ is unramified at y;

b) the support of $\Omega_{Y/X}$ does not contain y;

c) the diagonal morphism $\Delta : Y \to Y \times_X Y$ is a local isomorphism at y. ∎

3.3.8. *Corollary.* — *Unramified (resp. étale) morphisms are preserved under compositions, under any base change functor, and under any ground field extension functor.* ∎

3.3.9. *Corollary.* — *Let $\psi : Z \to Y$ and $\varphi : Y \to X$ be quasifinite morphisms and suppose that $\varphi\psi$ is étale and φ is unramified. Then ψ is étale.*

Proof. — The morphism ψ is a composition of the graph morphism $\Gamma_\psi : Z \to Z \times_X Y$ with the projection $p_2 : Z \times_X Y \to Y$. The first morphism is an open immersion because it is a base change of the open immersion $Y \to Y \times_X Y$ with respect to the evident morphism $Z \times_X Y \to Y \times_X Y$, and the second one is étale because it is a base change of the étale morphism $\varphi\psi : Z \to X$. ■

Let $\text{Ét}(X)$ denote the category of étale morphisms $U \to X$. Corollary 3.3.9 implies that any morphism in the category $\text{Ét}(X)$ is étale.

3.3.10. *Proposition. — Suppose that $\varphi : Y \to X$ is a quasifinite morphism or is a closed morphism of good k-analytic spaces. Then the set of points $y \in Y$ such that φ is not unramified (resp. étale) at y is Zariski closed.*

Proof. — We can replace Y by the complement to the support of the coherent $\mathcal{O}_{Y_G}$-module Ω_{Y_G/X_G} and assume that $\Omega_{Y_G/X_G} = 0$. In the second case the latter equality implies that φ has discrete fibres, and therefore φ is quasifinite, by Proposition 3.1.4. Hence our statement follows from Propositions 3.3.7 and 3.2.8. ■

3.3.11. *Proposition. — A morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ between schemes of locally finite type over $\mathcal{X} = \text{Spec}(\mathcal{A})$, where $\mathcal{A}$ is a k-affinoid algebra, is unramified (resp. étale) if and only if the corresponding morphism $\varphi^{an} : \mathcal{X}^{an} \to \mathcal{Y}^{an}$ is unramified (resp. étale).*

Proof. — The unramifiedness statement follows from Corollary 3.3.7 and the simple fact that $\Omega_{\mathcal{X}^{an}/\mathcal{Y}^{an}} = (\Omega_{\mathcal{X}/\mathcal{Y}})^{an}$. The étaleness statement now follows from Proposition 3.2.10. ■

3.4. Germs of analytic spaces

A *germ of k-analytic space* (or simply a *k-germ*) is a pair (X, S), where X is a k-analytic space, and S is a subset of the underlying topological space $|X|$. (S is said to be the *underlying topological space* of the k-germ (X, S).) If $S = \{x\}$, then (X, S) is denoted by (X, x). The k-germs form a category in which morphisms from (Y, T) to (X, S) are the morphisms $\varphi : Y \to X$ with $\varphi(T) \subset S$. The category k-$\mathcal{G}erms$ we are going to work with is the category of fractions of the latter category with respect to the system of morphisms $\varphi : (Y, T) \to (X, S)$ such that φ induces an isomorphism of Y with an open neighborhood of S in X. This system obviously admits a calculus of right fractions, and so the set of morphisms Hom((Y, T), (X, S)) in k $\mathcal{G}erms$ is the inductive limit of the set of morphisms $\varphi : \mathcal{V} \to X$ with $\varphi(T) \subset S$, where $\mathcal{V}$ runs through a fundamental system of open neighborhoods of T in Y. (Such a morphism $\varphi : \mathcal{V} \to X$ is said to be a *representative* of a morphism $(Y, T) \to (X, S)$.) It is easy to see that a morphism $(Y, T) \to (X, S)$ is an isomorphism in k-$\mathcal{G}erms$ if and only if it induces an isomorphism between some open neighborhoods of T and S. We remark that the correspondence $X \mapsto (X, |X|)$ induces a fully faithful functor k-$\mathcal{A}n \to k$-$\mathcal{G}erms$.

The category k-$\mathcal{G}erms$ admits fibre products. Indeed, let $(Y, T) \to (X, S)$ and $(X', S') \to (X, S)$ be two morphisms. If $\varphi : \mathcal{V} \to X$ and $f : \mathcal{U}' \to X$ are their representatives, then $(\mathcal{V} \times_X \mathcal{U}', \pi^{-1}(T \times_S S'))$ is a fibre product of (Y, T) and (X', S') over (X, S), where π is the canonical map $| \mathcal{V} \times_X \mathcal{U}' | \to | \mathcal{V} | \times_{|X|} | \mathcal{U}' |$. Thus, for any morphism $\varphi : (Y, T) \to (X, S)$ and a point $x \in S$ one can define the *fibre of* φ *at* x in the category k-$\mathcal{G}erms$ as the fibre product $(Y, T) \times_{(X, S)} (X, x)$ which is actually isomorphic to the k-germ $(Y, \varphi^{-1}(x))$, where $\varphi^{-1}(x)$ is the inverse image of x in T. In particular, for a morphism of k-analytic spaces $\varphi : Y \to X$ and a point $x \in X$, one has the fibre $(Y, \varphi^{-1}(x))$ of φ at x in the category k-$\mathcal{G}erms$. (Recall that in § 1.4 we defined the fiber Y_x of φ at x in the category $\mathcal{A}n_k$ of analytic spaces over k.)

Furthermore, for a non-Archimedean field K over k there is a *ground field extension functor* k-$\mathcal{G}erms \to$ K-$\mathcal{G}erms$: $(X, S) \mapsto (X \hat{\otimes} K, \pi^{-1}(S))$, where π is the canonical map $X \hat{\otimes} K \to X$. Similarly to $\mathcal{A}n_k$ one can define the category $\mathcal{G}erms_k$ of *germs of analytic spaces over* k (or simply *germs over* k). Its objects are pairs $(K, (X, S))$, where K is a non-Archimedean field over k and (X, S) is a K-germ. A morphism $(L, (Y, T)) \to (K, (X, S))$ is a pair consisting of an isometric embedding $K \hookrightarrow L$ and a morphism of L-germs $(Y, T) \to (X, S) \hat{\otimes}_K L$. As above, there is a fully faithful functor

$$\mathcal{A}n_k \to \mathcal{G}erms_k : (K, X) \mapsto (K, (X, | X |)).$$

For a k-germ (X, S), let $\text{Ét}(X, S)$ denote the category of the morphism $(Y, T) \to (X, S)$ that have an étale representative $\varphi : \mathcal{V} \to X$ with $T = \varphi^{-1}(S)$. It is clear that for $X \in k$-$\mathcal{A}n$ there is an equivalence of categories $\text{Ét}(X) \xrightarrow{\sim} \text{Ét}(X, | X |)$. For a point $x \in X$, let $\text{Fét}(X, x)$ denote the full subcategory of $\text{Ét}(X, x)$ consisting of the morphisms $(Y, T) \to (X, x)$ that have an étale representative $\varphi : \mathcal{V} \to X$ such that the morphism $\mathcal{V} \to \varphi(\mathcal{V})$ is finite. (Equivalently, $\text{Fét}(X, x)$ consists of the morphisms $(Y, T) \to (X, x)$ with finite set T that have an étale separated representative $\varphi : \mathcal{V} \to X$.) For a field K, let $\text{Fét}(K)$ denote the category of schemes finite and étale over the spectrum of K.

3.4.1. *Theorem. — Let* X *be a* k*-analytic space. Then for any point* $x \in X$ *there is an equivalence of categories* $\text{Fét}(X, x) \xrightarrow{\sim} \text{Fét}(\mathcal{H}(x))$.

Proof. — Consider first the case when the point x has an affinoid neighborhood. Let $\text{Fét}(\mathcal{X}(x))$ denote the category of schemes finite and étale over the affine scheme $\mathcal{X}(x) = \text{Spec}(\mathcal{O}_{X,x})$. Then the functor considered is a composition of the three evident functors

$$\text{Fét}(X, x) \to \text{Fét}(\mathcal{X}(x)) \to \text{Fét}(\kappa(x)) \to \text{Fét}(\mathcal{H}(x)).$$

The third functor is an equivalence of categories because the field $\kappa(x)$ is quasicomplete. The second one is an equivalence because the ring $\mathcal{O}_{X,x}$ is Henselian. We now verify that the first functor is faithful. Let $\varphi, \psi : (Y, y) \to (X, x)$ be two morphisms that induce the same homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ (we do not need here the étaleness of φ and ψ). We may assume that $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ are k-affinoid, and φ and ψ are induced by two homomorphisms of k-affinoid algebras $\alpha, \beta : \mathcal{A} \to \mathcal{B}$. Consider an admissible epimorphism $\gamma : k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{A}$ and set $f_i = \gamma(T_i)$. Since the images of $\alpha(f_i)$ and $\beta(f_i)$ in $\mathcal{O}_{Y,y}$ coincide, we can find an affinoid neighborhood V of y such that $\alpha(f_i)|_V = \beta(f_i)|_V$. By [Ber], 2.1.5, the induced morphisms $\varphi|_V, \psi|_V : V \to X$ coincide, and therefore the first functor is faithful. Furthermore, we claim that a morphism in $\mathrm{F\acute{e}t}(X, x)$ that becomes an isomorphism in $\mathrm{F\acute{e}t}(\mathcal{H}(x))$ is an isomorphism. Indeed, let $\varphi : Y \to X$ be an étale morphism with $x = \varphi(y)$ such that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. We can shrink X and Y and assume that $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ are k-affinoid, $\varphi^{-1}(x) = \{y\}$ and $\mathcal{B}$ is a free $\mathcal{A}$-module. From Lemma 2.1.6 it follows that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{Y,y}$, and therefore the rank of $\mathcal{B}$ over $\mathcal{A}$ is one, i.e., $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$. Finally, that any finite étale morphism over $\mathcal{H}(x)$ comes from an étale morphism over (X, x) is obtained by the construction from the proof of Theorem 2.1.5. That the first functor is fully faithful now follows from the fact that any morphism in the categories $\mathrm{F\acute{e}t}(X, x)$ and $\mathrm{F\acute{e}t}(\mathcal{H}(x))$ is étale.

Suppose now that the point x is arbitrary. We may assume that the space X Hausdorff, and we take affinoid domains $U_1, \ldots, U_n$ such that $x \in U_1 \cap \ldots \cap U_n$ and $U_1 \cup \ldots \cup U_n$ is a neighborhood of x. First we verify that the functor considered is faithful. Indeed, let $\varphi : Y \to X$ and $\psi : Z \to X$ be two étale morphisms with $\varphi^{-1}(x) = \{y\}$ and $\varphi^{-1}(x) = \{z\}$, and suppose that $f, g : Z \to Y$ are two morphisms over X with $f(z) = g(z) = y$ that give rise to the same embedding of fields $\mathcal{H}(y) \hookrightarrow \mathcal{H}(z)$. By the first case, we can find for each $1 \leqslant i \leqslant n$ an affinoid neighborhood W_i of z in $\psi^{-1}(U_i)$ such that $f|_{W_i} = g|_{W_i}$. Then the analytic domain $W = W_1 \cup \ldots \cup W_n$ is a neighborhood of the point z and $f|_W = g|_W$. It follows that the morphism from (Z, z) to (Y, y) induced by f and g coincide. In the same way one shows that a morphism in $\mathrm{F\acute{e}t}(X, x)$ that becomes an isomorphism in $\mathrm{F\acute{e}t}(\mathcal{H}(x))$ is an isomorphism. Since any morphism in the category $\mathrm{F\acute{e}t}(X, x)$ is étale, to prove the theorem it remains to show that the functor considered is essentially surjective.

Let K be a finite separable extension of the field $\mathcal{H}(x)$. By the first case, we can shrink all U_i and find finite étale morphism $\varphi_i : V_i \to U_i$ such that $\varphi_i^{-1}(x) = \{y_i\}$ and there are isomorphisms of fields $K \xrightarrow{\sim} \mathcal{H}(y_i)$ over $\mathcal{H}(x)$. (We fix such isomorphisms.) Suppose first that X is separated at x. Then we may assume that X is separated, and therefore $U_i \cap U_j$ are affinoid domains. Setting $V_{ij} = \varphi_i^{-1}(U_i \cap U_j)$, we have two finite étale morphisms $V_{ij} \to U_i \cap U_j$ and $V_{ji} \to U_i \cap U_j$ and, for the points $y_i \in V_{ij}$ and $y_j \in V_{ji}$, an isomorphism of fields $\mathcal{H}(y_j) \xrightarrow{\sim} \mathcal{H}(y_i)$ over $\mathcal{H}(x)$ induced by the isomorphisms $K \xrightarrow{\sim} \mathcal{H}(y_i)$ and $K \xrightarrow{\sim} \mathcal{H}(y_j)$. By the first case, we can shrink all U_i and assume that there exist isomorphisms $\nu_{ij} : V_{ij} \xrightarrow{\sim} V_{ji}$ over X that give rise to the above isomorphisms of fields. By construction, $V_{ii} = V_i$ and $\nu_{ij}(V_{ij} \cap V_{il}) = V_{ji} \cap V_{jl}$. We now can shrink all U_i and assume that $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $V_{ij} \cap V_{il}$. By Proposition 1.3.3, we can glue all V_i along V_{ij} and get a k-analytic space Y with a morphism

$\varphi : Y \to X$. By Proposition 3.1.8, the morphism φ is finite at the point y (that corresponds to the points y_i). It is clear that the point y is not contained in the support of Ω_{Y_G/X_G}, i.e., φ is unramified at y. It is flat at y, by Proposition 3.2.3.

Suppose now that X is arbitrary (and Hausdorff). The intersections $U_i \cap U_j$ are not affinoid now, but they are separated compact k-analytic spaces, and therefore we can apply the above construction using the fact that everything is already verified for separated spaces. The theorem is proved. ∎

3.4.2. *Corollary.* — *Let* $\varphi : Y \to X$ *be a morphism of analytic spaces over* k, *and let* $y \in Y$, $x = \varphi(y)$. *Suppose that the maximal purely inseparable extension of* $\mathcal{H}(x)$ *in* $\mathcal{H}(y)$ *is dense in* $\mathcal{H}(y)$. *Then the correspondence* $U \mapsto U \times_X Y$ *induces an equivalence of categories* $\mathrm{Fét}(X, x) \xrightarrow{\sim} \mathrm{Fét}(Y, y)$. ∎

3.5. Smooth morphisms

For a k-analytic spaces X we set $\mathbf{A}^d_X = \mathbf{A}^d \times X$ (the *d-dimensional affine spaces over* X).

3.5.1. *Definition.* — A morphism of k-analytic spaces $\varphi : Y \to X$ is said to be *smooth at a point* $y \in Y$ if there exists an open neighborhood $\mathcal{V}$ of y such that the induced morphism $\mathcal{V} \to X$ can be represented as a composition of an étale morphism $\mathcal{V} \to \mathbf{A}^d_X$ with the canonical morphism $\mathbf{A}^d_X \to X$; φ is said to be *smooth* if it is smooth at all points $y \in Y$. If the canonical morphism $X \to \mathcal{M}(k)$ is smooth, then X is said to be *smooth*.

We remark that the number d is equal to the *dimension of* φ *at the point* y, i.e., to the dimension of the fibre Y_x, where $x = \varphi(y)$, at the point y. If this number is independent of y, we say that φ is *of pure dimension d*. For example, smooth morphisms of pure dimension 0 are exactly étale morphisms. We remark that smooth morphisms are locally separated and closed. The following proposition follows easily from the definition of smooth morphisms and Corollary 3.3.8.

3.5.2. *Proposition.* — *Smooth morphisms are preserved under compositions, under any base change functor, and under extensions of the ground field.* ∎

3.5.3. *Proposition.* — *Suppose we are given a commutative diagram of morphisms of* k*-analytic spaces*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ & {}_g\searrow \quad \swarrow_f & \\ & S & \end{array}$$

(i) *If* φ *is étale, then* $\varphi_G^*(\Omega_{X_G/S_G}) \xrightarrow{\sim} \Omega_{Y_G/S_G}$.
(ii) *If* f *and* g *are smooth and* $\varphi_G^*(\Omega_{X_G/S_G}) \xrightarrow{\sim} \Omega_{Y_G/S_G}$, *then* φ *is étale.*

Since $\Omega_{\mathbf{A}^d_{X_G}/X_G}$ is a free $\mathcal{O}_{\mathbf{A}^d_{X_G}}$-module of rank d and, if X is good, the canonical morphism $\mathbf{A}^d_X \to X$ is flat, then the statement (i) implies

3.5.4. *Corollary. — Let $\varphi : Y \to X$ be a smooth morphism between good k-analytic spaces. Then φ is flat, and $\Omega_{Y/X}$ is a locally free $\mathcal{O}_Y$-module whose rank at a point $y \in Y$ is equal to the dimension of φ at y.* ∎

3.5.5. *Lemma. — Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

where φ is a G-locally closed immersion and f is flat quasifinite. Then $f'^*(\mathcal{N}_{Y_G/X_G}) = \mathcal{N}_{Y'_G/X'_G}$.

Proof. — We may assume that all the spaces are k-affinoid, φ is a closed immersion, and f is a finite morphism. Let $X = \mathcal{M}(\mathcal{A})$, $Y = \mathcal{M}(\mathcal{B})$, $X' = \mathcal{M}(\mathcal{A}')$ and $Y' = \mathcal{M}(\mathcal{B}')$, where $\mathcal{B}' = \mathcal{B} \hat{\otimes}_{\mathcal{A}} \mathcal{A}'$. One has exact admissible sequences $0 \to J \to \mathcal{A} \to \mathcal{B} \to 0$ and $0 \to J' \to \mathcal{A}' \to \mathcal{B}' \to 0$. Since $\mathcal{A}'$ is a flat finite $\mathcal{A}$-algebra, then the second sequence is obtained by tensoring of the first one with $\mathcal{A}'$ over $\mathcal{A}$. In particular, $J' = J \hat{\otimes}_{\mathcal{A}} \mathcal{A}' = J\mathcal{A}'$. It follows that $J'/J'^2 = J/J^2 \otimes_{\mathcal{A}} \mathcal{A}' = J/J^2 \hat{\otimes}_{\mathcal{A}} \mathcal{A}'$. ∎

Proof of Proposition **3.5.3.** — (i) Consider the diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\Delta_{X/S}} & X \times_S X \\ & & \uparrow{\scriptstyle \psi'} & & \uparrow{\scriptstyle \psi} \\ Y & \xrightarrow{\Delta_{Y/X}} & Y \times_X Y & \longrightarrow & Y \times_S Y \end{array}$$

where $Y \times_X Y$ is identified with the fibre product of X and $Y \times_S Y$ over $X \times_S X$. The morphism ψ is flat quasifinite. By Lemma 3.5.5, the conormal sheaf of the G-locally closed immersion $Y \times_X Y \to Y \times_S Y$ coincides with $\psi'^*(\Omega_{X_G/S_G})$. Since $\Delta_{Y/X}$ is an open immersion, then $\Omega_{Y_G/S_G} = \varphi_G^*(\Omega_{X_G/S_G})$.

(ii) Consider first the case when the space S is good. (Then X and Y are also good.) From the exact sequence 3.3.2 (i) it follows that $\Omega_{Y/X} = 0$, and therefore the morphism φ has discrete fibres. By Proposition 3.1.4, φ is quasifinite. Since $\Omega_{Y/X} = 0$, it is unramified. Let $y \in Y$, $x = \varphi(y)$ and $s = g(y)$. We have to verify that $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-algebra. Suppose first that $[\mathcal{H}(y) : k] < \infty$. Since $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ are flat $\mathcal{O}_{S,s}$-algebras (Corollary 3.5.4), then, by Corollary 5.9 from [SGA1], Exp. IV, it suffices to verify that $\mathcal{O}_{Y,y}/\mathfrak{m}_s \mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x}$-algebra. Since $[\mathcal{H}(y) : k) < \infty$, we have $\mathcal{O}_{Y,y}/\mathfrak{m}_s \mathcal{O}_{Y,y} = \mathcal{O}_{Y_s,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x} = \mathcal{O}_{X_s,x}$. Therefore we may assume that $S = \mathcal{M}(k)$. In this case the ring $\mathcal{O}_{X,x}$ is regular and, in particular, normal. Since $\mathcal{O}_{Y,y}$ is a finite unramified $\mathcal{O}_{X,x}$-algebra, it suffices by Theorem 9.5 (ii) from [SGA1], Exp. I, to verify that the canonical homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is injective or, equivalently, that $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same dimensions. But these dimensions are equal to the ranks of $\Omega_{X/k}$ and $\Omega_{Y/k}$ at the points x and y, respectively. Since $\varphi^*(\Omega_{X/k}) \tilde{\to} \Omega_{Y/k}$, they are equal.

Suppose now that the point y is arbitrary. Let K be a big enough non-Archimedean field such that there exists a point $y' \in Y' = Y \hat{\otimes} K$ with $[\mathscr{H}(y') : K] < \infty$ and $\pi(y') = y$, where π is the canonical mapping $Y' \to Y$. From Proposition 3.3.3 it follows that $\varphi'^*(\Omega_{X'/S'}) \tilde{\to} \Omega_{Y'/S'}$, where φ' is the induced morphism $Y' \to X' = X \hat{\otimes} K$. By the previous case, $\mathcal{O}_{Y',y'}$ is a flat $\mathcal{O}_{X',x'}$-algebra, where $x' = \varphi'(y')$. By Corollary 2.1.3, $\mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y',y'}$ are faithfully flat over $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$, respectively. It follows that $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$.

Consider now the general case. If U is an affinoid domain in S, then, by the first case, the induced morphism $g^{-1}(U) \to f^{-1}(U)$ is étale and, in particular, it is quasifinite. From Proposition 3.1.8 it follows that the morphism φ is quasifinite. This implies immediately that it is étale. ■

3.5.6. *Corollary.* — *In the situation of Proposition* 3.3.2 (i) *suppose that* φ *is smooth. Then there is an exact sequence*

$$0 \to \varphi_G^*(\Omega_{X_G/S_G}) \to \Omega_{Y_G/S_G} \to \Omega_{Y_G/X_G} \to 0. \quad ■$$

3.5.7. *Corollary.* — *Let* $\varphi : Y \to X$ *be a smooth morphism, and let* $f : Y \to \mathbf{A}^d_X$ *be an* X-*morphism defined by some functions* $f_1, \ldots, f_d \in \mathcal{O}(Y)$. *Then* f *is étale at a point* $y \in Y$ *if and only if for some affinoid domain* $U \subset X$ *that contains the point* $x = \varphi(y)$ *the elements* $df_1, \ldots, df_d$ *form a base of* $\Omega_{\varphi^{-1}(U)/U}$ *at* y.

Proof. — The direct implication is trivial. Suppose that $df_1, \ldots, df_d$ form a base of $\Omega_{\varphi^{-1}(U)/U}$ at y. By Proposition 3.5.3, the induced morphism $\varphi^{-1}(U) \to \mathbf{A}^d_U$ is étale at y. In particular, the point y is isolated in the fibre $f^{-1}(f(y))$. From Proposition 3.1.4 it follows that for any affinoid domain $V \subset X$ that contains the point x the induced morphism $\varphi^{-1}(V) \to \mathbf{A}^d_V$ is finite at y, and therefore, by Proposition 3.1.8, the morphism f is finite at y. It is étale because the elements $df_1, \ldots, df_d$ form a base of $\Omega_{\varphi^{-1}(V)/V}$ at y for any V as above. ■

3.5.8. *Proposition.* — *A morphism* $\varphi : \mathcal{X} \to \mathcal{Y}$ *between schemes of locally finite type over* $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$, *where* $\mathcal{A}$ *is a k-affinoid algebra, is smooth if and only if the corresponding morphism* $\varphi^{an} : \mathcal{X}^{an} \to \mathcal{Y}^{an}$ *is smooth.*

Proof. — The direct implication follows from Proposition 3.3.11. Suppose that φ^{an} is smooth. By Corollary 3.5.4, φ^{an} is flat, and $\Omega_{\mathcal{X}^{an}/\mathcal{Y}^{an}}$ is a locally free $\mathcal{O}_{\mathcal{X}^{an}}$-module. Since the morphisms $\mathcal{Y}^{an} \to \mathcal{Y}$ and $\mathcal{X}^{an} \to \mathcal{X}$ are faithfully flat, it follows that φ is flat. Since $\Omega_{\mathcal{X}^{an}/\mathcal{Y}^{an}} = (\Omega_{\mathcal{X}/\mathcal{Y}})^{an}$, it follows that $\Omega_{\mathcal{X}/\mathcal{Y}}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module. Therefore φ is smooth. ■

3.5.9. *Proposition.* — *Suppose we are given a commutative diagram of morphisms of good k-analytic spaces*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

where φ is a closed immersion and f is smooth. Then the following are equivalent:

a) g is smooth;

b) for any point $y \in Y$ there exist an open neighborhood X_1 of y in X and an étale morphism $h: X_1 \to \mathbf{A}^d_S$ over S such that $Y_1 = Y \cap X_1$ is the inverse image of the closed k-analytic subset of $\mathbf{A}^d_S$ defined by the equations $T_1 = \ldots = T_c = 0$ ($T_1, \ldots, T_d$ are the coordinate functions on $\mathbf{A}^d$).

Proof. — The implication *b)* $\Rightarrow$ *a)* is trivial. Suppose that g is smooth. Then the $\mathcal{O}_Y$-modules $\varphi^*(\Omega_{X/S}) = \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ and $\Omega_{Y/S}$ from the exact sequence 3.3.2 (ii) are locally free. We may decrease X and assume that they are free. Since the elements dh, $h \in \mathcal{O}(X)$, generate $\Omega_{X/S}$ over $\mathcal{O}_X$, then we can find $h_{c+1}, \ldots, h_d \in \mathcal{O}(X)$ such that the restrictions of $dh_{c+1}, \ldots, dh_d$ to Y form a base of $\Omega_{Y/S}$. After that we can decrease Y and find $h_1, \ldots, h_c \in \mathcal{J}(X)$ such that $dh_1, \ldots, dh_d$ form a base of $\Omega_{X/S}$, where $\mathcal{J}$ is the subsheaf of ideals in $\mathcal{O}_X$ that corresponds to Y. By Corollary 3.5.7, the induced morphism $h: X \to \mathbf{A}^d_S$ is étale. Let Z be the closed k-analytic subset of $\mathbf{A}^d_S$ defined by the equations $T_1 = \ldots = T_c = 0$, and let Y' be the inverse image of Z in X. By construction, Y is a closed k-analytic subset of Y'. Corollary 3.5.7 implies that the induced morphism $Y \to Z$ is étale. Therefore the closed immersion $i: Y \to Y'$ is étale (Corollary 3.3.9). Since it is an open map (Proposition 3.2.7), it follows that we can decrease X and assume that i is a homeomorphism. Finally, since the sheaf $i_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_{Y'}$-module and the homomorphism $\mathcal{O}_{Y'} \to i_*(\mathcal{O}_Y)$ is surjective, we have $\mathcal{O}_{Y'} \tilde{\to} i_*(\mathcal{O}_Y)$. It follows that i is an isomorphism. ∎

3.5.10. *Corollary.* — *In the situation of Proposition* 3.3.2 (ii) *suppose that f and g are smooth. Then there is an exact sequence*

$$0 \to \mathcal{J}_G/\mathcal{J}^2_G \to \varphi^*_G(\Omega_{X_G/S_G}) \to \Omega_{Y_G/S_G} \to 0.$$

In particular, $\mathcal{J}_G/\mathcal{J}^2_G$ is a locally free $\mathcal{O}_{Y_G}$-module whose rank is the codimension of Y *in* X. ∎

3.6. Smooth elementary curves

In this subsection we recall some results from [Ber] on the structure of the k-analytic curve $X = \mathcal{X}^{an}$ associated with a smooth geometrically connected projective curve $\mathcal{X}$ over k of genus $g \geqslant 0$, and we recall the Stable Reduction Theorem of Bosch and Lütkehbomert from [BL] which is actually the most important ingredient in the study of X. Furthermore, we introduce the notion of an *elementary triple* (X, Y, x) where Y is an open neighborhood of a point $x \in X$. (A k-analytic curve Y and pairs (X, Y) and (Y, x) for which such a triple exists will be called *elementary*.) And we show that any point of a smooth k-analytic curve has, after a finite separable extension of k, an elementary open neighborhood. First we consider the case of trivial valuation on k because it is very simple. But we remark that everything considered in the case of nontrivial valuation has the same meaning in the trivial valuation case.

Thus, *suppose that the valuation on k is trivial*. Then points of X are of the following three types. First, there is a canonical embedding of the set $\mathscr{X}_0$ of closed points of $\mathscr{X}$ in X, $\mathscr{X}_0 \stackrel{\sim}{\to} X_0 = \{ x \in X \mid [\mathscr{H}(x) : k] < \infty \}$. For $x \in X_0$ one has $\mathcal{O}_{X,x} = \widehat{\mathcal{O}_{\mathscr{X},\mathbf{x}}}$ and $\kappa(x) = \mathscr{H}(x) = k(\mathbf{x})$ ($\mathbf{x}$ is the image of x in $\mathscr{X}$). Furthermore, there is a *generic point* which corresponds to the trivial valuation on the field of rational functions $k(\mathscr{X})$. For this point x one has $\mathcal{O}_{X,x} = \kappa(x) = \mathscr{H}(x) = k(\mathbf{x}) = k(\mathscr{X})$. (The one-element set $\{x\}$ is denoted by $\Delta(X)$ and called the *skeleton* of X.) Finally, for any closed point a there is an interval which connects a with the generic point and which is parametrized by the unit interval [0, 1]. Namely, the point x associated with a number $0 < r < 1$ corresponds to the valuation on $k(\mathscr{X})$ which takes the value $r^{[\kappa(a):k]}$ on a local paremeter f at $\mathbf{a}$, i.e., $|f(x)| = r^{[\kappa(a):k]}$. This point x is denoted by $p(E(a, r))$. One has $\mathcal{O}_{X,x} = \kappa(x) = \mathscr{H}(x)$, and this field coincides with the fraction field of the ring $\widehat{\mathcal{O}_{\mathscr{X},\mathbf{a}}}$. We also set $E(a, r) = \{ y \in X \mid |f(y)| \leqslant r^{[\kappa(a):k]} \}$, $D(a, r) = \{ y \in X \mid |f(y)| < r^{[\kappa(a):k]} \}$ and

$$B(a; r, R) = \{ y \in X \mid r^{[\kappa(a):k]} < |f(y)| < R^{[\kappa(a):k]} \}.$$

The topology of X induces the usual topology on each of the intervals, and a basis of open neighborhoods of the generic point is formed by sets of the form $X \setminus \bigcup_{i=1}^m E(a_i, r_i)$, where $a_i \in X_0$ and $0 < r_i < 1$.

Let Y be an open neighborhood of a point $x \in X$. We say that the triple (X, Y, x) is *elementary* if one of the following is true:

a) $g = 0$, $x = a \in X(k)$ and $Y = D(a, r)$, where $0 < r < 1$;

b) $g = 0$, $x = p(E(a, r))$, where $a \in X(k)$ and $0 < r < 1$, and $Y = B(a; r', r'')$, where $0 < r' < r < r'' < 1$;

c) x is the generic point of X and $Y = X \setminus \coprod_{i=1}^m E(a_i, r_i)$, $m \geqslant 1$, where $a_i \in X(k)$ and $0 < r_i < 1$.

It is clear that for any open neighborhood Y of an arbitrary point $x \in X$ one can find a finite separable extension K of k and an open subset $Y'' \subset Y' = Y \hat{\otimes} K$ such that the point x has a unique preimage x' in Y'' and the triple (X', Y'', x') is elementary where $X' = X \hat{\otimes} K$.

Suppose that the valuation on k is nontrivial. As above, one has $\mathscr{X}_0 \stackrel{\sim}{\to} X_0$ and $\kappa(x) = \mathscr{H}(x) = k(\mathbf{x})$ for $x \in X_0$. But in this case the set X_0 is everywhere dense in X. For a point $x \in X \setminus X_0$ one has $\mathcal{O}_{X,x} = \kappa(x)$.

Consider first the case $\mathscr{X} = \mathbf{P}^1$. Let $a \in \mathbf{A}^1_0$ and $r > 0$. Then the real valued function on $k[\mathrm{T}]$

$$f \mapsto \max_i |\partial_i f(a)| r^i,$$

where $\partial_i(\Sigma\,\alpha_\nu \mathrm{T}^\nu) = \Sigma \binom{\nu+i}{i} \alpha_{\nu+i} \mathrm{T}^i$ $\left(\partial_i \text{ is the operator } \frac{1}{i!}\frac{d^i}{d\mathrm{T}^i}\right)$, is a multiplicative norm, and therefore it defines a point $x \in \mathbf{A}^1$ which is denoted by $p(\mathrm{E}(a, r))$. The notation tells that the point depends only on the closed disc $\mathrm{E}(a, r) = \{y \in \mathbf{A}^1 \mid |f(y)| \leqslant r^n\}$, where $f = \mathrm{T}^n + \alpha_1 \mathrm{T}^{n-1} + \ldots + \alpha_n$ is the monic generator of the maximal ideal of $k[\mathrm{T}]$ which corresponds to a. (We also define the open disc $\mathrm{D}(a, r) = \{y \in \mathbf{A}^1 \mid |f(y)| < r^n\}$ and the open annulus $\mathrm{B}(a; r, \mathrm{R}) = \{y \in \mathbf{A}^1 \mid r^n < |f(y)| < \mathrm{R}^n\}$. We remark that the radius r of the disc $\mathrm{E}(a, r)$ does not depend on the choice of the center a. If $r \in \sqrt{|k^*|}$ (such a point is said to be of *type (2)*), then the extension $\widetilde{\mathscr{H}(x)}/\widetilde{k}$ is finitely generated of transcendence degree one, and the group $|\mathscr{H}(x)^*|/|k^*|$ is finite. If $r \notin \sqrt{|k^*|}$ (such a point is said to be of *type (3)*), then the extension $\widetilde{\mathscr{H}(x)}/\widetilde{k}$ is finite and the group $|\mathscr{H}(x)^*|$ is generated by $|\kappa(a)^*|$ and r.

A more general construction of points of $\mathbf{A}^1$ is as follows. Let $\mathscr{E} = \{\mathrm{E}\}$ be a decreasing family of closed discs in $\mathbf{A}^1$. Then the real valued function on $k[\mathrm{T}]$

$$f \mapsto \inf_{\mathrm{E} \in \mathscr{E}} |f(p(\mathrm{E}))|$$

is a multiplicative seminorm, and therefore it defines a point $x = p(\mathscr{E}) \in \mathbf{A}^1$. By [Ber], 1.4.4, any point of $\mathbf{A}^1$ is obtained in this way. We set $\sigma = \bigcap_{\mathrm{E} \in \mathscr{E}} (\mathrm{E} \cap \mathbf{A}^1_0)$ and $r = \inf_{\mathrm{E} \in \mathscr{E}} r(\mathrm{E})$. Suppose first that $\sigma \neq \emptyset$. Then one does not obtain a new point. Namely, if $r = 0$, then $x \in \mathbf{A}^1_0$, and if $r > 0$, then $x = p(\mathrm{E}(a, r))$ for any $a \in \sigma$. Suppose therefore that $\sigma = \emptyset$. Then one obtains a new point. If $r = 0$, then x is the image of an element from $\widehat{k}^a \backslash k^a$ under the mapping $\mathbf{A}^1_{\widehat{k}^a} \to \mathbf{A}^1$. Points with $r > 0$ (they are said to be of *type (4)*) exist if and only if the field $\widehat{k}^a$ is not maximally complete. Points with $r = 0$ (for arbitrary σ) are said to be of *type (1)*. For a point x of type (1) or (4) the extension $\widetilde{\mathscr{H}(x)}/\widetilde{k}$ is algebraic and the group $|\mathscr{H}(x)^*|/|k^*|$ is torsion.

A basis of topology on $\mathbf{A}^1$ is formed by open sets of the form $\mathrm{D}(a, r) \backslash \bigcup_{i=1}^m \mathrm{E}(a_i, r_i)$. (We recall that any affinoid domain in $\mathbf{A}^1$ is a disjoint union of the standard affinoid domains which have the form $\mathrm{E}(a, r) \backslash \bigcup_{i=1}^m \mathrm{D}(a_i, r_i)$.) Let Y be an open neighborhood of a point $x \in \mathbf{A}^1$. We say that the triple $(\mathbf{P}^1, \mathrm{Y}, x)$ is *elementary* if one of the following is true:

a) x is of type (1) or (4) and $\mathrm{Y} = \mathrm{D}(a, r)$, where $a \in k$ and $r > 0$;

b) $x = p(\mathrm{E}(a, r))$, where $a \in k$ and $r \notin \sqrt{|k^*|}$, and $\mathrm{Y} = \mathrm{B}(a; r', r'')$, where $0 < r' < r < r''$;

c) $x = p(\mathrm{E}(a, r))$, where $a \in k$ and $r \in |k^*|$, and $\mathrm{Y} = \mathrm{D}(a, r') \backslash \coprod_{i=1}^m \mathrm{E}(a_i, r_i)$, $m \geqslant 0$, where $0 < r_i < r < r'$, $a_i \in k$, $|a_i - a| \leqslant r$ and $|a_i - a_j| = r$ for $i \neq j$.

It is clear that if $\mathscr{X}$ is of genus zero, then for any open neighborhood Y of a point $x \in X$ one can find a finite separable extension K of k and an open subset $Y'' \subset Y' = Y \hat{\otimes} K$ such that $\mathscr{X} \otimes K \xrightarrow{\sim} \mathbf{P}^1_K$, the point x has a unique preimage x' in Y'', and the triple $(\mathbf{P}^1_K, Y'', x')$ is elementary.

Consider now the case of an arbitrary smooth geometrically connected projective curve $\mathscr{X}$. Assume that the k-analytic curve X admits a distinguished formal covering $\mathscr{U}$ by strictly affinoid domains with $\widetilde{k}$-split semistable reduction $\widetilde{X} = \widetilde{X}_{\mathscr{U}}$, and let $\pi : X \to \widetilde{X}$ be the reduction map (see [Ber], § 4.3). Furthermore, let $\{ \mathscr{X}_i \}_{i \in I}$ be the irreducible components of $\widetilde{X}$. One has $g = b + \Sigma_{i \in I} g(\mathscr{X}_i)$, where b is the Betti number of the incidence graph $\Delta(\widetilde{X})$ of $\widetilde{X}$. For a point $\widetilde{x} \in \widetilde{X}$ one of the following possibilities holds ([Ber], 4.3.1).

(i) If $\widetilde{x}$ is the generic point of $\mathscr{X}_i$, then there exists a unique point $x_i \in X$ with $\pi(x_i) = \widetilde{x}$. One has $\widetilde{\mathscr{H}(x_i)} = \widetilde{k}(\mathscr{X}_i)$ ([Ber], 2.4.4 (ii)).

(ii) If $\widetilde{x}$ is a smooth closed point belonging to $\mathscr{X}_i$, then $\pi^{-1}(\widetilde{x})$ is a connected open set which becomes isomorphic, after a finite separable extension of k, to a disjoint union of a finite number of copies of the open unit disc with center at zero, and $\overline{\pi^{-1}(\widetilde{x})} = \pi^{-1}(\widetilde{x}) \cup \{ x_i \}$. If $\widetilde{x} \in \widetilde{X}(\widetilde{k})$, then $\pi^{-1}(\widetilde{x}) \xrightarrow{\sim} D(0, 1)$.

(iii) If $\widetilde{x}$ is a double point belonging to components $\mathscr{X}_i$ and $\mathscr{X}_j$ (which may coincide), then $\pi^{-1}(\widetilde{x}) \xrightarrow{\sim} B(0; r, 1)$, where $r \in | k^* |$ and $r < 1$, and $\overline{\pi^{-1}(\widetilde{x})} = \pi^{-1}(\widetilde{x}) \cup \{ x_i, x_j \}$.

One constructs as follows a closed subset $\Delta_{\mathscr{U}}(X) \subset X$ which has the structure of a finite graph and is isomorphic to the incidence graph $\Delta(\widetilde{X})$ (it is called the *skeleton of* X *with respect to the covering* $\mathscr{U}$). The vertices of $\Delta_{\mathscr{U}}(X)$ are the points x_i, $i \in I$. The edges of $\Delta_{\mathscr{U}}(X)$ correspond to the double points of $\widetilde{X}$ as follows. If $\widetilde{x}$ is a double point belonging to components $\mathscr{X}_i$ and $\mathscr{X}_j$, then the subset $\ell_{\widetilde{x}} \subset \pi^{-1}(\widetilde{x})$, which is the preimage of the set $\{ p(E(0, t)) \mid r < t < 1 \}$ under the isomorphism $\pi^{-1}(\widetilde{x}) \xrightarrow{\sim} B(0; r, 1)$, does not depend on the choice of the isomorphism, and the set $\bar{\ell}_{\widetilde{x}} = \ell_{\widetilde{x}} \cup \{ x_i, x_j \}$ is an edge of $\Delta_{\mathscr{U}}(X)$. There is a canonical (deformational) retraction $\tau : X \to \Delta_{\mathscr{U}}(X)$.

The reduction is said to be *good* if $\widetilde{X}$ is smooth. In this case $\Delta_{\mathscr{U}}(X)$ consists of one point x (the *generic point*) for which $\widetilde{\mathscr{H}(x)} = \widetilde{k}(\widetilde{X})$. The reduction is said to be *stable* if $\widetilde{X}$ is a stable curve. In this case, if $g \geqslant 2$ or if $g = 1$ and the reduction is good, then any other distinguished formal covering of X with stable reduction is equivalent to $\mathscr{U}$, and therefore the reduction map $X \to \widetilde{X}$ and the finite graph $\Delta(X) = \Delta_{\mathscr{U}}(X)$ do not depend on $\mathscr{U}$. If $g = 1$ and the reduction is bad, then the set $\Delta(X) = \Delta_{\mathscr{U}}(X)$ (without the structure of a graph) does not depend on $\mathscr{U}$. The graph $\Delta(X)$ is called the *skeleton of* X. The complement $X \backslash \Delta(X)$ is the set of points $x \in X$ which have an open neighborhood such that it becomes isomorphic, after a finite separable extension of k, to a disjoint union of a finite number of copies of the open unit disc with center at zero.

Assume that $g \geqslant 1$, and let Y be an open neighborhood of a point $x \in X$. We say that the triple (X, Y, x) is *elementary* if X has good reduction, x is the generic point of X, and $Y = X \backslash \coprod_{i=1}^m E_i$, $m \geqslant 1$, where E_i is an affinoid domain in $\pi^{-1}(\widetilde{x}_i)$, $\widetilde{x}_i \in \widetilde{X}(\widetilde{k})$, isomorphic to a closed disc $E(0, r_i)$, and the points $\widetilde{x}_1, \ldots, \widetilde{x}_m$ are pairwise different.

The Stable Reduction Theorem asserts that, for every smooth geometrically connected projective curve $\mathcal{X}$ over k of genus $g \geqslant 1$, there exists a finite separable extension K of k such that the K-analytic space $(\mathcal{X} \otimes \mathrm{K})^{\mathrm{an}}$ has stable reduction.

Finally, *we drop any assumptions*. A triple (X, Y, x) is said to be *elementary* if it is isomorphic to one of the elementary triples which were defined above. The following Proposition 3.6.1 is a particular case of the main result of the next subsection, Theorem 3.7.2, on the local structure of a smooth morphism of pure dimension one. The proof of Theorem 3.7.2 is essentially a generalization of the proof of Proposition 3.6.1.

3.6.1. *Proposition. — Let* Y *be a smooth k-analytic curve. Then for any point* $y \in \mathrm{Y}$ *there exist a finite separable extension* K *of k and an open subset* $\mathrm{Y}'' \subset \mathrm{Y}' = \mathrm{Y} \hat{\otimes} \mathrm{K}$ *such that the point* y *has a unique preimage* y' *in* Y'' *and the pair* (Y'', y') *is elementary.*

The following statement will be used also in § 7.3.

3.6.2. *Lemma. — Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a smooth morphism of pure dimension one with k-affinoid* $\mathrm{X} = \mathcal{M}(\mathcal{A})$. *Then for any point* $y \in \mathrm{Y}$ *there exist an open neighborhood* Y' *of* y *and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Y}' & \overset{j}{\hookrightarrow} & \mathcal{Y}^{\mathrm{an}} \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle \psi^{\mathrm{an}}} \\ & & \mathrm{X} \end{array}$$

where $\psi : \mathcal{Y} \to \mathcal{X} = \mathrm{Spec}(\mathcal{A})$ *is a smooth affine curve of finite type over* $\mathcal{X}$, *and* j *is an open immersion.*

Proof. — We can shrink Y and find an étale morphism $g : \mathrm{Y} \to \mathbf{A}^1_{\mathrm{X}}$. Let $\mathbf{z}$ denote the image of the point $z = g(y)$ in $\mathrm{A}^1_{\mathcal{X}}$. The field $k(\mathbf{z})$ is everywhere dense in $\kappa(z)$, and $\kappa(y)$ is a finite separable extension of $\kappa(z)$. By Proposition 2.4.1, there exists a finite separable extension K of $k(\mathbf{z})$ which embeds in $\kappa(y)$ and is everywhere dense in it. Take an arbitrary étale morphism of finite type between affine schemes $h : \mathcal{Y} \to \mathrm{A}^1_{\mathcal{X}}$ for which there exists a point $\mathbf{y} \in \mathcal{Y}$ with $h(\mathbf{y}) = \mathbf{z}$ and $k(\mathbf{y}) = \mathrm{K}$.

$$\begin{array}{ccccccc} \mathrm{Y} & \overset{g}{\longrightarrow} & \mathbf{A}^1_{\mathrm{X}} & \longrightarrow & \mathrm{A}^1_{\mathcal{X}} & \overset{h}{\longleftarrow} & \mathcal{Y} \\ & \underset{\varphi}{\searrow} & \downarrow & & \downarrow & \underset{\psi}{\swarrow} & \\ & & \mathrm{X} & \longrightarrow & \mathcal{X} & & \end{array}$$

The embedding of K in $\kappa(y)$ defines a point $y' \in \mathcal{Y}^{\mathrm{an}}$. Since K is everywhere dense in $\kappa(y)$, we have $\kappa(y) = \kappa(y')$. We get two étale morphisms of k-germs $(\mathrm{Y}, y) \to (\mathbf{A}^1_{\mathrm{X}}, z)$ and $(\mathcal{Y}^{\mathrm{an}}, y') \to (\mathbf{A}^1_{\mathrm{X}}, z)$ such that $\kappa(y) = \kappa(y')$. From Theorem 3.4.1 it follows that the k-germs (Y, y) and $(\mathcal{Y}^{\mathrm{an}}, y')$ are isomorphic. ■

Proof of Proposition **3.6.1.** — By Lemma 3.6.2, we can shrink Y and find an open embedding of Y in the analytification $\mathcal{X}'^{\mathrm{an}}$ of a smooth affine curve $\mathcal{X}'$ of finite type over k. Let $\mathscr{X}$ be the smooth projectivization of $\mathscr{X}'$. Increasing the field k, we may assume that $\mathscr{X}$ is geometrically connected. If the valuation on k is trivial or if the genus g of $\mathscr{X}$ is zero, then the statement is clear. Thus, assume that the valuation on k is nontrivial and $g \geqslant 1$. By the Stable Reduction Theorem, we can increase the field k and assume that $X = \mathscr{X}^{an}$ has $\widetilde{k}$-split stable reduction. If the point y is not a vertex of the skeleton $\Delta(X)$, then, after increasing of k, y has an open neighborhood in X (and therefore in Y) isomorphic to an open subset of $\mathbf{P}^1$. Assume therefore that y is a vertex of $\Delta(X)$.

First, we want to reduce the situation to the case when $\Delta(X) = \{y\}$. Let L be a connected open neighborhood of the point y in $\Delta(X)$ which does not contain loops and other vertices of $\Delta(X)$. One has $L = \{y\} \cup \bigcup_{i=1}^{m} \ell_i$, where ℓ_i is homeomorphic to an open interval. Furthermore, for each $1 \leqslant i \leqslant m$ there is an isomorphism $\tau^{-1}(\ell_i) \xrightarrow{\sim} B(0; r_i, 1)$, where $0 < r_i < 1$. We fix such an isomorphism so that the point $p(E(0, t))$ tends to the point y for $t \to 1$. We now glue the open set $\tau^{-1}(L)$ with m copies of $D(0, 1)$ via the isomorphisms $\tau^{-1}(\ell_i) \xrightarrow{\sim} B(0; r_i, 1) \subset D(0, 1)$. We get a new proper smooth k-analytic curve X which is the analytification of a smooth geometrically connected projective curve $\mathscr{X}$ of (new) genus $g \geqslant 0$. Suppose that $g \geqslant 1$. Increasing the field k, we may assume that X has stable reduction. Since any point $x \neq y$ has an open neighborhood such that, after a finite separable extension of k, it becomes isomorphic to a disjoint union of open discs, it follows that the reduction of X is good and y is the generic point of X.

Consider the reduction map $\pi : X \to \widetilde{X}$. Since y is a unique preimage of the generic point of $\widetilde{X}$, there exist $m \geqslant 1$ closed points $\widetilde{x}_1, \ldots, \widetilde{x}_m \in \widetilde{X}$ with $\pi^{-1}(\widetilde{X} \backslash \{\widetilde{x}_1, \ldots, \widetilde{x}_m\}) \subset Y$. Increasing the field k, we may assume that $\widetilde{x}_i \in \widetilde{X}(\widetilde{k})$, and therefore $\pi^{-1}(\widetilde{x}_i) \xrightarrow{\sim} D(0, 1)$. It follows that we can replace Y by a smaller open neighborhood of y of the form $X \backslash \bigcup_{i=1}^{m} E_i$, where $E_i \subset \pi^{-1}(\widetilde{x}_i)$ and $E_i \xrightarrow{\sim} E(0, r_i)$. ■

3.6.3. *Remark.* — (i) Let (Y, y) be an elementary pair. Then Y contains a disjoint union of $m \geqslant 1$ open annuli $B(r_i, R_i)$ such that the set $Y \backslash \bigcup_{i=1}^{m} B(r_i, R_i)$ is a connected compact neighborhood of the point y (it is actually an affinoid domain).

(ii) Suppose that the field k is algebraically closed. If (Y, y) is an elementary pair such that Y is not isomorphic to an open disc, then the open set $Y \backslash \{y\}$ is isomorphic to a disjoint union of a finite number of open annuli and an infinite number of open discs. From Proposition 3.6.1 it follows that any smooth k-analytic curve has a covering by elementary open subsets.

(iii) A broader class of smooth k-analytic curves is that of *standard* curves which are isomorphic to an open subset of the analytification of a smooth geometrically connected projective curve such that its complement is a disjoint union of $m \geqslant 1$ closed discs with center at zero. We remark that a standard curve is connected. We remark also that an elementary (resp. standard) curve remains elementary (resp. standard) after any extension of the ground field.

3.7. The local structure of a smooth morphism

3.7.1. *Definitions.* — (i) A morphism $\varphi : Y \to X$ is said to be an *elementary fibration of pure dimension one* (resp. *at a point* $y \in Y$) if it can be included to a commutative diagram

$$\begin{array}{ccccc} Y & \hookrightarrow & Z & \longleftarrow & V = \coprod_{i=1}^{m} (X \times E_i) \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\psi} & \swarrow{\scriptstyle pr} & \\ & & X & & \end{array}$$

such that

a) $\psi : Z \to X$ is a smooth proper morphism whose geometric fibres are irreducible curves of genus $g \geqslant 0$;

b) Y is an open subset of Z, and $V = Z \backslash Y$;

c) V is an analytic domain in Z isomorphic to a disjoint union $\coprod_{i=1}^{m} (X \times E_i)$, $m \geqslant 1$, where E_i are closed discs in $\mathbf{A}^1$ with center at zero, and *pr* is the canonical projection;

d) there exists an analytic domain $V \subset V'$ such that the isomorphism

$$V \xrightarrow{\sim} \coprod_{i=1}^{m}(X \times E_i)$$

from *c)* extends to an isomorphism $V' \xrightarrow{\sim} \coprod_{i=1}^{m}(X \times E'_i)$, where E'_i is a closed disc in $\mathbf{A}^1$ which contains E_i and has a bigger radius;

e) the pair (Z_x, Y_x) (resp. the triple (Z_x, Y_x, y)), where $x = \varphi(y)$, is elementary.

(ii) A morphism is said to be an *elementary fibration* if it is a composition of elementary fibrations of pure dimension one.

(iii) A morphism $\varphi : Y \to X$ is said to be *standard of pure dimension one* if everything from (i), except the property *e)*, is true for it. A composition of standard morphisms of pure dimension one is said to be *standard.*

We remark that the geometric fibers of a standard morphism are nonempty and connected. Furthermore, elementary fibrations (resp. standard morphisms) are preserved under any base change functor and under any ground field extension functor.

3.7.2. *Theorem.* — *Let* $\varphi : Y \to X$ *be a smooth morphism of pure dimension one, and suppose that* X *(and therefore* Y*) is good. Then for any point* $y \in Y$ *there exist an étale morphism* $f : X' \to X$ *and an open subset* $Y'' \subset Y' = Y \times_X X'$

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \\ \uparrow & \nearrow{\scriptstyle\varphi''} & \\ Y'' & & \end{array}$$

such that y *has a unique preimage* y' *in* Y'' *and* $\varphi'' : Y'' \to X'$ *is an elementary fibraiion of pure dimension one at the point* y'.

3.7.3. *Corollary.* — *Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a smooth morphism of good k-analytic spaces. Then for any point* $y \in \mathrm{Y}$ *there exist étale morphisms* $f : \mathrm{X}' \to \mathrm{X}$ *and* $g : \mathrm{Y}'' \to \mathrm{Y}' = \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}'$

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\varphi} & \mathrm{X} \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ \mathrm{Y}' & \xrightarrow{\varphi'} & \mathrm{X}' \\ \uparrow{\scriptstyle g} & \nearrow{\scriptstyle \varphi''} & \\ \mathrm{Y}'' & & \end{array}$$

such that $y \in f'(g(\mathrm{Y}''))$ *and* φ'' *is an elementary fibration.* ■

3.7.4. *Corollary.* — *A smooth morphism is an open map.* ■

Proof of Theorem **3.7.2.** — All the analytic spaces considered in the proof are assumed to be good. For numbers $0 < r' \leqslant r''$ (resp. $0 < r' < r''$) we denote by $\mathrm{A}(r', r'')$ (resp. $\mathrm{B}(r', r'')$) the closed (resp. open) annulus $\mathrm{E}(0, r'')\backslash\mathrm{D}(0, r')$ (resp. $\mathrm{D}(0, r'')\backslash\mathrm{E}(0, r')$) with center at zero.

3.7.5. *Proposition.* — *Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a separated smooth morphism of pure dimension one, and let* $x \in \mathrm{X}$. *Suppose that the fibre* Y_x *is isomorphic to the open annulus* $\mathrm{B}(r, \mathrm{R})_{\mathscr{H}(x)}$. *Then there exist numbers* $r < r' < \mathrm{R}' < \mathrm{R}$, *an open neighborhood* $\mathscr{U}$ *of* x *and an open subset* $\mathscr{V} \subset \varphi^{-1}(\mathscr{U})$ *such that*

a) $\mathscr{V}_x$ *coincides with* $\mathrm{B}(r', \mathrm{R}')_{\mathscr{H}(x)}$ *under the identification of* Y_x *with* $\mathrm{B}(r, \mathrm{R})_{\mathscr{H}(x)}$;
b) $\mathscr{V}$ *is isomorphic to the direct product* $\mathscr{U} \times \mathrm{B}(r', \mathrm{R}')$ *over* $\mathscr{U}$.

3.7.6. *Lemma.* — *Let* Y *be an open subset of* $\mathbf{A}^1_{\mathrm{X}}$, *and suppose that*

$$\mathrm{Y}_x = \mathrm{D}(0, r)_{\mathscr{H}(x)}\backslash \bigcup_{i=1}^{m} \mathrm{E}(a_i, r_i)_{\mathscr{H}(x)}, \quad m \geqslant 0,$$

where $r_i < r$ *and* $|\mathrm{T}(a_i)| < r$ *(*T *is the coordinate function on* $\mathbf{A}^1$*). Then*

(i) *if* $m = 0$, *then for any number* $0 < r' < r$ *there exists an open neighborhood* $\mathscr{U}$ *of* x *such that* Y *contains the direct product* $\mathscr{U} \times \mathrm{D}(0, r')$.

(ii) *if* $m \geqslant 1$, *then for any numbers* $\max_i(|\mathrm{T}(a_i)|, r_i) < r' < r'' < r$, *there exists an open neighborhood* $\mathscr{U}$ *of* x *such that* Y *contains the direct product* $\mathscr{U} \times \mathrm{B}(r', r'')$.

Proof. — In the case (i) (resp. (ii)) the intersection of all compact sets of the form $\mathrm{U} \times \mathrm{E}(0, r')$ (resp. $\mathrm{U} \times \mathrm{A}(r', r'')$), where U runs through compact neighborhoods of the point x, coincides with $\mathrm{E}(0, r')_{\mathscr{H}(x)}$ (resp. $\mathrm{A}(r', r'')_{\mathscr{H}(x)}$), and therefore it is contained in the open set Y. It follows that there exists a compact neighborhood U of x with $\mathrm{U} \times \mathrm{E}(0, r') \subset \mathrm{Y}$ (resp. $\mathrm{U} \times \mathrm{A}(r', r'') \subset \mathrm{Y}$). ■

3.7.7. *Lemma.* — *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow^{\varphi} \quad \swarrow^{\psi} & \\ & X & \end{array}$$

with smooth φ *and* ψ *of the same pure dimension, and let* $y \in Y$ *and* $x = \varphi(y)$. *Suppose that the induced morphism* $f_x : Y_x \to Z_x$ *is a local isomorphism at the point* y. *Then* f *is a local isomorphism at* y.

Proof. — Since Y_x is the fibre product of Y and Z_x over Z, the inverse image of the sheaf of differentials of f on Y_x coincides with that of f_x. But the latter sheaf is equal to zero at the point y. Therefore we may decrease Y and assume that f is unramified. By Proposition 3.3.2 (i), the canonical homomorphism $f^*(\Omega_{Z/X}) \to \Omega_{Y/X}$ is surjective. Since both $\mathcal{O}_Y$-modules are locally free and of the same rank, we have $f^*(\Omega_{Z/X}) \widetilde{\to} \Omega_{Y/X}$. From Proposition 3.5.3 (ii) it follows that f is étale at the point y. By hypothesis, f induces an isomorphism of fields $\mathcal{H}(z) \widetilde{\to} \mathcal{H}(y)$. By Proposition 2.4.1, $\kappa(z) \widetilde{\to} \kappa(y)$, and therefore f is a local isomorphism at y, by Theorem 3.4.1. ∎

Proof of Proposition **3.7.5.** — We may assume that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. Take a number $r < t < R$ and set $y = p(E(0, t)) \in Y_x$. Let V be an affinoid neighborhood of y in Y. Shrinking X and Y, we may assume that the fibre V_x is connected. Then

$$V_x = E(0, R')_{\mathcal{H}(x)} \setminus \bigcup_{i=1}^{m} D(a_i, r_i)_{\mathcal{H}(x)},$$

where $t < R' < R$, $r_i < R'$ and $|T(a_i)| < R'$ (T is the coordinate function on $B(r, R)_{\mathcal{H}(x)}$). Let $V = \mathcal{M}(\mathcal{B})$. Since the image of $\mathcal{B} \otimes_{\mathcal{A}} \kappa(x)$ in $\mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{H}(x)$ is everywhere dense, we can shrink X and assume that there exists an element $f \in \mathcal{B}$ such that the norm of $f - T$ in $\mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{H}(x)$ is less than $\min_i (r_i, t)$. It follows that the morphism $f: V \to \mathbf{A}^1_X$ induces an isomorphism $V_x \widetilde{\to} V_x$ and $f(y) = p(E(0, t))$. Take an open neighborhood $\mathcal{V}$ of the point y such that $\mathcal{V} \subset V$. Then the induced morphism $f: \mathcal{V} \to \mathbf{A}^1_X$ satisfies the hypothesis of Lemma 3.7.7, and therefore f is a local isomorphism at the point y. The required statement now follows from Lemma 3.7.6. ∎

3.7.8. *Proposition.* — *Let* $\varphi : Y \to X$ *be a smooth morphism of pure dimension one, and let* $x \in X$. *Suppose that the fibre* Y_x *is isomorphic to the open disc* $D(0, R)_{\mathcal{H}(x)}$ *and that the morphism* φ *is a composition of an open immersion* $Y \hookrightarrow Z$ *with a compact morphism* $\psi : Z \to X$. *Then for any* $0 < r < R$ *there exist an open neighborhood* $\mathcal{U}$ *of* x *and an open subset* $\mathcal{V} \subset \varphi^{-1}(\mathcal{U})$ *such that*

a) $\mathcal{V}_x$ *coincides with* $D(0, r)_{\mathcal{H}(x)}$ *under the identification of* $\mathcal{W}_x$ *with* $D(0, R)_{\mathcal{H}(x)}$;

b) $\mathcal{V}$ *is isomorphic to* $\mathcal{U} \times D(0, r)$ *over* $\mathcal{U}$.

Let P and Q be Hausdorff topological spaces, and let $P_1 \subset P$ and $Q_1 \subset Q$ be their open subsets for which there is a homeomorphism $f: P_1 \widetilde{\to} Q_1$. The topological space, which is obtained by gluing P and Q via f, will be denoted by $P \cup_f Q$. The following statement is trivial.

3.7.9. *Lemma.* — *Suppose that there exists an open subset* $P' \subset P$ *such that* $\overline{P}_1 \subset P'$ *and the space* $P' \cup_f Q$ *is Hausdorff. Then the space* $P \cup_f Q$ *is Hausdorff. Furthermore, suppose in addition that* $P \backslash P_1$ *is compact and* $Q \xrightarrow{\sim} P_1 \cup_f Q$ *is relatively compact in* $P' \cup_f Q$. *Then the space* $P \cup_f Q$ *is compact.* ∎

Proof of Proposition **3.7.8.** — We may assume that X is k-affinoid. In particular, Z is Hausdorff and compact. From Proposition 3.7.5 it follows that we can shrink X and find an open subset $\mathscr{V} \subset Y$ such that $\mathscr{V}_x = B(r', R')_{\mathscr{H}(x)}$ and $\mathscr{V} \xrightarrow{\sim} X \times B(r', R')$ for some $r < r' < R' < R$. Take numbers $r' < r'' < t < R'' < R'$ and denote by $\mathscr{W}$ the open subset of $\mathscr{V}$ which corresponds to $X \times B(r'', R'')$. The subset $\Sigma \subset \mathscr{W}$, which corresponds to $X \times A(t, t)$, is compact because X is compact, and therefore $Z \backslash \Sigma$ is an open subset of Z. We apply the above gluing procedure to the spaces $P = Z \backslash \Sigma$, $P' = \mathscr{V} \backslash \Sigma$, $P_1 = \mathscr{W} \backslash \Sigma$, $Q = X \times (\mathbf{P}^1 \backslash E(0, r'')) \amalg X \times D(0, R'')$,

$$Q_1 = X \times B(r'', t) \amalg X \times B(t, R''),$$

and to the canonical isomorphism $f : P_1 \xrightarrow{\sim} Q_1$. One has

$$P' \cup_f Q \xrightarrow{\sim} X \times (\mathbf{P}^1 \backslash E(0, r')) \amalg X \times D(0, R').$$

From Lemma 3.7.9 it follows that the space $Z' = P \cup_f Q$ is Hausdorff and compact. In particular, the morphism $\psi' : Z' \to X$ is compact.

Furthermore, the above construction restricted to the fibre at x gives a compact $\mathscr{H}(x)$-analytic space Z'_x such that the connected component of $D(0, R)_{\mathscr{H}(x)}$ is isomorphic to the projective line $\mathbf{P}^1_{\mathscr{H}(x)}$. Moreover, the morphism ψ' is smooth at all points of this connected component. Since ψ' is compact, we can shrink X and replace Z' by the connected component of $D(0, R)_{\mathscr{H}(x)}$ in Z' so that the morphism ψ' becomes proper smooth of pure dimension one. It has a section $\sigma : X \to Z'$ defined by the point infinity of the disc $\mathbf{P}^1 \backslash E(0, r')$. One has $\dim_{\mathscr{H}(x)} H^0(Z'_x, \mathcal{O}_{Z'_x}) = 1$ and $H^1(Z'_x, \mathcal{O}_{Z'_x}) = 0$. By the Semi-continuity Theorem ([Ber], 3.3.11), we can shrink X and assume that the same is true for all $x' \in X$. It follows that all the fibres of ψ' are isomorphic to the projective line.

Finally, let L be the invertible sheaf on Z' which corresponds to $\sigma(X)$, and let $L_{x'}$ denote the inverse image of L on the fibre $Z'_{x'}$, $x' \in X$. Then $H^1(Z'_{x'}, L_{x'}) = 0$ and $\dim H^0(Z'_{x'}, L_{x'}) = 2$. It follows that $\psi'_*(L)$ is a locally free $\mathcal{O}_X$-module of rank two. Shrinking X, we may assume that $\psi'_*(L)$ is free. In this case it defines a morphism $Z' \to \mathbf{P}^1_X$ over X which is evidently an isomorphism. The required statement now follows from Lemma 3.7.6 (i). ∎

We are now ready to prove the theorem. First of all we remark that if K is a finite separable extension of the field $\kappa(x)$, then there exists an étale morphism $f : X' \to X$ such that $f^{-1}(x) = \{x'\}$ and $\kappa(x') = K$. The preimage of the point y under the morphism

$f' : Y' = Y \times_X X' \to Y$ is a nonempty finite set of points. Fixing one of them, say y', we can replace φ by a morphism $Y'' \to X'$, where Y'' is an open neighborhood of y' which does not contain other points from $f'^{-1}(y)$. If we make the above procedure, we say for brevity that we increase the field $\kappa(x)$. Furthermore, we may assume that the space X is k-affinoid.

Step 1. — *One can increase the field* $\kappa(x)$ *so that* φ *is included in a commutative diagram*

$$\begin{array}{ccc} Y & \overset{j}{\hookrightarrow} & Z \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle \psi} \\ & & X \end{array}$$

where

a) ψ *is a smooth proper morphism of pure dimension one whose geometric fibres are irreducible curves of genus* $g \geq 0$, *and* j *is an open immersion;*

b) if $g \geq 1$, *then* Z_x *has good reduction and* y *is the generic point of* Z_x.

Let Z be an affinoid neighborhood of the point y. Replacing Y by a small open neighborhood of y which is contained in Z, we may assume that the morphism $\varphi : Y \to X$ is a composition of an open immersion $Y \hookrightarrow Z$ with a compact morphism $\psi : Z \to X$. By Proposition 3.6.1, we can increase the field $\kappa(x)$ and assume that the pair (Y_x, y) is elementary. In particular, Y_x contains a disjoint union of $m \geq 1$ open annuli $B(r_i, R_i)_{\mathscr{H}(x)}$ such that the set $Y_x \backslash \coprod_{i=1}^m B(r_i, R_i)_{\mathscr{H}(x)}$ is a connected compact neighborhood of the point y (Remark 3.6.3 (i)). By Proposition 3.7.5, we can shrink X, Y and the annuli and assume that there are pairwise disjoint open subsets $\mathscr{V}_i \subset Y$, $1 \leq i \leq m$, such that $\mathscr{V}_{i,x} = B(r_i, R_i)_{\mathscr{H}(x)}$ and $\mathscr{V}_i \xrightarrow{\sim} X \times B(r_i, R_i)$. We take numbers $r_i < r'_i < t_i < R'_i < R_i$ and denote by $\mathscr{W}_i$ the open subset of $\mathscr{V}_i$ which corresponds to $X \times B(r'_i, R'_i)$. The set $\Sigma_i \subset \mathscr{W}_i$, which corresponds to $X \times A(t_i, t_i)$, is compact because X is compact, and therefore $Z \backslash \bigcup_{i=1}^m \Sigma_i$ is an open subset of Z. We apply the gluing procedure to the spaces $P = Z \backslash \bigcup_{i=1}^m \Sigma_i$, $P' = \bigcup_{i=1}^m (\mathscr{V}_i \backslash \Sigma_i)$, $P_1 = \bigcup_{i=1}^m (\mathscr{W}_i \backslash \Sigma_i)$,

$$Q = \coprod_{i=1}^m (X \times (\mathbf{P}^1 \backslash E(0, r'_i)) \amalg X \times D(0, R'_i)),$$

$$Q_1 = \coprod_{i=1}^m (X \times B(r'_i, t_i) \amalg X \times B(t_i, R'_i)),$$

and to the canonical isomorphism $f : P_1 \xrightarrow{\sim} Q_1$. One has

$$P' \cup_f Q \xrightarrow{\sim} \coprod_{i=1}^m (X \times (\mathbf{P}^1 \backslash E(0, r_i)) \amalg X \times D(0, R_i)).$$

From Lemma 3.7.9 it follows that the space $Z' = P \cup_f Q$ is Hausdorff and compact. In particular, the morphism $\psi' : Z' \to X$ is compact.

Furthermore, the above construction restricted to the fibre at x gives a compact $\mathscr{H}(x)$-analytic space Z'_x such that the connected component of the point y is a smooth geometrically connected proper $\mathscr{H}(x)$-analytic curve. Moreover, ψ' is smooth at all points of this connected component. Since ψ' is compact, we can shrink X and replace Z' by the connected component of the point y so that the morphism ψ' becomes proper smooth of pure dimension one and the fibre Z'_x becomes a smooth geometrically connected proper $\mathscr{H}(x)$-analytic curve. The point infinity of $\mathbf{P}^1\backslash E(0, r'_i)$ or the point zero of $E(0, R'_i)$ defines a section $\sigma : X \to Z'$. From the construction it follows that we can increase the field $\kappa(x)$ and assume that the curve Z'_x has good reduction. Moreover, if the genus g of Z'_x is positive, then y is the generic point of Z'_x.

Finally, one has $\dim_{\mathscr{H}(x)} H^0(Z'_x, \mathcal{O}_{Z'_x}) = 1$ and $\dim_{\mathscr{H}(x)} H^1(Z'_x, \mathcal{O}_{Z'_x}) = g$. By the Semicontinuity Theorem, we can shrink X and assume that the above equalities hold for all points $x' \in X$. In particular, the fibres of ψ' are connected. They are geometrically connected because ψ' is smooth and has a section.

Step 2. — *One can increase the field* $\kappa(x)$ *and shrink* Y *so that the morphism* $\varphi : Y \to X$ *is an elementary fibration at* y.

We can increase the field $\kappa(x)$ and assume that the triple (Z_x, Y_x, y) is elementary. In particular, $Y_x = Z_x \backslash \coprod_{i=1}^m E_i$, where $E_i \xrightarrow{\sim} E(0, R_i)_{\mathscr{H}(x)}$, $R_i < 1$, and there are bigger open subsets $E_i \subset D_i \subset Z_x$ with $D_i \xrightarrow{\sim} D(0, R'_i)_{\mathscr{H}(x)}$, $R_i < R'_i < 1$. From Proposition 3.7.8 it follows that we can shrink X and find pairwise disjoint open subsets $\mathscr{V}_i \subset Z$ such that $\mathscr{V}_{i,x} = D(0, t'_i)_{\mathscr{H}(x)}$ and $\mathscr{V}_i \xrightarrow{\sim} X \times D(0, t'_i)$, where $R_i < t'_i < R'_i$. Take numbers $R_i < t_i < t'_i$ and set $\mathscr{W}_i = X \times D(0, t_i)$. Since $(Z\backslash \bigcup_{i=1}^m \mathscr{W}_i)_x \subset Y_x$, we can shrink X and assume that $Z \backslash \bigcup_{i=1}^m \mathscr{W}_i \subset Y$. Finally, we take numbers $R_i < r_i < t_i$ and set $V = \coprod_{i=1}^m (X \times E(0, r_i))$. By construction, $Z\backslash V \subset Y$. Therefore we can replace Y by $Z\backslash V$ so that all the conditions of Definition 3.7.1 (i) hold. (The condition *d)* holds for $V' = \coprod_{i=1}^m (X \times E(0, r'_i))$, where $r_i < r'_i < t_i$.) The theorem is proved. ■

The following remarks will not be used in the sequel.

3.7.10. *Remarks.* — (i) One can add to the definition of an elementary fibration 3.7.1 (i) the condition that all the fibres of the morphism ψ have stable reduction. Indeed, taking an integer $n \geqslant 3$ which is prime to char $(\widetilde{k})$, one can increase the field $\kappa(x)$ and assume that all the points of order n on the Jacobian of the curve Z_x are rational. One can then shrink X so that, for any $x' \in X$, all the points of order n on the Jacobian of $Z_{x'}$ are rational, and therefore $Z_{x'}$ has stable reduction.

(ii) One can show that if $X = \mathscr{M}(\mathscr{A})$ is k-affinoid, then the space Z from Definition 3.7.1 (i) is the analytification $\mathscr{Z}^{\mathrm{an}}$ of a smooth projective curve $\mathscr{Z}$ over $\mathscr{X} = \mathrm{Spec}(\mathscr{A})$.

(iii) One can give a similar local description of a flat morphism of good k-analytic spaces $\varphi : Y \to X$ at a point $y \in Y$ such that the fibre Y_x, $x = \varphi(y)$, is a curve smooth outside the point y.

(iv) We think that the theorem 3.7.2 is true without the assumption that the space X (and therefore Y) is good. At least this is so if each point of X has an open neighborhood which is isomorphic to an analytic domain in a k-affinoid space (see Remark 1.4.3 (ii)). Indeed, shrinking X and Y, we may assume that X is a special domain in a k-affinoid space X′ and that φ factorizes through an étale morphism $Y \to \mathbf{A}^1_X$. Since $\mathbf{A}^1_X$ is an analytic domain in $\mathbf{A}^1_{X'}$, then, by Corollary 3.4.2, we can shrink Y and assume that the étale morphism $Y \to \mathbf{A}^1_X$ is a base change of an étale morphism $Y' \to \mathbf{A}^1_{X'}$, i.e., the morphism φ is the base change of a smooth morphism $\varphi' : Y' \to X'$ with respect to $X \to X'$. Then the validity of the theorem 3.7.2 for φ' implies its validity for φ.

§ 4. Étale cohomology

4.1. Étale topology on an analytic space

The *étale topology* on a k-analytic space X is the Grothendieck topology on the category Ét(X) generated by the pretopology for which the set of coverings of $(U \to X) \in \text{Ét}(X)$ is formed by the families $\{ U_i \xrightarrow{f_i} U \}_{i \in I}$ such that $U = \bigcup_{i \in I} f_i(U_i)$. We denote by $X_{\text{ét}}$ the site obtained in this way (the *étale site* of X) and by $X^{\sim}_{\text{ét}}$ the category of sheaves of sets on $X_{\text{ét}}$ (the *étale topos* of X). Furthermore, we denote by S(X) (resp. $S(X, \Lambda)$) the category of abelian sheaves (resp. sheaves of Λ-modules) on $X_{\text{ét}}$ and by D(X) (resp. $D(X, \Lambda)$) the corresponding derived category. For a sheaf F on $X_{\text{ét}}$ we often say that F is a sheaf on X. The cohomology groups of an abelian sheaf $F \in S(X)$ will be denoted by $H^q(X, F)$.

Any morphism $\varphi : Y \to X$ of analytic spaces over k induces a morphism of sites $Y_{\text{ét}} \to X_{\text{ét}}$. If $\mathcal{X}$ is a scheme of locally finite type over $\text{Spec}(\mathcal{A})$, where $\mathcal{A}$ is a k-affinoid algebra, then Propositions 2.6.8 and 3.3.11 imply that there is a morphism of sites $(\mathcal{X}^{\text{an}})_{\text{ét}} \to \mathcal{X}_{\text{ét}}$, where $\mathcal{X}_{\text{ét}}$ is the étale site of the scheme $\mathcal{X}$. The inverse image of a sheaf $\mathcal{F} \in \mathcal{X}^{\sim}_{\text{ét}}$ on $\mathcal{X}^{\text{an}}$ will be denoted by $\mathcal{F}^{\text{an}}$.

Our first purpose is to show that certain reasonable presheaves on $X_{\text{ét}}$ are actually sheaves. For this we introduce a (big) *flat quasifinite site* X_{fqf} of X. This is the site with the underlying category $k\text{-}\mathcal{A}n_{/X}$ of k-analytic spaces over X and with the Grothendieck topology generated by the pretopology for which the set of coverings of $(Y \to X) \in k\text{-}\mathcal{A}n_{/X}$ is formed by the families $\{ Y_i \xrightarrow{f_i} Y \}_{i \in I}$ such that the f_i are flat quasifinite and $Y = \bigcup_{i \in I} f_i(Y_i)$. There is an evident morphism of sites $X_{\text{fqf}} \to X_{\text{ét}}$. It is clear that if a presheaf on X_{fqf} is a sheaf, then its restriction on $X_{\text{ét}}$ is also a sheaf.

4.1.1. *Lemma. — The following conditions are sufficient for a presheaf* F *on* X_{fqf} *to be a sheaf*:

(1) *for any k-analytic space* Y *over* X *the restriction of* F *to the* G*-topology on* Y *is a sheaf*;

(2) *for any finite faithfully flat morphism of k-affinoid spaces* $Z \to Y$ *over* X *the sequence* $F(Y) \to F(Z) \rightrightarrows F(Z \times_Y Z)$ *is exact.*

Proof. — Let $\mathcal{U} = (U_i \xrightarrow{\varphi_i} Y)_{i \in I}$ be a covering in X_{fqf}. We have to show that the sequence

$$(*) \qquad F(Y) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_Y U_j)$$

is exact. For every $i \in I$ we take an open covering $\{V_j\}_{j \in J_i}$ of U_i such that the induced morphisms $V_j \to \varphi_i(V_j)$ are finite. From (1) it follows that if $(*)$ is exact for the covering $\{V_j\}_{j \in J}$, where $J = \bigcup_{i \in I} J_i$, then $(*)$ is exact for $\mathcal{U}$. So we may assume that all the morphisms $U_i \to \varphi_i(U_i)$ are finite. Furthermore, since $(*)$ is exact for the open covering $\{\varphi_i(U_i)\}_{i \in I}$ of Y, it suffices to show that the sequence

$$F(Y) \to F(Z) \rightrightarrows F(Z \times_Y Z)$$

is exact for any finite faithfully flat morphism $\varphi : Z \to Y$ over X. Finally, if $\mathcal{V} = \{V_i\}_{i \in I}$ is a quasinet of affinoid domains on Y, then $\mathcal{W} = \{\varphi^{-1}(V_i)\}_{i \in I}$ is a quasinet of affinoid domains on Z and, by (1), the sequences $(*)$ for the coverings $\mathcal{V}$ and $\mathcal{W}$ of Y and Z, respectively, are exact. Thus, the situation is reduced to the case of k-affinoid spaces. But the required fact in this case is guaranteed by (2). ■

4.1.2. *Corollary. — Let* F *be a coherent sheaf on* X_G. *Then the presheaf* $\tilde{F}$, *which assigns to a* k*-analytic space* Y *over* X *the group of global sections of the inverse image of* F *on* Y_G, *is a sheaf on* X_{fqf} *and therefore on* $X_{\mathrm{\acute{e}t}}$. ■

4.1.3. *Proposition. — A presheaf representable by a* k*-analytic space good over* X *is a sheaf on* X_{fqf} *and therefore on* $X_{\mathrm{\acute{e}t}}$.

Proof. — Let F be representable by a k-analytic space X′ over X, i.e., $F(Y) = \mathrm{Hom}_X(Y, X')$. The condition (1) holds for F, by Proposition 1.3.2. Let $Y = \mathcal{M}(\mathcal{B})$ and $Z = \mathcal{M}(\mathcal{C})$. Then $\mathcal{C}$ is a faithfully flat $\mathcal{B}$-algebra. It is well known that in this situation the sequence $0 \to \mathcal{B} \to \mathcal{C} \rightrightarrows \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$ is exact. Since $\mathcal{C}$ is a finite Banach $\mathcal{B}$-algebra, we have $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} = \mathcal{C} \hat{\otimes}_{\mathcal{B}} \mathcal{C}$, and the above sequence is admissible. It follows that the condition (2) holds, at least, for k-affinoid X′. In the general case we have to show that for any $g : Z \to X'$ over X with $g \circ p_1 = g \circ p_2$, where p_i are the canonical projections $Z \times_Y Z$, there exists a unique $f: Y \to X'$ over X with $g = f \circ \varphi$ (φ is $Z \to Y$).

Uniqueness of f. (Here the assumption that X′ is good over X is not used.) Assume that we are given $f_1, f_2 : Y \to X'$ with $f_1 \circ \varphi = f_2 \circ \varphi$. Since φ is surjective, f_1 and f_2 coincide as maps of topological spaces. Let U be an affinoid domain in X′. Then $f_1^{-1}(U) = f_2^{-1}(U) = \bigcup_{i=1}^n V_i$ for some affinoid domains $V_i \subset Y$. Applying the particular case to the morphisms $\varphi^{-1}(V_i) \to V_i$ and the k-affinoid space U, we get $f_1|_{V_i} = f_2|_{V_i}$. Since Y is covered by a finite number of such V_i, it follows that $f_1 = f_2$.

Existence of f. We can replace X by Y and X′ by $Y \times_X X'$ and assume that X′ is good. By the uniqueness, it suffices to construct f locally. Let $y \in Y$, $z \in \varphi^{-1}(y)$, U an affinoid neighborhood of $g(z)$ in X′. Since φ is an open map, $\varphi(g^{-1}(U))$ contains an affinoid neighborhood V of the point y. We claim that $\varphi^{-1}(V) \subset g^{-1}(U)$. Indeed, if $\varphi(z_1) = \varphi(z_2)$, then there exists $z' \in Z \times_Y Z$ with $p_1(z') = z_1$ and $p_2(z') = z_2$. If $z_1 \in g^{-1}(U)$, then $g(z_1) = gp_1(z') = gp_2(z') = g(z_2) \in U$. And so the situation is reduced to the morphism $\varphi^{-1}(V) \to V$ and the k-affinoid space U. ■

4.1.4. *Corollary.* — (i) *There is a fully faithful functor* $\text{Ét}(X) \to X^{\sim}_{\text{ét}}$. *In particular, the étale topology on* Ét(X) *is weaker than the canonical one.*

(ii) *If a sheaf* F *is representable by an* X*-space* T *étale over* X, *then for any morphism* $\varphi : Y \to X$ *the sheaf* $\varphi^* F$ *on* Y *is representable by the space* $T \times_X Y$. ■

4.1.5. *Remark.* — The Proposition 4.1.3 certainly should be true without the assumption that the space X′ is good over X. For this it would be enough to know that for a finite étale morphism of k-affinoid spaces $\varphi : Y \to X$ and for any affinoid domain $V \subset Y$ the image $\varphi(V)$ is a finite union of affinoid domains in X. If everything is strictly k-affinoid, this is a particular case of a result of Raynaud that gives the same fact for an arbitrary flat morphism.

4.1.6. *Example.* — (i) Let Λ be a set. Then the k-analytic space $\coprod_{\lambda \in \Lambda} X$ over X represents the constant sheaf Λ_X.

(ii) The *sheaf of abelian groups* $G_{a,X}$ is defined by $G_{a,X}(Y) = \mathcal{O}(Y)$.

(iii) The *sheaf of multiplicative groups* $G_{m,X}$ is defined by $G_{m,X}(Y) = \mathcal{O}(Y)^*$.

(iv) The *sheaf of* n-th *roots of unity* $\mu_{n,X}$ is defined by $\mu_{n,X}(Y) = \{f \in \mathcal{O}(Y)^* \mid f^n = 1\}$. If the field k contains all n-th roots of unity and n is prime to char(k), then the sheaf $\mu_{n,X}$ is isomorphic to the constant sheaf $(\mathbf{Z}/n\mathbf{Z})_X$.

4.1.7. *Proposition.* — (i) *The* Kummer sequence

$$0 \to \mu_{n,X} \to G_{m,X} \xrightarrow{f \mapsto f^n} G_{m,X} \to 0$$

is exact in $S(X_{\text{fqf}})$. *If n is prime to* char(k), *then it is exact also in* S(X).

(ii) *If* $p = \text{char}(k) > 0$, *then the* Artin-Schreier sequence

$$0 \to (\mathbf{Z}/p^n\,\mathbf{Z})_X \to G_{a,X} \xrightarrow{f \mapsto f^{p^n} - f} G_{a,X} \to 0$$

is exact in S(X) *(and therefore in* $S(X_{\text{fqf}})$*).*

Proof. — (i) It suffices to show that for a compact k-analytic space X and an element $f \in \mathcal{O}(Y)^*$ there exists a flat quasifinite morphism $Y \to X$ such that the image of f in $\mathcal{O}(Y)^*$ is in $\mathcal{O}(Y)^{*n}$. Let $\{U_i\}_{i \in I}$ be a finite affinoid covering of X. Then $\mathcal{B}_i := \mathcal{A}_{U_i}[T]/(T^n - 1)$ is a finite Banach $\mathcal{A}_{U_i}$-algebra, and the morphism $\varphi_i : V_i = \mathcal{M}(\mathcal{B}_i) \to U_i$ is finite flat (étale if n is prime to char(k)). Furthermore, for any pair $i, j \in I$, there is a canonical isomorphism of special domains $\nu_{ij} : V_{ij} := \varphi_i^{-1}(U_i \cap U_j) \xrightarrow{\sim} V_{ji} := \varphi_j^{-1}(U_i \cap U_j)$, and one has $V_{ii} = V_i$,

$$\nu_{ij}(V_{ij} \cap V_{il}) = V_{ji} \cap V_{jl}$$

and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $V_{ij} \cap V_{il}$. Therefore we can glue all V_i along V_{ij}, and we get a compact k-analytic space Y and a flat (étale if n is prime to char(k)) finite morphism $Y \to X$. The image of T in each $\mathcal{B}_i$ defines an element $g \in \mathcal{O}(Y)^*$ with $g^n = f$. (ii) is proved in the same way. ■

The cohomology groups of Λ_X (if Λ is an abelian group), $G_{a,X}$, $G_{m,X}$ and $\mu_{n,X}$ will be denoted by $H^q(X, \Lambda)$, $H^q(X, G_a)$, $H^q(X, G_m)$ and $H^q(X, \mu_n)$, respectively. The first Čech cohomology set $\check{H}^1(X, F)$ can be defined for any sheaf of groups F on X. This set contains a marked element that corresponds to the trivial cocycle. (If F is abelian, then $\check{H}^1(X, F) = H^1(X, F)$.) The set $\check{H}^1(X, F)$ has the usual interpretation as the set of sheaves on X that are principal homogeneous spaces of F over X. On the other hand, if F is representable by an X-group G (a group object in k-$\mathcal{A}n_{/X}$), then one has the set PHS(G/X) of isomorphism classes of principal homogeneous spaces of G in the category k-$\mathcal{A}_{/X}$, and there is an evident mapping $\mathrm{PHS}(G/X) \to \check{H}^1(X, G)$.

4.1.8. *Proposition. — If* G *is an* X-*group étale over* X, *then* $\mathrm{PHS}(G/X) \tilde{\to} \check{H}^1(X, G)$.

Proof. — It suffices to verify that a sheaf F, which is a principal homogeneous space of G over X, is representable by an étale X-space. Furthermore, by Corollary 4.1.4, it suffices to show that F is representable locally in the usual topology of X. In particular, we may assume that X is paracompact and there is a finite étale surjective morphism $U \to X$ for which $F|_U$ is representable. If X is k-affinoid, then U is also k-affinoid and the representability of F follows from the descent theory of schemes. In the general case we take a locally finite net τ of affinoid domains of X. For $V \in \tau$, let $f_V : Y_V \to V$ be an étale morphism that represents the inverse image of F on V. For a pair V, W $\in \tau$, there is a canonical isomorphism $\alpha_{V,W} : Y_{V,W} := f_V^{-1}(V \cap W) \to Y_{W,V} := f_W^{-1}(V \cap W)$, and the system of isomorphisms $\alpha_{V,W}$ satisfies the necessary conditions for gluing of Y_V along $Y_{V,W}$. In this way we get an X-space Y. It is easy to see that the canonical morphism $Y \to X$ is étale and that it represents the sheaf F. ■

If G is an abstract group, then the principal homogeneous spaces of the constant X-group G_X are called *étale Galois coverings of* X *with the group* G.

4.1.9. *Corollary. — Let* G *be an abstract group. Then there is a bijection between* $\check{H}^1(X, G)$ *and the set of isomorphism classes of étale Galois coverings of* X *with the Galois group* G *and with a given action of* G. ■

We remark that if $\varphi : Y \to X$ is an étale Galois covering with the Galois group G, then for any abelian sheaf F on X there is a spectral sequence

$$H^p(G, H^q(Y, F)) \Rightarrow H^{p+q}(X, F).$$

The group $H^1(X, G_m)$ can be interpreted as the group of invertible $\mathcal{O}_{X_{ét}}$-modules, where $\mathcal{O}_{X_{ét}}$ is the sheaf of rings on $X_{ét}$ associated with the structural sheaf $\mathcal{O}_{X_G}$ (see Corollary 4.1.2). In particular, there is an injective homomorphism $\mathrm{Pic}(X) \to H^1(X, G_m)$.

4.1.10. *Proposition.* — *(Hilbert Theorem 90). If* X *is good, then* $\mathrm{Pic}(X) \xrightarrow{\sim} H^1(X, G_m)$.

Proof. — Let $\mathscr{L}$ be an invertible $\mathcal{O}_{X_{ét}}$-module. Since the homomorphism considered is injective (for an arbitrary X), it suffices to show that any point $x \in X$ has an open neighborhood $\mathscr{U}$ such that $\mathscr{L}|_{\mathscr{U}}$ comes from $\mathrm{Pic}(\mathscr{U})$. Therefore we can shrink X and assume that there is a finite étale morphism $Y \to X$ such that $\mathscr{L}|_Y$ comes from $\mathrm{Pic}(Y)$. If V is an affinoid neighborhood of x, then $\mathscr{L}$ defines an invertible $\mathcal{O}_{\mathscr{V}_{ét}}$-module on the affine scheme $\mathscr{V} = \mathrm{Spec}(\mathscr{A}_V)$. By the descent theory for schemes, the latter module comes from an invertible $\mathcal{O}_{\mathscr{V}}$-module. Thus, if $\mathscr{U}$ is an open neighborhood of x that is contained in V, then $\mathscr{L}|_{\mathscr{U}}$ comes from an invertible $\mathcal{O}_{\mathscr{U}}$-module. ■

For an arbitrary k-analytic spaces X we can prove only the following

4.1.11. *Corollary.* — *There is a canonical homomorphism* $H^1(X, G_m) \to \mathrm{Pic}(X_G)$ *such that its composition with* $\mathrm{Pic}(X) \to H^1(X, G_m)$ *coincides with the canonical homomorphism* $\mathrm{Pic}(X) \to \mathrm{Pic}(X_G)$.

Proof. — Let $\mathscr{L}$ be an invertible $\mathcal{O}_{X_{ét}}$-module. Then it defines for any affinoid domain $V \subset X$ an invertible $\mathcal{O}_V$-module L_V, and these modules are glued together to an invertible $\mathcal{O}_{X_G}$-module L. The correspondence $\mathscr{L} \mapsto L$ gives the required homomorphism. ■

The group $H^1(X, \mu_n)$ (n is prime to char(k)) can be interpreted as the group of isomorphism classes of the pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is an invertible $\mathcal{O}_{X_{ét}}$-module and φ is an isomorphism $\mathcal{O}_{X_{ét}} \xrightarrow{\sim} \mathscr{L}^{\otimes n}$. From Proposition 4.1.10 it follows that if X is good, then the latter group coincides with the group of isomorphism classes of the pairs (L, φ), where $L \in \mathrm{Pic}(X)$ and φ is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} L^{\otimes n}$. In the general case Corollary 4.1.11 gives a canonical homomorphism from $H^1(X, \mu_n)$ to the group of isomorphism classes of the pairs (L, φ), where $L \in \mathrm{Pic}(X_G)$ and φ is an isomorphism $\mathcal{O}_{X_G} \xrightarrow{\sim} L^{\otimes n}$. (The latter is the group $H^1(X, \mu_n)$ introduced by Drinfeld in [Drl].) In § 4.3 we'll show that this is an isomorphism.

4.2. Stalks of a sheaf

For technical reason we consider the *étale topology* on a k-germ (X, S). It is the Grothendieck topology on the category Ét(X, S) (see § 3.4) generated by the pretopology for which the set of coverings of $((U, T) \to (X, S)) \in$ Ét(X, S) is formed by the families $\{(U_i, T_i) \xrightarrow{f_i} (U, T)\}_{i \in I}$ such that $T = \bigcup_{i \in I} f_i(T_i)$. We denote by $(X, S)_{ét}$ the corresponding site (the *étale site* of (X, S)) and by $(X, S)^{\sim}_{ét}$ the category of sheaves of sets on $(X, S)_{ét}$ (the *étale topos* of (X, S)). The category of abelian sheaves (resp. presheaves) on $(X, S)_{ét}$ will be denoted by S(X, S) (resp. **P**(X, S)). There is an evident morphism of sites $i_{(X,S)} : (X, S)_{ét} \to X_{ét}$. (If $S = |X|$, it is an isomorphism.) For a sheaf F on X we set $F_{(X,S)} = i^*_{(X,S)} F$ and $F(X, S) = F_{(X,S)}(X, S)$.

For a field K we denote by $K_{ét}$ the étale site of the spectrum of K and by $K_{ét}^{\sim}$ the category of sheaves of sets on $K_{ét}$. (The latter is equivalent to the category of discrete $G_{\mathscr{H}(x)}$-sets.) The following statement follows immediately from Theorem 3.4.1.

4.2.1. *Proposition. — For any point x of a k-analytic space* X *there is an equivalence of categories* $\mathscr{H}(x)_{ét}^{\sim} \xrightarrow{\sim} (X, x)_{ét}^{\sim}$. ∎

Let (X, S) be a k-germ. For a point $x \in S$ let i_x denote the canonical morphism of sites $\mathscr{H}(x)_{ét} \to (X, S)_{ét}$. The inverse image of a sheaf F with respect to i_x is denoted by F_x and is called the *stalk of* F *at x*. (We identify F_x with the corresponding discrete $G_{\mathscr{H}(x)}$-set.) The image of an element $f \in F(X, S)$ in F_x is denoted by f_x. If F is an abelian sheaf, then the *support* of an element $f \in F(X, S)$ is the set $\mathrm{Supp}(f) = \{ x \in S \mid f_x \neq 0 \}$ (from the following Proposition 4.2.2 it follows that this is a closed subset of S.) Furthermore, for a subextension $\mathscr{H}(x) \subset K \subset \mathscr{H}(x)^s$ we denote by $F_x(K)$ the subset of $G(\mathscr{H}(x)^s/K)$-invariant elements. (For example, $F_x(\mathscr{H}(x)) = F_x^{G_{\mathscr{H}(x)}}$ and $F_x(\mathscr{H}(x)^s) = F_x$.)

4.2.2. *Proposition. — For a sheaf* F *on* (X, S) *and a point* $x \in X$, *one has*

$$F_x(\mathscr{H}(x)) = \varinjlim_{\mathscr{U} \ni x} F(\mathscr{U}, S \cap \mathscr{U}),$$

where $\mathscr{U}$ *runs through open neighborhoods of x in* X.

Proof. — The set $F_x(\mathscr{H}(x))$ is the inductive limit of the sets F(Y, T) over all $((Y, T) \xrightarrow{f} (X, S)) \in \mathrm{Ét}(X, S)$ with a fixed point $y \in T$ over x such that $\mathscr{H}(x) \xrightarrow{\sim} \mathscr{H}(y)$. By Theorem 3.4.1, the latter implies that the morphism f induces an isomorphism of k-germs $(Y, y) \xrightarrow{\sim} (X, x)$, and the required statement follows. ∎

4.2.3. *Corollary. — A morphism of sheaves* $F \to G$ *on* (X, S) *is a mono/epi/isomorphism if and only if for all* $x \in X$ *the induced maps* $F_x \to G_x$ *possess the same properties.* ∎

Let (X, S) be a k-germ. For each open subset $U \subset S$ we fix an open subset $\mathscr{U} \subset X$ with $\mathscr{U} \cap S = U$. Then the correspondence $U \mapsto \mathscr{U}$ defines a functor from the category of open subsets of S to the category Ét(X, S), and this functor does not depend (up to a canonical isomorphism) on the choice of the sets $\mathscr{U}$. In this way we get a morphism of sites $\pi : (X, S)_{ét} \to S$, where S is the site induced by the usual topology of S.

4.2.4. *Proposition. — For an abelian sheaf* F *on* (X, S) *and a point* $x \in S$, *one has* $(R^q \pi_* F)_x \xrightarrow{\sim} H^q(G_{\mathscr{H}(x)}, F_x)$, $q \geqslant 0$.

Proof. — The case $q = 0$ follows from Proposition 4.2.2. Therefore in the general case it suffices to verify that if F is a flabby sheaf on (X, S), then $F_{(X,x)}$ is a flabby sheaf on (X, x). For this it suffices to show that $\check{H}^q(\mathscr{W}, F_{(X,x)}) = 0$, $q \geqslant 1$, for any covering $\mathscr{W}$ of (X, x) of the form $((U, u) \xrightarrow{f} (X, x))$ where f is finite. By Proposition 4.2.2, the Čech complex of the covering $\mathscr{W}$ is an inductive limit of the Čech complexes of the coverings $((\mathscr{V}, T \cap \mathscr{V}) \to (\mathscr{U}, S \cap \mathscr{U}))$, where $\mathscr{U}$ runs through sufficiently small open neighborhoods of the point x, $T = f^{-1}(S)$ and $\mathscr{V} = f^{-1}(\mathscr{U})$. Since F is flabby, those Čech complexes are acyclic. ■

In the previous proof and in the sequel, an abelian sheaf F on a site C is called *flabby* if its q-dimensional cohomology groups on any object of C are trivial for all $q \geqslant 1$. In [SGA4], Exp. V, 4.1, such a sheaf is called C-acyclic. Recall (*loc. cit.*, 4.3) that F is flabby if and only if the Čech cohomology groups $\check{H}^q$ of any object of C (resp. for all coverings of any object of C) are trivial for $q \geqslant 1$. We remark that if C is the site of a topological space, then the notion of a flabby sheaf is not related to the notion of a flasque sheaf from [God]. If G is a profinite group, then a discrete G-module M is flabby if $H^q(H, M) = 0$ for all open subgroups $H \subset G$ and all $q \geqslant 1$.

4.2.5. *Corollary.* — *An abelian sheaf* F *on* (X, S) *is flabby if and only if*

(1) *for any point* $x \in S$, F_x *is a flabby* $G_{\mathscr{H}(x)}$*-module*;

(2) *for any* $((Y, T) \to (X, S)) \in \acute{E}t(X, S)$, *the restriction of* F *to the usual topology of* T *is a flabby sheaf.* ■

We now obtain first applications of the above results. They are obtained using the spectral sequence

$$(*) \qquad E_2^{p,q} = H^p(|X|, R^q \pi_* F) \Rightarrow H^{p+q}(X, F)$$

of the morphism of sites $\pi : X_{\acute{e}t} \to |X|$.

Let l be a prime integer. The *l-cohomological dimension* $cd_l(X)$ of a k-analytic space X is the minimal integer n (or ∞) such that $H^q(X, F) = 0$ for all $q > n$ and for all abelian l-torsion sheaves F on X. (F is said to be l-torsion if all its stalks are l-torsion.) For example, if $X = \mathscr{M}(k)$, then $cd_l(X) = cd_l(k)$.

4.2.6. *Theorem.* — *Let* X *be a paracompact k-analytic space, and let l be a prime integer. Then* $cd_l(X) \leqslant cd_l(k) + 2\dim(X)$. *If* $l = char(k)$ *then* $cd_l(X) \leqslant 1 + \dim(X)$.

Proof. — Let F be an abelian l-torsion sheaf. From Proposition 1.2.18 it follows that the member $E_2^{p,q}$ of the spectral sequence $(*)$ is zero for $p > \dim(X)$. Since $(R^q \pi_* F)_x = H^q(G_{\mathscr{H}(x)}, F_x)$, then Theorem 2.5.1 implies that $E_2^{p,q} = 0$ for

$$q > cd_l(k) + \dim(X)$$

(resp. $q > 1$ if $l = char(k)$). The required fact now follows from the spectral sequence. ■

4.2.7. *Theorem.* — *If* X *is a good k-analytic space, then for any coherent* $\mathcal{O}_X$*-module* F *there is a canonical isomorphism*

$$H^q(|X|, F) \xrightarrow{\sim} H^q(X, \widetilde{F}), \quad q \geqslant 0.$$

4.2.8. *Lemma.* — *Let* $x \in X$. *Then* $(G_{a,X})_x = \mathcal{O}^{sh}_{X,x}$, *where* $\mathcal{O}^{sh}_{X,x}$ *is the strict Henselization of the local Henselian ring* $\mathcal{O}_{X,x}$.

Proof. — The assertion straightforwardly follows from Proposition 4.2.2. ∎

Proof of Theorem. — It is clear that $\pi_*(\widetilde{F}) = F$. It suffices to show that $R^q \pi_*(\widetilde{F}) = 0$ for all $q \geqslant 1$. For this it suffices to verify that $H^q(G_{\kappa(x)}, (\widetilde{F})_x) = 0$ for all $x \in X$ and $q \geqslant 1$. We set $A = \mathcal{O}_{X,x}$. From Lemma 4.2.8 it follows that $(\widetilde{F})_x = F_x \otimes_A A^{sh}$, where F_x is the stalk of the coherent module F at the point x (this is a finitely generated A-module). We claim that $H^q(G(K/\kappa(x)), F_x \otimes_A B) = 0$, $q \geqslant 1$, for any finite Galois extension K of $\kappa(x)$, where B is the finite extension of A which corresponds to the extension $K/\kappa(x)$. For this we remark that the cohomology groups considered coincide with the cohomology groups of the Čech complex

$$F_x \otimes_A B \to F_x \otimes_A B \otimes_A B \to \dots \to F_x \otimes_A B^{\otimes n} \to \dots$$

Since B is a faithfully flat A-algebra, this complex is exact. ∎

4.3. Quasi-immersions of analytic spaces

Let $\varphi : (Y, T) \to (X, S)$ be a morphism of germs over k. Then for any pair of points $y \in T$ and $x \in S$ with $x = \varphi(y)$ there is an isometric embedding of fields $\mathcal{H}(x) \hookrightarrow \mathcal{H}(y)$. We always fix for such a pair an extension of the above embedding to an embedding of separable closures $\mathcal{H}(x)^s \hookrightarrow \mathcal{H}(y)^s$. It induces a homomorphism of Galois groups $G_{\mathcal{H}(y)} \to G_{\mathcal{H}(x)}$. The following statement follows straightforwardly from the definitions and Proposition 4.2.2.

4.3.1. *Proposition.* — (i) *For any sheaf* F *on* (X, S) *and any pair of points* $y \in T$ *and* $x \in S$ *with* $x = \varphi(y)$, *there is a canonical bijection* $F_x \xrightarrow{\sim} (\varphi^* F)_y$ *that is compatible with the action of the groups* $G_{\mathcal{H}(x)}$ *and* $G_{\mathcal{H}(y)}$.

(ii) *For any sheaf* F *on* (Y, T) *and any point* $x \in S$, *one has*

$$(\varphi_* F)_x (\mathcal{H}(x)) = \varinjlim_{\mathcal{U} \ni x} F(\varphi^{-1}(\mathcal{U}), T \cap \varphi^{-1}(\mathcal{U})),$$

where $\mathcal{U}$ *runs through open neighborhoods of* x *in* X. ∎

4.3.2. *Corollary.* — *Let* $\varphi : Y \to X$ *be a finite morphism of k-analytic spaces. Then the functor* $\varphi_* : \mathbf{S}(Y) \to \mathbf{S}(X)$ *is exact. In particular, for any abelian sheaf* F *on* Y *one has* $H^q(X, \varphi_* F) \xrightarrow{\sim} H^q(Y, F)$, $q \geqslant 0$.

Proof. — From Proposition 4.3.1 (ii) it follows easily that for any point $x \in X$ the stalk $(\varphi_* F)_x$ is isomorphic to the direct sum over all $y \in \varphi^{-1}(x)$ of the induced $G_{\mathcal{H}(x)}$-modules $\mathrm{Ind}^{\mathcal{H}(y)}_{\mathcal{H}(x)}(F_y)$. It follows that the functor φ_* is exact. ∎

4.3.3. *Definition.* — A morphism $\varphi : (Y, T) \to (X, S)$ of germs over k is said to be a *quasi-immersion* if it induces a homeomorphism of T with its image $\varphi(T)$ in S and, for every pair of points $y \in T$ and $x \in S$ with $x = \varphi(y)$, the maximal purely inseparable extension of $\mathscr{H}(x)$ in $\mathscr{H}(y)$ is everywhere dense in $\mathscr{H}(y)$.

For example, analytic domains, closed immersions and the morphisms of the form $X \hat{\otimes} K \to X$, where K is a purely inseparable extension of k, are quasi-immersions. Furthermore, if $\varphi : Y \to X$ is a morphism of k-analytic spaces, then for any point $x \in X$ the canonical morphism $Y_x \to Y$ is a quasi-immersion. We remark that quasi-immersions are preserved under compositions and under any base change in the category $\mathcal{G}erm_k$ (when it is well defined).

4.3.4. *Proposition (Rigidity Theorem).* — *Let* $\varphi : (Y, T) \to (X, S)$ *be a quasi-immersion of germs over k. Then*

(i) φ *induces an equivalence of categories* $(Y, T)^{\sim}_{\text{ét}} \xrightarrow{\sim} (X, \varphi(T))^{\sim}_{\text{ét}}$;

(ii) *if* $\varphi(T)$ *is closed in* S, *then* φ *induces an equivalence between the category* $\mathbf{S}(Y, T)$ *and the full subcategory of* $\mathbf{S}(X, S)$ *which consists of such* F *that* $F_x = 0$ *for all* $x \in S \backslash \varphi(T)$.

Proof. — (i) We may assume that $S = \varphi(T)$. Let $y \in T$ and $x = \varphi(y)$. By hypothesis, there is an isomorphism $G_{\mathscr{H}(y)} \xrightarrow{\sim} G_{\mathscr{H}(x)}$. We claim that for any sheaf F on (Y, T) there is a bijection of $G_{\mathscr{H}(y)} = G_{\mathscr{H}(x)}$-sets $(\varphi_* F)_x \xrightarrow{\sim} F_y$. It suffices to verify that

$$(\varphi_* F)_x (\mathscr{H}(x)) = F_y(\mathscr{H}(y)).$$

By Proposition 4.3.1 (ii), one has

$$(\varphi_* F)_x (\mathscr{H}(x)) = \varinjlim_{\mathscr{U} \ni x} F(\varphi^{-1}(\mathscr{U}), T \cap \varphi^{-1}(\mathscr{U})).$$

Since φ induces a homeomorphism of T with S, the limit coincides with $F_y(\mathscr{H}(y))$. It follows that the functor φ_* is fully faithful.

Let now F be a sheaf on (X, S). Then there is a bijection of $G_{\mathscr{H}(y)} = G_{\mathscr{H}(x)}$-sets $F_x = (\varphi^* F)_y$. We have $(\varphi^* F)_y = (\varphi_* \varphi^*(F))_x$. Therefore, $F \xrightarrow{\sim} \varphi_* \varphi^*(F)$. The required statement follows.

(ii) By (i), we may assume that T is closed subset of S, and φ is the canonical morphism $(X, T) \to (X, S)$. It is clear that for an abelian sheaf F on (X, T) one has $(\varphi_* F)_x = F_x$ if $x \in T$ and $(\varphi_* F)_x = 0$ if $x \in S \backslash T$. If now F is a sheaf on (X, S) such that $F_x = 0$ for all $x \in S \backslash T$, then $(\varphi_* \varphi^*(F))_x = (\varphi^* F)_x = F_x$ if $x \in T$, and therefore $F \xrightarrow{\sim} \varphi_* \varphi^*(F)$. ∎

We remark that from Proposition 4.3.4 it follows that if $\varphi : (Y, T) \to (X, S)$ is a quasi-immersion of germs over k, then the cohomology of (Y, T) with coefficients in an abelian sheaf coincides with the cohomology of $(X, \varphi(T))$ with coefficients in the corresponding sheaf.

A k-germ (X, S) is said to be *paracompact* if S has a basis of paracompact neighborhoods (in this case S is evidently paracompact). For example, k-germs of the form (X, x), where $x \in X$, are paracompact. From this it follows that if X is Hausdorff, then any k-germ (X, S) with compact S is paracompact. Finally, if X is paracompact and S is closed, then (X, S) is paracompact.

4.3.5. *Proposition (Continuity Theorem).* — *Let* (X, S) *be a paracompact k-germ, and let* F *be a sheaf of sets on* X. *Then the following map*

$$\varinjlim_{\mathcal{U}\supset S} H^q(\mathcal{U}, F) \xrightarrow{\sim} H^q((X, S), F_{(X,S)}),$$

where $\mathcal{U}$ *runs through open neighborhoods of* S *in* X, *is bijective in each of the following cases*:

(1) $q = 0$;

(2) F *is an abelian sheaf and* $q \geqslant 0$;

(3) S *is locally closed in* X, F *is a sheaf of groups,* $q = 1$ *and one takes* $\check{H}^1$ *instead of* H^1.

Proof. — Consider the following commutative diagram of morphisms of sites

$$\begin{array}{ccc} X_{\text{ét}} & \xrightarrow{\pi} & |X| \\ \uparrow{\scriptstyle i_{(X,S)}} & & \uparrow{\scriptstyle i_S} \\ (X, S)_{\text{ét}} & \xrightarrow{\pi_S} & S \end{array}$$

From Proposition 4.2.2 (resp. 4.2.4) it follows that in the case (1) (resp. (2)) there is an isomorphism of sheaves $i_S^*(\pi_* F) \xrightarrow{\sim} \pi_{S*}(F_{(X,S)})$ (resp. $i_S^*(R^q \pi_* F) \xrightarrow{\sim} R^q \pi_{S*}(F_{(X,S)})$). Therefore the case (1) (resp. (2)) follows from the corresponding topological fact [God], II.3.3.1 (resp. [Gro], 3.10.2).

(3) We may assume that S is closed in X. Let $\alpha \in \check{H}^1((X, S), F)$, and let $\{(V_i, T_i) \xrightarrow{f_i} (X, S)\}_{i\in I}$ be an étale covering such that α is induced by a cocycle of this covering. We can refine the covering and assume that the morphisms f_i induce finite morphisms $V_i \to \mathcal{U}_i := f_i(V_i)$, $T_i = f_i^{-1}(S)$, the sets $\mathcal{U}_i$ are paracompact, and $\{\mathcal{U}_i\}_{i\in I}$ is a locally finite covering of X. The sets $S_i := S \cap \mathcal{U}_i$ and their finite intersections have a basis of paracompact neighborhoods. It follows that we can find, for each pair $i, j \in I$, an open neighborhood $\mathcal{U}_{ij}$ of $S_i \cap S_j$ in $\mathcal{U}_i \cap \mathcal{U}_j$ and, for each triple $i, j, l \in I$, an open neighborhood $\mathcal{U}_{ijl}$ of $S_i \cap S_j \cap S_l$ in $\mathcal{U}_{ij} \cap \mathcal{U}_{jl} \cap \mathcal{U}_{il}$ such that α is defined by elements $\alpha_{ij} \in F(f_{ij}^{-1}(\mathcal{U}_{ij}))$, where $f_{ij} : V_i \times_X V_j \to X$, that satisfy the cocycle condition in $F(f_{ijl}^{-1}(\mathcal{U}_{ijl}))$, where $f_{ijl} : V_i \times_X V_j \times_X V_l \to X$. Let now $\{\mathcal{U}'_i\}_{i\in I}$ be an open covering of X with compact $\overline{\mathcal{U}'_i}$ and $\overline{\mathcal{U}'_i} \subset \mathcal{U}_i$. Since S is closed, the sets $S_i \cap \overline{\mathcal{U}'_i}$ are compact. Therefore there exist open neighborhoods $\mathcal{U}''_i$ of $S_i \cap \overline{\mathcal{U}'_i}$ in $\mathcal{U}_i$ such that $\mathcal{U}''_i \cap \mathcal{U}''_j \subset \mathcal{U}_{ij}$ and $\mathcal{U}''_i \cap \mathcal{U}''_j \cap \mathcal{U}''_l \subset \mathcal{U}_{ijl}$. Then α comes from the group $\check{H}^1(\mathcal{U}, F)$, where $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}''_i$. ∎

4.3.6. *Corollary.* — *Let $\varphi : Y \to X$ be a quasi-immersion of analytic spaces over k. Suppose that the image $\varphi(Y)$ of Y has a basis of paracompact neighborhoods in X. Then for any abelian sheaf F on X there is a canonical isomorphism*

$$\varinjlim_{\mathcal{U} \supset \varphi(Y)} H^q(\mathcal{U}, F) \xrightarrow{\sim} H^q(Y, \varphi^* F), \quad q \geqslant 0,$$

where $\mathcal{U}$ runs through open neighborhoods of $\varphi(Y)$ in X. ∎

For a Hausdorff k-analytic space X, let $\mathbf{P}(X_{G_{cl}})$ denote the category of abelian presheaves on the category of closed analytic domains in X. Then any abelian presheaf $P \in \mathbf{P}(X_{G_{cl}})$ and any covering $\mathcal{V} = \{V_i\}_{i \in I}$ of X by closed analytic domains define a Čech complex $\mathcal{C}(\mathcal{V}, P)$. Its cohomology groups are denoted by $\check{H}^n(\mathcal{V}, P)$. One has $\check{H}^q(\mathcal{V}, P) = R^q L_{\mathcal{V}}(P)$, where $L_{\mathcal{V}} : \mathbf{P}(X_{G_{cl}}) \to \mathcal{A}b$ is the left exact functor defined as follows

$$L_{\mathcal{V}}(P) = \mathrm{Ker}(\prod_i P(V_i) \rightrightarrows \prod_{i,j} P(V_i \cap V_j)).$$

4.3.7. *Theorem (Leray spectral sequence).* — *Let X be a paracompact k-analytic space. For $F \in \mathbf{S}(X)$ and $q \geqslant 0$, let $\mathcal{H}^q(F)$ denote the abelian preasheaf $V \mapsto H^q(V, F|_V)$. Then for any locally finite covering $\mathcal{V} = \{V_i\}_{i \in I}$ of X by closed analytic domains there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{V}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(X, F).$$

Proof. — We will show that the required spectral sequence is the Grothendieck spectral sequence ([Gro], 2.4.1) of the composition of functors

$$\mathbf{S}(X) \xrightarrow{Q} \mathbf{P}(X_{G_{cl}}) \xrightarrow{L_{\mathcal{V}}} \mathcal{A}b,$$

where Q is the composition of functors

$$\mathbf{S}(X) \xrightarrow{i} \mathbf{P}(X) \xrightarrow{\pi_p} \mathbf{P}(|X|) \xrightarrow{j^p} \mathbf{P}(X_{G_{cl}}),$$

and the functor j^p is defined as follows

$$j^p P(V) = \varinjlim_{\mathcal{U} \supset V} P(\mathcal{U}).$$

(Here $\mathbf{P}(X)$ and $\mathbf{P}(|X|)$ are the categories of abelian presheaves on the étale and the usual topologies of X, respectively, and π_p is the restriction functor.)

We have to verify the following three facts:

(1) $R^q Q(F) = \mathcal{H}^q(F)$;

(2) if F is injective, then the presheaf $Q(F)$ is $L_{\mathcal{V}}$-acyclic;

(3) $(L_{\mathcal{V}} \circ Q)(F) = H^0(X, F)$.

(1) From Corollary 4.3.6 it follows that for any closed analytic domain $V \subset X$ one has

$$\mathcal{H}^q(F)(V) = \varinjlim_{\mathcal{U} \supset V} H^q(\mathcal{U}, F).$$

This means that $\mathcal{H}^q(F) = j^p(\pi_p(\mathcal{H}^q(F)))$. But the functors π_p and j^p are exact, and the functors i and π_p send injectives to injectives. Therefore $R^q Q(F) = \mathcal{H}^q(F)$.

(2) Consider the commutative diagram of functors

$$\begin{array}{ccc} \mathbf{S}(X) & \xrightarrow{i} & \mathbf{P}(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_p} \\ \mathbf{S}(|X|) & \xrightarrow{i'} & \mathbf{P}(|X|) \end{array}$$

If F is injective, then $\pi_*(F)$ is an injective sheaf on $|X|$. By Theorem II.5.2.3 *c)* from [God], one has

$$\check{H}^q(\mathcal{V}, j^p \circ i' \circ \pi_*(F)) = 0 \text{ for } q \geqslant 1.$$

From the above diagram it follows that $Q = j^p \circ \pi_p \circ i = j^p \circ i' \circ \pi_*$, i.e., the sheaf $Q(F)$ is $L_{\mathcal{V}}$-acyclic.

(3) follows from the above diagram and Theorem II.5.2.2 from [God]. ■

4.3.8. *Corollary. — Let n be an integer prime to* char(k). *Then for any k-analytic space* X *the group* $H^1(X, \mu_n)$ *is canonically isomorphic to the group of isomorphism classes of pairs* (L, φ), *where* $L \in \mathrm{Pic}(X_G)$ *and* φ *is an isomorphism* $\mathcal{O}_{X_G} \xrightarrow{\sim} L^{\otimes n}$.

Proof. — The homomorphism from the first group to the second one (let us denote it by $'H^1(X, \mu_n)$) was constructed in the end of § 4.1, and we know that it is an isomorphism if X is good. Suppose that X is paracompact, and let $\mathcal{V}$ be a locally finite affinoid covering of X. The Leray spectral sequence for $\mathcal{V}$ gives an exact sequence

$$(*) \qquad 0 \to E_2^{1,0} \to H^1(X, \mu_n) \to E_2^{0,1} \to E_2^{2,0}.$$

If we set $'H^0(X, \mu_n) := H^0(X, \mu_n)$ and define groups $'E_2^{p,q}$, $q = 0, 1$, in terms of the groups $'H^i(X, \mu_n)$, $i = 0, 1$, in the same way as $E_2^{p,q}$, $q = 0, 1$, are defined in terms of the groups $H^i(X, \mu_n)$, $i = 0, 1$, then one can show directly that the similar exact sequence $('*)$ takes place even in the more general situation when X is arbitrary and $\mathcal{V}$ is a quasinet of analytic domains in X. In particular, the homomorphism considered is an isomorphism when X is paracompact. If X is Hausdorff, we use the analogous spectral sequences $(*)$ and $('*)$ for a covering $\mathcal{V}$ of X by open paracompact subsets. If X is arbitrary, we use the same reasoning for a covering of X by open Hausdorff subsets. ■

4.4. Quasiconstructible sheaves

Let (X, S) be a k-germ. A sheaf F on (X, S) is said to be *locally constant* if there exists an étale covering $\{(U_i, T_i) \to (X, S)\}_{i \in I}$ such that the restriction $F_{(U_i, T_i)}$ of F to every (U_i, T_i) is a constant sheaf. A locally constant sheaf F is said to be *finite* if all the above sheaves $F_{(U_i, T_i)}$ are defined by finite sets. There is a one-to-one correspondence between the set of isomorphism classes of finite locally constant sheaves on (X, S) such that their stalks consist of n elements and the set $\check{H}^1(X, \Sigma_n)$, where Σ_n is the symmetric group of degree n. If F is finite locally constant, then for any sheaf F′ on (X, S) and any point $x \in S$ one has $\mathcal{H}om(F, F')_x \xrightarrow{\sim} \mathcal{H}om(F_x, F'_x)$. Furthermore, the full subcategory of $\mathbf{S}(X, S)$ consisting of finite locally constant abelian sheaves is abelian and preserved under extensions. (Everything above holds in any topos.) We remark that if $\varphi : (Y, T) \to (X, S)$ is a quasi-immersion of germs over k, then the categories of finite locally constant sheaves on (Y, T) and $(X, \varphi(Y))$ are equivalent.

4.4.1. *Proposition.* — *Let* (X, S) *be a paracompact k-germ, and suppose that* S *is locally closed in* X. *Then any finite locally constant* F *on* (X, S) *comes from a finite locally constant sheaf on an open neighborhood of* S *in* X.

Proof. — For an integer $n \geq 0$, let S_n denote the set of all points $x \in S$ such that the stalk F_x consists of n elements. Then the sets S_n are disjoint and open in S, and therefore the k-germs (X, S_n) are also paracompact. Replacing S by S_n, we may assume that all stalks of F consist of n elements. In this case the required statement follows from Proposition 4.3.5 (the case (3)). ■

4.4.2. *Definition.* — A sheaf F on a k-germ (X, S) is said to be *quasiconstructible* if there is a finite decreasing sequence of closed subsets $S = S_0 \supset S_1 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset$ such that for all $0 \leq i \leq n$ the sheaves $F_{(X, S_i \setminus S_{i+1})}$ are finite locally constant.

It is clear that the inverse image of a quasiconstructible sheaf under a morphism of germs over k is quasiconstructible.

4.4.3. *Proposition.* — *The full subcategory of* $\mathbf{S}(X, S)$ *consisting of quasiconstructible sheaves is abelian and preserved under extensions. Furthermore, any quotient sheaf (and therefore any subsheaf) of a quasiconstructible sheaf is quasiconstructible.*

Proof. — The first statement follows from the corresponding properties of finite locally constant sheaves. To verify the second statement, it suffices to show that if $\alpha : F \to G$ is an epimorphism of abelian sheaves on (X, S) and F is finite locally constant, then G is quasiconstructible. It suffices to assume that F is locally isomorphic in the étale topology to $(\mathbf{Z}/p^n\mathbf{Z})_{(X, S)}$, where p is a prime integer. The set T of the points $x \in S$ such that α induces an isomorphism $F_x \xrightarrow{\sim} G_x$ is closed in S. Over the germ $(X, S \setminus T)$, α goes through the quotient sheaf $F/p^{n-1}F$. By induction there is a decreasing sequence of closed subsets $S \setminus T = S'_0 \supset S'_1 \supset \ldots \supset S'_m \supset S'_{m+1} = \emptyset$ such that the sheaves $G_{(X, S'_i \setminus S'_{i+1})}$ are finite locally constant. We get a decreasing sequence of closed subsets

$$S = S_0 \supset S_1 \supset \ldots \supset S_{m+1} \supset S_{m+2} = \emptyset,$$

where $S_i = S'_i \cup T$ for $0 \leq i \leq m+1$. Since $S_i \setminus S_{i+1} = S'_i \setminus S'_{i+1}$ for $0 \leq i \leq m$ and $S_{m+1} = T$, the sheaves $G_{(X, S_i \setminus S_{i+1})}$ are finite locally constant. ■

Here is an example of a quasiconstructible sheaf. Let T be a locally closed subset of S, and denote by j the canonical morphism of k-germs $(X, T) \to (X, S)$. Then for any sheaf G on (X, T) one can define the following subsheaf of $j_! G$ of $j_* G$ (this is a particular case of a construction from § 5.1). If $(X', S') \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$, then $j_! G(X', S')$ consists of all elements of $G(X', f^{-1}(T))$ whose support is closed in S'. It is clear that $(j_! G)_{(X,T)} = G$ and $(j_! G)_{(X, S\setminus T)} = 0$. Therefore if the sheaf G is finite locally constant, then the sheaf $j_! G$ is quasiconstructible. Moreover, if G is quasiconstructible, then so is $j_! G$.

4.4.4. *Proposition. — Any quasiconstructible abelian sheaf* F *on* (X, S) *has a finite filtration whose subsequent quotients are of the form* $j_! G$, *where* j *is the morphism of* k*-germs* $(X, T) \to (X, S)$ *defined by a locally closed subset* $T \subset S$ *and* G *is a finite locally constant abelian sheaf on* (X, T).

Proof. — Let $S = S_0 \supset S_1 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset$ be a decreasing sequence of closed subsets such that the sheaves $F_{(X, S_i \setminus S_{i+1})}$ are finite locally constant. The set $T = S \setminus S_1$ is open in S, and the canonical morphism $j : (X, T) \to (X, S)$ induces a monomorphism of sheaves $j_! F_{(X,T)} \to F$. The required statement is obtained by applying the induction to the pullback of the quotient sheaf on the germ (X, S_1). ∎

4.4.5. *Proposition. — Any abelian torsion sheaf* F *on* (X, S) *is a filtered inductive limit of quasiconstructible sheaves.*

4.4.6. *Lemma. — Let* $\varphi : Y \to X$ *be a finite étale morphism of* k*-analytic spaces. Then for any finite locally constant sheaf* F *on* Y *the sheaf* $\varphi_* F$ *is finite locally constant.*

Proof. — We may assume that X is connected. If V is a connected affinoid domain in X, then the rank of the finite morphism of k-affinoid spaces $\varphi^{-1}(V) \to V$ does not depend on V. Let $\mathrm{rk}(\varphi)$ denote this rank. We prove the statement by induction on $\mathrm{rk}(\varphi)$. If $\mathrm{rk}(\varphi) = 1$, then φ is an isomorphism, and the statement is trivial. In the general case we consider φ as an étale covering of X and reduce the situation to the morphism $Y \times_X Y \to X' := Y$. The latter morphism has a section σ which is an open immersion. Therefore, $Y \times_X Y = \sigma(X') \amalg Y'$. Replacing X' by a connected component, we reduce the situation to the morphism $\varphi' : Y' \to X'$ whose rank is less than $\mathrm{rk}(\varphi)$. ∎

Proof of Proposition **4.4.5.** — To simplify notation, we consider only the case of a k-analytic space X (instead of the k-germ (X, S)). If $\{ G_i \}_{i \in I}$ is a finite family of quasiconstructible subsheaves of F, then the sheaf $G := \mathrm{Im}(\bigoplus_i G_i \to F)$ is also quasiconstructible (Proposition 4.4.3). Therefore it suffices to show that for any point $x \in S$ and any element $f \in F_x$ there exists a quasiconstructible subsheaf $G \subset F$ with $f \in G_x$. Shrinking X, we may assume that there is a finite étale surjective morphism $\varphi : U \to X$

such that f comes from a torsion element of $F(U)$. Let $nf = 0$, $n \geqslant 1$, and let $(\mathbf{Z}/n\mathbf{Z})_U \to F|_U$ be the homomorphism that takes 1 to f. It induces a homomorphism $\varphi_*(\mathbf{Z}/n\mathbf{Z})_U \to F$. By Lemma 4.4.6, the first sheaf is finite locally constant, and therefore, by Proposition 4.4.3, its image in F is quasiconstructible. The required statement follows. ■

§ 5. Cohomology with compact support

5.1. Cohomology with support

A family Φ of closed subsets of a topological space S is said to be a *family of supports* if it is preserved under finite unions and contains all closed subsets of any set from Φ. The family of supports Φ is said to be *paracompactifying* if any $A \in \Phi$ is paracompact and has a neighborhood $B \in \Phi$. Furthermore, for a continuous map $\varphi : T \to S$ and families of supports Φ and Ψ in S and T, respectively, we denote by $\Phi\Psi$ the family supports in T which consists of all closed subsets $A \subset T$ such that $A \in \Psi$ and $\overline{\varphi(A)} \in \Phi$. For example, if Ψ is the family of all closed subsets of T, then $\Phi\Psi = \varphi^{-1}(\Phi)$, where $\varphi^{-1}(\Phi)$ consists of all closed subsets of the sets $\varphi^{-1}(B)$ for $B \in \Phi$.

5.1.1. *Example.* — Let S be a Hausdorff topological space. Then the family C_S of all compact subsets of S is a family of supports. If S is locally compact, then C_S is paracompactifying. More generally, let $\varphi : T \to S$ be a Hausdorff continuous map and assume that each point of S has a compact neighborhood. Then the family C_φ of all closed subsets $A \subset T$ such that the induced map $A \to S$ is compact is a family of supports. If S is paracompact and T is locally compact, then the family C_φ is paracompactifying.

Let (X, S) be a k-germ, and let Φ be a family of supports in S. Then one can define the following left exact functor $\Gamma_\Phi : \mathbf{S}(X, S) \to \mathscr{A}b$:

$$\Gamma_\Phi(F) = \{ s \in F(X, S) \mid \mathrm{Supp}(s) \in \Phi \}.$$

The values of its right derived functors are denoted by $H^n_\Phi((X, S), F)$, $n \geqslant 0$. For example, if Φ is the family of all closed subsets of S, then we get the groups $H^n((X, S), F)$. If Φ is the family of all closed subsets of a fixed closed subset $\Sigma \subset S$, we get the *cohomology groups with support in* Σ denoted by $H^n_\Sigma((X, S), F)$. If S is Hausdorff and $\Phi = C_S$, then we get the *cohomology groups with compact support* $H^n_c((X, S), F)$. We remark that for any family of supports Φ in S and any $F \in \mathbf{S}(X, S)$ there is a spectral sequence

$$H^p_\Phi(S, R^q \pi_*(F)) \Rightarrow H^{p+q}_\Phi((X, S), F),$$

where π is the morphism of sites $(X, S)_{\acute{e}t} \to S$.

Let now $\varphi : (Y, T) \to (X, S)$ be a morphism of germs over k. For

$$((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$$

and a morphism $(V, P) \xrightarrow{g} (U, R)$ in $\acute{E}t(X, S)$ we introduce notations for germs and morphisms by the following diagram with cartesian squares

$$\begin{array}{ccc} (Y, T) & \xrightarrow{\varphi} & (X, S) \\ \uparrow f_\varphi & & \uparrow f \\ (Y_f, T_f) & \xrightarrow{\varphi_f} & (U, R) \\ \uparrow g_\varphi & & \uparrow g \\ (Y_{fg}, T_{fg}) & \xrightarrow{\varphi_{fg}} & (V, P) \end{array}$$

5.1.2. *Definition.* — A *φ-family of supports* is a system Φ of families of supports $\Phi(f)$ in T_f for all $((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$ such that it satisfies the following conditions:

(1) for any morphism $(V, P) \xrightarrow{g} (U, R)$ in $\acute{E}t(X, S)$, one has $g_\varphi^{-1}(\Phi(f)) \subset \Phi(fg)$;

(2) if for a closed subset $A \subset T_f$ there exists a covering $\{(V_i, P_i) \xrightarrow{g_i} (U, R)\}_{i \in I}$ in $\acute{E}t(X, S)$ such that $g_{i,\varphi}^{-1}(A) \in \Phi(fg_i)$ for all $i \in I$, then $A \in \Phi(f)$.

(ii) The φ-family of supports Φ is said to be *paracompactifying* if, for any $((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$, each point of R has a neighborhood $(V, P) \xrightarrow{g} (U, R)$ in $\acute{E}t(X, S)$ such that the family of supports $\Phi(fg)$ is paracompactifying.

For a family of supports Φ in S and a φ-family of supports Ψ, we denote by $\Phi\Psi$ the family of supports $\Phi\Psi(\mathrm{id})$ in T. Let now $\psi : (Z, P) \to (Y, T)$ be a second morphism, and let Φ (resp. Ψ) be a φ-family (resp. ψ-family) of supports. For $((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$, we set $(\Phi\Psi)(f) = \Phi(f)\, \Psi(f_\varphi)$. Then $\Phi\Psi = \{(\Phi\Psi)(f)\}$ is a φψ-family of supports. Furthermore, if Φ is a φ-family of supports and

$$((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S),$$

then, for any $A \in \Phi(f)$, each point $u \in R$ has an open neighborhood $\mathcal{U}$ in R such that the closure of the set $f_\varphi(A \cap \varphi_f^{-1}(\mathcal{U}))$ belongs to $\Phi(\mathrm{id})$.

5.1.3. *Example.* — Let $\varphi : (Y, T) \to (X, S)$ be a morphism of germs over k such that the induced map $T \to S$ is Hausdorff. For $((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$, we set $\mathscr{C}_\varphi(f) = C_{\varphi_f}$. Then $\mathscr{C}_\varphi = \{\mathscr{C}_\varphi(f)\}$ is a φ-family of supports. If S and T are locally closed in X and Y, respectively, then the family $\mathscr{C}_\varphi$ is paracompactifying. Furthermore, if S is Hausdorff, then $\mathscr{C}_S\, \mathscr{C}_\varphi = C_T$. If $\psi : (Z, P) \to (Y, T)$ is a second morphism of germs such that the induced map $P \to T$ is also Hausdorff, then $\mathscr{C}_\varphi\, \mathscr{C}_\psi = \mathscr{C}_{\varphi\psi}$.

A φ-family of supports Φ defines a left exact functor $\varphi_\Phi : \mathbf{S}(Y, T) \to \mathbf{S}(X, S)$ as follows. If $F \in \mathbf{S}(Y, T)$ and $((U, R) \xrightarrow{f} (X, S)) \in \acute{E}t(X, S)$, then

$$(\varphi_\Phi F)(U, R) = \{ s \in F(Y_f, T_f) \mid \mathrm{Supp}(s) \in \Phi(f) \}.$$

For example, if Φ is the family of all closed subsets, then we get the functor φ_*. If the map $T \to S$ is Hausdorff and $\Phi = \mathscr{C}_\varphi$, then we get a left exact functor $\mathbf{S}(Y, T) \to \mathbf{S}(X, S)$ which is denoted by $\varphi_!$. Furthermore, if Ψ is a family of supports in S, then there is an isomorphism of functors $\Gamma_\Psi \circ \varphi_\Phi \xrightarrow{\sim} \Gamma_{\Psi\Phi}$. If $\psi : (Z, P) \to (Y, T)$ is a second morphism of germs and Ψ is a ψ-family of supports, then there is an isomorphism of functors $\varphi_\Phi \circ \psi_\Psi \xrightarrow{\sim} (\varphi\psi)_{\Phi\Psi}$.

Finally, let (X, S) be a k-germ, and let i be the canonical morphism of k-germs $(X, R) \to (X, S)$ defined by a closed subset $R \subset S$. Then one can define a left exact functor

$$i^! : \mathbf{S}(X, S) \to \mathbf{S}(X, R)$$

as follows. Let j denote the canonical morphism $(X, T) \to (X, S)$, where $T = S\backslash R$, and let $F \in \mathbf{S}(X, S)$. The stalks of the sheaf $F' = \mathrm{Ker}(F \to j_* j^* F)$ are equal to zero outside R. By Proposition 4.3.4 (ii), $F' \xrightarrow{\sim} i_*(i^* F')$. We set $i^! F = i^* F'$. Since there is an exact sequence $0 \to i_*(i^! F) \to F \to j_*(j^* F)$, the functor $i^!$ is left exact. Its right derived functors are denoted by $\mathscr{H}^q_R((X, S), F)$. It is easy to see that the functor $i^!$ is right adjoint to the functor i_*. It follows that the functor $i^!$ takes injectives to injectives.

5.2. Properties of cohomology with support

5.2.1. *Proposition. — Let* $\varphi : (Y, T) \to (X, S)$ *be a morphism of germs over* k, *and let* Φ *be a* φ-*family of supports. Then for any* $F \in \mathbf{S}(Y, T)$ *and any* $n \geq 0$ *the sheaf* $R^n \varphi_\Phi(F)$ *is associated with the presheaf* $((U, R) \xrightarrow{f} (X, S)) \mapsto H^n_{\Phi(f)}((Y_f, T_f), F)$.

Proof. — Let $P^n F$ denote the presheaf considered. We have an exact ∂-functor $\{ P^n \}_{n \geq 0} : \mathbf{S}(Y, T) \to \mathbf{P}(X, S)$ (see [Gro], § 2.1). If $S^n F$ denotes the sheaf associated with the presheaf $P^n F$, then we get an exact ∂-functor $\{ S^n \}_{n \geq 0} : \mathbf{S}(Y, T) \to \mathbf{S}(X, S)$. Since $\varphi_\Phi \xrightarrow{\sim} S^0$ and $\{ R^n \varphi_\Phi \}_{n \geq 0}$ is a universal ∂-functor that extends φ_Φ, there is a morphism of ∂-functors $R^n \varphi_\Phi \to S^n$, $n \geq 0$. It is an isomorphism because the ∂-functor $\{ S^n \}_{n \geq 0}$ is exact, and $P^n F = 0$ (and therefore $S^n F = 0$) for any injective sheaf $F \in \mathbf{S}(Y, T)$ and $n \geq 1$. ■

5.2.2. *Theorem (Leray spectral sequence). — Let* $\varphi : (Y, T) \to (X, S)$ *be a morphism of germs over* k. *Let* Φ *be a family of supports in* S, *and let* Ψ *be a* φ-*family of supports. Suppose that the family* Φ *is paracompactifying, or that* Ψ *is the family of all closed subsets of* T. *Then for any abelian sheaf* F *on* (Y, T) *there is a spectral sequence*

$$E_2^{pq} = H^p_\Phi((X, S), R^q \varphi_\Psi(F)) \Rightarrow H^{p+q}_{\Phi\Psi}((Y, T), F).$$

Proof. — Since $\Gamma_\Phi(\varphi_\Psi F) = \Gamma_{\Phi\Psi}(F)$, to apply Theorem 2.4.1 from [Gro], it suffices to verify that the functor φ_Ψ sends injective sheaves to Γ_Φ-acyclic sheaves. If Ψ is the family of all closed subsets of T, then this is evident because $\varphi_\Psi = \varphi_*$ sends injectives to injectives.

5.2.3. *Lemma.* — *If* F *is an injective sheaf on* (Y, T), *then the stalk* $(\varphi_\Psi F)_x$ *of* $\varphi_\Psi F$ *at a point* $x \in S$ *is a flabby* $G_{\mathscr{H}(x)}$*-module.*

Proof. — Since our statement is local, we can shrink X and assume that it is Hausdorff. For a subset $Q \subset T$ we denote by Z_Q the analytic space over k which is a disjoint union of the spaces $\mathscr{M}(\mathscr{H}(y))$ over all points $y \in Q$. The category $\mathbf{S}(Z_Q)$ is equivalent to the category of families $\{M(y)\}_{y \in Q}$ of discrete $G_{\mathscr{H}(y)}$-modules $M(y)$. Let ℓ_Q denote the canonical morphism $Z_Q \to (Y, T)$ of germs over k. Furthermore, for every point $y \in T$ we take an embedding of F_y in an injective $G_{\mathscr{H}(y)}$-module $M(y)$. Let M_Q denote the sheaf on $\mathbf{S}(Z_Q)$ which corresponds to the family $\{M(y)\}_{y \in Q}$, and set $N_Q = \ell_{Q*}(M_Q)$. It is clear that the sheaves M_Q and N_Q are injective, and there is a canonical embedding of sheaves $F \hookrightarrow N_T$. Since F and N_T are injective, F is a direct summand of N_T, and therefore it suffices to verify our statement for the sheaf N_T.

We claim that the canonical embeddings of sheaves $N_Q \to N_T$ induce an isomorphism

$$\varinjlim_{Q \in \Psi(\mathrm{Id})} \varphi_*(N_Q) \xrightarrow{\sim} \varphi_\Psi(N_T).$$

Indeed, let $s \in (\varphi_\Psi N_T)(U, R)$, where $((U, R) \xrightarrow{f} (X, S))$ is a neighborhood of the point x in $\acute{E}t(X, S)$, i.e., $s \in N_T(Y_f, T_f)$ and $\mathrm{Supp}(s) \in \Psi(f)$. For a point $u \in R$ we take an open neighborhood $\mathscr{U}$ such that the closure of $f_\varphi(\mathrm{Supp}(s) \cap \varphi_f^{-1}(\mathscr{U}))$ in T belongs to Ψ. Denoting this closure by Q, we see that the restriction of s to $\varphi_f^{-1}(\mathscr{U}, R \cap \mathscr{U})$ comes from $N_Q(\varphi_f^{-1}(\mathscr{U}, R \cap \mathscr{U}))$.

The lemma now follows from the fact that the filtered inductive limit of flabby $G_{\mathscr{H}(x)}$-modules is a flabby $G_{\mathscr{H}(x)}$-module. ∎

Suppose that the family Φ is paracompactifying, and let F be an injective sheaf on (Y, T). From Lemma 5.2.3 and Proposition 4.2.4 it follows that $R^q \pi_*(\varphi_\Psi F) = 0$ for all $q \geqslant 1$, where π is the morphism of sites $(X, S)_{\acute{e}t} \to S$. From the spectral sequence

$$H^p_\Phi(S, R^q \pi_*(\varphi_\Psi F)) \Rightarrow H^{p+q}_\Phi((X, S), \varphi_\Psi F)$$

it follows that $H^p_\Phi((X, S), \varphi_\Psi F) = H^p_\Phi(S, \pi_*(\varphi_\Psi F))$. Since the family Φ is paracompactifying, the restriction of the sheaf $\varphi_\Psi(F)$ to the usual topology of S is Γ_Φ-acyclic, by Lemma 3.7.1 from [Gro]. The theorem is proved. ∎

Applying Theorem 5.2.2 and Proposition 5.2.1, we get the following

5.2.4. *Corollary.* — *Let* $(Z, R) \xrightarrow{\psi} (Y, T) \xrightarrow{\varphi} (X, S)$ *be morphisms of germs over* k. *Let* Φ *be a* φ*-family of supports, and let* Ψ *be a* ψ*-family of supports. Suppose that the family* Φ *is paracompactifying, or that* Ψ *is the family of all closed subsets of* R. *Then for any abelian sheaf* F *on* (Z, R) *there is a spectral sequence*

$$R^p \varphi_\Phi(R^q \psi_\Psi(F)) \Rightarrow R^{p+q}(\varphi\psi)_{\Phi\Psi}(F). \quad ∎$$

Let $j : (X, T) \to (X, S)$ be the morphism of k-germs defined by a locally closed subset $T \subset S$. The functor $j_!$ is exact because the stalk $(j_! F)_x$ coincides with F_x if $x \in T$

and is equal to zero if $x \notin T$. If T is closed in S, then $j_! = j_*$. If T is open in S, then the functor $j_!$ is left adjoint to j^*. We remark that for any family of supports Φ in S, one has $\Phi\mathscr{C}_j = \Phi_T := \Phi|_T$.

5.2.5. *Corollary.* — *Let* $j : (X, T) \to (X, S)$ *be the morphism of k-germs defined by a locally closed subset* $T \subset S$, *and let* Φ *be a paracompactifying family of supports in* S. *Then for any abelian sheaf* F *on* (X, T) *and any* $q \geqslant 0$ *there is a canonical isomorphism*

$$H^q_\Phi((X, S), j_! F) \xrightarrow{\sim} H^q_{\Phi_T}((X, T), F). \quad \blacksquare$$

If Ψ is the family of all closed subsets of T, then $\Phi\Psi = \Phi \cap T$, where $\Phi \cap T = \{A \cap T \mid A \in \Phi\}$.

5.2.6. *Proposition.* — *Let* (X, S) *be a k-germ,* F *an abelian sheaf on* (X, S), Φ *a family of supports in* S, T *an open subset of* S, *and* $R = S \backslash T$.

(i) *There is an exact sequence*

$$\dots \to H^{q-1}_\Phi((X, S), F) \to H^{q-1}_{\Phi\cap T}((X, T), F_{(X,T)}) \to$$
$$\to H^q_{\Phi_R}((X, S), F) \to H^q_\Phi((X, S), F) \to H^q_{\Phi\cap T}((X, T), F_{(X,T)}) \to \dots$$

(ii) *If* Φ *is paracompactifying, then there is an exact sequence*

$$\dots \to H^{q-1}_\Phi((X, S), F) \to H^{q-1}_{\Phi_R}((X, R), F_{(X,R)}) \to$$
$$\to H^q_{\Phi_T}((X, T), F_{(X,T)}) \to H^q_\Phi((X, S), F) \to H^q_{\Phi_R}((X, R), F_{(X,R)}) \to \dots$$

Proof. — Consider the following morphisms of k-germs

$$(X, T) \overset{j}{\hookrightarrow} (X, S) \overset{i}{\leftarrow} (X, R).$$

(ii) The long exact sequence is obtained from the following exact sequence of sheaves on (X, S)

$$0 \to j_! F_{(X,T)} \to F \to i_* F_{(X,R)} \to 0,$$

using the isomorphisms

$$H^q_\Phi((X, S), j_! F_{(X,T)}) = H^q_{\Phi_T}((X, T), F_{(X,T)}),$$
$$H^q_\Phi((X, S), i_* F_{(X,R)}) = H^q_{\Phi_R}((X, R), F_{(X,R)}).$$

(i) Recall that for any abelian category $\mathscr{A}$ the category $\mathscr{L}(\mathscr{A})$ of covariant left exact functors $\mathscr{A} \to \mathscr{A}b$ is abelian (see, for example, [Mit]). Namely, a morphism of functors $F \to G$ is surjective if for any $A \in \mathscr{A}$ and an element $\alpha \in G(A)$ there exist a monomorphism $A \to B$ and an element $\beta \in F(B)$ such that the images of the elements α and β in G(B) coincide. We claim that there is the following short exact sequence of left exact functors on $\mathbf{S}(X, S)$

$$0 \to \Gamma_{\Phi_R} \to \Gamma_\Phi \to \Gamma_{\Phi\cap T} \circ j^* \to 0.$$

For this we use the construction from the proof of Lemma 5.2.3. Let $F \to N_S$ be the monomorphism constructed there. It suffices to show that the canonical mapping

$$\Gamma_\Phi(N_S) \to \Gamma_{\Phi \cap T}(N_S|_{(X,T)})$$

is surjective. By the proof of Lemma 5.2.3, the first (resp. second) group coincides with the inductive limit over $A \in \Phi$ of the groups $\Gamma_A(N_S)$ (resp. $\Gamma_{A \cap T}(N_S|_{(X,T)})$). But $\Gamma_A(N_S) = M_S(A)$, $\Gamma_{A\cap T}(N_S|_{(X,T)}) = M_S(A \cap T)$, and the mapping $M_S(A) \to M_S(A \cap T)$ is evidently surjective.

The required exact sequence is induced by the above short exact sequence. ■

5.2.7. *Corollary. — Suppose that* (X, S) *is a k-germ,* R *is a closed subset of* S, $T = S \backslash R$, $i : (X, R) \to (X, S)$ *and* $j : (X, T) \to (X, S)$ *are the canonical morphisms, and* $F \in \mathbf{S}(X, S)$. *Then*

(i) *there is an exact sequence*

$$0 \to i_*(i^! F) \to F \to j_* j^* F \to i_* \mathscr{H}^1_R((X, S), F) \to 0;$$

(ii) *for any* $q \geqslant 1$ *there is a canonical isomorphism*

$$R^q j_*(j^* F) \xrightarrow{\sim} \mathscr{H}^{q+1}_R((X, S), F).$$

Proof. — Let $((U, Q) \xrightarrow{f} (X, S)) \in \text{Ét}(X, S)$. Applying the exact sequence 5.2.6 (i) to the k-germs (U, Q), $(U, f^{-1}(R))$, $(U, f^{-1}(T))$, and to the family of all closed subsets of S, we get a long exact sequence of presheaves on (X, S). It induces a long exact sequence of the associated sheaves. It remains to remark that the sheaf associated with the presheaf $(U, Q) \mapsto H^q((U, Q), F)$, $q \geqslant 1$, is equal to zero, and the sheaf associated with the presheaf $(U, Q) \mapsto H^q_{f^{-1}(R)}((U, Q), F)$, $q \geqslant 0$, coincides with $i_* \mathscr{H}^q_R((X, S), F)$. ■

5.2.8. *Proposition. — Let* (X, S) *be a k-germ, and let* Φ *be a paracompactifying family of supports in* S. *Then for any abelian sheaf* F *on* (X, S) *and any* $q \geqslant 0$ *there is a canonical isomorphism*

$$\varinjlim H^q_{\Phi_T}((X, T), F_{(X,T)}) \xrightarrow{\sim} H^q_\Phi((X, S), F),$$

where the limit is taken over the family of all open subsets $T \subset S$ *whose closure belong to* Φ.

Prooof. — The assertion is evidently true for $q = 0$. The general case is obtained using Proposition 3.10.1 from [Gro]. ■

5.2.9. *Proposition. — Let* (X, S) *be a k-germ with Hausdorff* X *and locally closed* S, *and let* F *be an abelian sheaf on* (X, S) *which is a filtered inductive limit of abelian sheaves, i.e.,* $F = \varinjlim F_i$. *Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$\varinjlim H^q_c((X, S), F_i) \xrightarrow{\sim} H^q_c((X, S), F).$$

Proof. — The statement is evidently true for $q = 0$. Since an inductive system of sheaves has a resolution by inductive systems of injective sheaves, it suffices to verify that if all the sheaves F_i are injective, then $H^q_c((X, S), F) = 0$ for all $q \geqslant 1$. But this follows from Proposition 4.2.4 and the corresponding fact for the usual cohomology with compact supports (see [God], II.4.12.1). ∎

5.3. The stalks of the sheaf $R^q \varphi_! F$

5.3.1. *Theorem (Weak Base Change Theorem).* — *Let* $\varphi : Y \to X$ *be a Hausdorff morphism of k-analytic spaces, and let* $F \in \mathbf{S}(Y)$ *and* $x \in X$. *We set* $Y_{\bar{x}} = Y_x \widehat{\otimes}_{\mathscr{H}(x)} \widehat{\mathscr{H}(x)^a}$ *and denote by* F_x *(resp.* $F_{\bar{x}}$*) the inverse image of* F *on* Y_x *(resp.* $Y_{\bar{x}}$*). Then for any* $q \geqslant 0$ *there is an isomorphism* $G_{\mathscr{H}(x)}$*-modules*

$$(R^q \varphi_! F)_x \xrightarrow{\sim} H^q_c(Y_{\bar{x}}, F_{\bar{x}}).$$

Proof. — Since our statement is local, we can decrease X and assume that X and Y are Hausdorff.

5.3.2. *Lemma.* — *There is an isomorphism* $(\varphi_! F)_x (\mathscr{H}(x)) \xrightarrow{\sim} H^0_c(Y_x, F_x)$.

Proof. — From Proposition 4.2.2 it follows that

$$(\varphi_! F)_x(\mathscr{H}(x)) = \varinjlim_{\mathscr{U} \ni x} \{s \in F(\varphi^{-1}(\mathscr{U})) \mid \text{the map } \operatorname{Supp}(s) \to \mathscr{U} \text{ is compact}\}.$$

Furthermore, since any compact subset of Y has a basis of paracompact open neighborhoods, from Propositions 4.3.4 and 4.3.5 it follows that

$$H^0_c(Y_x, F_x) = \varinjlim_{\mathscr{W} \supset \varphi^{-1}(x)} \{s \in F(\mathscr{W}) \mid \text{the set } \operatorname{Supp}(s) \cap \varphi^{-1}(x) \text{ is compact}\}.$$

Therefore our statement follows from the well-known topological fact. ∎

5.3.3. *Corollary.* — *One has*

$$(\varphi_! F)_x = \varinjlim_{K/\mathscr{H}(x)} H^0_c(Y_x \widehat{\otimes}_{\mathscr{H}(x)} K, F_x),$$

where K *runs through finite extensions of* $\mathscr{H}(x)$ *in* $\mathscr{H}(x)^s$. ∎

5.3.4. *Lemma.* — *Let* X *be a Hausdorff k-analytic space, and let* $F \in \mathbf{S}(X)$. *We set* $X' = X \widehat{\otimes} \widehat{k}^a$ *and denote by* F' *the inverse image of* F *on* X'. *Then*

$$\varinjlim_{K/k} H^0_c(X \widehat{\otimes} K, F) \xrightarrow{\sim} H^0_c(X', F'),$$

where K *runs through finite extensions of* k *in* k^s.

Proof. — Let $x' \in X'$. For a finite extension K of k in k^s we denote by x_K the image of x' in $X \widehat{\otimes} K$. We set $x = x_k$ and fix an embedding of fields $\mathcal{H}(x)^s \hookrightarrow \mathcal{H}(x')^s$ over the embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$. Since $\mathcal{H}(x)\, k^a$ is everywhere dense in $\mathcal{H}(x')^a$, $G_{\mathcal{H}(x')} \widetilde{\to} G_{\mathcal{H}(x)k}$. Therefore there is an exact sequence of Galois groups

$$0 \to G_{\mathcal{H}(x')} \to G_{\mathcal{H}(x)} \to G(\mathcal{H}(x)\, k^s/\mathcal{H}(x)) \to 0.$$

We remark that $G(\mathcal{H}(x)\, k^s/\mathcal{H}(x))$ is a closed subgroup of G_k. It follows that

$$F'_{x'}(\mathcal{H}(x')) = \varinjlim_{K/k} F_x(\mathcal{H}(x_K)).$$

We now claim that for any open neighborhood $\mathcal{U}'$ of x' there exist a finite extension K of k in k^s and an open neighborhood $\mathcal{U}$ of the point x_K such that the preimage of $\mathcal{U}$ in X' is contained in $\mathcal{U}'$. Indeed, since the point x has a neighborhood which is a finite union of affinoid domains, the situation is reduced to the case when $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. Shrinking $\mathcal{U}'$, we may assume that

$$\mathcal{U}' = \{ y \in X' \mid |f_i(y)| < a_i,\ |g_j(y)| > b_j,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m \},$$

where $f_i, g_j \in \mathcal{A} \otimes K'$ for some finite extension K' of k in k^a. Let K be the maximal subextension of K' separable over k. If $p = \mathrm{char}(k) > 0$, then $(K')^{p^l} \subset K$ for some $l \geqslant 0$, and therefore replacing f_i, g_j, a_i, b_j by $f_i^{p^l}, g_j^{p^l}, a_i^{p^l}, b_j^{p^l}$, respectively, we may assume that $f_i, g_j \in \mathcal{A} \otimes K$. Then the same inequalities define an open neighborhood $\mathcal{U}$ of x_K in $X \widehat{\otimes} K$ whose preimage in X' is $\mathcal{U}'$.

Finally, let $s \in H^0_c(X', F')$. Then for any point $x' \in \mathrm{Supp}(s)$ there exist a finite subextension $k \subset K \subset k^s$ and an open neighborhood $\mathcal{U}$ of the point x_K such that the restriction of s to the preimage of $\mathcal{U}$ is induced by an element of $H^0(\mathcal{U}, F)$. Since $\mathrm{Supp}(s)$ is compact, we can find a finite extension K of k in k^s such that s is induced by an element of $H^0_c(X \widehat{\otimes} K, F)$. ∎

5.3.5. *Corollary.* — *In the situation of Lemma* 5.3.4, *for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$\varinjlim_{K/k} H^q_c(X \widehat{\otimes} K, F) \widetilde{\to} H^q_c(X', F').$$

Proof. — It suffices to show that if F is an injective sheaf on X, then $H^q_c(X', F') = 0$ for all $q \geqslant 1$.

First of all, for any point $x' \in X'$, $F'_{x'}$ is a flabby $G_{\mathcal{H}(x')}$-module. Indeed, if x is the image of x' in X, then F_x is a flabby $G_{\mathcal{H}(x)}$-module. It follows that F_x is a flabby H-module for any closed subgroup $H \subset G_{\mathcal{H}(x)}$. Since $G_{\mathcal{H}(x')}$ is a closed subgroup of $G_{\mathcal{H}(x)}$ and $F'_{x'} = F_x$, $F'_{x'}$ is a flabby $G_{\mathcal{H}(x')}$-module.

Consider now the morphism of sites $X'_{\text{ét}} \to |X'|$. From the previous fact it follows that $R^q \pi'_* F' = 0$ for all $q \geqslant 1$, and therefore $H^q_c(X', F') = H^q_c(|X'|, \pi'_* F')$. To verify that the latter group is trivial, it suffices to show that for any compact subset $\Sigma' \subset X'$

and any element $s' \in F'(\Sigma')$ there exists an element $t' \in F'(X')$ which induces s'. By the reasoning from the proof of Lemma 5.3.4, we can find a finite extension K of k in k^s and an element $s \in F(\Sigma)$, where Σ is the image of Σ' in $X \hat{\otimes} K$, which give rise to s'. Since F is injective, there exists an element $t \in F(X \hat{\otimes} K)$ that induces s, and the required fact follows. ∎

The case $q = 0$ of our theorem follows from Corollary 5.3.3 and Lemma 5.3.4. So it suffices to show that if F is an injective sheaf on Y, then $H^q_c(Y_{\bar{x}}, F_{\bar{x}}) = 0$ for all $q \geqslant 1$ and $x \in X$. By Corollary 5.3.5, it suffices to show that $H^q_c(Y_x, F) = 0$. This is proved, using the reasoning from the proof of Corollary 5.3.5. ∎

We say that a morphism of analytic spaces over k, $f: X' \to X$, is *restricted* if for any pair of points $x' \in X'$ and $x \in X$ with $f(x') = x$ the canonical embedding of fields $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$ extends to an embedding $\mathcal{H}(x)^a \hookrightarrow \mathcal{H}(x')^a$ whose image is everywhere dense. For example, quasi-immersions and morphisms of the form $X \hat{\otimes} \hat{k}^a \to X$ are restricted.

5.3.6. *Corollary. — Let* $\varphi : Y \to X$ *be a Hausdorff morphism of k-analytic spaces, and let* $f: X' \to X$ *be a restricted morphism of analytic spaces over k, which give rise to a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

Then for any abelian sheaf F *on* Y *and any* $q \geqslant 0$ *there is a canonical isomorphism*

$$f^*(R^q \varphi_! F) \widetilde{\to} R^q \varphi'_!(f'^* F). \quad ∎$$

The following is a consequence of Corollary 5.3.5.

5.3.7. *Corollary (Hochschild-Serre Spectral Sequence). — Let* X *be a Hausdorff k-analytic space, and let* $F \in \mathbf{S}(X)$. *We set* $X' = X \hat{\otimes} \hat{k}^a$ *and denote by* F′ *the inverse image of* F *on* X′. *Then there is a spectral sequence*

$$E_2^{p,q} = H^p(G_k, H^q_c(X', F')) \Rightarrow H^{p+q}_c(X, F). \quad ∎$$

5.3.8. *Corollary. — Let* $\varphi : Y \to X$ *be a Hausdorff morphism of k-analytic spaces, and let* F *be an abelian torsion sheaf on* Y. *Then* $R^q \varphi_! F = 0$ *for all* $q > 2d$, *where d is the dimension of* φ.

Proof. — By Theorem 5.3.1, it suffices to show that if the field k is algebraically closed, and X is a Hausdorff k-analytic space of dimension d, then $H^q_c(X, F) = 0$ for all $q > 2d$. By Proposition 5.2.8 and Corollary 5.2.5, one has

$$H^q_c(X, F) = \varinjlim_{\mathcal{U} \subset X} H^q((X, \bar{\mathcal{U}}), (j_!(F|_{\mathcal{U}}))_{(X, \bar{\mathcal{U}})}),$$

where $\mathcal{U}$ runs through open subsets with compact closure, j is the canonical embedding $\mathcal{U} \hookrightarrow X$. By Proposition 4.3.5, the latter group coincides with the inductive limit of the groups $H^q(\mathcal{V}, j_! F|_{\mathcal{U}})$ over all open paracompact neighborhoods $\mathcal{V}$ of the compact set $\overline{\mathcal{U}}$. The required fact follows from Theorem 4.2.6. ■

Let n be a positive integer, and let $\varphi : Y \to X$ be a Hausdorff morphism of finite dimension. By Corollary 5.3.8, the derived functor $R\varphi_! : D^+(Y, \mathbf{Z}/n\mathbf{Z}) \to D^+(X, \mathbf{Z}/n\mathbf{Z})$ takes $D^b(Y, \mathbf{Z}/n\mathbf{Z})$ to $D^b(X, \mathbf{Z}/n\mathbf{Z})$ and extends to an exact functor

$$R\varphi_! : D(Y, \mathbf{Z}/n\mathbf{Z}) \to D(X, \mathbf{Z}/n\mathbf{Z})$$

which takes $D^-(Y, \mathbf{Z}/n\mathbf{Z})$ to $D^-(X, \mathbf{Z}/n\mathbf{Z})$.

5.3.9. *Theorem. — Suppose that* $F^\cdot \in D^-(X, \mathbf{Z}/n\mathbf{Z})$ and $G^\cdot \in D^-(Y, \mathbf{Z}/n\mathbf{Z})$ *or that* $F^\cdot \in D^b(X, \mathbf{Z}/n\mathbf{Z})$ *has finite* Tor-*dimension and* $G^\cdot \in D(Y, \mathbf{Z}/n\mathbf{Z})$. *Then there is a canonical isomorphism*

$$F^\cdot \overset{\mathbf{L}}{\otimes} R\varphi_!(G^\cdot) \xrightarrow{\sim} R\varphi_!(\varphi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot).$$

Proof. — First of all, for arbitrary abelian sheaves F on X and G on Y there is a canonical homomorphism $F \otimes \varphi_!(G) \to \varphi_!(\varphi^*(F) \otimes G)$. From Theorem 5.3.1 it follows easily that it is an isomorphism. We claim that if F is flat and G is $\varphi_!$-acyclic, then the sheaf $\varphi^*(F) \otimes G$ is also $\varphi_!$-acyclic. Indeed, by Theorem 5.3.1, we may assume that $X = \mathcal{M}(k)$, where k is algebraically closed. In this case F is a constant sheaf associated with a flat $\mathbf{Z}/n\mathbf{Z}$-module M. If M is free of finite rank, then for any $q \geqslant 1$

$$H^q_c(Y, \varphi^*(F) \otimes G) = H^q_c(Y, M_Y \otimes G) = H^q_c(Y, G) \otimes M = 0.$$

It follows that the same is true if M is projective of finite rank. Our claim now follows from the facts that any flat module is a filtered inductive limit of projective modules of finite rank, and the functor $G \mapsto H^q_c(Y, G)$ commutes with filtered inductive limits (Proposition 5.2.9).

In the situation of the theorem we take a flat resolution $P^\cdot \to F^\cdot$ of $F^\cdot$ and a $\varphi_!$-acyclic resolution $G^\cdot \to I^\cdot$ of $G^\cdot$. In the second case we may assume that $P^\cdot$ is bounded. Then $\varphi^*(P^\cdot) \to \varphi^*(F^\cdot)$ is a flat resolution of $\varphi^*(F^\cdot)$. By the previous claim, the complex $\varphi^*(P^\cdot) \otimes I^\cdot$ is $\varphi_!$-acyclic, and therefore

$$\begin{aligned} F^\cdot \overset{\mathbf{L}}{\otimes} R\varphi_!(G^\cdot) &= P^\cdot \otimes \varphi_!(I^\cdot) \xrightarrow{\sim} \varphi_!(\varphi^*(P^\cdot) \otimes I^\cdot) \\ &= R\varphi_!(\varphi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot). \end{aligned}$$

The required statement follows. ■

5.3.10. *Corollary. — If* $G^\cdot \in D^b(Y, \mathbf{Z}/n\mathbf{Z})$ *is of finite* Tor-*dimension, then* $R\varphi_!(G^\cdot)$ *is also of finite* Tor-*dimension, and for any* $F^\cdot \in D(X, \mathbf{Z}/n\mathbf{Z})$ *there is a canonical isomorphism*

$$F^\cdot \overset{\mathbf{L}}{\otimes} R\varphi_!(G^\cdot) \xrightarrow{\sim} R\varphi_!(\varphi^*(F^\cdot) \overset{\mathbf{L}}{\otimes} G^\cdot).$$

Proof. — Suppose that $H^{-i}(G' \overset{L}{\otimes} G^{\bullet}) = 0$ for all $i > m$ and $G' \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$. It follows that $H^{-i}(\varphi^*(F) \overset{L}{\otimes} G^{\bullet}) = 0$ and therefore $R^{-i}\,\varphi_!(\varphi^*(F) \overset{L}{\otimes} G^{\bullet}) = 0$ for all $i > m$ and $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$. By Theorem 4.3.9, $H^{-i}(F \overset{L}{\otimes} R\varphi_!(G^{\bullet})) = 0$. The second assertion follows from this. ∎

If n is prime to $\operatorname{char}(k)$, we define for $m \in \mathbf{Z}$ a sheaf $\mu^m_{n,X}$ as follows. If $m \geqslant 0$, then $\mu^m_{n,X} = \mu^{\otimes m}_{n,X}$. If $m < 0$, then $\mu^m_{n,X} = (\mu^{-m}_{n,X})^{\vee}$, where $F^{\vee} = \mathcal{H}om(F, \mathbf{Z}/n\mathbf{Z})$. For $F^{\bullet} \in D(X, \mathbf{Z}/n\mathbf{Z})$ we set $F^{\bullet}(m) = F^{\bullet} \overset{L}{\otimes} \mu^m_{n,X}$. Since $\mu^m_{n,X}$ is a locally free $\mathbf{Z}/n\mathbf{Z}$-module, $F^{\bullet}(m) = F^{\bullet} \otimes \mu^m_{n,X}$. One has $F^{\bullet}(m)\,(m') = F^{\bullet}(m + m')$ and $\varphi^*(F^{\bullet}(m)) = (\varphi^* F^{\bullet})\,(m)$.

5.3.11. *Corollary.* — *Suppose that n is prime to* $\operatorname{char}(k)$. *Then for* $F^{\bullet} \in D(X, \mathbf{Z}/n\mathbf{Z})$, $G^{\bullet} \in D(Y, \mathbf{Z}/n\mathbf{Z})$ *and* $m \in \mathbf{Z}$ *there are canonical isomorphisms*

$$R\varphi_!(G^{\bullet}(m)) \xrightarrow{\sim} (R\varphi_!\, G^{\bullet})\,(m) \quad \textit{and} \quad R\varphi_!(\varphi^*\, F^{\bullet}(m)) \xrightarrow{\sim} F^{\bullet} \overset{L}{\otimes} R\varphi_!(\mu^m_{n,Y}).$$ ∎

5.4. The trace mapping for flat quasifinite morphisms

In this subsection, for every separated flat quasifinite morphism $\varphi : Y \to X$ of analytic spaces over k and every abelian sheaf F on X, we construct a *trace mapping*

$$\mathrm{Tr}_{\varphi} : \varphi_!\, \varphi^*(F) \to F.$$

Suppose that such mappings are already constructed. We say that Tr_{φ} are compatible with base change if for any separated flat quasifinite morphism of k-analytic spaces $\varphi : Y \to X$ and any morphism $f : X' \to X$ of analytic spaces over k which give rise to a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

the following diagram is commutative

$$\begin{array}{ccccc} \varphi'_!\, \varphi'^*(f^*\, F) & = & \varphi'_! f'^*\, \varphi^*\, F & = & f^*(\varphi_!\, \varphi^*\, F) \\ & {\scriptstyle \mathrm{Tr}_{\varphi'}}\searrow & & \swarrow{\scriptstyle \mathrm{Tr}_{\varphi}} & \\ & & f^*\, F & & \end{array}$$

where Tr_{φ} is the evident homomorphism induced by $\mathrm{Tr}_{\varphi} : \varphi_!\, \varphi^*\, F \to F$ (here we use the canonical isomorphism $f^*\, \varphi_!\, G \xrightarrow{\sim} \varphi'_! f'^*\, G$, $G \in \mathbf{S}(Y)$, which is easily obtained from the proof of Corollary 4.3.2).

Furthermore, we say that the Tr_φ are compatible with composition if for any separated flat quasifinite morphisms $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$ the following diagram is commutative

$$\begin{array}{ccc}
(\varphi\psi)_!\,(\varphi\psi)^*\,F & = & \varphi_!(\psi_!\,\psi^*)\,\varphi^*(F) \\
 & & \big\downarrow \mathrm{Tr}_\psi \\
 & \mathrm{Tr}_{\varphi\psi}\searrow & \varphi_!\,\varphi^*(F) \\
 & & \big\downarrow \mathrm{Tr}_\varphi \\
 & & F
\end{array}$$

where Tr_ψ is the evident homomorphism induced by $\mathrm{Tr}_\psi : \psi_!\,\psi^*(\varphi^*\,F) \to \varphi^*\,F$.

5.4.1. *Theorem. — To every separated flat quasifinite morphism* $\varphi : Y \to X$ *and every abelian sheaf on* X *one can assign a trace mapping*

$$\mathrm{Tr}_\varphi : \varphi_!\,\varphi^*(F) \to F.$$

These mappings have the following properties and are uniquely determined by them:

a) the Tr_φ *are functional on* F;

b) the Tr_φ *are compatible with base change*;

c) the Tr_φ *are compatible with composition*;

d) if φ *is finite of constant rank d, then the composition homomorphism*

$$F \longrightarrow \varphi_*\,\varphi^*(F) = \varphi_!\,\varphi^*(F) \xrightarrow{\mathrm{Tr}_\varphi} F$$

is the multiplication by d.

Proof. — 1) Suppose that $X = \mathscr{M}(K)$, where K is a non-Archimedean field over k. Then $Y = \mathscr{M}(L)$, where L is a finite product $\prod_{i \in I} L_i$ of finite local Artinien K-algebras. Let K_i be the maximal separable extension of K in L_i. We set $Y_i = \mathscr{M}(L_i)$ and $Y'_i = \mathscr{M}(K_i)$. The categories $\mathbf{S}(Y_i)$ and $\mathbf{S}(Y'_i)$ are canonically equivalent. Let φ_i denote the morphism $Y'_i \to X$. From *b)-d)* it follows that the following equality should hold

$$\mathrm{Tr}_\varphi = \sum_{i \in I} [L_i : K_i]\,\mathrm{Tr}_{\varphi_i}.$$

If now L is a finite separable extension of K, then F can be regarded as a G_K-module, and $\varphi_*\,\varphi^*(F)$ is the induced module $\mathrm{Ind}_{G_K}^{G_L}(F)$ which is the set of all continuous maps $f : G_K \to F$ such that $f(hx) = hf(x)$ for $h \in G_L$ with the action $(gf)\,(x) = f(xg)$ for $g \in G_K$. In this case from *b)* it follows that Tr_φ should coincide with the homomorphism of G_K-modules

$$\mathrm{Ind}_{G_K}^{G_L}(F) \to F : f \mapsto \sum_{x \in G_K/G_L} x f(x^{-1}).$$

We remark that if φ is arbitrary and some mapping $T : \varphi_!\,\varphi^*(F) \to F$ induces on stalks the above homomorphisms, then T should coincide with Tr_φ.

2) Suppose that φ is finite. For $(U \xrightarrow{f} X) \in \acute{E}t(X)$, let $\{ V_i \}_{i \in I}$ be the connected components of $Y_f = Y \times_X U$. Then the induced morphisms $\varphi_i : V_i \to U$ are finite. We set $P(U) = \bigoplus_{i \in I} F(U)$. The correspondence $U \mapsto P(U)$ defines an abelian presheaf on X. Furthermore, the canonical homomorphisms

$$P(U) = \bigoplus_{i \in I} F(U) \to \bigoplus_{i \in I} (\varphi_{i_*} \varphi_i^* F)(U) = (\varphi_* \varphi^* F)(U)$$

define a homomorphism of presheaves $P \to \varphi_* \varphi^* F$. We claim that it induces an isomorphism of sheaves $aP \xrightarrow{\sim} \varphi_* \varphi^* F$. Indeed, it suffices to verify that it induces an isomorphism on stalks, but this is evident.

Let now d_i be the rank of φ_i (recall that V_i is connected). For

$$(s_i)_{i \in I} \in \bigoplus_{i \in I} F(U) = P(U)$$

we set

$$T_U((s_i)_{i \in I}) = \sum_{i \in I} d_i s_i \in F(U).$$

It is easy to see that the mappings T_U define a homomorphism of presheaves $P \to F$, and therefore a mapping $\varphi_! \varphi^*(F) \to F$. Considering its stalks, we see that it should coincide with Tr_φ.

3) If φ is an open embedding, then the functor $\varphi_!$ is left adjoint to φ^*. It is clear that Tr_φ should coincide with the adjunction mapping $\varphi_! \varphi^*(F) \to F$. More generally, suppose that φ can be represented as a composition

$$Y \overset{j'}{\hookrightarrow} Y' \xrightarrow{\varphi'} X' \overset{j}{\hookrightarrow} X,$$

where j and j' are open embeddings and φ' is finite. Then we define Tr_φ as the unique mapping which satisfies *c)*. Considering the stalks, we see that Tr_φ does not depend on j, j' and φ'.

4) Let φ be arbitrary. Take an open covering $\{ \mathcal{V}_i \}_{i \in I}$ of Y such that φ induces finite morphisms $\mathcal{V}_i \to \varphi(\mathcal{V}_i)$. We denote by φ_i the morphism $\mathcal{V}_i \to X$ and by ν_i the canonical embedding $\mathcal{V}_i \hookrightarrow Y$. There is an exact sequence of sheaves on Y

$$\bigoplus_{i, j \in I} \nu_{ij!}((\varphi^* F)|_{\mathcal{V}_{ij}}) \to \bigoplus_{i \in I} \nu_{i!}((\varphi^* F)|_{\mathcal{V}_i}) \to \varphi^* F \to 0,$$

where $\mathcal{V}_{ij} = \mathcal{V}_i \cap \mathcal{V}_j$ and ν_{ij} is the canonical embedding $\mathcal{V}_{ij} \hookrightarrow Y$. Since the functor $\varphi_!$ is exact, there is an exact sequence

$$\bigoplus_{i, j \in I} \varphi_{ij!} \varphi_{ij}^*(F) \to \bigoplus_{i \in I} \varphi_{i!} \varphi_i^*(F) \to \varphi_! \varphi^*(F) \to 0,$$

where φ_{ij} are the induced morphisms $\mathcal{V}_{ij} \to X$. From 3) it follows that the mapping

$$\bigoplus_{i \in I} Tr_{\varphi_i} : \bigoplus_{i \in I} \varphi_{i!} \varphi_i^*(F) \to F$$

is zero on the image of $\bigoplus_{i,j\in I}\varphi_{ij_!}\,\varphi_{ij}^*(F)$. Therefore it induces a mapping $\mathrm{Tr}_\varphi:\varphi_!\,\varphi^*(F)\to F$. Considering its stalks, we see that it does not depend on the choice of the covering. Moreover, the mapping constructed has all the properties *a)-d)*. ■

All the mappings which are induced by the trace mapping Tr_φ will also be denoted by Tr_φ. For example, the induced homomorphisms

$$H^q_c(Y,\varphi^*F)=H^q_c(X,\varphi_!\,\varphi^*(F))\to H^q_c(X,F)$$

are examples of such mappings.

5.4.2. *Remarks.* — (i) Let $\varphi:Y\to X$ be a flat finite morphism. Then $\varphi_*(\mathcal{O}_Y)$ is a locally free sheaf of $\mathcal{O}_X$-algebras. Therefore one can define in a standard way the norm homomorphism

$$N:\varphi_*(\mathcal{O}_Y^*)\to\mathcal{O}_X^*.$$

This homomorphism extends naturally to a homomorphism of sheaves on the étale site of X

$$N:\varphi_*(G_{m,Y})\to G_{m,X}.$$

From Theorem 5.4.1 it follows easily that, for any $n\geqslant 1$ prime to char(k), there is a commutative diagram of abelian sheaves on X

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi_*(\mu_{n,Y}) & \longrightarrow & \varphi_*(G_{m,Y}) & \longrightarrow & \varphi_*(G_{m,Y}) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \mathrm{Tr}_\varphi} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \\ 0 & \longrightarrow & \mu_{n,X} & \longrightarrow & G_{m,X} & \longrightarrow & G_{m,X} & \longrightarrow & 0 \end{array}$$

(ii) If $\varphi:Y\to X$ is an étale morphism, then $\varphi_!$ is left adjoint to the functor φ^*. In this case the trace mapping Tr_φ coincides with the adjunction mapping $\varphi_!\,\varphi^*(F)\to F$.

§ 6. Calculation of cohomology for curves

6.1. The Comparison Theorem for projective curves

Recall that a scheme $\mathcal{X}$ over k is called *compactifiable* if there exists an open immersion $j:\mathcal{X}\hookrightarrow\overline{\mathcal{X}}$ of $\mathcal{X}$ in a proper scheme $\overline{\mathcal{X}}$ over k. For such a scheme $\mathcal{X}$ and an abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ one defines the étale cohomology groups with compact support as follows:

$$H^q_c(\mathcal{X},\mathcal{F})=H^q(\overline{\mathcal{X}},j_!\,\mathcal{F})$$

(this definition does not depend on the choice of j). Recall also that, by Nagata's Theorem, any separated scheme of finite type over k is compactifiable. The following statement is the starting point for the induction in the proof of the Comparison Theorem for

Cohomology with Compact Support 7.1.1 (The proof of the latter theorem may be read immediately after the proof of 6.1.1.)

6.1.1. *Theorem. — Let $\mathscr{X}$ be a separated algebraic curve of finite type over k, and let $\mathscr{F}$ be an abelian torsion sheaf on $\mathscr{X}$. Then for any $q \geq 0$ there is a canonical isomorphism*

$$\mathrm{H}^q_c(\mathscr{X}, \mathscr{F}) \tilde{\to} \mathrm{H}^q_c(\mathscr{X}^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}).$$

Proof. — Denote the homomorphism considered by θ^q. First we want to reduce the situation to the case when k is algebraically closed, $\mathscr{X}$ is projective, and $\mathscr{F}$ is finite constant.

1) *We may assume that $\mathscr{X}$ is projective.* — Indeed, if $j: \mathscr{X} \hookrightarrow \overline{\mathscr{X}}$ is an open embedding in a projective curve, then $\mathrm{H}^q_c(\mathscr{X}, \mathscr{F}) = \mathrm{H}^q(\overline{\mathscr{X}}, j_! \mathscr{F})$, $\mathrm{H}^q_c(\mathscr{X}^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}) = \mathrm{H}^q(\overline{\mathscr{X}}^{\mathrm{an}}, j_!^{\mathrm{an}} \mathscr{F}^{\mathrm{an}})$ and $(j_! \mathscr{F})^{\mathrm{an}} \tilde{\to} j_!^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}$.

2) *We may assume that $\mathscr{F}$ is constructible.* — This is because any abelian torsion sheaf on $\mathscr{X}$ is a filtered inductive limit of constructible sheaves and the cohomology of $\mathscr{X}$ and $\mathscr{X}^{\mathrm{an}}$ commutes with filtered inductive limits (Proposition 5.2.9).

3) *We may assume that $\mathscr{F}$ is finite constant.* — Indeed, assume that θ^q are isomorphisms for such sheaves. Then θ^q are isomorphisms for any sheaf of the form $\mathscr{F} = \varphi_*((\mathbf{Z}/n\mathbf{Z})_{\mathscr{Y}})$ for some finite morphism $\varphi: \mathscr{Y} \to \mathscr{X}$ because in this case $\mathrm{H}^q(\mathscr{X}, \mathscr{F}) = \mathrm{H}^q(\mathscr{Y}, \mathbf{Z}/n\mathbf{Z})$ and $\mathrm{H}^q(\mathscr{X}^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}) = \mathrm{H}^q(\mathscr{Y}^{\mathrm{an}}, \mathbf{Z}/n\mathbf{Z})$ (Corollary 4.3.2). Furthermore, an arbitrary constructible sheaf $\mathscr{F}$ can be embedded in a finite direct sum of sheaves of the above form. It follows that there is an exact sequence $0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$ such that θ^q are isomorphisms for each $\mathscr{F}^i$, $i \geq 0$. Therefore θ^q are isomorphisms for $\mathscr{F}$.

4) *We may assume that k is algebraically closed.* — Indeed, let $\mathscr{X}' = \mathscr{X} \otimes k^s$ and $\mathscr{X}'' = \mathscr{X} \otimes \widehat{k^a}$, and let $\mathscr{F}'$ and $\mathscr{F}''$ denote the pullbacks of $\mathscr{F}$ on $\mathscr{X}'$ and $\mathscr{X}''$, respectively. Then $\mathrm{H}^q(\mathscr{X}', \mathscr{F}') \tilde{\to} \mathrm{H}^q(\mathscr{X}'', \mathscr{F}'')$ because the fields k^s and $\widehat{k^a}$ are separably closed. Since $\mathscr{X}''^{\mathrm{an}} = \mathscr{X}^{\mathrm{an}} \widehat{\otimes} \widehat{k^a}$, the required fact follows from the homomorphism of Hochschield-Serre spectral sequences

$$\begin{array}{ccc} \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q(\mathscr{X}', \mathscr{F}')) & \Longrightarrow & \mathrm{H}^{p+q}(\mathscr{X}, \mathscr{F}) \\ \downarrow & & \downarrow \\ \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q(\mathscr{X}''^{\mathrm{an}}, \mathscr{F}''^{\mathrm{an}})) & \Longrightarrow & \mathrm{H}^{p+q}(\mathscr{X}^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}) \end{array}$$

We remark that since the cohomological dimension of $\mathscr{X}$ and $\mathscr{X}^{\mathrm{an}}$ is at most two, then the θ^q are isomorphisms for $q > 2$.

If $\mathscr{F} = (\mathbf{Z}/p^n\mathbf{Z})_{\mathscr{X}}$, where $p = \mathrm{char}(k)$, then the Artin-Schreier exact sequences for $\mathscr{X}$ and $\mathscr{X}^{\mathrm{an}}$, the coincidence of the étale cohomology with the usual cohomology for coherent sheaves (Theorem 4.2.7) and GAGA ([Ber], 3.4.10) imply that the θ^q are isomorphisms.

Suppose that $\mathscr{F} = (\mathbf{Z}/n\mathbf{Z})_{\mathscr{X}}$, where n is prime to $\mathrm{char}(k)$. This sheaf is isomorphic to $\mu_{n,\mathscr{X}}$. Then θ^0 is an isomorphism because $\pi_0(\mathscr{X}) = \pi_0(\mathscr{X}^{\mathrm{an}})$, by [Ber], 3.4.8 and 3.5.1.

Furthermore, the Kummer exact sequences for $\mathscr{X}$ and $\mathscr{X}^{an}$, the Hilbert Theorem 90 (Proposition 4.1.10) and GAGA imply that θ^1 is an isomorphism and that for the verification of the fact that θ^2 is an isomorphism it suffices to show that the n-torsion of the group $H^2(\mathscr{X}^{an}, G_m)$ is trivial. This follows from the following lemma.

6.1.2. *Lemma. — Suppose that k is algebraically closed. Let* X *be a paracompact good one-dimensional k-analytic space. Then the torsion of the group* $H^2(X, G_m)$ *is p-torsion, where* $p = \mathrm{char}(k)$. *If* $\mathrm{char}(k) = 0$, *then* $H^2(X, G_m) = 0$.

Proof. — The spectral sequence of the morphism of sites $\pi : X_{ét} \to |X|$, Lemma 4.2.8 and Proposition 1.2.18 imply that $H^2(X, G_m) = H^0(|X|, R^2\pi_* G_{m,X})$. Thus, to prove the lemma it suffices to show that the sheaf $R^2\pi_* G_{m,X}$ is p-torsion. By Proposition 4.2.4 and Lemma 4.2.8, for a point $x \in X$ one has

$$(R^2\pi_* G_{m,X})_x = H^2(G_{\kappa(x)}, (\mathcal{O}^{sh}_{X,x})^*).$$

The latter is the Brauer group of the local Henselian ring $\mathcal{O}_{X,x}$ and is isomorphic to the Brauer group of $\kappa(x)$ (see [Gro2]). It is equal to zero, by Theorem 2.5.1. ∎

The following fact is a corollary of Lemma 6.1.2. It will be used in § 6.2 and § 6.3 and will be proved in § 6.4 for arbitrary one-dimensional affinoid spaces and for integers n prime to $\mathrm{char}(\widetilde{k})$.

6.1.3. *Corollary. — Suppose that k is algebraically closed. Let* X *be an affinoid domain in the analytification* $\mathscr{X}^{an}$ *of a projective curve* $\mathscr{X}$. *Then for any integer n prime to* $\mathrm{char}(k)$ *one has* $H^2(X, \mu_n) = 0$.

Proof. — The group $H^2(X, \mu_n)$ does not change if we replace X by its inverse image in the normalization of the reduction of $\mathscr{X}$. Therefore we may assume that $\mathscr{X}$ is smooth connected and X is connected. By Lemma 6.1.2 and the Kummer exact sequence, one has $H^2(X, \mu_n) = \mathrm{Pic}(X)/n\,\mathrm{Pic}(X)$. We claim that the group Pic(X) is divisible. Since X is connected, there exists a point $x \in \mathscr{X}(k) = \mathscr{X}^{an}(k)$ which does not belong to X. Then $\mathscr{X}' = \mathscr{X}\backslash\{x\}$ is an irreducible affine curve, and it is known that the group $\mathrm{Pic}(\mathscr{X}')$ is divisible. Therefore it suffices to show that the canonical map $\mathrm{Pic}(\mathscr{X}') \to \mathrm{Pic}(X)$ is surjective. One has

$$\mathrm{Pic}(\mathscr{X}') = \mathrm{Div}(\mathscr{X}')/\mathrm{Div}_0(\mathscr{X}'),$$

where $\mathrm{Div}(\mathscr{X}')$ (resp. $\mathrm{Div}_0(\mathscr{X}')$) is the group of divisors (resp. principal divisors) on $\mathscr{X}'$. Similarly, if $\mathscr{Y} = \mathrm{Spec}(\mathscr{A})$, where $X = \mathscr{M}(\mathscr{A})$, then $\mathrm{Pic}(X) = \mathrm{Pic}(\mathscr{Y})$, by Kiehl's Theorem, and one has

$$\mathrm{Pic}(\mathscr{Y}) = \mathrm{Div}(\mathscr{Y})/\mathrm{Div}_0(\mathscr{Y}).$$

Our claim follows from the evident fact that the canonical map $\mathrm{Div}(\mathscr{X}') \to \mathrm{Div}(\mathscr{Y})$ is surjective. ∎

6.2. The trace mapping for curves

For brevity a separated k-analytic space of pure dimension one will be called a *k-analytic curve.*

6.2.1. *Theorem.* — *Suppose that k is algebraically closed, and let n be an integer prime to* $\mathrm{char}(k)$. *Then one can assign to every smooth k-analytic curve* X *a trace mapping*

$$\mathrm{Tr}_{\mathrm{X}} : \mathrm{H}^2_c(\mathrm{X}, \mu_n) \to \mathbf{Z}/n\mathbf{Z}.$$

These mappings have the following properties and are uniquely determined by them:

a) for any flat quasifinite morphism $\varphi : \mathrm{Y} \to \mathrm{X}$ *the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{H}^2_c(\mathrm{Y}, \mu_n) & \xrightarrow{\mathrm{Tr}_\varphi} & \mathrm{H}^2_c(\mathrm{X}, \mu_n) \\ & {\scriptstyle \mathrm{Tr}_{\mathrm{Y}}} \searrow \quad \swarrow {\scriptstyle \mathrm{Tr}_{\mathrm{X}}} & \\ & \mathbf{Z}/n\mathbf{Z} & \end{array}$$

b) $\mathrm{Tr}_{\mathbf{P}^1}$ *is the canonical mapping* $\mathrm{H}^2(\mathbf{P}^1, \mu_n) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$ *which is induced by the degree homomorphism* $\deg : \mathrm{Pic}(\mathbf{P}^1) \xrightarrow{\sim} \mathbf{Z}$.

Furthermore, the Tr_{X} *are compatible with algebraically closed extensions of the ground field and are surjective. If n is prime to* $\mathrm{char}(\widetilde{k})$ *and* X *is connected, then* Tr_{X} *is an isomorphism.*

Proof. — Let A be the closed annulus $\mathrm{A}(a; r, r) = \{ x \in \mathbf{A}^1 \mid |(\mathrm{T} - a)(x)| = r \}$, and let $f = \sum_{i=-\infty}^{\infty} a_i(\mathrm{T} - a)^i \in \mathcal{O}(\mathrm{A})$.

6.2.2. *Lemma.* — $f \in \mathcal{O}(\mathrm{A})^*$ *if and only if there exists m with* $|a_m| r^m > |a_i| r^i$ *for all* $i \neq m$.

Proof. — If $r \notin |k^*|$, then $\mathcal{O}(\mathrm{A})$ is a field, and our statement is evident. Suppose that $r \in |k^*|$. In this case we may assume that $r = 1$, $||f|| = 1$ and $a = 0$. Since $||f^{-1}|| = 1$, then the element $\widetilde{f} \in \widetilde{k}\left[\mathrm{T}, \frac{1}{\mathrm{T}}\right]$ is invertible. It follows that $\widetilde{f} = \alpha \mathrm{T}^m$ for some $\alpha \in \widetilde{k}$, $m \in \mathbf{Z}$, and we are done. ∎

The complement of A in $\mathbf{P}^1$ is a disjoint union of the open discs $\mathrm{D}(a, r)$ and $\mathbf{P}^1 \backslash \mathrm{E}(a, r)$ which are called the *complementary* open discs of A. Let D be one of these discs. For $f \in \mathcal{O}(\mathrm{A})^*$ we set $\deg_{\mathrm{D}}(f) = m$, if $\mathrm{D} = \mathrm{D}(a, r)$, and $\deg_{\mathrm{D}}(f) = -m$, if $\mathrm{D} = \mathbf{P}^1 \backslash \mathrm{E}(a, r)$, where m is from Lemma 6.2.2. We get a homomorphism

$$\deg_{\mathrm{D}} : \mathcal{O}(\mathrm{A})^* \to \mathbf{Z}.$$

The notation can be motivated as follows. There is a rational function g with $||f - g|| < ||f||$. For such a g one has $g \in \mathcal{O}(\mathrm{A})^*$, and therefore the divisor (g) is concentrated outside A. If $(g)_D$ denotes the part of the divisor which is concentrated on D, then $\deg_D(f) = \deg(g)_D$. We remark that if D′ is the second complementary disc, then $\deg_D(f) + \deg_{D'}(f) = 0$.

6.2.3. *Lemma.* — *Let* $f \in \mathcal{O}(A)$. *Then* $f(A)$ *is an annulus if and only if there exists* $b \in k^*$ *such that* $f - b \in \mathcal{O}(A)^*$ *and* $\deg_D(f - b) \neq 0$.

Proof. — Assume that $f(A) = A(b; R, R)$. Then the function $f - b$ is invertible and, by Lemma 6.2.2, one has $f - b = a_m(T - a)^m (1 + \varphi)$, where $\varphi \in \mathcal{O}(A)$ with $\| \varphi \| < 1$ and $R = | a_m | r^m = \| f - b \|$. If $m = 0$, then $f - b = a_0(1 + \varphi)$ and that is impossible because in this case $f(A) \subset D(b, R)$. Conversely, if $f = b + a_m(T - a)^m (1 + \varphi)$, where $\| \varphi \| < 1$ and $m \neq 0$, then $f(A) = A(b, | a_m | r^m)$. ∎

Let $f \in \mathcal{O}(A)$ and assume that $A' = f(A)$ is an annulus. For a complementary open disc D of A we denote by $f(D)$ the complementary open disc of A′ which is of the same type as D if $m > 0$ (resp. of the opposite type if $m < 0$), where m is from Lemma 6.2.2. (We say that two open discs in $\mathbf{P}^1$ are of the same type if they contain or do not contain the infinity simultaneously.) For example, if f comes from $\mathcal{O}(E)$, where $E = D \cup A$, then $f(D)$ is the usual image of D under f. We remark that the induced morphism $f: A \to A'$ is flat and finite. Let N be the norm homomorphism $\mathcal{O}(A)^* \to \mathcal{O}(A')^*$.

6.2.4. *Lemma.* — *For any* $g \in \mathcal{O}(A)^*$ *one has*

$$\deg_{f(D)}(N(g)) = \deg_D(g).$$

Proof. — Since both sides of the equality do not change under extensions of the ground field, we may assume that $r = 1$ and $\|f\| = 1$. Of course, we may also assume that $a = b = 0$ and $\| g \| = 1$. The morphism f induces a flat finite morphism of reductions $\widetilde{f}: \widetilde{A} = \operatorname{Spec}\left(\widetilde{k}\left[T, \frac{1}{T}\right]\right) \to \widetilde{A}' = \operatorname{Spec}\left(\widetilde{k}\left[T', \frac{1}{T'}\right]\right)$, and the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O}(A)^{0*} & \xrightarrow{N} & \mathcal{O}(A')^{0*} \\ \downarrow & & \downarrow \\ \mathcal{O}(\widetilde{A})^* & \xrightarrow{\widetilde{N}} & \mathcal{O}(\widetilde{A}')^* \end{array}$$

If $\deg_D(g) = 0$, then $\widetilde{g}$ is constant on $\widetilde{A}$. This implies that $\widetilde{N}(\widetilde{g}) = \widetilde{N(g)}$ is constant on $\widetilde{A}'$, and therefore $\deg_{f(D)}(N(g)) = 0$. Since both sides of the equality are additive with respect to g, we may assume that $g = T$. One has $T' = a_m T^m(1 + \varphi)$, where $| a_m | = 1$, $\| \varphi \| < 1$ and $m \neq 0$. If $m > 0$, then $f(D) = D$ and $\widetilde{N(T)} = \widetilde{a}_m^{-1} T'$. If $m < 0$, then $f(D)$ is the second complementary disc and $\widetilde{N(T)} = \widetilde{a}_m^{-1} T'^{-1}$. In both cases we have the required equality. ∎

Let γ_D denote the composition of the following surjective homomorphisms

$$\mathcal{O}(A)^* \to H^1(A, \mu_n) \to H^2_c(D, \mu_n) \to H^2(\mathbf{P}^1, \mu_n) \xrightarrow[\sim]{\mathrm{Tr}_{\mathbf{P}^1}} \mathbf{Z}/n\mathbf{Z}.$$

The first homomorphism is obtained from the Kummer exact sequence (it is surjective because $\mathrm{Pic}(A) = 0$). The second one is obtained from the cohomological exact sequence associated with the embeddings $D \hookrightarrow E = D \cup A \leftarrow A$ (it is surjective because $H^2(E, \mu_n) = 0$, by Corollary 6.1.3). The third one is surjective because $H^2(\mathbf{P}^1 \setminus D, \mu_n) = 0$. We remark that if n is prime to $\mathrm{char}(\widetilde{k})$, then $\mathcal{O}(A)^*/\mathcal{O}(A)^{*n} \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$.

6.2.5. *Lemma. — For any $g \in \mathcal{O}(A)^*$ one has*

$$\gamma_D(g) \equiv -\deg_D(g) \pmod{n}.$$

Proof. — It suffices to consider the case $D = D(a, r)$. First we claim that it suffices to verify the equality only for the function $g = T - a$. Indeed, take a rational function h with sufficiently small $\| g - h \|$ so that $g \equiv h (\mathrm{mod}\ \mathcal{O}(A)^{*n})$ and $\deg_D(g) = \deg_D(h)$. Then $\gamma_D(g) = \gamma_D(h)$, and therefore we may assume that g is a rational function. Since $\gamma_D(g)$ and $\deg_D(g)$ are additive with respect to g, it suffices to assume that $g = T - \alpha$ for some $\alpha \in k$. If $\alpha \notin D$, then g comes from $\mathcal{O}(E)^*$. Therefore the image of $T - \alpha$ in $H^1(A, \mu_n)$ comes from $H^1(E, \mu_n)$ in the exact sequence $H^1(E, \mu_n) \to H^1(A, \mu_n) \to H^2_c(D, \mu_n)$ associated with the embeddings $D \overset{j}{\hookrightarrow} E \overset{i}{\leftarrow} A$. It follows that $\gamma_D(T - \alpha) = 0$. Thus, we may assume that $g = T - a$. We remark that this function belongs to $\mathcal{O}_E(A)^* = H^0(A, i^* G_{m,E})$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! \mu_{n,D} & \longrightarrow & \mu_{n,E} & \longrightarrow & i_* \mu_{n,A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! G_{m,D} & \longrightarrow & G_{m,E} & \longrightarrow & i_* i^* G_{m,E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! G_{m,D} & \longrightarrow & G_{m,E} & \longrightarrow & i_* i^* G_{m,E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose columns are Kummer exact sequences. It induces the *anticommutative* diagram

$$\begin{array}{ccc}
\mathcal{O}_E(A)^* & \xrightarrow{\delta} & H^1_c(D, G_m) \\
\downarrow & & \downarrow \\
H^1(A, \mu_n) & \longrightarrow & H^2_c(D, \mu_n)
\end{array}$$

Therefore it suffices to verify that the image of the element $\delta(T - a)$ in

$$\mathrm{Pic}(\mathbf{P}^1) = H^1(\mathbf{P}^1, G_m)$$

has degree one.

The group $H^1_c(D, G_m)$ has the following two descriptions. It is the group of equivalences classes of pairs (L, φ), where L is an invertible sheaf on E (resp. $\mathbf{P}^1$) and φ is an isomorphism $\mathcal{O}_E \to L$ in a neighborhood of A (resp. an isomorphism $\mathcal{O}_{\mathbf{P}^1} \to L$ in a neighborhood of $\mathbf{P}^1 \backslash D$). If $h \in \mathcal{O}_E(A)^*$, then $\delta(h) = (\mathcal{O}_E, \mathcal{O}_E \xrightarrow{h} \mathcal{O}_E)$. On the other hand one has a commutative diagram

$$\begin{array}{ccc} H^0_c(D, \mathcal{D}iv_D) & \xrightarrow{\nu} & H^1_c(D, G_m) \\ \downarrow & & \downarrow \\ H^0(\mathbf{P}^1, \mathcal{D}iv_{\mathbf{P}^1}) & \longrightarrow & H^1(\mathbf{P}^1, G_m) \end{array}$$

where $\mathcal{D}iv_D$ and $\mathcal{D}iv_{\mathbf{P}^1}$ are the sheaves of Cartier divisors. If $d = \Sigma n_i(a_i) \in H^0_c(D, \mathcal{D}iv_D)$, then $\nu(d) = (\mathcal{O}_E(d), \mathcal{O}_E \to \mathcal{O}_E(d))$ (the latter is an isomorphism in a neighborhood of A which does not meet the support of d). Thus we have $\delta(T - a) = (\mathcal{O}_E, \mathcal{O}_E \xrightarrow{T-a} \mathcal{O}_E)$ and $\nu(a) = (\mathcal{O}_E(a), \mathcal{O}_E \to \mathcal{O}_E(a))$. These pairs are equivalent because there is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_E & \xrightarrow{T-a} & \mathcal{O}_E \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \frac{1}{T-a}} \\ \mathcal{O}_E & \longrightarrow & \mathcal{O}_E(a) \end{array}$$

The lemma is proved. ∎

Let now B be the open annulus $B(a; r, R) = \{ x \in \mathbf{A}^1 \mid r < |(T - a)(x)| < R \}$. We set $\overline{A} = A(a; r, R)$, $A_1 = A(a; R, R)$ and $A_2 = A(a; r, r)$. Let γ_B denote the composition of the following surjective homomorphisms

$$\mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^* \longrightarrow H^1(A_1, \mu_n) \oplus H^1(A_2, \mu_n) \longrightarrow H^2(B, \mu_n) \longrightarrow H^2(\mathbf{P}^1, \mu_n) \xrightarrow[\sim]{\mathrm{Tr}_{\mathbf{P}^1}} \mathbf{Z}/n\mathbf{Z}.$$

The first homomorphism is obtained from the Kummer exact sequence. The second one is obtained from the cohomological exact sequence associated with the embeddings $B \hookrightarrow \overline{A} \leftarrow A_1 \amalg A_2$ (it is surjective because $H^2(\overline{A}, \mu_n) = 0$, by Corollary 6.1.3). The third one is surjective because $H^2(\mathbf{P}^1 \backslash B, \mu_n) = 0$. We also set $D_1 = D(a, R)$ and $D_2 = \mathbf{P}^1 \backslash E(a, r)$. One has $B = D_1 \cap D_2$.

6.2.6. *Lemma. — For* $(g_1, g_2) \in \mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^*$ *one has*

$$\gamma_B(g_1, g_2) \equiv -\deg_{D_1}(g_1) - \deg_{D_2}(g_2) \pmod n.$$

Proof. — We set $E_1 = E(a, R)$, $E_2 = \mathbf{P}^1 \backslash D(a, r)$, $A^1 = E_1 \backslash E(a, r)$, and $A^2 = D_1 \backslash D(a, r)$.

For $i = 1, 2$ one has a commutative diagram of embeddings

$$\begin{array}{ccccc} B & \hookrightarrow & \overline{A} & \longleftarrow & A_1 \amalg A_2 \\ \| & & \uparrow & & \uparrow \\ B & \hookrightarrow & A^i & \longleftarrow & A_i \\ \downarrow & & \downarrow & & \| \\ D_i & \hookrightarrow & E_i & \longleftarrow & A_i \end{array}$$

It induces a commutative diagram of homomorphisms

$$\begin{array}{ccccccc} \mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^* & \longrightarrow & H^1(A_1, \mu_n) \oplus H^1(A_2, \mu_n) & \longrightarrow & H^2_c(B, \mu_n) & \longrightarrow & H^2(\mathbf{P}^1, \mu_n) \\ \uparrow & & \uparrow & & \| & & \| \\ \mathcal{O}(A_i)^* & \longrightarrow & H^1(A_i, \mu_n) & \longrightarrow & H^2_c(B, \mu_n) & \longrightarrow & H^2(\mathbf{P}^1, \mu_n) \\ \| & & \| & & \downarrow & & \| \\ \mathcal{O}(A_i)^* & \longrightarrow & H^1(A_i, \mu_n) & \longrightarrow & H^2_c(D_i, \mu_n) & \longrightarrow & H^2(\mathbf{P}^1, \mu_n) \end{array}$$

The required statement now follows from Lemma 6.2.5. ■

For an open subset $X \subset \mathbf{A}^1$ we define

$$\mathrm{Tr}_X = \mathrm{Tr}_{\mathbf{P}^1} \circ \mathrm{Tr}_j : H^2_c(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z},$$

where j is the embedding $X \hookrightarrow \mathbf{P}^1$.

6.2.7. *Lemma.* — *Let* X *be an open disc or an open annulus in* $\mathbf{A}^1$. *Then for any nonconstant function* $f \in \mathcal{O}(X)$ *the following diagram is commutative*

$$\begin{array}{ccc} H^2_c(X, \mu_n) & \xrightarrow{\mathrm{Tr}_f} & H^2(\mathbf{P}^1, \mu_n) \\ & {}_{\mathrm{Tr}_X}\searrow \quad \swarrow_{\mathrm{Tr}_{\mathbf{P}^1}} & \\ & \mathbf{Z}/n\mathbf{Z} & \end{array}$$

Proof. — Consider first the case $X = D = D(a, r)$. Then $f = \sum_{i=0}^{\infty} a_i (T - a)^i$. Since the group $H^2_c(D, \mu_n)$ is an inductive limit of the groups $H^2_c(D(a, r'), \mu_n)$, it suffices to verify the statement for $D(a, r')$ instead of D, where r' is sufficiently close to r. Therefore we may assume that the function f satisfies the following conditions:

a) $|a_i| r^i \to 0$ for $i \to \infty$;

b) there exists $m \geqslant 1$ with $|a_m| r^m > |a_i| r^i$ for all $i \geqslant 1$ with $i \neq m$.

The property *a)* means that f extends to the closed disc $E = E(a, r)$, and *b)* means that, for $A = A(a; r, r)$, $A' = f(A)$ is an annulus and $D' = f(D)$ is a complementary open disc of A'. Since the homomorphism $\mathcal{O}(A)^* \to H^2_c(D, \mu_n)$ is surjective, it suffices to verify the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}(A)^* & \xrightarrow{N} & \mathcal{O}(A')^* \\ & {\scriptstyle\gamma_D}\searrow \quad \swarrow{\scriptstyle\gamma_{D'}} & \\ & \mathbf{Z}/n\mathbf{Z} & \end{array}$$

For $g \in \mathcal{O}(A)^*$ one has

$$\gamma_{D'}(N(g)) \equiv -\deg_{D'}(N(g)) = -\deg_D(g) \equiv \gamma_D(g) \pmod n.$$

Consider now the case $X = B = B(a; r, R)$. Then $f = \sum_{i=-\infty}^{\infty} a_i(T-a)^i$. As above we may assume that f extends to the closed annulus $\bar{A} = A(a; r, R)$ and, if $A_1 = A(a; R, R)$ and $A_2 = A(a; r, r)$, then $A'_1 = f(A_1)$ and $A'_2 = f(A_2)$ are annuli. It follows that one can find $r < t_1 < \ldots < t_l < R$, $l \geq 0$, such that for each $0 \leq j \leq l$ there is m with $|a_m| t^m > |a_i| t^i$ for all $t \in]t_j, t_{j+1}[$ and $i \neq m$ (we set $t_0 = r$ and $t_{l+1} = R$). Let $x_j = p(E(a, t_j))$ and $\Sigma = \{x_1, \ldots, x_l\}$. Then the open set $\mathcal{U} = X \backslash \Sigma$ is a disjoint union of the open annuli $B_j = B(a; t_j, t_{j+1})$, $0 \leq j \leq l$, and of an infinite number of open discs. By Theorem 2.5.1, one has

$$H^2((B, \Sigma), \mu_n) = \bigoplus_{j=1}^{l} H^2(G_{\mathcal{K}(x_j)}, \mu_n) = 0.$$

Therefore the embeddings $\mathcal{U} \hookrightarrow B \leftarrow (B, \Sigma)$ induce a surjection $H^2_c(\mathcal{U}, \mu_n) \to H^2_c(B, \mu_n)$. Since the lemma is true for open discs, it suffices to verify it for the annuli B_j.

Thus, we may assume in addition that for some $m \in \mathbf{Z}$ one has

$$f = a_m(T-a)^m (1 + \varphi),$$

where $\varphi \in \mathcal{O}(\bar{A})$ and $\|\varphi\|_{\bar{A}} < 1$. In particular, $B' = f(B)$ and $\bar{A}' = f(\bar{A})$ are also annuli. Furthermore, if $D_1 = D(a, R)$ and $D_2 = \mathbf{P}^1 \backslash E(a, r)$, then $B' = D'_1 \cap D'_2$, where $D'_i = f(D_i)$. Since the homomorphism $\mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^* \to H^2_c(B, \mu_n)$ is surjective, it suffices to verify the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^* & \xrightarrow{N} & \mathcal{O}(A'_1)^* \oplus \mathcal{O}(A'_2)^* \\ & {\scriptstyle\gamma_B}\searrow \quad \swarrow{\scriptstyle\gamma_{B'}} & \\ & \mathbf{Z}/n\mathbf{Z} & \end{array}$$

For $(g_1, g_2) \in \mathcal{O}(A_1)^* \oplus \mathcal{O}(A_2)^*$ one has

$$\gamma_{B'}(N(g_1), N(g_2)) = \gamma_{D'_1}(N(g_1)) + \gamma_{D'_2}(N(g_2)) = \gamma_{D_1}(g_1) + \gamma_{D_2}(g_2) = \gamma_B(g_1, g_2).$$

The lemma is proved. ■

Suppose that X is an elementary k-analytic curve (see § 3.6), and let f be a non-constant analytic function on it. It gives rise to a flat quasifinite morphism $f: X \to \mathbf{P}^1$. We set $\mathrm{Tr}_X = \mathrm{Tr}_{\mathbf{P}^1} \circ \mathrm{Tr}_f : H^2_c(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$.

6.2.8. *Corollary.* — *The mapping* Tr_X *does not depend on the choice of* f.

Proof. — Assume that X is not isomorphic to an open disc or an open annulus. Then one can find a point $x \in X$ such that the open set $\mathscr{U} = X\backslash\{x\}$ is a disjoint union of a finite number of open annuli and of an infinite number of open discs (see Remark 3.6.3 (ii)). Since the homomorphism $H^2_c(\mathscr{U}, \mu_n) \to H^2_c(X, \mu_n)$ is surjective, the required statement follows from Lemma 6.2.7. ∎

Let now X be an arbitrary smooth k-analytic curve. By Proposition 3.6.1, we can find an open covering $\{\mathscr{U}_i\}_{i\in I}$ of X by elementary open subsets and, for each pair, $i, j \in I$, an open covering $\{\mathscr{U}_{ijl}\}_{l\in I_{ij}}$ of $\mathscr{U}_i \cap \mathscr{U}_j$ also by elementary open subsets. If ν_i and ν_{ijl} denote the open embeddings $\mathscr{U}_i \hookrightarrow X$ and $\mathscr{U}_{ijl} \hookrightarrow X$, then one has an exact sequence

$$\bigoplus_{i,j,l} \nu_{ijl!}(\mu_{n,\mathscr{U}_{ijl}}) \to \bigoplus_i \nu_{i!}(\mu_{n,\mathscr{U}_i}) \to \mu_{n,X} \to 0$$

which induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} \bigoplus_{i,j,l} H^2_c(\mathscr{U}_{ijl}, \mu_n) & \longrightarrow & \bigoplus_i H^2_c(\mathscr{U}_i, \mu_n) & \longrightarrow & H^2_c(X, \mu_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_X} & & \\ \bigoplus_{i,j,l} \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \bigoplus_i \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & 0 \end{array}$$

The right vertical arrow is the definition of Tr_X. By Corollary 6.2.8, it does not depend on the choice of the coverings. Thus, the trace mappings are defined. The verification of the necessary properties of Tr_X is now trivial. ∎

6.2.9. *Corollary.* — *If* $X = \mathscr{X}^{\mathrm{an}}$, *where* $\mathscr{X}$ *is a separated smooth algebraic curve of finite type over* k, *then the diagram*

$$\begin{array}{ccc} H^2_c(\mathscr{X}, \mu_n) & \xrightarrow{\sim} & H^2_c(X, \mu_n) \\ & {\scriptstyle \mathrm{Tr}_{\mathscr{X}}}\searrow \quad \swarrow{\scriptstyle \mathrm{Tr}_X} & \\ & \mathbf{Z}/n\mathbf{Z} & \end{array}$$

is commutative. ∎

6.2.10. *Remark.* — The last statement of the theorem is not true without the assumption that n is prime to $\mathrm{char}(\widetilde{k})$. For example, assume that $\mathrm{char}(k) = 0$ and $p = \mathrm{char}(\widetilde{k}) > 0$, and let $X = D$ be the open unit disc in $\mathbf{A}^1$. Then the embeddings $D \hookrightarrow \mathbf{P}^1 \leftarrow E = \mathbf{P}^1\backslash D$ induce an exact sequence

$$0 \to H^1(E, \mu_p) \to H^2_c(D, \mu_p) \to H^2(\mathbf{P}^1, \mu_p) \to 0.$$

The group $H^1(E, \mu_p)$ is huge, its cardinality is at least the cardinality of $\widetilde{k}$ (see Remark 6.4.2).

6.3. Tame étale coverings of curves

In this subsection the ground field k is assumed to be algebraically closed, and the characteristic of the residue field $\widetilde{k}$ of k is denoted by p.

Let $\varphi : Y \to X$ be an étale morphism. The *geometric ramification index of* φ *at a point* $y \in Y$ is the number

$$\nu_\varphi(y) = [\mathscr{H}(y) : \mathscr{H}(x)],$$

where $x = \varphi(y)$. The morphism φ is said to be *tame at* y if $\nu_\varphi(y)$ is not divisible by p. It is said to be *tame* if it is tame at all points of Y. We remark that if X (and therefore Y) is good, then, by Proposition 2.4.1, one has $\nu_\varphi(y) = [\kappa(y) : \kappa(x)]$.

6.3.1. *Remarks.* — (i) It follows from the definition that if y belongs to an analytic subdomain $X' \subset X$, then $\nu_\varphi(y) = \nu_{\varphi'}(y)$, where φ' is the induced morphism $\varphi^{-1}(X') \to X'$. We note also that if φ is tame at y, then

$$\nu_\varphi(y) = [\widetilde{\mathscr{H}(y)} : \widetilde{\mathscr{H}(x)}]\,[|\mathscr{H}(y)^*| : |\mathscr{H}(x)^*|]$$

(Proposition 2.4.7 and Lemma 2.4.8).

(ii) If φ is an étale Galois covering with Galois group G, then $\nu_\varphi(y)$ is equal to the order of the stabilizer of y in G. In particular, if the order of G is not divisible by p, then φ is tame.

(iii) Suppose that X is connected and φ is finite. Then the sheaf $\varphi_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_X$-module. Its rank is said to be the *degree* $\deg(\varphi)$ of φ. In this case for any $x \in X$ one has

$$\deg(\varphi) = \sum_{y \in \varphi^{-1}(x)} \nu_\varphi(y).$$

(iv) The use of the word " geometric " can be explained as follows. Suppose that the set of k-points $X(k)$ is everywhere dense in X (this is so, for example, if the valuation of k is nontrivial and X is strictly k-analytic). Then $\nu_\varphi(y)$ is equal to the maximal integer n such that for any open neighborhood $\mathscr{V}$ of y there exists a point $x \in X$ having at least n inverse images in $\mathscr{V}$.

6.3.2. *Theorem.* — *Any tame finite étale Galois covering of the one-dimensional disc is trivial.*

Proof. — If the valuation of k is trivial, the statement is easily verified. So we assume that the valuation of k is nontrivial.

Let $\varphi : Y \to X = E(0, r)$ be a tame finite étale Galois covering. We assume that Y is connected and the number $n = \deg(\varphi)$ is bigger than one. Since X is simply connected (and even contractible), it suffices to show that any point $x \in X$ has exactly n inverse images. Suppose that this is not so, and let Σ denote the set of points $x \in X$ having at most $n - 1$ inverse images. It is clear that Σ is closed. We now use the classification of points of $E(0, r)$ from [Ber], 1.4.4 (see § 3.6). If a point x is of type (1) or (4), then the field $\widetilde{\mathscr{H}(x)} = \widetilde{k}$ is algebraically closed, and the group $|\mathscr{H}(x)^*| = |k^*|$ is divisible. From Remark 6.3.1 (i) it follows that $x \notin \Sigma$. Hence, Σ may consists only of points of types (2) or (3).

Consider the following partial ordering on $X : x \leqslant y$ if $|f(x)| \leqslant |f(y)|$ for all $f \in k[T]$. The restriction of this ordering to Σ satisfies the conditions of Zorn's Lemma, and therefore there exists a minimal point $x \in \Sigma$. Let $x = p(E(a, r'))$ for some $a \in k$ and $r' > 0$. Since $E(a, r') = \{ x' \in X \mid x' \leqslant x \}$, we may replace X by $E(a, r')$ and assume that x is the maximal point of X and $\Sigma = \{ x \}$.

We claim that the preimage of x in Y consists of one point. Indeed, the set $X \backslash \{ x \}$ is a disjoint union of open discs. Let D be such an open disc. Since D is simply connected, $\varphi^{-1}(D)$ is a disjoint union $\coprod_{i=1}^n D_i$, where all D_i are isomorphic to D. For the closure $\overline{D}_i$ of D_i in Y one has $\overline{D}_i = D_i \cup \{ y_i \}$ (see Remark 6.3.4 (i)). It is clear that $\varphi^{-1}(x) = \{ y_1, \ldots, y_n \}$. Since Y is arcwise connected, it follows that $y_1 = \ldots = y_n = y$. One has

$$n = [\mathscr{H}(y) : \mathscr{H}(x)] = [\widetilde{\mathscr{H}(y)} : \widetilde{\mathscr{H}(x)}] \, [| \mathscr{H}(y)^* | : | \mathscr{H}(x)^* |].$$

Suppose first that x is of type (3), i.e., $r \notin |k^*|$. In this case $X = D \cup \{ x \}$, where $D = D(a, r)$, and the group $|\mathscr{H}(x)^*|$ is generated by $|k^*|$ and r. Since $\widetilde{\mathscr{H}(x)} = \widetilde{k}$, we have $[|\mathscr{H}(y)^*| : |\mathscr{H}(x)^*|] = n$. We now remark that the group $|\mathscr{H}(y)^*|$ is generated by the values of the spectral norm on $\mathscr{B}$, where $Y = \mathscr{M}(\mathscr{B})$, because y is the maximal point of Y. But for any $f \in \mathscr{B}$ one has

$$\rho(f) = \sup_{y' \in \varphi^{-1}(D)} |f(y')|,$$

and we know that $\varphi^{-1}(D)$ is a disjoint union of n copies of $D = D(a, r)$. Since a nonzero analytic function on Y has at most a finite number of zeroes, it follows that the number in the right hand side of the equality belongs to the group generated by $|k^*|$ and r. This contradicts to the equality $[|\mathscr{H}(y)^*| : |\mathscr{H}(x)^*|] = n$.

Suppose now that x is of type (2), i.e., $r \in |k^*|$. We then may assume that $r = 1$. In this case $\widetilde{X} = \mathrm{Spec}(\widetilde{\mathscr{A}})$, where $\mathscr{A} = k\{ T \}$, is the affine line over $\widetilde{k}$, and $\widetilde{\mathscr{H}(x)} = \widetilde{k}(T)$. By Remark 6.3.1 (i), $[\widetilde{\mathscr{H}(y)} : \widetilde{\mathscr{H}(x)}] = n$. The field $\widetilde{\mathscr{H}(y)}$ is the field of rational functions of the affine curve $\widetilde{Y} = \mathrm{Spec}(\widetilde{\mathscr{B}})$ (see [Ber], 2.4.4). We claim that the induced morphism $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ is étale. For this it suffices to show that the any $\widetilde{k}$-point $\widetilde{x} \in \widetilde{X}(\widetilde{k})$ has exactly n inverse images in $\widetilde{Y}(\widetilde{k})$. Let π (resp. π') denote the reduction map $X \to \widetilde{X}$ (resp. $Y \to \widetilde{Y}$). Then $\pi^{-1}(\widetilde{x})$ is the unit open disc D, and $\varphi^{-1}(D)$ is a disjoint union of n copies of D. But $\varphi^{-1}(D) = \pi'^{-1}(\widetilde{\varphi}^{-1}(\widetilde{x}))$. By a result of Bosch ([Bos]), all the sets $\pi'^{-1}(\widetilde{y})$, where $\widetilde{y} \in \widetilde{Y}(k)$, are connected. It follows that $\widetilde{x}$ has exactly n inverse images. Thus we get a nontrivial finite étale Galois covering of the affine line over k whose degree is prime to p. This is impossible. ■

6.3.3. *Corollary.* — *Any finite étale Galois covering of the one-dimensional disc, whose degree is prime to p, is trivial.* ∎

6.3.4. *Remarks.* — (i) The following fact was used in the proof of 6.3.2 and will be used in the proof of 6.3.9. Let $\mathcal{U}$ be an open subset of a k-affinoid space X, and assume that there is an isomorphism $\varphi : \mathcal{U} \xrightarrow{\sim} \mathrm{D} = \mathrm{D}(0, r) \subset \mathbf{A}^1$. Then $\overline{\mathcal{U}} = \mathcal{U} \cup \{ x \}$, where $x \notin \mathrm{X}(k)$, and for any open neighborhood $\mathcal{V}$ of the point x the set $\varphi(\mathcal{U} \cap \mathcal{V})$ contains the annulus $\mathrm{B}(r', r) = \mathrm{D} \backslash \mathrm{E}(a, r')$ for some $0 < r' < r$. This fact easily follows from the description of k-analytic curves in [Ber], § 4, but here is its simple explanation. Let $x \in \overline{\mathcal{U}} \backslash \mathcal{U}$. A basis of open neighborhoods of the point x is formed by the sets of the form $\mathcal{V} = \{ y \in \mathrm{X} \mid |f_i(y)| < a_i, \, |g_j(y)| > b_j, \, 1 \leqslant i \leqslant n, \, 1 \leqslant j \leqslant m \}$. But it f is a nonzero analytic function on X, it has at most a finite number of zeroes on $\mathcal{U}$. It follows that the set $\varphi(\{ y \in \mathcal{U} \mid |f(y)| < a \})$ is a disjoint union of open discs in D, and the set $\varphi(\{ y \in \mathcal{U} \mid |f(y)| > a \})$ is the complement of a disjoint union of closed discs in D. The first set is relatively compact in D, and the second one contains the annulus $\mathrm{B}(r', r)$ for some $0 < r' < r$. It follows that the set $\varphi(\mathcal{U} \cap \mathcal{V})$ contains the annulus $\mathrm{B}(r', r)$ for some $0 < r' < r$. The point x does not belong to $\mathrm{X}(k)$ because a basis of open neighborhoods of a k-point is formed by sets of the form $\{ y \in \mathrm{X} \mid |f(y)| < a \}$. The required fact follows.

(ii) A morphism of k-analytic spaces $\varphi : \mathrm{Y} \to \mathrm{X}$ is said to be a *covering* if every point $x \in \mathrm{X}$ has an open neighborhood $\mathcal{U}$ such that $\varphi^{-1}(\mathcal{U})$ is a disjoint union of non-empty spaces $\mathcal{V}_i$ such that the induced morphisms $\mathcal{V}_i \to \mathcal{U}$ are finite. It is easy to deduce from Theorem 6.3.2 and its proof that any tame étale Galois covering of the one-dimensional disc is trivial.

For $0 < r \leqslant \mathrm{R} < \infty$ we denote by $\mathrm{A}(r, \mathrm{R})$ the annulus $\{ x \in \mathbf{A}^1 \mid r \leqslant |\mathrm{T}(x)| \leqslant \mathrm{R} \}$. Let φ_n denote the finite morphism $\mathrm{A}(r^{1/n}, \mathrm{R}^{1/n}) \to \mathrm{A}(r, \mathrm{R}) : z \mapsto z^n$. If n is prime to $\mathrm{char}(k)$, then φ_n is a finite étale Galois covering. A finite étale covering of $\mathrm{A}(r, \mathrm{R})$ is said to be *standard* if it is isomorphic to φ_n for some n. If n is prime to p, then φ_n is tame.

6.3.5. *Theorem.* — *Any tame finite étale Galois covering $\varphi : \mathrm{Y} \to \mathrm{X} = \mathrm{A}(r, \mathrm{R})$ with connected* Y *is standard.*

Proof. — We set $n = \deg(\varphi)$. Suppose first that $r = \mathrm{R}$ and set $x = p(\mathrm{E}(0, r))$. If $r \notin |k^*|$, then $\mathrm{X} = \{ x \}$, and our statement follows from Proposition 2.4.4. If $r \in |k^*|$, we may assume that $r = 1$. In this case $\mathrm{X} = \mathcal{M}(\mathcal{A})$, where $\mathcal{A} = k\left\{ \mathrm{T}, \frac{1}{\mathrm{T}} \right\}$, and the reduction $\widetilde{\mathrm{X}}$ is the complement to zero in the affine line over $\widetilde{k}$. The set $\mathrm{X} \backslash \{ x \}$ is a disjoint union of open unit discs. By Theorem 6.3.2, for such a disc D the space $\varphi^{-1}(\mathrm{D})$ is a disjoint union of n spaces isomorphic to D. Since Y is arcwise connected, we have $\varphi^{-1}(x) = \{ y \}$. Let $\mathrm{Y} = \mathcal{M}(\mathcal{B})$. We claim that $\sqrt[n]{\mathrm{T}} \in \mathcal{B}$. Indeed, as in the proof of Theorem 6.3.2 one shows that the induced morphism $\widetilde{\varphi} : \widetilde{\mathrm{Y}} \to \widetilde{\mathrm{X}}$ is étale and

$\deg(\widetilde{\varphi}) = n$. Therefore $\widetilde{\mathscr{B}} = \widetilde{\mathscr{A}}[\sqrt[n]{\mathrm{T}}]$, and there exists an element $f \in \mathscr{B}$ with $||f|| = 1$ and $||f^n - \mathrm{T}|| < 1$. The element f is invertible in $\mathscr{B}$ because T is invertible. We have

$$\mathrm{T} = f^n\left(1 + \frac{\mathrm{T} - f^n}{f^n}\right).$$

Since $||(\mathrm{T} - f^n)/f^n|| < 1$ and $p \nmid n$, $\sqrt[n]{\mathrm{T}} \in \mathscr{B}$. From this it follows that φ is isomorphic to φ_n.

In the general case we denote by m the product of all integers between 1 and n which are prime to p, and set $\mathrm{X}' = \mathrm{A}(r^{1/m}, \mathrm{R}^{1/m})$. It suffices to show that there exists a finite morphism $\psi : \mathrm{X}' \to \mathrm{Y}$ such that $\varphi\psi = \varphi_m$. For this we consider the induced morphism $\varphi' : \mathrm{Y}' = \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}' \to \mathrm{X}'$. Since X′ is simply connected (and even contractible), it suffices to show that $\nu_{\varphi'}(y') = 1$ for any point $y' \in \mathrm{Y}'$. We set

$$\ell = \{ p(\mathrm{E}(0, t)) \mid r \leqslant t \leqslant \mathrm{R} \} \subset \mathrm{X}.$$

If $y \notin (\varphi_m \varphi')^{-1}(\ell)$, then the required fact follows from Theorem 6.3.2 because $\mathrm{X}\backslash\ell$ is a disjoint union of open discs. Let $x' = \varphi'(y')$ and suppose that the point $x = \varphi_m(x')$ belongs to ℓ, i.e., $x = p(\mathrm{E}(0, t))$, $r \leqslant t \leqslant \mathrm{R}$. By Remark 6.3.1 (i), $\nu_{\varphi'}(y')$ does not change if we replace X by the annulus $\mathrm{A}(t, t)$. But for such annuli the required fact is already established. ∎

6.3.6. *Corollary.* — *Let* D *be an open disc with center at zero, and set* $\mathrm{D}^* = \mathrm{D}\backslash\mathrm{E}(0, r)$, *where* $0 \leqslant r < r(\mathrm{D})$. *Then any tame finite étale Galois covering* $\varphi^* : \mathrm{Y}^* \to \mathrm{D}^*$ *extends to a finite flat covering* $\varphi : \mathrm{Y} \to \mathrm{D}$, *which is étale outside zero.* ∎

6.3.7. *Theorem (Riemann Existence Theorem).* — *Let* $\mathscr{X}$ *be an algebraic curve of locally finite type over* k. *Then the functor* $\mathscr{Y} \mapsto \mathscr{Y}^{\mathrm{an}}$ *defines an equivalence between the category of finite étale Galois coverings of* $\mathscr{X}$, *whose degree is prime to* p, *and the category of similar coverings of* $\mathscr{X}^{\mathrm{an}}$.

6.3.8. *Remark.* — If the valuation on k is trivial, then the Riemann Existence Theorem is true for arbitrary schemes of locally finite type over k and for arbitrary finite étale coverings. This follows from [Ber], 3.5.1 (iii).

Proof. — 1) *The functor is fully faithful* (this is true for schemes of arbitrary dimension and for arbitrary finite étale coverings). Let $\mathscr{Y}' \to \mathscr{X}$ and $\mathscr{Y}'' \to \mathscr{X}$ be finite étale coverings of $\mathscr{X}$. We may assume that $\mathscr{Y}'$ is connected. Then the set $\mathrm{Hom}_{\mathscr{X}}(\mathscr{Y}', \mathscr{Y}'')$ corresponds bijectively to the set of connected components $\mathscr{Y}_i$ of $\mathscr{Y}' \times_{\mathscr{X}} \mathscr{Y}''$ such that the canonical morphism $\mathscr{Y}_i \to \mathscr{Y}'$ is an isomorphism. The similar fact is true for the set $\mathrm{Hom}_{\mathscr{X}^{\mathrm{an}}}(\mathscr{Y}'^{\mathrm{an}}, \mathscr{Y}''^{\mathrm{an}})$. Therefore the bijectivity of the map

$$\mathrm{Hom}_{\mathscr{X}}(\mathscr{Y}', \mathscr{Y}'') \to \mathrm{Hom}_{\mathscr{X}^{\mathrm{an}}}(\mathscr{Y}'^{\mathrm{an}}, \mathscr{Y}''^{\mathrm{an}})$$

follows from [Ber], 3.4.6 (9) and 3.4.8 (iii).

2) *The functor is essentially surjective.* — We may assume that $\mathscr{X}$ is separated, reduced and irreducible. If $\mathscr{X}$ is projective, the assertion follows from GAGA (see [Ber], 3.4.14).

In the general case, let $\mathscr{X} \hookrightarrow \mathscr{X}'$ be an open immersion of $\mathscr{X}$ in a projective curve $\mathscr{X}'$ such that all the points $x_1, \ldots, x_n$ from the complement to $\mathscr{X}$ are smooth. Then we can find sufficiently small open neighborhoods D_i of x_i, which are isomorphic to open discs and disjoint. Let $Y \to \mathscr{X}^{an}$ be a tame finite étale covering of $\mathscr{X}^{an}$. Applying Corollary 6.3.6, we can construct a finite flat covering $\varphi' : Y' \to \mathscr{X}'^{an}$. By GAGA, φ' comes from a finite flat covering $\psi : \mathscr{Y}' \to \mathscr{X}'$. Hence φ comes from the covering $\psi^{-1}(\mathscr{X}) \to \mathscr{X}$. ∎

6.3.9. *Theorem.* — *Let $\mathscr{X}$ be a projective curve over k, and let X be an affinoid subdomain of $\mathscr{X}^{an}$ such that $\mathscr{X}^{an} \backslash X$ is a disjoint union $\bigcup_{i=1}^{n} D_i$, where each D_i is isomorphic to an open disc. Pick points $x_i \in D_i(k)$, $1 \leqslant i \leqslant n$, and set $\mathscr{X}' = \mathscr{X} \backslash \{x_1, \ldots, x_n\}$. Then the functor $\mathscr{Y}' \mapsto \mathscr{Y}'^{an} \times_{\mathscr{X}'^{an}} X$ defines an equivalence between the category of finite étale Galois coverings of $\mathscr{X}'$, whose degree is prime to p, and the category of similar coverings of X.*

6.3.10. *Remark.* — It is very likely that any pure one-dimensional reduced k-affinoid space X can be identified with an affinoid subdomain of the analytification $\mathscr{X}^{an}$ of a projective curve $\mathscr{X}$ such that the condition of Theorem 6.3.9 holds. This is true at least in the following cases:

1) the valuation on k is nontrivial, and X is a normal strictly k-affinoid space (M. Van der Put [Put]);

2) the valuation on k is trivial, and X is irreducible and contains a point x for which the field $\mathscr{H}(x)$ is bigger than k and has trivial valuation (see the proof of Theorem 6.4.1).

Proof. — For a fixed i, take a point $x \in \mathscr{X}(k)$ which does not belong to $\bar{D}_i$ (such a point evidently exists). If f is a nonconstant rational function on $\mathscr{X}$ regular outside x, then the set $\{z \in \mathscr{X}^{an} \mid |f(z)| < a\}$ is an affinoid domain in $\mathscr{X}^{an}$ and, for a sufficiently large a, contains $\bar{D}_i$. Therefore we can apply Remark 6.3.4 (i) to D_i. It follows that $\bar{D}_i = D_i \cup \{z_i\}$, where $z_i \in X \backslash X(k)$, and for a sufficiently small open neighborhood of the point z_i its intersection with D_i is the annulus $D_i \backslash E_i$, where E_i is a closed disc in D_i with center at x_i. (We remark that some of the points z_i may coincide.) Let now $\varphi : Y \to X$ be a finite étale Galois covering, whose degree is prime to p. By Corollary 3.4.2 applied to the points z_i, φ can be extended to a finite étale Galois covering $\varphi' : Y' \to X'$, where $X' = X \cup \bigcup_{i=1}^{n}(D_i \backslash E_i)$ and E_i is a closed disc in D_i with center at x_i. From Corollary 6.3.6 it follows that φ' extends to a finite étale Galois covering of $\mathscr{X}'^{an}$. The required statement now follows from Theorem 6.3.7. ∎

The following statement is a particular case of the Comparison Theorem 7.5.1 (it will not be used in the sequel).

6.3.11. *Corollary. (Comparison Theorem for curves).* — *Let $\mathscr{X}$ be an algebraic curve of locally finite type over k, and let $\mathscr{F}$ be an abelian constructible sheaf on $\mathscr{X}$ with torsion orders prime to p. Then for any $q \geqslant 0$ there is a canonical isomorphism*

$$H^q(\mathscr{X}, \mathscr{F}) \xrightarrow{\sim} H^q(\mathscr{X}^{an}, \mathscr{F}^{an}).$$

Proof. — Using the spectral sequences of an open affine covering $\{\mathscr{X}_i\}_{i\in I}$ of $\mathscr{X}$ and of the corresponding open covering $\{\mathscr{X}_i^{an}\}_{i\in I}$ of $\mathscr{X}^{an}$, we reduce the statement to the case when $\mathscr{X}$ is affine of finite type over k. Furthermore, we may assume that $\mathscr{X}$ is reduced. Finally, we may assume that $\mathscr{F} = (\mathbf{Z}/n\mathbf{Z})_{\mathscr{X}}$, $p \nmid n$ (see the proof of Theorem 6.1.1). The homomorphism considered is an isomorphism for $q = 0$, because $\pi_0(\mathscr{X}) = \pi_0(\mathscr{X}^{an})$, and for $q = 1$, by the Riemann Existence Theorem 6.3.7. Since $H^q(\mathscr{X}, \mathbf{Z}/n\mathbf{Z}) = 0$ for $q \geqslant 2$, it remains to show that $H^2(\mathscr{X}^{an}, \mathbf{Z}/n\mathbf{Z}) = 0$. Let $\mathscr{X} \hookrightarrow \bar{\mathscr{X}}$ be an open embedding of $\mathscr{X}$ in a projective curve $\bar{\mathscr{X}}$ such that $\bar{\mathscr{X}}\backslash\mathscr{X} = \{x_1, \dots, x_m\}$ are smooth k-points. Then the k-analytic space $\mathscr{X}^{an}$ is a union of an increasing sequence of affinoid domains X_i, $i \geqslant 1$, whose complements in $\bar{\mathscr{X}}^{an}$ are disjoint unions of m open discs with centers at the point $x_1, \dots, x_m$. By Theorem 6.3.9, $H^1(\mathscr{X}, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^1(X_i, \mathbf{Z}/n\mathbf{Z})$ and, by Corollary 6.1.3, $H^2(X_i, \mathbf{Z}/n\mathbf{Z}) = 0$ for all $i \geqslant 1$. Therefore the required fact follows from the following lemma which is an analog of Proposition 3.10.2 from [Gro] (the field k is not assumed to be algebraically closed).

6.3.12. *Lemma.* — *Let* X *be a paracompact k-analytic space, and suppose that* X *is a union of an increasing sequence of closed or open analytic domains* X_i, $i \geqslant 1$. *Let* F *be an abelian sheaf on* X, *and let* $q \geqslant 1$. *Assume that for each* $i \geqslant 1$ *the image of the group* $H^{q-1}(X_{i+1}, F)$ *in* $H^{q-1}(X_i, F)$ *under the restriction homomorphism coincides with the image of the group* $H^{q-1}(X_{i+2}, F)$. *Then there is a canonical isomorphism*

$$H^q(X, F) \xrightarrow{\sim} \varprojlim H^q(X_i, F).$$

Proof. — First of all we remark that if J is an injective abelian sheaf on X and Y is a closed or open analytic domain in X, then the pullback of J on Y is acyclic and the homomorphism $J(X) \to J(Y)$ is surjective. Take an injective resolution of F, $0 \to F \to J^0 \to J^1 \to \dots$, and consider, for $i \geqslant 1$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^0(X) & \longrightarrow & J^1(X) & \longrightarrow & J^2(X) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^0(X_i) & \longrightarrow & J^1(X_i) & \longrightarrow & J^2(X_i) & \longrightarrow & \dots \end{array}$$

The first row gives the cohomology groups of X, and from the above remark it follows that the second row gives the cohomology groups of X_i and the vertical arrows are surjections. That the homomorphism considered is surjective is easy. Suppose that $\alpha \in J^q(X)$ is such that $d\alpha = 0$ and the image of α in each $H^q(X_i, F)$ is zero. We have to construct an element $\beta \in J^{q-1}(X)$ with $d\beta = \alpha$. Since $J^{q-1}(X) = \varprojlim J^{q-1}(X_i)$, it suffices to construct a system of elements $\beta_i \in J^{q-1}(X_i)$, $i \geqslant 1$, such that $\alpha|_{X_i} = d\beta_i$ and $\beta_{i+1}|_{X_i} = \beta_i$. Suppose that, for some $i \geqslant 1$, we already constructed elements $\beta_j \in J^{q-1}(X_j)$, $1 \leqslant j \leqslant i$, and $\beta'_{i+1} \in J^{q-1}(X_{i+1})$ with $\beta'_{i+1}|_{X_j} = \beta_j$ for $j \leqslant i$ and $\alpha|_{X_{i+1}} = d\beta'_{i+1}$. Take an element $\beta''_{i+2} \in J^{q-1}(X_{i+2})$ with $\alpha|_{X_{i+2}} = d\beta''_{i+2}$. Then the element $\beta''_{i+2}|_{X_{i+1}} - \beta'_{i+1}$ gives rise to an element of $H^{q-1}(X_{i+1}, F)$. By hypothesis, we can find elements $\gamma_{i+2} \in J^{q-1}(X_{i+2})$ with $d\gamma_{i+2} = 0$ and $\delta_i \in J^{q-2}(X_i)$ (we set $J^{-1} = 0$) such that

$$\gamma_{i+2}|_{X_i} = \beta''_{i+2}|_{X_i} - \beta_i + d\delta_i.$$

Furthermore, since the homomorphism $J^{q-2}(X_{i+2}) \to J^{q-2}(X_i)$ is surjective, there exists an element $\delta_{i+2} \in J^{q-2}(X_{i+2})$ with $\delta_{i+2}|_{X_i} = \delta_i$. Setting $\beta'_{i+2} = \beta''_{i+2} - \gamma_{i+2} + d\delta_{i+2}$ and $\beta_{i+1} = \beta'_{i+2}|_{X_{i+1}}$, we get $\alpha|_{X_{i+2}} = d\beta'_{i+2}$ and $\beta'_{i+2}|_X = \beta_j$ for all $j \leqslant i+1$. ■

6.4. Cohomology of affinoid curves

In this subsection we continue to assume that k is algebraically closed, and we set $p = \mathrm{char}(\widetilde{k})$.

6.4.1. *Theorem. — Let* X *be a one-dimensional k-affinoid space, and let n be an integer prime to p. Then*

(i) *the group* $H^q(X, \mathbf{Z}/n\mathbf{Z})$ *is finite for* $q = 0, 1$ *and equal to zero for* $q \geqslant 2$;

(ii) *for any algebraically closed non-Archimedean field* K *over k one has*

$$H^q(X, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^q(X \widehat{\otimes} K, \mathbf{Z}/n\mathbf{Z}), \quad q \geqslant 0.$$

Proof. — We may assume that X is pure one-dimensional, connected and reduced. Furthermore, we may assume that X is normal. Indeed, let $\varphi : Y \to X$ be the normalization of X. Then there is an exact sequence of sheaves

$$0 \to (\mathbf{Z}/n\mathbf{Z})_X \to \varphi_*(\mathbf{Z}/n\mathbf{Z})_Y \to F \to 0,$$

where F is a sky-scraper sheaf. Since $H^q(X, \varphi_*(\mathbf{Z}/n\mathbf{Z})_Y) = H^q(Y, \mathbf{Z}/n\mathbf{Z})$ for $q \geqslant 0$, $H^q(X, F) = 0$ for $q \geqslant 1$, and the group $H^0(X, F)$ is finite and does not change under extensions of the ground field, it suffices to prove the theorem for Y instead of X.

Consider first the case when X can be identified with an affinoid subdomain of the analytification $\mathscr{X}^{an}$ of a projective curve $\mathscr{X}$ such that $\mathscr{X}^{an} \setminus X$ is a disjoint union $\bigcup_{i=1}^m D_i$, where all the D_i are isomorphic to open discs. (For example, by the result of Van der Put mentioned in Remark 6.3.10 this is the case when X is strictly k-affinoid.) The curve $\mathscr{X}$ is evidently connected. Therefore for any algebraically closed non-Archimedean field K over k the curve $\mathscr{X} \otimes K$ is connected. It follows that $X \widehat{\otimes} K$ is connected, and hence

$$H^0(X, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^0(X \widehat{\otimes} K, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}.$$

Furthermore, pick points $x_i \in D_i(k)$, $1 \leqslant i \leqslant m$, and set $\mathscr{X}' = \mathscr{X} \setminus \{x_1, \ldots, x_m\}$. Theorem 6.3.9 implies that

$$H^1(X, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^1(\mathscr{X}', \mathbf{Z}/n\mathbf{Z}).$$

From this it follows that the statements (i) and (ii) are true for $q = 1$. They are true for $q \geqslant 2$, by Corollary 6.1.3.

Consider now the general case. Let K be a bigger algebraically closed non-Archimedean field K over k, and let π denote the canonical morphism $X' = X \hat{\otimes} K \to X$. To prove the theorem, it suffices to show that for any point $x \in X$ the following property holds

$$(*) \qquad (\pi_*(\mathbf{Z}/n\mathbf{Z})_{X'})_x = 0 \quad \text{and} \quad (R^1 \pi_*(\mathbf{Z}/n\mathbf{Z})_{X'})_x = 0.$$

We remark that if, for any étale morphism $g : Y \to X$ with $g^{-1}(x) = \{y\}$, the point y has a basis of affinoid neighborhoods such that the theorem is true for them, then the property $(*)$ holds for the point x. For example, this is the case when the valuation on k is nontrivial and X is strictly k-affinoid. Furthermore, we remark that if Y is an affinoid domain in X and $x \in Y$, then $(R^q \pi_*(\mathbf{Z}/n\mathbf{Z})_{X'})_x \xrightarrow{\sim} (R^q \pi_{Y*}(\mathbf{Z}/n\mathbf{Z})_{Y'})_x$. Therefore the validity of the property $(*)$ does not change if we replace X by a smaller or a bigger k-affinoid space.

There are the following two possibilities:

(1) $|\mathscr{H}(x)^*| = |k^*|$;

(2) $\widetilde{\mathscr{H}(x)} = \tilde{k}$, and the group $|\mathscr{H}(x)^*|$ is generated by $|k^*|$ and a number $r \notin |k^*|$.

1) If $x \in X(k)$, then $(*)$ evidently holds. Suppose that $x \notin X(k)$. Take an admissible epimorphism

$$\varphi : k\{r_1^{-1} T_1, \dots, r_m^{-1} T_m\} \to \mathscr{A},$$

where $X = \mathscr{M}(\mathscr{A})$, and set $f_i = \varphi(T_i)$. We assume that $f_i \neq 0$.

Suppose first that the valuation on k is nontrivial. In this case we can find numbers $r'_i \in |k^*|$ with $|f_i(x)| \leqslant r'_i \leqslant r_i$, $1 \leqslant i \leqslant m$. Then the Weierstrass domain $X(r_1'^{-1} f_1, \dots, r_m'^{-1} f_m)$ is strictly k-affinoid and contains the point x. By the above remark, the property $(*)$ holds for the point x.

Suppose now that the valuation on k is trivial. Then the valuation on $\mathscr{H}(x)$ is also trivial, and therefore $|f_i(x)| \geqslant 1$. In particular, $r_i \geqslant 1$ and the algebra $\mathscr{A}$ is finitely generated over k. We replace X by the Weierstrass domain $X(f_1, \dots, f_m)$. If $\mathscr{X}$ is the projectivization of the affine curve $\mathrm{Spec}(\mathscr{A})$, then X is an affinoid domain in $\mathscr{X}^{an}$ and the complement of X is a finite disjoint union of open discs. Moreover, the point x has a basis of affinoid neighborhoods of the same type. Finally, if $g : Y \to X$ is an étale morphism with $g^{-1}(x) = \{y\}$, then the similar facts are true for the point y. By the above remark, the property $(*)$ holds for the point x.

2) Shrinking X, we can find $f \in \mathscr{A}$ with $|f(x)| = r$. Consider the induced morphism $f : X \to \mathbf{A}^1$. The image of the point x is the point $y = p(E(0, r))$ because $r \notin |k^*|$. Since the fibres of X are discrete, we may shrink X and assume that $f^{-1}(y) = \{x\}$. But $Y = \{y\}$ is an affinoid domain in $\mathbf{A}^1$ (it is the annulus $A(r, r)$). Therefore $\{x\} = f^{-1}(y)$ is an affinoid domain in X. Thus, replacing X by the reduction of $\{x\}$, we get a morphism $f : X = \{x\} \to Y = \{y\}$. The field $\mathscr{H}(x)$ is a zero-dimensional $\mathscr{H}(y)$-affinoid algebra.

It follows that the extension $\mathscr{H}(x)/\mathscr{H}(y)$ is finite, and therefore f is a finite morphism. One has $\mathscr{H}(y) = \{ \sum_{i=-\infty}^{\infty} a_i \mathrm{T}^i \mid |a_i| r^i \to 0 \text{ as } i \to \infty \}$ and $|\mathscr{H}(y)^*| = |\mathscr{H}(x)^*|$.

Suppose first that the valuation on k is trivial. Then $|\mathscr{H}(y)^*|$ is a discrete subgroup of $\mathbf{R}_+^*$, and from this it follows that $\mathscr{H}(y) \xrightarrow{\sim} \mathscr{H}(x)$. This means that f is an isomorphism $\mathrm{X} \xrightarrow{\sim} \mathrm{A}(r, r)$. Moreover, any finite étale covering of X is of the same type. Therefore the property $(*)$ holds for the point x.

Suppose now that the valuation on k is nontrivial. Then the extension $\mathscr{H}(x)/\mathscr{H}(y)$ is separable. Indeed, suppose that $\varphi = \sum_{i=-\infty}^{\infty} a_i \mathrm{T}^i \in \mathscr{H}(y)$ and $\varphi^{1/p} \in \mathscr{H}(x) \backslash \mathscr{H}(y)$. We may assume that $a_i = 0$ for i divisible by p. Then $|\varphi| = |a_n| r^n$ for some n prime to p. We get $|\varphi^{1/p}| = (|a_n| r)^{1/p} \notin |\mathscr{H}(x)^*| = |\mathscr{H}(y)^*|$. Thus, $f: \mathrm{X} \to \mathrm{Y}$ is a finite étale morphism. By Corollary 3.4.2, f extends to a finite étale morphism $\bar{f}: \overline{\mathrm{X}} \to \overline{\mathrm{Y}} = \mathrm{A}(r', r'')$ for some $r' < r < r''$ with $r', r'' \in |k^*|$. Since $\overline{\mathrm{Y}}$ is strictly k-affinoid, then so is $\overline{\mathrm{X}}$, and we are done. ∎

6.4.2. *Remark.* — Both statements of Theorem 6.4.1 are not true without the assumption that n is prime to $\mathrm{char}(\widetilde{k})$. For example, assume that $\mathrm{char}(k) = 0$ and $p = \mathrm{char}(\widetilde{k}) > 0$, and let X be the closed unit disc in $\mathbf{A}^1$. Since $\mathrm{Pic}(\mathrm{X}) = 0$, the Kummer exact sequence implies that

$$\mathrm{H}^1(\mathrm{X}, \mu_p) = k\{\mathrm{T}\}^* / k\{\mathrm{T}\}^{*p}.$$

One has $k\{\mathrm{T}\}^* = \{ f \in k\{\mathrm{T}\} \mid \|f - f(0)\| < \|f\| \}$. Therefore the correspondence $f \mapsto f'(0)/f(0)$ gives a surjective homomorphism from $k\{\mathrm{T}\}^*$ to the maximal ideal k^{00} of the ring of integers k^0. It induces a surjective homomorphism

$$\mathrm{H}^1(\mathrm{X}, \mu_p) \to k^{00}/pk^{00}.$$

If now K is an algebraically closed non-Archimedean field over k for which $\mathrm{K}^{00}/p\mathrm{K}^{00}$ is bigger than k^{00}/pk^{00}, then the group $\mathrm{H}^1(\mathrm{X} \widehat{\otimes} \mathrm{K}, \mu_p)$ does not coincide with $\mathrm{H}^1(\mathrm{X}, \mu_p)$. By the way, in Drinfeld's calculation of the group $\mathrm{H}^1(\mathrm{X}, \mu_n)$ for a standard affinoid domain in $\mathbf{A}^1$ ([Dr], 10.1) one also should assume that n is prime to $\mathrm{char}(k)$, otherwise the result stated is not true. (The same is repeated in [FrPu], V. 3.7.)

§ 7. Main Theorems

7.1. The Comparison Theorem for Cohomology with Compact Support

Recall that a morphism of schemes $\varphi : \mathscr{Y} \to \mathscr{X}$ is called *compactifiable* if there is a commutative diagram

$$\begin{array}{ccc} \mathscr{Y} & \overset{j}{\hookrightarrow} & \overline{\mathscr{Y}} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\bar{\varphi}} & \\ & \mathscr{X} & \end{array}$$

where $\bar{\varphi}$ is a proper morphism and j is an open immersion. For a sheaf $\mathscr{G}$ on $\mathscr{Y}$ one defines

$$R^q \varphi_! \mathscr{G} = R^q \bar{\varphi}_*(j_! \mathscr{G}).$$

By Nagata's Theorem, any separated morphism of finite type is compactifiable.

7.1.1. *Theorem. — Let $\mathscr{X}$ be a compactifiable scheme over k, and let $\mathscr{F}$ be an abelian torsion sheaf on $\mathscr{X}$. Then for any $q \geqslant 0$ there is a canonical isomorphism*

$$H^q_c(\mathscr{X}, \mathscr{F}) \xrightarrow{\sim} H^q_c(\mathscr{X}^{an}, \mathscr{F}^{an}).$$

Proof. — Let $\varphi : \mathscr{Y} \to \mathscr{X}$ be a compactifiable morphism between schemes of locally finite type over $\mathrm{Spec}(\mathscr{A})$, where $\mathscr{A}$ is a k-affinoid algebra (the situation is slightly more general for a further use). For a point $\mathbf{x} \in \mathscr{X}$ we denote by $\mathscr{Y}_{\mathbf{x}}$ the fibre of φ at $\mathbf{x}$ and set $\mathscr{Y}_{\bar{\mathbf{x}}} = \mathscr{Y}_{\mathbf{x}} \otimes k(\mathbf{x})^s$, and for an abelian torsion sheaf $\mathscr{G}$ on $\mathscr{Y}$, we denote by $\mathscr{G}_{\mathbf{x}}$ and $\mathscr{G}_{\bar{\mathbf{x}}}$ the pullbacks of $\mathscr{G}$ on $\mathscr{Y}_{\mathbf{x}}$ and $\mathscr{Y}_{\bar{\mathbf{x}}}$, respectively. By the Base Change Theorem for Cohomology with Compact Support for schemes, one has

$$(R^q \varphi_! \mathscr{G})_{\bar{\mathbf{x}}} = H^q_c(\mathscr{Y}_{\bar{\mathbf{x}}}, \mathscr{G}_{\bar{\mathbf{x}}}).$$

Furthermore, for a point $x \in \mathscr{X}^{an}$ over $\mathbf{x}$, we fix an embedding of fields $k(\mathbf{x})^s \hookrightarrow \mathscr{H}(x)^s$. It gives rise to an isomorphism

$$(\mathscr{Y}_{\bar{\mathbf{x}}} \otimes_{k(\mathbf{x})^s} \widehat{\mathscr{H}(x)^a})^{an} \xrightarrow{\sim} \mathscr{Y}^{an}_{\bar{x}}.$$

Since the cohomology with compact support of schemes are preserved under separably closed extensions of the ground field, one has

$$H^q_c(\mathscr{Y}_{\bar{\mathbf{x}}}, \mathscr{G}_{\bar{\mathbf{x}}}) \xrightarrow{\sim} H^q_c(\mathscr{Y}_{\bar{\mathbf{x}}} \otimes_{k(\mathbf{x})^s} \widehat{\mathscr{H}(x)^a}, \mathscr{G}_{\bar{\mathbf{x}}}).$$

Finally, the Weak Base Change Theorem 5.3.1 tells that

$$R^q \varphi^{an}_!(\mathscr{G}^{an})_{\bar{x}} = H^q_c(\mathscr{Y}^{an}_{\bar{x}}, \mathscr{G}^{an}_{\bar{x}}).$$

We use the above remarks to establish the following two facts.

7.1.2. *Lemma. — In the above situation assume that the dimension of φ is at most one. Then for any $q \geqslant 0$ one has*

$$(R^q \varphi_! \mathscr{G})^{an} \xrightarrow{\sim} R^q \varphi^{an}_! \mathscr{G}^{an}.$$

Furthermore, if in addition $\mathscr{X}$ and $\mathscr{Y}$ are compactifiable over k and the theorem is true for $\mathscr{X}$, then it is also true for $\mathscr{Y}$.

Proof. — The first statement follows from Theorem 6.1.1 and the above remarks, and the homomorphism of Leray spectral sequences

$$\begin{array}{ccc} H^p_c(\mathscr{X}, R^q\varphi_! \mathscr{G}) & \Longrightarrow & H^{p+q}_c(\mathscr{Y}, \mathscr{G}) \\ \downarrow & & \downarrow \\ H^p_c(\mathscr{X}^{an}, R^q\varphi^{an}_! \mathscr{G}^{an}) & \Longrightarrow & H^{p+q}_c(\mathscr{Y}^{an}, \mathscr{G}^{an}) \end{array}$$

shows that the second statement is also true. ■

7.1.3. *Lemma. — In the above situation assume that the morphism φ is finite and surjective. Then if $\mathscr{X}$ and $\mathscr{Y}$ are compactifiable over k and the theorem is true for $\mathscr{Y}$, then it is also true for $\mathscr{X}$.*

Proof. — Let $\mathscr{F}$ be an abelian torsion sheaf on $\mathscr{X}$. Since φ is finite and surjective, the canonical homomorphism $\mathscr{F} \to \varphi_*(\varphi^* \mathscr{F})$ is injective. Furthermore, by Corollary 4.3.2, for an abelian sheaf $\mathscr{G}$ on $\mathscr{Y}$ one has $H^q_c(\mathscr{X}^{an}, \varphi^{an}_* \mathscr{G}^{an}) \xrightarrow{\sim} H^q_c(\mathscr{Y}^{an}, \mathscr{G}^{an})$. In particular, the hypothesis implies that the theorem is true for all sheaves of the form $\varphi_* \mathscr{G}$. Thus, one can construct a resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$, such that the theorem is true for all $\mathscr{F}^i$. The homomorphism of spectral sequences

$$\begin{array}{ccc} H^p(H^q_c(\mathscr{X}, \mathscr{F}^\cdot)) & \Longrightarrow & H^{p+q}_c(\mathscr{X}, \mathscr{F}) \\ \downarrow & & \downarrow \\ H^p(H^q_c(\mathscr{X}^{an}, \mathscr{F}^{an\cdot})) & \Longrightarrow & H^{p+q}_c(\mathscr{X}^{an}, \mathscr{F}^{an}) \end{array}$$

shows that the theorem is also true for $\mathscr{F}$. ■

We are now ready to prove the theorem.

1) *The theorem is true for the direct product* $(P^1)^n$, *where* P^1 *is a the projective line over k.* — This follows from Lemma 7.1.2.

2) *The theorem is true for the projective space* P^n *over k.* — Indeed, there exists a finite surjective morphism $(P^1)^n \to P^n$, and we can use Lemma 7.1.3.

3) *The theorem is true for any projective scheme over k.* — This follows from Lemma 7.1.2.

4) *The theorem is true for any affine scheme of finite type over k.* — This is so because such a scheme is isomorphic to an open subscheme of a projective scheme.

5) *The theorem is true for any proper scheme over k.* — For a proper scheme $\mathscr{X}$ we can find an open everywhere dense affine subscheme $\mathscr{U}$, and so the dimension of the closed subscheme $\mathscr{Y} = \mathscr{X} \backslash \mathscr{U}$ is strictly less than the dimension of $\mathscr{X}$. Therefore the statement is obtained by induction using the exact cohomological sequences associated with the embeddings $\mathscr{U} \hookrightarrow \mathscr{X} \leftarrow \mathscr{Y}$ and $\mathscr{U}^{an} \hookrightarrow \mathscr{X}^{an} \leftarrow \mathscr{Y}^{an}$ and the five-lemma.

6) *The theorem is true for any compactifiable scheme.* — This is already clear. ■

7.1.4. *Corollary. — Let $\varphi : \mathscr{Y} \to \mathscr{X}$ be a compactifiable morphism between schemes of locally finite type over* $\mathrm{Spec}(\mathscr{A})$, *where $\mathscr{A}$ is a k-affinoid algebra, and let $\mathscr{G}$ be an abelian torsion sheaf on $\mathscr{Y}$. Then for any $q \geq 0$ there is a canonical isomorphism*

$$(R^q\varphi_! \mathscr{G})^{an} \xrightarrow{\sim} R^q\varphi^{an}_! \mathscr{G}^{an}.$$

Proof. — The statement is obtained from Theorem 7.1.1, using the remarks from the beginning of its proof. ∎

Let $\varphi : Y \to X$ be a morphism of k-analytic spaces, and let G be a sheaf on Y. We say that the pair (φ, G) is *quasialgebraic* if, for each point $x \in X$, one has $Y_x = \mathscr{X}^{an}$ and $G_x = \mathscr{G}^{an}$, where $\mathscr{X}$ is a compactifiable scheme over the field $\mathscr{H}(x)$ and $\mathscr{G}$ is a sheaf on $\mathscr{X}$. For example, for the canonical projection $\mathrm{pr} : Y = X \times \mathscr{Y}^{an} \to X$, where $\mathscr{Y}$ is a compactifiable scheme over k, the pair $(\mathrm{pr}, \mu_{n, Y})$ is quasialgebraic.

7.1.5. *Corollary.* — *Let $\varphi : Y \to X$ be a Hausdorff morphism of k-analytic spaces, and let $f : X' \to X$ be a morphism of analytic spaces over k, which give rise to a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

Furthermore, let G be an abelian torsion sheaf on Y and assume that the pair (φ, G) is quasialgebraic. Then for any $q \geqslant 0$ there is a canonical isomorphism

$$f^*(R^q \varphi_! G) \tilde{\to} R^q \varphi'_!(f'^* G).$$

Proof. — Let $x \in X$ and $x' \in X'$ be a pair of points with $x = f(x')$. By hypothesis, one can find a compactifiable scheme $\mathscr{X}$ over $\mathscr{H}(x)$ and a sheaf $\mathscr{G}$ on $\mathscr{X}$ with $Y_x = \mathscr{X}^{an}$ and $G_x = \mathscr{G}^{an}$. One has $Y'_{x'} = (\mathscr{X} \otimes_{\mathscr{H}(x)} \mathscr{H}(x'))^{an}$. The statement follows from the Weak Base Change Theorem 5.3.1, Theorem 7.1.1 and the fact that the cohomology with compact support of schemes are preserved under separably closed extensions of the ground field. ∎

7.2. The trace mapping

In this subsection we fix an integer n which is prime to the characteristic of the field k.

Our goal is to extend the construction of the trace mapping from § 5.4 and § 6.2 to any separated smooth morphism $\varphi : Y \to X$ of pure dimension d, i.e., to construct a canonical homomorphism of sheaves

$$\mathrm{Tr}_\varphi : R^{2d} \varphi_!(\mu_{n, Y}^d) \to (\mathbf{Z}/n\mathbf{Z})_X.$$

As in the Theorems 5.4.1 and 6.2.1, we will characterize the trace mapping by certain properties.

Let $\varphi : Y \to X$ be a Hausdorff morphism of k-analytic spaces, and let $f : X' \to X$ be a morphism of analytic spaces over k. They give rise to a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

Suppose we are given two mappings $\alpha : R^{2d}\varphi_!(\mu^d_{n,Y}) \to (\mathbf{Z}/n\mathbf{Z})_X$ and

$$\alpha' : R^{2d}\varphi'_!(\mu^d_{n,Y'}) \to (\mathbf{Z}/n\mathbf{Z})_{X'}.$$

We say that α and α' are compatible with base change if the following diagram is commutative

$$\begin{array}{ccc} f^*(R^{2d}\varphi_!(\mu^d_{n,Y})) & \longrightarrow & R^{2d}\varphi'_!(\mu^d_{n,Y'}) \\ \downarrow{\scriptstyle f^*(\alpha)} & & \downarrow{\scriptstyle \alpha'} \\ f^*((\mathbf{Z}/n\mathbf{Z})_X) & \xrightarrow{\sim} & (\mathbf{Z}/n\mathbf{Z})_{X'} \end{array}$$

Here the upper arrow is the base change morphism, and the lower isomorphism is the canonical one.

Furthermore, let $\varphi : Y \to X$ and $\psi : Z \to Y$ be Hausdorff morphisms whose dimensions are at most d and e, respectively. Suppose we are given three homomorphisms $\alpha : R^{2d}\varphi_!(\mu^d_{n,Y}) \to (\mathbf{Z}/n\mathbf{Z})_X$, $\beta : R^{2e}\psi_!(\mu^e_{n,Z}) \to (\mathbf{Z}/n\mathbf{Z})_Y$ and

$$\gamma : R^{2(d+e)}(\varphi\psi)_!\,(\mu^{d+e}_{n,Z}) \to (\mathbf{Z}/n\mathbf{Z})_X.$$

Using the Leray spectral sequence 5.2.2 and Corollary 5.3.8, we get an isomorphism

$$R^{2(d+e)}(\varphi\psi)_!\,(\mu^{d+e}_{n,Z}) \xrightarrow{\sim} R^{2d}\varphi_!(R^{2e}\psi_!(\mu^{d+e}_{n,Z})).$$

Corollary 5.3.11 gives an isomorphism

$$R^{2e}\psi_!(\mu^{d+e}_{n,Z}) \xrightarrow{\sim} R^{2e}\psi_!(\mu^e_{n,Z}) \otimes \mu^d_{n,Y}.$$

We get a mapping

$$\begin{aligned} R^{2(d+e)}(\varphi\psi)_!\,(\mu^{d+e}_{n,Z}) &\xrightarrow{\sim} R^{2d}\varphi_!(R^{2e}\psi_!(\mu^e_{n,Z}) \otimes \mu^d_{n,Y}) \\ &\to R^{2d}\varphi_!(\mu^d_{n,Y}) \to (\mathbf{Z}/n\mathbf{Z})_X \end{aligned}$$

The composition mapping is denoted by $\alpha \mathbin{\square} \beta$. We remark that if α and β are isomorphisms (resp. epimorphisms), then $\alpha \mathbin{\square} \beta$ is also an isomorphism (resp. epimorphism). We say that the mappings α, β and γ are compatible with composition if $\gamma = \alpha \mathbin{\square} \beta$. Note that the operation $\square$ is transitive.

7.2.1. *Theorem. — One can assign to every separated smooth morphism $\varphi : Y \to X$ of pure dimension d a trace mapping*

$$\mathrm{Tr}_\varphi : R^{2d}\varphi_!(\mu^d_{n,Y}) \to (\mathbf{Z}/n\mathbf{Z})_X.$$

These mappings have the following properties and are uniquely determined by them:

a) Tr_φ *are compatible with base change;*

b) Tr_φ *are compatible with composition;*

c) if $d = 0$ *(i.e.,* φ *is étale), then* Tr_φ *is the trace mapping* $\varphi_!(\mathbf{Z}/n\mathbf{Z})_Y \to (\mathbf{Z}/n\mathbf{Z})_X$ *from* § 5.4;

d) if $X = \mathscr{M}(k)$, *k is algebraically closed and* $d = 1$, *then* Tr_φ *is the trace mapping* $\mathrm{Tr}_Y : H^2_c(Y, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$ *from* § 6.2.

Furthermore, if the fibres of φ *are nonempty, then* Tr_φ *is an epimorphism. If in addition the geometric fibres of* φ *are nonempty and connected and n is prime to* $\mathrm{char}(\widetilde{k})$, *then* Tr_φ *is an isomorphism.*

Proof. — First of all, let φ be the morphism $\pi^d : \mathbf{A}^d \to \mathscr{M}(k)$. It is the analytification of the morphism of schemes $\psi : A^d \to \mathrm{Spec}(k)$. One has a trace mapping $\mathrm{Tr}_\psi : R^{2d}\,\psi_!(\mu^d_{n,A^d}) \to (\mathbf{Z}/n\mathbf{Z})_{\mathrm{Spec}(k)}$ (which is an isomorphism). Its analytification (Corollary 7.1.4) gives rise to a trace mapping $\mathrm{Tr}_{\pi^d} : R^{2d}\,\pi^d_!(\mu^d_{n,\mathbf{A}^d}) \to (\mathbf{Z}/n\mathbf{Z})_{\mathscr{M}(k)}$ (which is also an isomorphism). Furthermore, if φ is the morphism $\pi^d_X : \mathbf{A}^d_X = X \times \mathbf{A}^d \to X$, then we define $\mathrm{Tr}_{\pi^d_X}$ as the base change of Tr_{π^d} (Corollary 7.1.5).

7.2.2. *Lemma.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism which can be represented as a composition of an étale morphism* $f : Y \to \mathbf{A}^d_X$ *with the projection* $\pi^d_X : \mathbf{A}^d_X \to X$. *Then the mapping*

$$\mathrm{Tr}_\varphi = \mathrm{Tr}_{\pi^d_X} \circ \mathrm{Tr}_f : R^{2d}\,\varphi_!(\mu^d_{n,Y}) \to (\mathbf{Z}/n\mathbf{Z})_X$$

does not depend on the representation.

Proof. — We may increase the field k and assume that its valuation is nontrivial. If $d = 1$, the statement is obtained from the case of curves (Theorem 6.2.1), using the Weak Base Change Theorem 5.3.1. Suppose that $d \geqslant 2$. The Weak Base Change Theorem 5.3.1 reduces the situation to the case when we are given a separated connected smooth k-analytic space X for algebraically closed k and elements $f_1, \ldots, f_d \in \mathcal{O}(X)$ such that the morphism $f : X \to \mathbf{A}^d$ that they define is étale. We have to verify that the mapping

$$\mathrm{Tr}_X^{(f_1, \ldots, f_d)} = \mathrm{Tr}_{\mathbf{A}^d} \circ \mathrm{Tr}_f : H^{2d}_c(X, \mu^d_n) \to H^{2d}_c(\mathbf{A}^d, \mu^d_n) \to \mathbf{Z}/n\mathbf{Z}$$

does not depend on the choice of the elements $f_1, \ldots, f_d$.

First of all, this mapping is independent of the ordering of the elements $f_1, \ldots, f_d$ since the group $\mathrm{GL}_d(k)$ acts trivially on $H^{2d}_c(\mathbf{A}^d, \mu^d_n) = H^{2d}_c(A^d, \mu^d_n)$.

Let $g_1, \ldots, g_d$ be another system of elements in $\mathcal{O}(X)$ for which the corresponding morphism $g : X \to \mathbf{A}^d$ is étale. Take an arbitrary point $x \in X(k)$ (such a point exists because the field k is algebraically closed, and its valuation is nontrivial). Replacing f_i by $f_i - f_i(x)$ and g_i by $g_i - g_i(x)$, we may assume that the elements $f_1, \ldots, f_d, g_1, \ldots, g_d$ are contained in the maximal ideal $\mathbf{m}_x$ of the local ring $\mathcal{O}_{X,x}$. But then $(f_1, \ldots, f_d)$ and $(g_1, \ldots, g_d)$ are regular systems of parameters for $\mathcal{O}_{X,x}$, i.e., they form two bases of the k-vector space $\mathbf{m}_x/\mathbf{m}_x^2$. Applying Steinitz Exchange Theorem for these bases, we can find a finite chain of systems of elements

$$(f_1, \dots, f_d) = (f_1^{(1)}, \dots, f_d^{(1)}), (f_1^{(2)}, \dots, f_d^{(2)}), \dots, (f_1^{(m)}, \dots, f_d^{(m)}) = (g_1, \dots, g_d)$$

such that

a) each $f_j^{(i)}$ is one of the elements $f_1, \dots, f_d, g_1, \dots, g_d$;

b) each system gives rise to a basis of $\mathbf{m}_x/\mathbf{m}_x^2$;

c) each system $(f_1^{(i+1)}, \dots, f_d^{(i+1)})$ arises from $(f_1^{(i)}, \dots, f_d^{(i)})$ by the replacement of just one element.

By Proposition 3.3.10, each of the systems $(f_1^{(i)}, \dots, f_d^{(i)})$ gives rise to an étale morphism $f^{(i)} : X' \to \mathbf{A}^d$, where X' is a nonempty Zariski open subset of X. We remark now that if $Y = X \backslash X'$, then $\dim(Y) \leqslant d - 1$, and therefore the canonical homomorphism $H_c^{2d}(X', \mu_n) \to H_c^{2d}(X, \mu_n)$ is bijective. Thus it suffices to show that if $f_1, \dots, f_{d-1}, g$ and $f_1, \dots, f_{d-1}, h$ are two systems of elements in $\mathcal{O}(X)$ which give rise to étale morphisms $\varphi : X \to \mathbf{A}^d$ and $\psi : X \to \mathbf{A}^d$, respectively, then $\mathrm{Tr}_X^{(f_1, \dots, f_{d-1}, g)} = \mathrm{Tr}_X^{(f_1, \dots, f_{d-1}, h)}$.

Let π be the projection $\mathbf{A}^d \to \mathbf{A}^{d-1}$ on the first $d-1$ coordinates. Since $\pi \circ \varphi = \pi \circ \psi$, it follows, by the case $d = 1$, that $\mathrm{Tr}_\pi \circ \mathrm{Tr}_\varphi = \mathrm{Tr}_\pi \circ \mathrm{Tr}_\psi$. We have

$$\begin{aligned} \mathrm{Tr}_X^{(f_1, \dots, f_{d-1}, g)} &= (\mathrm{Tr}_{\mathbf{A}^{d-1}} \square \mathrm{Tr}_\pi) \circ \mathrm{Tr}_\varphi = \mathrm{Tr}_{\mathbf{A}^{d-1}} \square (\mathrm{Tr}_\pi \circ \mathrm{Tr}_\varphi) \\ &= \mathrm{Tr}_{\mathbf{A}^{d-1}} \square (\mathrm{Tr}_\pi \circ \mathrm{Tr}_\psi) = (\mathrm{Tr}_{\mathbf{A}^{d-1}} \square \mathrm{Tr}_\pi) \circ \mathrm{Tr}_\psi \\ &= \mathrm{Tr}_X^{(f_1, \dots, f_{d-1}, h)}. \end{aligned}$$

The lemma is proved. ■

The construction of the trace mapping for arbitrary φ is obtained from Lemma 7.2.2 in the same way as the corresponding construction in Theorem 6.2.1 is obtained from Corollary 6.2.8.

It remains to show that if the geometric fibres of φ are nonempty and connected and n is prime to $\mathrm{char}(\widetilde{k})$, then Tr_φ is an isomorphism. By the Weak Base Change Theorem 5.3.1, it suffices to show that if k is algebraically closed and X is a separated connected smooth k-analytic space of dimension d, then $\mathrm{Tr}_X : H_c^{2d}(X, \mu_n^d) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$. We remark that it suffices to find for such a space X an étale covering $(U_i \to X)_{i \in I}$ with separated U_i such that all Tr_{U_i} are isomorphisms. This is verified by induction. To use the induction, it suffices to show that for any point $x \in X$ there exists a separated étale morphism $f : U \to X$ and a smooth morphism $\varphi : U \to V$ of pure dimension one to a separated smooth k-analytic space V such that $x \in f(U)$ and the geometric fibres of φ are nonempty and connected. For this we shrink X and take an étale morphism $X \to \mathbf{A}^d$. Let ψ be the composition of the latter morphism with the projection $\mathbf{A}^d \to \mathbf{A}^{d-1}$. The morphism $\psi : X \to \mathbf{A}^{d-1}$ is smooth of pure dimension one. Applying Theorem 3.7.2, we get the required morphisms f and φ. ■

7.2.3. *Corollary.* — *Let* $\varphi : \mathcal{Y} \to \mathcal{X}$ *be a separated smooth morphism of pure dimension* d *between schemes of locally finite type over* $\mathrm{Spec}(\mathcal{A})$, *where* $\mathcal{A}$ *is a* k*-affinoid algebra. Then the diagram*

$$\begin{array}{ccc} (\mathrm{R}^{2d}\,\varphi_!(\mu^d_{n,\mathcal{Y}}))^{\mathrm{an}} & \xrightarrow{\sim} & \mathrm{R}^{2d}\,\varphi^{\mathrm{an}}_!(\mu^d_{n,\mathcal{Y}^{\mathrm{an}}}) \\ & {\scriptstyle(\mathrm{Tr}_\varphi)^{\mathrm{an}}}\searrow \quad \swarrow{\scriptstyle\mathrm{Tr}_{\varphi^{\mathrm{an}}}} & \\ & (\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}^{\mathrm{an}}} & \end{array}$$

is commutative. ∎

Let $\varphi : \mathrm{Y} \to \mathrm{X}$ be a separated smooth morphism of pure dimension d. By Corollary 5.3.11, for any $\mathrm{F}^\bullet \in \mathrm{D}(\mathrm{X}, \mathbf{Z}/n\mathbf{Z})$ there is a canonical isomorphism

$$\mathrm{R}\varphi_!(\varphi^*\,\mathrm{F}^\bullet(d)\,[2d]) \xrightarrow{\sim} \mathrm{F}^\bullet \overset{\mathrm{L}}{\otimes} \mathrm{R}\varphi_!(\mu^d_{n,\mathrm{Y}})\,[2d].$$

Theorem 7.2.1 gives a morphism (in $\mathrm{D}(\mathrm{X}, \mathbf{Z}/n\mathbf{Z})$)

$$\mathrm{R}\varphi_!(\mu^d_{n,\mathrm{Y}})\,[2d] \to (\mathbf{Z}/n\mathbf{Z})_{\mathrm{X}}.$$

Therefore we get a morphism

$$\mathrm{Tr}_\varphi : \mathrm{R}\varphi_!(\varphi^*\,\mathrm{F}^\bullet(d)\,[2d]) \to \mathrm{F}^\bullet$$

which will also be called a trace mapping. For $\mathrm{F} \in \mathbf{S}(\mathrm{X}, \mathbf{Z}/n\mathbf{Z})$ the latter morphism is induced by a homomorphism of sheaves $\mathrm{R}^{2d}\,\varphi_!(\varphi^*\,\mathrm{F}(d)) \to \mathrm{F}$. It is an isomorphism if the geometric fibres of φ are nonempty and connected and n is prime to $\mathrm{char}(\widetilde{k})$ because $\mathrm{R}^{2d}\,\varphi_!(\varphi^*\,\mathrm{F}(d)) = \mathrm{F} \otimes \mathrm{R}^{2d}\,\varphi_!(\mu^d_{n,\mathrm{Y}})$.

7.3. Poincaré Duality

First of all we want to fix notation. Let X be a k-analytic space, and let Λ be a ring (in practice, $\Lambda = \mathbf{Z}/n\mathbf{Z}$). The functors of homomorphisms Hom and of germs of homomorphisms $\mathcal{H}om$ (between Λ_{X}-modules) have derived functors

$$\underline{\mathrm{Hom}} : \mathrm{D}(\mathrm{X}, \Lambda)^0 \times \mathrm{D}^+(\mathrm{X}, \Lambda) \to \mathrm{D}(\Lambda)$$

and

$$\underline{\mathcal{H}om} : \mathrm{D}(\mathrm{X}, \Lambda)^0 \times \mathrm{D}^+(\mathrm{X}, \Lambda) \to \mathrm{D}(\mathrm{X}, \Lambda),$$

where $\mathrm{D}(\Lambda)$ is the derived category of Λ-modules. Recall that if $\mathrm{G}^\bullet$ is bounded below and $\mathrm{G}^\bullet \to \mathrm{J}^\bullet$ is an injective resolution of $\mathrm{G}^\bullet$, then $\underline{\mathrm{Hom}}(\mathrm{F}^\bullet, \mathrm{G}^\bullet) = \mathrm{Hom}^\bullet(\mathrm{F}^\bullet, \mathrm{J}^\bullet)$ and $\underline{\mathcal{H}om}(\mathrm{F}^\bullet, \mathrm{G}^\bullet) = \mathcal{H}om^\bullet(\mathrm{F}^\bullet, \mathrm{J}^\bullet)$. One sets

$$\mathrm{Ext}^q(\mathrm{F}^\bullet, \mathrm{G}^\bullet) = \mathrm{H}^q(\underline{\mathrm{Hom}}(\mathrm{F}^\bullet, \mathrm{G}^\bullet))$$

and

$$\mathcal{E}xt^q(\mathrm{F}^\bullet, \mathrm{G}^\bullet) = \mathrm{H}^q(\underline{\mathcal{H}om}(\mathrm{F}^\bullet, \mathrm{G}^\bullet)).$$

For sheaves $F, G \in \mathbf{S}(X, \Lambda)$ these are the usual functors Ext and $\mathcal{E}xt$, and $\mathcal{E}xt^q(F, G)$ is the sheaf associated with the presheaf $(U \to X) \mapsto \mathrm{Ext}^q(F|_U, G|_U)$. We remark that if a Λ-module G is injective, then the sheaf $\mathcal{H}om(F, G)$ is flabby.

Let now $\varphi : Y \to X$ be a separated smooth morphism of pure dimension d, and let n be an integer. By functoriality of $R\varphi_!$, for any complexes $G^\cdot$ and $G'^\cdot$ of sheaves of $(\mathbf{Z}/n\mathbf{Z})_Y$-modules which are bounded above and below, respectively, there is a canonical morphism of complexes

$$\varphi_*(\mathcal{H}om^\cdot(G^\cdot, G'^\cdot)) \to \mathcal{H}om^\cdot(R\varphi_! G^\cdot, R\varphi_! G'^\cdot).$$

Let $G'^\cdot \to J^\cdot$ be an injective resolution of $G'^\cdot$. Then the complexes $\mathcal{H}om^\cdot(G^\cdot, J^\cdot)$ on Y and $\varphi_! J^\cdot$ on X consist of flabby sheaves. Therefore there is the following canonical morphisms in $D^+(X, \mathbf{Z}/n\mathbf{Z})$

$$R\varphi_*(\underline{\mathcal{H}om}(G^\cdot, G'^\cdot)) = \varphi_*(\mathcal{H}om^\cdot(G^\cdot, J^\cdot)) \to \mathcal{H}om^\cdot(R\varphi_! G^\cdot, R\varphi_! J^\cdot) = \underline{\mathcal{H}om}(R\varphi_! G^\cdot, R\varphi_! G'^\cdot).$$

Assume now that n is prime to $\mathrm{char}(k)$. Then applying this morphism to complexes $G'^\cdot$ of the form $\varphi^* F^\cdot(d)\,[2d]$ and using the trace mapping $R\varphi_!(\varphi^* F^\cdot(d)\,[2d]) \to F^\cdot$, we obtain for any $G^\cdot \in D^-(Y, \mathbf{Z}/n\mathbf{Z})$ and $F^\cdot \in D^+(X, \mathbf{Z}/n\mathbf{Z})$ a *duality morphism*

$$R\varphi_*(\underline{\mathcal{H}om}(G^\cdot, \varphi^* F^\cdot(d)\,[2d])) \to \underline{\mathcal{H}om}(R\varphi_! G^\cdot, F^\cdot).$$

7.3.1. *Theorem (Poincaré Duality Theorem). — Suppose that n is prime to* $\mathrm{char}(\widetilde{k})$. *Then the duality morphism is an isomorphism.*

We remark that the theorem is equivalent to the fact that, for all $q \in \mathbf{Z}$, the duality morphism induces isomorphisms

$$\mathrm{Ext}^q(G^\cdot, \varphi^* F^\cdot(d)\,[2d]) \xrightarrow{\sim} \mathrm{Ext}^q(R\varphi_! G^\cdot, F^\cdot).$$

By Remark 6.2.10, the theorem is not true without the assumption that n is prime to $\mathrm{char}(\widetilde{k})$.

Proof. — Fixing one of the complexes $G^\cdot$ or $F^\cdot$, one gets an exact functor with respect to the second complex which is way out right (see [Ha1], § I.7). Therefore it suffices to verify the theorem only for complexes of the form $G^\cdot = G$, where $G \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$, and $F^\cdot = F(-d)\,[-2d]$, where $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$. Thus it suffices to show that for all $q \geq 0$ the canonical mappings

$$\Phi^q(G, F) := \mathrm{Ext}^q(G, \varphi^* F) \to \Psi^q(G, F) := \mathrm{Hom}(R\varphi_! G, F(-d)\,[q-2d])$$

are isomorphisms. For example, in the case $d = 0$ the theorem follows from the fact that the functor $\varphi_!$ is left adjoint to the functor φ^*.

First of all we reduce the situation to the case when the space X (and therefore Y) is good. Indeed, assume that our statement is true in this case. Since the statement is local with respect to X, we can shrink X and assume that $X = \bigcup_{i=1}^{m} X_i$, where X_i are closed analytic domains isomorphic to open subsets of k-affinoid spaces (in particular, the X_i are good). Since any F has an embedding in the direct sum $\bigoplus_{i=1}^{m} \nu_{i*} F_i$, where $F_i = \nu_i^* F$ and ν_i is the canonical embedding $X_i \to X$, it suffices to verify that

$$\Phi^q(G, \nu_{i*} F_i) \tilde{\to} \Psi^q(G, \nu_{i*} F_i), \quad q \geqslant 0.$$

If ν_i' and φ_i' are defined by the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle \nu_i'} & & \uparrow{\scriptstyle \nu_i} \\ Y_i & \xrightarrow{\varphi_i'} & X_i \end{array}$$

there is a canonical isomorphism of sheaves $\varphi^* \nu_{i*} F_i \tilde{\to} \nu'_{i*} \varphi_i'^* F_i$ (the stalks of these sheaves are isomorphic). Therefore, $\Phi^q(G, \nu_{i*} F_i) = \Phi^q(\nu_i'^* G, F_i)$. Furthermore, the morphism $\nu_i : X_i \to X$ satisfies the condition of Corollary 5.3.6, and therefore $\nu_i^*(R\varphi_! G) \tilde{\to} R\varphi'_{i!}(\nu_i'^* G)$. It follows that $\Psi^q(G, \nu_{i*} F_i) = \Psi^q(\nu_i'^* G, F_i)$, and since the space X_i is good, the assumption gives an isomorphism $\Phi^q(G, \nu_{i*} F_i) \tilde{\to} \Psi^q(G, \nu_{i*} F_i)$.

Thus, we may assume that the spaces X and Y are good. Suppose that $d \geqslant 1$. We fix the sheaf F and set $\Phi^q(G) = \Phi^q(G, F)$ and $\Psi^q(G) = \Psi^q(G, F)$. Then $\{\Phi^q\}_{q \geqslant 0}$ and $\{\Psi^q\}_{q \geqslant 0}$ are exact contravariant ∂-functors from $\mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$ to $\mathcal{A}b$, the functors

$$\Phi^0(G) = \mathrm{Hom}(G, \varphi^* F) \quad \text{and} \quad \Psi^0(G) = \mathrm{Hom}(R^{2d} \varphi_! G, F(-d))$$

are left exact, and Φ^q are right satellites of the functor Φ^0. Thus to prove the theorem it suffices (and is necessary) to show that

a) $\Phi^0 \tilde{\to} \Psi^0$, and

b) the functors Ψ^q, $q \geqslant 1$, are right satellites of the functor Ψ^0.

The statement *b)* is equivalent to the fact that the functors Ψ^q, $q \geqslant 1$, are effaceable, i.e., for any $G \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$ and $\alpha \in \Psi^q(G)$ there exists an epimorphism of sheaves $G' \to G$ such that α goes to zero under the induced homomorphism $\Psi^q(G) \to \Psi^q(G')$.

Step 1. — $\Phi^0 \tilde{\to} \Psi^0$. Since the functors Φ^0 and Ψ^0 are left exact, it suffices to find a family of sheaves $\mathcal{M} \subset \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$ such that for any $G \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$ there exists an epimorphism $\bigoplus_{i \in I} M_i \to G$ with $M_i \in \mathcal{M}$ and, for any $M \in \mathcal{M}$, one has $\Phi^0(M) \tilde{\to} \Psi^0(M)$. We take for $\mathcal{M}$ the class of sheaves of the form $g_!(\mathbf{Z}/n\mathbf{Z})_V$, where $g : V \to Y$ is a separated étale morphism for which there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle f} \\ V & \xrightarrow{\psi} & U \end{array}$$

such that f is a separated étale morphism and ψ is a smooth morphism whose geometric fibres are nonempty and connected. From Corollary 3.7.3 (see also the proof of Proposition 4.4.5) it follows that any G is the epimorphic image of a direct sum of some sheaves from $\mathcal{M}$. Let $M = g_!(\mathbf{Z}/n\mathbf{Z})_V$. Then in the above notation we have

$$\Phi^0(M) = \mathrm{Hom}(g_!(\mathbf{Z}/n\mathbf{Z})_V, \varphi^* F) = \mathrm{Hom}((\mathbf{Z}/n\mathbf{Z})_V, \varphi^* F|_V) = \varphi^* F(V)$$

and

$$\begin{aligned}\Psi^0(M) &= \mathrm{Hom}(R^{2d}\varphi_!(g_!(\mathbf{Z}/n\mathbf{Z})_V), F(-d)) \\ &= \mathrm{Hom}(f_!(R^{2d}\psi_!(\mathbf{Z}/n\mathbf{Z})_V), F(-d)) \\ &= \mathrm{Hom}(R^{2d}\psi_!(\mathbf{Z}/n\mathbf{Z})_V, F(-d)|_U) \\ &= \mathrm{Hom}((\mathbf{Z}/n\mathbf{Z})_U(-d), F(-d)|_U) = F(U)\end{aligned}$$

because the geometric fibres of ψ are nonempty and connected and n is prime to $\mathrm{char}(\widetilde{k})$ (Theorem 7.2.1). Thus, it suffices to show that if $\varphi : Y \to X$ is a separated smooth morphism of pure dimension d with nonempty and connected geometric fibres, then for $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ the mapping

$$\begin{aligned}\chi : \varphi^* F(Y) &= \mathrm{Hom}((\mathbf{Z}/n\mathbf{Z})_Y, \varphi^* F) \\ &\to \mathrm{Hom}(R^{2d}\varphi_!(\mu_{n,Y}^d), R^{2d}\varphi_!\varphi^* F(d)) = F(X)\end{aligned}$$

is an isomorphism. Since $R^{2d}\varphi_!\varphi^* F(d) \tilde{\to} F \otimes R^{2d}\varphi_!(\mu_{n,Y}^d)$, the composition of the canonical mapping $F(X) \to \varphi^* F(Y)$ with χ is the identity on $F(X)$. Therefore the required fact follows from the following statement.

7.3.2. *Proposition. — Let $\varphi : Y \to X$ be a morphism of k-analytic spaces, and suppose that any étale base change of φ is an open map with nonempty and connected fibres. Then for any sheaf of sets F on X one has $F \tilde{\to} \varphi_* \varphi^* F$.*

7.3.3. *Lemma. — Any k-analytic space X contains a point x such that, for some embedding of fields $k^s \hookrightarrow \mathcal{H}(x)^s$, the image of the induced homomorphism of Galois groups $G_{\mathcal{H}(x)} \to G_k$ has finite index in G_k.*

Proof. — We may assume that X is k-affinoid. If X is strictly k-affinoid, then we take an arbitrary point x with $[\mathcal{H}(x) : k] < \infty$ (i.e., $x \in X_0$). Therefore it suffices to show that if $K = K_r$, where $r \notin \sqrt{|k^*|}$, and the statement is true for the K-affinoid space $X' = X \hat{\otimes} K$, then it is also true for X. For this we take a point $x' \in X'$ such that, for some embedding of fields $K^s \hookrightarrow \mathcal{H}(x')^s$, the image of the induced homomorphism $G_{\mathcal{H}(x')} \to G_K$ has finite index. Furthermore, we fix an embedding of fields $k^s \hookrightarrow K^s$. (We remark that the induced homomorphism $G_K \to G_k$ is surjective.) Let x be the image of the point x' in X. The above embeddings of fields induce an embedding $k^s \hookrightarrow \mathcal{H}(x)^s$, and we get a commutative diagram of homomorphisms

$$\begin{array}{ccc} G_{\mathcal{H}(x)} & \longrightarrow & G_k \\ \uparrow & & \uparrow \\ G_{\mathcal{H}(x')} & \longrightarrow & G_K \end{array}$$

where the right homomorphism is surjective and the image of the lower homomorphism has finite index. The required statement follows. ∎

Proof of Proposition **7.3.2.** — For a sheaf of sets G on Y and two elements $f, g \in G(Y)$ we set $\operatorname{Supp}(f-g) = \{y \in Y \mid f_y \neq g_y\}$. It is a closed subset of Y.

We claim that if $f, g \in \varphi^* F(Y)$, then $\operatorname{Supp}(f-g) = \varphi^{-1}(\Sigma)$ for some set $\Sigma \subset X$. (From this it follows that the set Σ is closed because φ is an open map.) Indeed, it suffices to show that if $X = \mathcal{M}(k)$ and $f \neq g$, then $\operatorname{Supp}(f-g) = Y$. Assume that $f_y \neq g_y$ at some point $y \in Y$. By the definition of $\varphi^* F$, there is an étale neighborhood $h : V \to Y$ of the point y such that V is a K-analytic space, where K is a finite separable extension of k, and the elements $f|_V$ and $g|_V$ come from some elements of F(K). Clearly, the latter elements are not equal. It follows that $f|_V$ and $g|_V$ do not coincide at all points of V, and therefore $f_{y'} \neq g_{y'}$ for all $y' \in h(V)$. Since h is an open map, it follows that the set $\operatorname{Supp}(f-g)$ is open, and therefore it coincides with Y because Y is connected.

Let now $f \in \varphi^* F(Y)$. It suffices to show that every point $x \in X$ has an étale neighborhood $X' \to X$ such that the restriction of f on $Y \times_X X'$ comes from an element $g \in F(X')$. By Lemma 7.3.3, we can replace X by an étale neighborhood of the point x and assume that there exists a point $y \in \varphi^{-1}(x)$ for which the homomorphism $G_{\mathscr{H}(y)} \to G_{\mathscr{H}(x)}$, induced by an embedding $\mathscr{H}(x)^s \hookrightarrow \mathscr{H}(y)^s$, is surjective. It follows that $F_x(\mathscr{H}(x)) \tilde{\to} (\varphi^* F)_y (\mathscr{H}(y))$. Therefore we can shrink X and find an element $g \in F(X)$ whose image in $(\varphi^* F)_y (\mathscr{H}(y))$ coincides with f_y. We have $\operatorname{Supp}(f-g) = \varphi^{-1}(\Sigma)$ for some closed set $\Sigma \subset X$. Since $x \notin \Sigma$, we can find an open neighborhood $\mathscr{U}$ of x such that $f|_{\varphi^{-1}(\mathscr{U})} = g$. ■

Step 2. — *The functors* Ψ^q, $q \geqslant 1$, *are effaceable for* $d = 1$. To show this we need the following fact which is an analog of the Fundamental Lemma 1.6.9 from [SGA4], Exp. XVIII.

7.3.4. *Fundamental Lemma.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism of pure dimension one, and suppose that* X *is good. Then for any point* $y \in Y$ *there exist separated étale morphisms* $f : X' \to X$ *and* $g : Y'' \to Y' = Y \times_X X'$

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \\ \uparrow{\scriptstyle g} & \nearrow{\scriptstyle \varphi''} & \\ Y'' & & \end{array}$$

such that $y \in f'(g(Y''))$ *and*

(i) *the homomorphism* $R^1 \varphi''_!(\mu_{n, Y''}) \to R^1 \varphi'_!(\mu_{n, Y'})$ *is zero*;

(ii) *the geometric fibres of* φ'' *are nonempty, noncompact and connected.*

Proof. — We replace X by an affinoid neighborhood of the point $x = \varphi(y)$. Let $X = \mathcal{M}(\mathscr{A})$.

1) *One can shrink* Y *and assume that there is a commutative diagram*

$$\begin{array}{ccc} Y & \overset{j}{\hookrightarrow} & \mathcal{Y}^{an} \\ & \underset{\varphi}{\searrow} & \downarrow \psi^{an} \\ & & X \end{array}$$

where $\psi : \mathcal{Y} \to \mathcal{X} = \mathrm{Spec}(\mathcal{A})$ *is a smooth affine curve of finite type over* $\mathcal{X}$, *and* j *is an open immersion.* This is Lemma 3.6.2.

We remark that from 1) it follows that $\varphi_!(\mu_{n,Y}) = 0$ and the same is true for any étale open subset of Y. Let $\mathcal{Y} = \mathrm{Spec}(B)$.

2) *One can shrink* Y *and* $\mathcal{Y}$ *and assume that there exist* $f_1, \ldots, f_m \in B$ *and* $\varepsilon_1, \ldots, \varepsilon_m > 0$ *such that*

$$Y = \{ y' \in \mathcal{Y}^{an} \mid |f_i(y')| < \varepsilon_i, \ 1 \leqslant i \leqslant m \}.$$

For this we remark that a basis of open sets in $\mathcal{Y}^{an}$ is formed by sets of the form

$$\{ y' \in \mathcal{Y}^{an} \mid |f_i(y')| < a_i, \ |g_i(y')| > b_j, \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m \},$$

where $f_i, g_j \in B$ and $a_i, b_j > 0$. Replacing B by the localization $B_{g_1 \ldots g_m}$, the second inequality can be rewritten as $|g_j^{-1}(y')| < b_j^{-1}$, and we are done.

3) *In the situation of* 2) *the canonical homomorphism*

$$R^1 \varphi_!(\mu_{n,Y}) \to R^1 \psi_!^{an}(\mu_{n,\mathcal{Y}^{an}})$$

is injective. (We remark that, by Corollary 7.1.4, the latter sheaf coincides with $(R^1 \psi_! \, \mu_{n,\mathcal{Y}})^{an}$.) By the Weak Base Change Theorem 5.3.1, it suffices to verify the following fact. Suppose that $\mathcal{X}$ is an affine curve of finite type over k, $f_1, \ldots, f_m \in \mathcal{O}(\mathcal{X})$, and $\mathcal{U} = \{ x \in \mathcal{X}^{an} \mid |f_i(x)| < \varepsilon_i, \ 1 \leqslant i \leqslant m \}$ for some $\varepsilon_i > 0$. Then the canonical homomorphism $H_c^1(\mathcal{U}, \mu_n) \to H_c^1(\mathcal{X}^{an}, \mu_n)$ is injective. We may assume that the set $S = \mathcal{X}^{an} \backslash \mathcal{U}$ is nonempty. By Proposition 5.2.6 (ii), there is an exact sequence

$$H_c^0((\mathcal{X}^{an}, S), \mu_n) \to H_c^1(\mathcal{U}, \mu_n) \to H_c^1(\mathcal{X}^{an}, \mu_n).$$

We claim that $H_c^0((\mathcal{X}^{an}, S), \mu_n) = 0$. Indeed, it suffices to verify that if S is contained in a disjoint union of open sets $\mathcal{V}$ and $\mathcal{W}$ and the set $S \cap \mathcal{V}$ is nonempty, then it is noncompact. Suppose that $S \cap \mathcal{V}$ is nonempty and compact. Then we can find its affinoid neighborhood V in $\mathcal{V}$ (see [Ber], 2.6.3). By the Maximum Modulus Principle [Ber], 2.5.20, every of the functions f_i takes its maximum at the topological boundary $\partial(V/\mathcal{V})$ of the set V in $\mathcal{V}$. Since $\partial(V/\mathcal{V}) \cap S = \emptyset$, then $\partial(V/\mathcal{V}) \subset \mathcal{U}$. From this it follows that $|f_i(x)| < \varepsilon_i$ for all $x \in V$ and $1 \leqslant i \leqslant m$. This contradicts to the supposition that $S \cap \mathcal{V} \neq \emptyset$.

We are now ready to prove the lemma. By the Fundamental Lemma 1.6.9 from [SGA4], Exp. XVIII, one can find a separated étale morphism of finite type $g : \mathcal{Y}' \to \mathcal{Y}$

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\psi} & \mathcal{X} \\ \uparrow{\scriptstyle g} & \nearrow{\scriptstyle \psi'} & \\ \mathcal{Y}' & & \end{array}$$

such that $\mathbf{y} \in g(\mathcal{Y}')$ ($\mathbf{y}$ is the image of the point y in $\mathcal{Y}$) and the homomorphism $R^1\,\psi'_!(\mu_{n,\mathcal{Y}'}) \to R^1\,\psi_!(\mu_{n,\mathcal{Y}})$ is zero. Let Y' be the inverse image of Y in $\mathcal{Y}'^{\mathrm{an}}$, and let φ' denote the restriction of ψ'^{an} to Y'. Consider the commutative diagram

$$\begin{array}{ccc} R^1\,\varphi_!(\mu_{n,Y}) & \longrightarrow & (R^1\,\psi_!\,\mu_{n,\mathcal{Y}})^{\mathrm{an}} \\ \uparrow & & \uparrow \\ R^1\,\varphi'_!(\mu_{n,Y'}) & \longrightarrow & (R^1\,\psi'_!\,\mu_{n,\mathcal{Y}'})^{\mathrm{an}} \end{array}$$

By 3), the upper arrow is injective. Since the right arrow is zero, it follows that the left arrow is also zero. Finally, to satisfy the condition (ii), it suffices to apply Theorem 3.7.2 to the morphism $\varphi' : Y' \to Y$. The Fundamental Lemma is proved. ∎

The following is Lemma 2.14.2 from [SGA4], Exp. XVIII.

7.3.5. *Lemma.* — *Suppose that* $\mathcal{A}$ *is an abelian category,* $m \geqslant 0$, $\{ F^\bullet_i \}_{0 \leqslant i \leqslant 2m}$ *are objects of* $D^b(\mathcal{A})$ *such that* $H^q(F^\bullet_i) = 0$ *for* $q \notin [0, m]$, $f_i : F^\bullet_i \to F^\bullet_{i+1}$ $(0 \leqslant i \leqslant 2m - 1)$ *are morphisms, and* f *is their composition* $F^\bullet_0 \to F^\bullet_{2m}$. *Assume that* $H^q(f_i) = 0$ *for* $q < m$. *Then there exists in* $D^b(\mathcal{A})$ *a morphism* $\varphi : H^m(F^\bullet_0)\,[-m] \to F^\bullet_{2m}$ *such that the diagram*

$$\begin{array}{ccc} F^\bullet_0 & \xrightarrow{f} & F^\bullet_{2m} \\ & \searrow \quad \nearrow{\scriptstyle \varphi} & \\ & H^m(F^\bullet_0)\,[-m] & \end{array}$$

is commutative. ∎

7.3.6. *Corollary.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism of pure dimension one, and suppose that* X *is good. Then for any point* $y \in Y$ *there exist separated étale morphisms* $f : X' \to X$ *and* $g : Y'' \to Y' = Y \times_X X'$

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \\ \uparrow{\scriptstyle g} & \nearrow{\scriptstyle \varphi''} & \\ Y'' & & \end{array}$$

such that $y \in f'(g(Y''))$ *and there is a factorization*

$$\begin{array}{ccc} R\varphi''_!(\mu_{n,Y''}) & \longrightarrow & R\varphi'_!(\mu_{n,Y'}) \\ & {\scriptstyle t}\searrow \quad \nearrow & \\ & (\mathbf{Z}/n\mathbf{Z})_{X'} & \end{array}$$

where t is induced by the trace mapping.

Proof (see [SGA4], Exp. XVIII, 2.14.4). — We set $\varphi_0 = \varphi : Y_0 = Y \to X_0 = X$ and construct by induction four diagrams

$$\begin{array}{ccc} Y_i & \xrightarrow{\varphi_i} & X_i \\ \uparrow & & \uparrow \\ Y'_{i+1} & \xrightarrow{\varphi'_{i+1}} & X_{i+1} \\ \uparrow & {\scriptstyle \varphi_{i+1}}\nearrow & \\ Y_{i+1} & & \end{array}$$

which satisfy the conclusion of the Fundamental Lemma. Furthermore, we set $X' = X_4$ and $Y''_i = Y_i \times_{X_i} X'$. If ψ_i denotes the canonical morphism $Y''_i \to X'$, then the mappings

$$R^q\, \psi_{i+1_!}(\mu_{n,Y''_{i+1}}) \to R^q\, \psi_{i!}(\mu_{n,Y''_i})$$

are zeroes for $q = 0, 1$ and the mapping

$$\mathrm{Tr} : R^2\, \psi_{4_!}(\mu_{n,Y''_4}) \to (\mathbf{Z}/n\mathbf{Z})_{X'}$$

is an isomorphism. By Lemma 7.3.5, the required statement is true for $Y'' = Y''_4$. ∎

We now prove the statement of Step 2. Let $\mathcal{M}$ be the family of sheaves on Y of the form $h_!(\mathbf{Z}/n\mathbf{Z})_V$, where $h : V \to Y$ is a separated étale morphism. Since any $G \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$ is the epimorphic image of a direct sum of some sheaves from $\mathcal{M}$, it suffices to show that for any $M = h_!(\mathbf{Z}/n\mathbf{Z})_V$ and $\alpha \in \Psi^q(M)$ there exists an epimorphism $\bigoplus_{i \in I} M_i \to M$ with $M_i \in \mathcal{M}$ such that the image of α in all $\Psi^q(M_i)$ is zero. Let ψ denote the morphism $h \circ \varphi : V \to X$. One has

$$\Psi^q(M) = \mathrm{Hom}(R\varphi_!\, M, F(-1)\,[q-2]) = \mathrm{Hom}(R\psi_!\, \mu_{n,V}, F[q-2]).$$

We apply Corollary 7.3.6 to the morphism ψ and an arbitrary point $y \in V$. It follows that one can find separated étale morphisms $f : X' \to X$ and $g : V'' \to V' = V \times_X X'$

$$\begin{array}{ccc} V & \xrightarrow{\psi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ V' & \xrightarrow{\psi'} & X' \\ \uparrow{\scriptstyle g} & {\scriptstyle \psi''}\nearrow & \\ V'' & & \end{array}$$

such that $y \in f'(g(V''))$ and there is a factorization

$$\begin{array}{ccc} R\psi''_!(\mu_{n,V''}) & \longrightarrow & R\psi'_!(\mu_{n,V'}) \\ & \searrow \quad \nearrow & \\ & (\mathbf{Z}/n\mathbf{Z})_{X'}[-2] & \end{array}$$

We set $M' = (hf'g)_!\, (\mathbf{Z}/n\mathbf{Z})_{V''}$. The image of α in

$$\Psi^q(M') = \mathrm{Hom}(R\psi''_!(\mu_{n,V''}), F|_{X'}[q-2])$$

is a morphism which goes through a morphism $(\mathbf{Z}/n\mathbf{Z})_{X'}[-2] \to F|_{X'}[q-2]$. The latter morphism is represented by an element $\beta \in \mathrm{Ext}^q((\mathbf{Z}/n\mathbf{Z})_{X'}, F|_{X'}) = H^q(X', F)$. Let $x' = \psi(y')$, where $y' \in V'$ and $f'(y') = y$. Since $q \geqslant 1$, we can find a separated étale neighborhood $X'' \to X'$ of the point x' such that the element β goes to zero under the canonical homomorphism

$$H^q(X', F) \to H^q(X'', F).$$

Thus, if we replace the above objects by their base change under the morphism $X'' \to X'$, the image of α in $\Psi^q(M')$ will be zero.

We remark that from Steps 1 and 2 it follows that the theorem is true for $d = 1$.

Step 3. — *The functors Ψ^q, $q \geqslant 1$, are effaceable for $d \geqslant 2$.* First of all we remark that to verify the statement for a morphism $\varphi : Y \to X$ it suffices to find an étale covering $\{ Y_i \xrightarrow{g_i} Y \}_{i \in I}$ such that it is true for all of the morphisms $\varphi_i = \varphi \circ g_i : Y_i \to X$. Indeed, suppose we have $G \in \mathbf{S}(Y, \mathbf{Z}/n\mathbf{Z})$, $q \geqslant 1$ and $\alpha \in \Psi^q(G)$. Let G_i be the pullback of G on Y_i and let α_i be the image of α in $\Psi^q_i(G_i) = \mathrm{Hom}(R\varphi_{i!}\, G_i, F(-d)\,[q-2d])$. By hypothesis, for any $i \in I$ there exists an epimorphism $G'_i \to G_i$ such that the image of α_i in $\Psi^q_i(G'_i)$ is zero. Then we get an epimorphism of sheaves $\bigoplus_{i \in I} g_{i!}(G'_i) \to G$ such that the image of α in any $\Psi^q(g_{i!}(G'_i)) = \Psi^q_i(G'_i)$ is zero.

We prove our statement by induction. Assume that it is true for $d - 1$. By the above remark it suffices to verify the statement for a morphism $\chi : Z \to X$ which is a composition of separated smooth morphisms $\psi : Z \to Y$ of pure dimension $d - 1$ and $\varphi : Y \to X$ of pure dimension one. Since the theorem is true for $d = 1$, we have

$$\begin{aligned} \Psi^q(G) &= \mathrm{Hom}(R\chi_!\, G, F(-d)\,[q-2d]) \\ &\qquad = \mathrm{Hom}(R\varphi_!(R\psi_!\, G(d-1)\,[2d-2]), F(-1)\,[q-2]) \\ &= \mathrm{Ext}^q(R\psi_!\, G(d-1)\,[2d-2], \varphi^* F) \\ &\qquad = \mathrm{Hom}(R\psi_!\, G, \varphi^* F(-d+1)\,[q-2d+2]) \\ &= \Psi^q_{(\psi, \varphi^* F)}(G) \end{aligned}$$

where the latter denotes the analogous functor for the morphism ψ and the sheaf $\varphi^* F$. By induction, there exists an epimorphism $G' \to G$ such that the image of α in $\Psi^q_{(\psi, \varphi^* F)}(G')$ is zero. Since the latter group coincides with $\Psi^q(G')$, we get the required statement. The theorem is proved. ∎

7.4. Applications of Poincaré Duality

Let n be an integer prime to char$(\widetilde{k})$.

7.4.1. *Theorem.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism of pure dimension d. Then for any* $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ *there is a canonical isomorphism*

$$R\varphi_*(\varphi^* F) \xrightarrow{\sim} \underline{\mathcal{H}om}(R\varphi_!(\mathbf{Z}/n\mathbf{Z})_Y, F(-d)[-2d]).$$

Proof. — Since $\mathbf{Z}/n\mathbf{Z}$ is a free $\mathbf{Z}/n\mathbf{Z}$-module, then $\mathcal{E}xt^q((\mathbf{Z}/n\mathbf{Z})_Y, \varphi^* F) = 0$ for all $q \geqslant 1$, and therefore $\underline{\mathcal{H}om}((\mathbf{Z}/n\mathbf{Z})_Y, \varphi^* F) \xrightarrow{\sim} \varphi^* F$. The required statement is obtained by applying Poincaré Duality to the complexes $G^\cdot = (\mathbf{Z}/n\mathbf{Z})_Y$ and $F^\cdot = F(-d)[-2d]$. ∎

We say that a morphism of k-analytic spaces $\varphi : Y \to X$ is *acyclic* if for any $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ one has $F \xrightarrow{\sim} \varphi_*(\varphi^* F)$ and $R^q \varphi_*(\varphi^* F) = 0$, $q \geqslant 1$.

7.4.2. *Corollary.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism of pure dimension d, and suppose that the geometric fibres of* φ *are nonempty and connected and have trivial cohomology groups* H^q_c *with coefficients in* $\mathbf{Z}/n\mathbf{Z}$ *for* $q < 2d$. *Then the morphism* φ *is acyclic. For example, the morphisms* $X \times \mathbf{A}^d \to X$ *and* $X \times D \to X$, *where* D *is an open disc in* $\mathbf{A}^d$, *are acyclic.* ∎

The following is a straightforward consequence of Poincaré Duality.

7.4.3. *Theorem.* — *Suppose that k is algebraically closed, and let* X *be a separated smooth k-analytic space of pure dimension d. Then for any* $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ *and* $q \geqslant 0$ *there is a canonical isomorphism*

$$\mathrm{Ext}^q(F, \mu^d_{n,Y}) \xrightarrow{\sim} H^{2d-q}_c(X, F)^\vee.$$

In particular, if F *is finite locally constant, then one has*

$$H^q(X, F^\vee(d)) \xrightarrow{\sim} H^{2d-q}_c(X, F)^\vee. \quad \blacksquare$$

7.4.4. *Corollary.* — *Suppose that k is algebraically closed, and let* X *be a proper smooth k-analytic space. Then for any finite locally constant sheaf* $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ *the groups* $H^q(X, F)$ *are finite.* ∎

Let S be a k-analytic space. A *smooth* S-*pair* (Y, X) is a commutative diagram of morphisms of k-analytic spaces

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

where f and g are smooth, and i is a closed immersion. The *codimension of* (Y, X) *at a point* $y \in \mathrm{Y}$ is the codimension at y of the fibre Y_s in X_s, where $s = g(y)$. One can associate with a smooth S-pair (Y, X) the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Y} & \xrightarrow{i} & \mathrm{X} & \xleftarrow{j} & \mathrm{U} \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ & & \mathrm{S} & & \end{array}$$

where $\mathrm{U} = \mathrm{X} \backslash \mathrm{Y}$, and j is the canonical open immersion.

7.4.5. *Theorem (Cohomological Purity Theorem). — Let* (Y, X) *be a smooth* S*-pair of codimension* c, *and let* F *be an abelian sheaf on* X *which is locally isomorphic (in the étale topology) to a sheaf of the form* $f^* \mathrm{F}'$, *where* $\mathrm{F}' \in \mathbf{S}(\mathrm{S}, \mathbf{Z}/n\mathbf{Z})$ *(for example,* F *is a locally constant* $(\mathbf{Z}/n\mathbf{Z})_{\mathrm{X}}$*-module). Then*

(i) $\mathscr{H}^q_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) = 0$ *for* $q \neq 2c$;

(ii) *there is a canonical isomorphism* $\mathscr{H}^{2c}_{\mathrm{Y}}(\mathrm{X}, \mathrm{F}) \xrightarrow{\sim} i^* \mathrm{F}(-c)$.

Proof. — We remark that if we are given a cartesian diagram

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{i} & \mathrm{X} \\ \uparrow & & \uparrow{\scriptstyle \varphi} \\ \mathrm{Y}' & \longrightarrow & \mathrm{X}' \end{array}$$

with étale φ, then for any $q \geqslant 0$ there is a canonical isomorphism

$$\mathscr{H}^q_{\mathrm{Y}}(\mathrm{X}, \mathrm{F})|_{\mathrm{Y}'} \xrightarrow{\sim} \mathscr{H}^q_{\mathrm{Y}'}(\mathrm{X}', \mathrm{F}|_{\mathrm{X}'}).$$

Therefore we may assume that $\mathrm{F} = f^* \mathrm{F}'$ for some $\mathrm{F}' \in \mathbf{S}(\mathrm{S}, \mathbf{Z}/n\mathbf{Z})$.

Let f be of pure dimension d. Then g is of pure dimension $e := d - c$. By Poincaré Duality, for any $\mathrm{G} \in \mathbf{S}(\mathrm{Y}, \mathbf{Z}/n\mathbf{Z})$ there is a canonical isomorphism

$$\mathrm{Hom}(\mathrm{G}, g^* \mathrm{F}'(e)\,[2e]) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{R}g_! \,\mathrm{G}, \mathrm{F}').$$

Since $\mathrm{R}g_! \,\mathrm{G} = \mathrm{R}f_!(i_* \mathrm{G})$, it follows by Poincaré Duality applied to the morphism f that one has

$$\mathrm{Hom}(\mathrm{R}g_! \,\mathrm{G}, \mathrm{F}') \xrightarrow{\sim} \mathrm{Hom}(i_* \mathrm{G}, f^* \mathrm{F}'(d)\,[2d]).$$

Furthermore, since the functor i_* is left adjoint to the functor $i^!$, the latter group is isomorphic to $\mathrm{Hom}(\mathrm{G}, \mathrm{R}i^!(f^* \mathrm{F}')\,(d)\,[2d])$. Thus, we get an isomorphism

$$\mathrm{Hom}(\mathrm{G}, g^* \mathrm{F}'(e)\,[2e]) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{G}, \mathrm{R}i^!(f^* \mathrm{F}')\,(d)\,[2d]),$$

and this isomorphism is functorial on G. It follows that it is induced by an isomorphism of complexes in $\mathrm{D}(\mathrm{Y}, \mathbf{Z}/n\mathbf{Z})$

$$g^* \mathrm{F}'(e)\,[2e] \xrightarrow{\sim} \mathrm{R}i^!(f^* \mathrm{F}')\,(d)\,[2d].$$

Hence, $\mathrm{R}i^!(\mathrm{F}) \xrightarrow{\sim} i^* \mathrm{F}(-c)\,[-2c]$, and the theorem follows. ■

In the following three corollaries, the situation is the same as in Theorem 7.4.5.

7.4.6. *Corollary.* — $H^q_Y(X, F) = 0$ *for* $0 \leqslant q \leqslant 2c - 1$ *and*

$$H^q_Y(X, F) \xrightarrow{\sim} H^{q-2c}(Y, i^* F(-c))$$

for $q \geqslant 2c$. ∎

7.4.7. *Corollary.* — $F \xrightarrow{\sim} j_*(F|_U)$, $R^{2c-1} j_*(F|_U) \xrightarrow{\sim} i_*(i^* F(-c))$, *and* $R^q j_*(F|_U) = 0$ *for* $q \neq 0, 2c - 1$. ∎

Applying the spectral sequence $R^p f_*(R^q j_*(F|_U)) \Rightarrow R^{p+q} h_*(F|_U)$, we get

7.4.8. *Corollary (Gysin sequence).* — *Suppose that the relative dimension of* f *is equal to* d.

(i) $R^q f_* F \xrightarrow{\sim} R^q h_*(F|_U)$ *for* $0 \leqslant q \leqslant 2c - 2$, *and there is an exact sequence*

$$0 \to R^{2c-1} f_* F \to R^{2c-1} h_*(F|_U) \to g_*(i^* F(-c)) \to$$
$$\to R^{2c} f_* F \to R^{2c} h_*(F|_U) \to R^1 g_*(i^* F(-c)) \to \dots$$

(ii) *Suppose that* k *is algebraically closed and* $S = \mathcal{M}(k)$. *Then* $H^q(X, F) \xrightarrow{\sim} H^q(U, F)$ *for* $0 \leqslant q \leqslant 2c - 2$, *and there is an exact sequence*

$$0 \to H^{2c-1}(X, F) \to H^{2c-1}(U, F) \to H^0(Y, i^* F(-c)) \to$$
$$\to H^{2c}(X, F) \to H^{2c}(U, F) \to H^1(Y, i^* F(-c)) \to \dots$$
$$\to H^{2(d-c)}(Y, i^* F(-c)) \to H^{2d}(X, F) \to H^{2d}(U, F) \to 0. \quad ∎$$

The following statement will be used in the proof of the Comparison Theorem 7.5.1.

7.4.9. *Theorem.* — *Let* $\varphi : Y \to X$ *be a separated smooth morphism of pure dimension* d, *and suppose that* F *is a finite locally constant* $(\mathbf{Z}/n\mathbf{Z})_Y$*-module such that all the sheaves* $R^q \varphi_!(F^\vee)$, $q \geqslant 0$, *are finite locally constant. Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$R^q \varphi_* F \xrightarrow{\sim} (R^{2d-q} \varphi_!(F^\vee))^\vee (-d).$$

7.4.10. *Lemma.* — *Let* X *be a* k*-analytic space. Suppose that the cohomology sheaves of a complex* $F^\bullet \in D^-(X, \mathbf{Z}/n\mathbf{Z})$ *are finite locally constant, and let* G *be a locally free sheaf of* $(\mathbf{Z}/n\mathbf{Z})_X$*-modules of finite rank. Then for any* $q \in \mathbf{Z}$ *there is a canonical isomorphism*

$$\mathcal{E}xt^q(F^\bullet, G) \xrightarrow{\sim} \mathcal{H}om(H^{-q}(F^\bullet), G).$$

Proof. — If F is finite locally constant, then $\mathcal{E}xt^q(F, G) = 0$ for all $q \geqslant 1$ because $\mathbf{Z}/n\mathbf{Z}$ is an injective $\mathbf{Z}/n\mathbf{Z}$-module. Our statement now follows from the spectral sequence (see [Gro], 2.4.2)

$$\mathcal{E}xt^p(H^{-q}(F^\bullet), G) \Rightarrow \mathcal{E}xt^{p+q}(F^\bullet, G). \quad ∎$$

Proof of Theorem **7.4.9.** — Since $G^\vee$ is also finite locally constant, Lemma 7.4.10 gives an isomorphism

$$\underline{\mathcal{H}om}(F^\vee, \mu^d_{n,Y}) = \mathcal{H}om(F^\vee, \mu^d_{n,Y}) = G(d).$$

By Poincaré Duality and Lemma 7.4.10, we have

$$(\mathrm{R}^q \varphi_* \mathrm{F})(d) = \mathrm{R}^q \varphi_*(\underline{\mathscr{H}om}(\mathrm{F}^\vee, \mu^d_{n,\mathrm{Y}}))$$
$$= \mathscr{E}xt^{q-2d}(\mathrm{R}\varphi_!(\mathrm{F}^\vee), (\mathbf{Z}/n\mathbf{Z})_\mathrm{X}) = (\mathrm{R}^{2d-q}\varphi_!(\mathrm{F}^\vee))^\vee.$$

The required statement follows. ■

7.5. The Comparison Theorem

7.5.1. *Theorem. — Suppose that $\mathscr{S}$ is a scheme of finite type over* $\mathrm{Spec}(\mathscr{A})$, *where $\mathscr{A}$ is a k-affinoid algebra, $f: \mathscr{X} \to \mathscr{S}$ and $\varphi: \mathscr{Y} \to \mathscr{X}$ are morphisms of finite type, and $\mathscr{F}$ is a constructible abelian sheaf on $\mathscr{Y}$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Then there exists an everywhere dense open subset $\mathscr{U} \subset \mathscr{S}$ such that the following properties hold.*

(i) *The sheaves $\mathrm{R}^q \varphi_* \mathscr{F}|_{f^{-1}(\mathscr{U})}$ are constructible and equal to zero except a finite number of them.*

(ii) *The formation of the sheaves $\mathrm{R}^q \varphi_* \mathscr{F}$ is compatible with any base change $\mathscr{S}' \to \mathscr{S}$ such that the image of $\mathscr{S}'$ is contained in $\mathscr{U}$.*

(iii) *In* (ii) *assume that $\mathscr{S}'$ is a scheme of locally finite type over* $\mathrm{Spec}(\mathscr{B})$, *where $\mathscr{B}$ is an affinoid $\mathscr{A}$-algebra, and that the morphism $\mathscr{S}' \to \mathscr{S}$ is a composition $\mathscr{S}' \to \mathscr{S} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{S}$. Let φ' be the morphism $\mathscr{Y}' = \mathscr{Y} \times_{\mathscr{S}} \mathscr{S}' \to \mathscr{X}' = \mathscr{X} \times_{\mathscr{S}} \mathscr{S}'$, and let $\mathscr{F}'$ be the inverse image of $\mathscr{F}$ on $\mathscr{Y}'$. Then for any $q \geqslant 0$ there is a canonical isomorphism*

$$(\mathrm{R}^q \varphi'_* \mathscr{F}')^{\mathrm{an}} \simeq \mathrm{R}^q \varphi'^{\mathrm{an}}_* \mathscr{F}'^{\mathrm{an}}.$$

The condition on the morphism $\mathscr{S}' \to \mathscr{S}$ implies that for any scheme $\mathscr{Z}$ of locally finite type over $\mathscr{S}$ one has

$$(\mathscr{Z} \times_{\mathscr{S}} \mathscr{S}')^{\mathrm{an}} \simeq \mathscr{Z}^{\mathrm{an}} \times_{\mathscr{S}^{\mathrm{an}}} \mathscr{S}'^{\mathrm{an}}.$$

The existence of an everywhere dense open subset $\mathscr{U} \subset \mathscr{S}$ which possesses the properties (i) and (ii) is guaranteed by Deligne's " generic " theorem 1.9 from [SGA4½], Th. finitude. In fact the proof of Theorem 7.5.1 follows closely the proof of Deligne's theorem and uses it. Moreover, the proof is a purely formal reasoning which works over the field of complex numbers $\mathbf{C}$ as well (of course, in this case one should assume that $\mathscr{A} = \mathscr{B} = \mathbf{C}$). The other main ingredients of the proof are the Comparison Theorem for Cohomology with Compact Support, Poincaré Duality for schemes and analytic spaces, and the constructibility of the sheaves $\mathrm{R}^q \varphi_! \mathscr{F}$ ([SGA4], Exp. XVII, 5.3.6).

Proof. — By Deligne's theorem, we can shrink $\mathscr{S}$ and assume that the properties (i) and (ii) hold for $\mathscr{U} = \mathscr{S}$. And so it remains to show that there exists an everywhere dense open subset $\mathscr{U} \subset \mathscr{S}$ for which the property (iii) holds.

Since the property (iii) is local with respect to $\mathscr{S}$ and $\mathscr{X}$, we may assume that they are separated. Furthermore, if $(\mathscr{Y}_i \xrightarrow{g_i} \mathscr{Y})_{i \in \mathrm{I}}$ is a *finite* étale covering of $\mathscr{Y}$, then it suffices to verify (iii) for all of the morphisms $\varphi g_i : \mathcal{Y}_i \to \mathcal{X}$. Therefore we may assume that $\mathcal{Y}$ is also separated. Finally, since the scheme $\mathcal{Y}$ is Noetherian, we can find an integer n prime to $\mathrm{char}(\widetilde{k})$ with $n\mathcal{F} = 0$. Therefore we can work with the categories of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules.

7.5.2. *Lemma.* — *The property* (iii) *holds when* $\mathcal{X} = \mathcal{S}$, φ *is smooth, and* $\mathcal{F}$ *is locally constant.*

Proof. — We may assume that φ is smooth of pure dimension d. Since the sheaf $\mathcal{F}^\vee$ is also locally constant, all the sheaves $\mathrm{R}^q\,\varphi_!(\mathcal{F}^\vee)$, $q \geqslant 0$, are constructible. Furthermore, since they are equal to zero for $q > 2d$, we can shrink $\mathcal{S}$ and assume that all the sheaves $\mathrm{R}^q\,\varphi_!(\mathcal{F}^\vee)$, $q \geqslant 0$, are actually locally constant. Theorem 7.4.9 and its analog for schemes give isomorphisms

$$\mathrm{R}^q\,\varphi_*(\mathcal{F}) \stackrel{\sim}{\to} (\mathrm{R}^{2d-q}\,\varphi_!(\mathcal{F}^\vee))^\vee\,(-d),$$

$$\mathrm{R}^q\,\varphi_!^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}}) \stackrel{\sim}{\to} (\mathrm{R}^{2d-q}\,\varphi_!^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}\vee}))^\vee\,(-d).$$

The required statement now follows from the Comparison Theorem for Cohomology with Compact Support (Corollaries 7.1.4 and 7.1.5). ∎

We remark that it suffices to prove the theorem in the case when φ is an open immersion with everywhere dense image. Indeed, we may assume that $\mathcal{Y}$ is affine. Then there is a factorization $\varphi = \psi j$, where $j : \mathcal{Y} \hookrightarrow \mathcal{Z}$ is an open immersion, and $\psi : \mathcal{Z} \to \mathcal{X}$ is a proper morphism. Hence $\mathrm{R}\varphi_* = \mathrm{R}\psi_*\,\mathrm{R}j_*$ and $\mathrm{R}\varphi_*^{\mathrm{an}} = \mathrm{R}\psi_*^{\mathrm{an}}\,\mathrm{R}j_*^{\mathrm{an}}$, and the required statement follows from the Comparison Theorem for Cohomology with Compact Support.

We remark also that it suffices to assume that $\mathcal{S}$ is the spectrum of an integral domain. Let η be the generic point of $\mathcal{S}$. We prove the following statement by induction.

$(*)_d$ *The property* (iii) *holds when* $\dim(\mathcal{Y}_\eta) \leqslant d$ *and* φ *is an open immersion with everywhere dense image.*

1) $(*)_0$ *is true.* It suffices to show that there exists an everywhere dense open subset $\mathcal{U} \subset \mathcal{S}$ with $f^{-1}(\mathcal{U}) \subset \mathcal{Y}$. For this we may assume that $\mathcal{X}$ is affine, and we can find an open immersion with everywhere dense image $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ in a proper $\mathcal{S}$-scheme $\overline{\mathcal{X}}$. It follows that $\mathcal{Y}$ is everywhere dense in $\overline{\mathcal{X}}$, and therefore $\overline{\mathcal{X}}_\eta = \mathcal{Y}_\eta$. The latter means that the point η is not contained in the closed set $\{ s \in \mathcal{S} \mid \dim(\overline{\mathcal{X}}_s) \geqslant 1 \}$. Thus, we can shrink $\mathcal{S}$ and assume that $\overline{\mathcal{X}}$ is finite over $\mathcal{S}$. Finally, since $\overline{\mathcal{X}}_\eta = \mathcal{Y}_\eta$, the image of the closed set $\overline{\mathcal{X}} \backslash \mathcal{Y}$ is closed and nowhere dense in $\mathcal{S}$. Therefore, shrinking $\mathcal{S}$, we get $\mathcal{Y} = \mathcal{X} = \overline{\mathcal{X}}$.

Suppose now that $d \geqslant 1$ and assume that $(*)_{d-1}$ is true.

2) *One can shrink* $\mathcal{S}$ *and find an open subset* $\mathcal{Z} \subset \mathcal{X}$ *such that its complement* $\mathcal{X}_1$ *is finite over* $\mathcal{S}$ *and the property* (iii) *holds for the open immersion* $\mathcal{Y} \cap \mathcal{Z} \to \mathcal{Z}$.

Since the statement is local with respect to $\mathscr{X}$, we may assume that $\mathscr{X}$ is a closed subscheme of the affine space $\mathbf{A}^m_{\mathscr{S}}$. Let π_i denote the i-th projection $\mathscr{X} \to \mathbf{A}^1_{\mathscr{S}}$. The induction hypotheses applied to the diagrams

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\varphi} & \mathscr{X} \\ & \searrow \quad \swarrow_{\pi_i} & \\ & \mathbf{A}^1_{\mathscr{S}} & \end{array}$$

gives open subsets $\mathscr{U}_i \subset \mathbf{A}^1_{\mathscr{S}}$ such that the property (iii) holds for the open immersions $\mathscr{Y} \cap \pi_i^{-1}(\mathscr{U}_i) \to \pi_i^{-1}(\mathscr{U}_i)$ and the sets $\mathscr{U}_i$. It follows that if we set $\mathscr{Z} = \bigcup_{i=1}^m \pi_i^{-1}(\mathscr{U}_i)$, then it also holds for the open immersion $\mathscr{Y} \cap \mathscr{Z} \to \mathscr{Z}$ and the set $\mathscr{S}$. Finally, the reasoning from 1) shows that we can shrink $\mathscr{S}$ and assume that the morphism $\mathbf{A}^1_{\mathscr{S}} \backslash \mathscr{U}_i \to \mathscr{S}$ are finite. It follows that the morphism $\mathscr{X}_1 = \bigcap_{i=1}^m \pi_i^{-1}(\mathbf{A}^1_{\mathscr{S}} \backslash \mathscr{U}_i) \to \mathscr{S}$ is also finite.

3) $(*)_d$ *is true if* $\mathscr{Y}$ *is smooth over* $\mathscr{S}$ *and* $\mathscr{F}$ *is locally constant.*

Since the statement is local with respect to $\mathscr{X}$, we may assume that $\mathscr{X}$ is a closed subscheme of the affine space $\mathbf{A}^m_{\mathscr{S}}$. After that we can replace $\mathscr{X}$ by its closure in the projective space $\mathbf{P}^m_{\mathscr{S}}$. In particular, we may assume that the morphism $f : \mathscr{X} \to \mathscr{S}$ is proper.

We shrink $\mathscr{S}$ and take the open immersion $j : \mathscr{Z} \hookrightarrow \mathscr{X}$ and the closed immersion $i : \mathscr{X}_1 \to \mathscr{X}$ which are guaranteed by 2). Consider the commutative diagram

$$\begin{array}{ccccc} \mathscr{Y} & \overset{\varphi}{\hookrightarrow} & \mathscr{X} & \xleftarrow{i} & \mathscr{X}_1 \\ & {}_g\searrow & \downarrow f & \swarrow_{f_1} & \\ & & \mathscr{S} & & \end{array}$$

By construction, f is proper, f_1 is finite, and the property (iii) holds for the open immersion $\mathscr{Y} \cap \mathscr{Z} \to \mathscr{Z}$ and the set $\mathscr{U} = \mathscr{S}$. By hypotheses, g is smooth, and therefore, by Lemma 7.5.2, we can shrink $\mathscr{S}$ and assume that the property (iii) holds for the morphism $g : \mathscr{Y} \to \mathscr{S}$.

To verify (iii), we may assume that the base change is already done. (The only thing we should take care of is that our further manipulations do not change the scheme $\mathscr{S}$.)

Consider the exact sequence

$$(1) \qquad 0 \to j_! j^* \mathrm{R}\varphi_* \mathscr{F} \to \mathrm{R}\varphi_* \mathscr{F} \to i_* i^* \mathrm{R}\varphi_* \mathscr{F} \to 0$$

and the induced morphisms of exact sequences $(1)^{\mathrm{an}} \to (1^{\mathrm{an}})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!^{\mathrm{an}}(j^* \mathrm{R}\varphi_* \mathscr{F})^{\mathrm{an}} & \longrightarrow & (\mathrm{R}\varphi_* \mathscr{F})^{\mathrm{an}} & \longrightarrow & i_*^{\mathrm{an}}(i^* \mathrm{R}\varphi_* \mathscr{F})^{\mathrm{an}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta(\mathscr{F})} & & \downarrow & & \\ 0 & \longrightarrow & j_!^{\mathrm{an}}(j^{\mathrm{an}*} \mathrm{R}\varphi_*^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}) & \longrightarrow & \mathrm{R}\varphi_*^{\mathrm{an}} \mathscr{F}^{\mathrm{an}} & \longrightarrow & i_*^{\mathrm{an}}(i^{\mathrm{an}*} \mathrm{R}\varphi_*^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}) & \longrightarrow & 0 \end{array}$$

Since the first vertical arrow is an isomorphism, then to show that $\theta(\mathscr{F})$ is an isomorphism, it suffices to verify that

$$(i^* \mathrm{R}\varphi_* \mathscr{F})^{\mathrm{an}} \xrightarrow{\sim} i^{\mathrm{an}*} \mathrm{R}\varphi_*^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}.$$

For this we apply Rf_* and Rf_*^{an} to the sequences (1) and (1^{an}), respectively. We obtain a morphism of exact triangles

$$\begin{array}{ccccccc} \longrightarrow & (Rf_* j_! j^* R\varphi_* \mathscr{F})^{an} & \longrightarrow & (Rg_* \mathscr{F})^{an} & \longrightarrow & f_{1*}^{an}(i^* R\varphi_* \mathscr{F})^{an} & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & Rf_*^{an} j_!^{an}(j^{an*} R\varphi_*^{an} \mathscr{F}^{an}) & \longrightarrow & Rg_*^{an} \mathscr{F}^{an} & \longrightarrow & f_{1*}^{an}(i^{an*} R\varphi_*^{an} \mathscr{F}^{an}) & \longrightarrow \end{array}$$

The first and the second vertical arrows are isomorphisms. Therefore the third arrow is also an isomorphism. Since f_1 is finite, it follows that $(i^* R\varphi_* \mathscr{F})^{an} \stackrel{\sim}{\to} i^{an*} R\varphi_*^{an} \mathscr{F}^{an}$.

4) $(*)_d$ *is true in the general case*. Decreasing $\mathscr{S}$, we can find a finite radicial surjective morphism $\mathscr{S}' \to \mathscr{S}$ such that the scheme $(\mathscr{Y} \times_{\mathscr{S}} \mathscr{S}')_{red}$ has an everywhere dense open subset which is smooth over $\mathscr{S}'$. Since such a manipulation does not change the étale cohomology, we may assume that $\mathscr{Y}$ has an everywhere dense subset $\mathscr{Z}$ which is smooth over $\mathscr{S}$. Shrinking $\mathscr{Z}$, we may assume that $\mathscr{F}$ is locally constant over $\mathscr{Z}$. Let j denote the open immersion $\mathscr{Z} \hookrightarrow \mathscr{Y}$. By 3), we can shrink $\mathscr{S}$ and assume that the theorem is true for the pairs of morphisms $(f : \mathscr{X} \to \mathscr{S}, \psi = \varphi j : \mathscr{Z} \to \mathscr{X})$ and $(g : \mathscr{Y} \to \mathscr{S}, j : \mathscr{Z} \to \mathscr{Y})$ and the sheaf $j^* \mathscr{F}$. It follows that if we define $\mathscr{G}^\cdot$ by the exact triangle

(2) $$\to \mathscr{F} \to Rj_* j^* \mathscr{F} \to \mathscr{G}^\cdot \to$$

then there is an exact triangle

(2^{an}) $$\to \mathscr{F}^{an} \to Rj_*^{an} j^{*an} \mathscr{F}^{an} \to \mathscr{G}^{an\cdot} \to$$

Furthermore, the sheaves $H^q(\mathscr{G}^\cdot)$ are constructible and equal to zero except a finite number of them, and the formation of $H^q(\mathscr{G}^\cdot)$ is compatible with any base change. Finally, the sheaves $H^q(\mathscr{G}^\cdot)$ have support in the closed subset $\mathscr{W} = \mathscr{Y} \backslash \mathscr{Z}$, and one has $\dim(\mathscr{W}_\eta) \leqslant d - 1$. Since the canonical morphism $\mathscr{W} \to \mathscr{X}$ is a composition of the open immersion $\mathscr{W} \hookrightarrow \overline{\mathscr{W}}$, where $\overline{\mathscr{W}}$ is the Zariski closure of $\mathscr{W}$ in $\mathscr{X}$, and the closed immersion $\overline{\mathscr{W}} \to \mathscr{X}$, it follows, by inductional hypotheses, that we can shrink $\mathscr{S}$ and assume that the property (iii) holds for the morphism $\varphi : \mathscr{Y} \to \mathscr{X}$ and the sheaves $H^q(\mathscr{G}^\cdot)$.

As in the proof of 3), to verify (iii) we may assume that the base change is already done. Applying $R\varphi_*$ and $R\varphi_*^{an}$ to the triangles (2) and (2^{an}), respectively, we obtain a morphism of triangles

$$\begin{array}{ccccccc} \longrightarrow & (R\varphi_* \mathscr{F})^{an} & \longrightarrow & (R\psi_* j^* \mathscr{F})^{an} & \longrightarrow & (R\varphi_* \mathscr{G}^\cdot)^{an} & \longrightarrow \\ & \downarrow{\scriptstyle \theta(\mathscr{F})} & & \downarrow & & \downarrow & \\ \longrightarrow & R\varphi_*^{an} \mathscr{F}^{an} & \longrightarrow & R\psi_*^{an} j^{an*} \mathscr{F}^{an} & \longrightarrow & R\varphi_*^{an} \mathscr{G}^{an\cdot} & \longrightarrow \end{array}$$

Since the second vertical arrow is an isomorphism, to show that $\theta(\mathscr{F})$ is an isomorphism, it suffices to verify that

$$(R\varphi_* \mathscr{G}^\cdot)^{an} \stackrel{\sim}{\to} R\varphi_*^{an} \mathscr{G}^{an\cdot}.$$

But this follows from the homomorphism of spectral sequences

$$\begin{array}{ccc} (R^p\,\varphi_*(H^q(\mathscr{G}^\bullet)))^{an} & \Longrightarrow & (R^{p+q}\,\varphi_*\,\mathscr{G}^\bullet)^{an} \\ \downarrow & & \downarrow \\ R^p\,\varphi_*^{an}(H^q(\mathscr{G}^{an\bullet})) & \Longrightarrow & R^{p+q}\,\varphi_*^{an}\,\mathscr{G}^{an\bullet} \end{array}$$

because the left arrows are isomorphisms. The theorem is proved. ∎

7.5.3. *Corollary. — Let* $\varphi : \mathscr{Y} \to \mathscr{X}$ *be a morphism of finite type between schemes of locally finite type over* k, *and let* $\mathscr{F}$ *be a constructible abelian sheaf on* $\mathscr{Y}$ *with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$(R^q\,\varphi_*\,\mathscr{F})^{an} \xrightarrow{\sim} R^q\,\varphi_*^{an}\,\mathscr{F}^{an}.$$

Proof. — Since the statement is local with respect to $\mathscr{X}$, we may assume that $\mathscr{X}$ (and therefore $\mathscr{Y}$) is of finite type over k, and therefore we can apply Theorem 7.5.1 for $\mathscr{A} = k$ and $\mathscr{S} = \mathrm{Spec}(k)$. ∎

7.5.4. *Corollary. — Let* $\mathscr{X}$ *be a scheme of locally finite type over* k, *and let* $\mathscr{F}$ *be a constructible abelian sheaf on* $\mathscr{X}$ *with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$H^q(\mathscr{X}, \mathscr{F}) \xrightarrow{\sim} H^q(\mathscr{X}^{an}, \mathscr{F}^{an}).$$

Proof. — First of all we remark that it suffices to consider the case when the scheme is affine and of finite type over k. Indeed, if this is so, then in the general case we can take a covering $\mathscr{U} = \{U_i\}_{i\in I}$ of $\mathscr{X}$ by open affine subschemes of finite type over k and use the homomorphism of spectral sequences

$$\begin{array}{ccc} \check{H}^p(\mathscr{U}/\mathscr{X}, \mathscr{H}^q(\mathscr{F})) & \Longrightarrow & H^{p+q}(\mathscr{X}, \mathscr{F}) \\ \downarrow & & \downarrow \\ \check{H}^p(\mathscr{U}^{an}/\mathscr{X}^{an}, \mathscr{H}^q(\mathscr{F}^{an})) & \Longrightarrow & H^{p+q}(\mathscr{X}^{an}, \mathscr{F}^{an}) \end{array}$$

Thus, we may assume that $\mathscr{X}$ is affine and of finite type over k. In this case we apply Corollary 7.5.3, the homomorphism of the Leray spectral sequences associated with the morphisms $\mathscr{X} \to \mathrm{Spec}(k)$ and $\mathscr{X}^{an} \to \mathscr{M}(k)$, and the fact that the required statement is true for $\mathscr{X} = \mathrm{Spec}(k)$. ∎

7.6. The invariance of cohomology under extensions of the ground field

7.6.1. *Theorem. — Let* K/k *be an extension of algebraically closed non-Archimedean fields. Let* X *be a* k*-analytic space, and let* F *be an abelian torsion sheaf on* X *with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *We set* $X' = X \hat{\otimes} K$ *and denote by* F' *the inverse image of* F *on* X'. *Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$H^q(X, F) \xrightarrow{\sim} H^q(X', F').$$

Proof. — All the sheaves considered below are supposed to be abelian torsion with torsion orders prime to $\mathrm{char}(\widetilde{k})$.

We remark that if the theorem is true for k-affinoid spaces, it is also true for arbitrary k-analytic spaces. Indeed, if X is separated paracompact, we can find a locally finite affinoid covering $\mathcal{V} = \{ V_i \}_{i \in I}$ of X and apply the homomorphism of the spectral sequences 4.3.7

$$\begin{array}{ccc} \check{H}^p(\mathcal{V}, \mathcal{H}^q(F)) & \Longrightarrow & H^{p+q}(X, F) \\ \downarrow & & \downarrow \\ \check{H}^p(\mathcal{V}', \mathcal{H}^q(F')) & \Longrightarrow & H^{p+q}(X', F') \end{array}$$

where $\mathcal{V}' = \{ V_i \hat{\otimes} K \}_{i \in I}$. If X is arbitrary paracompact, we use the same reasoning and the fact that the intersection of two affinoid domains is a compact analytic domain. If X is Hausdorff, we use the similar reasoning for a covering of X by open paracompact subsets and the fact that the intersection of two open paracompact subsets in a locally compact space is paracompact. If X is arbitrary, we use the same reasoning for a covering of X by open Hausdorff subsets.

Furthermore, we remark that it suffices to prove the theorem for one-dimensional k-affinoid spaces. Indeed, let $Y = \mathcal{M}(\mathcal{B})$ be a k-affinoid space of dimension $d \geqslant 2$, and suppose that the theorem is true for k-affinoid spaces of smaller dimension. Take an element $f \in \mathcal{B}$ which is not constant at any irreducible component of Y, and consider the induced morphism $f: Y \to \mathbf{A}^1$. Let $X = \mathcal{M}(\mathcal{A})$ be a closed disc in $\mathbf{A}^1$ which contains the image of Y. We get a morphism of k-affinoid spaces $\varphi : Y \to X$ such that $\dim(X) = 1$ and $\dim(Y_x) < d$ for all points $x \in X$. Consider the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle \pi'} & & \uparrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

where $X' = X \hat{\otimes} K$ and $Y' = Y \hat{\otimes} K = Y \times_X X'$. To prove the theorem for Y, it suffices to show that for any sheaf F on Y and any $q \geqslant 0$ there is a canonical isomorphism

$$(R^q \varphi_* F)' \xrightarrow{\sim} R^q \varphi'_* F'.$$

But this is obtained, by induction, from the Weak Base Change Theorem 5.3.1. Indeed, if $x \in X$ and $x' \in X'$ is a pair of points with $x = \pi(x')$, then a fixed embedding of fields $\mathcal{H}(x)^a \hookrightarrow \mathcal{H}(x')^a$ induces an isomorphism of analytic spaces

$$Y_{\bar{x}} \hat{\otimes}_{\widehat{\mathcal{H}(x)^a}} \widehat{\mathcal{H}(x')^a} \xrightarrow{\sim} Y'_{\bar{x}'}.$$

Thus, we may assume that X in the theorem is a one-dimensional k-affinoid space. To prove the theorem for X, we consider the following more general situation. Let S be a locally closed subset of X, and F be a sheaf on the k-germ (X, S). We set

$(X', S') = (X, S) \hat{\otimes} K$ and denote by F' the inverse image of F on (X', S'). Furthermore we denote by $\theta^q((X, S), F)$ and $\theta^q_c((X, S), F)$, respectively, the canonical homomorphisms

$$H^q((X, S), F) \to H^q((X', S'), F')$$

and

$$H^q_c((X, S), F) \to H^q_c((X', S'), F').$$

Our starting point is Theorem 6.4.1 (ii) which implies that $\theta^q(X, F)$ is an isomorphism if the sheaf F is finite constant. And here is a continuation.

1) $\theta^q(X, F)$ *is an isomorphism, when* F *is finite locally constant.* — Indeed, one can find a finite étale covering $Y \to X$ such that the sheaf $F|_Y$ is constant, and therefore one can use the spectral sequences associated with the étale coverings $(Y \to X)$ and $(Y \hat{\otimes} K \to X')$.

2) $\theta^q((X, S), F)$ *is an isomorphism, when* S *is closed and* F *is finite locally constant.* — Indeed, by Proposition 4.4.1, the sheaf F extends to a finite locally constant sheaf on an open neighborhood of S. Our statement now follows from Proposition 4.3.5 because affinoid neighborhoods of S form a basis of its neighborhoods.

3) $\theta^q_c((X, S), F)$ *is an isomorphism when* F *extends to a finite locally constant sheaf on* $(X, \overline{S})$ ($\overline{S}$ is the closure of S in X). Indeed, we can apply 2), the exact cohomological sequences associated with the embeddings

$$(X, S) \overset{j}{\hookrightarrow} (X, \overline{S}) \overset{i}{\leftarrow} (X, \overline{S}\backslash S)$$

and

$$(X', S') \overset{j'}{\hookrightarrow} (X', (\overline{S})') \overset{i'}{\leftarrow} (X', (\overline{S})'\backslash S'),$$

where $(X', (\overline{S})') = (X, \overline{S}) \hat{\otimes} K$, and the five-lemma.

4) $\theta^q_c((X, S), F)$ *is an isomorphism for an arbitrary finite locally constant sheaf* F. Indeed, by Proposition 5.2.8, one has

$$H^q_c((X, S), F) = \varinjlim H^q_c((X, T), F)$$

and

$$H^q_c((X', S'), F') = \varinjlim H^q_c((X', T'), F'),$$

where T runs through open subsets of S with compact closure. Therefore we can apply 3).

5) $\theta^q(X, F)$ *is an isomorphism, when* F *is of the form* $j_! G$, *where* j *is the morphism of germs* $(X, S) \to X$ *determined be a locally closed subset* $S \subset X$, *and* G *is a finite locally constant sheaf on* (X, S). From Corollary 5.2.5 it follows that

$$H^q(X, j_! G) = H^q_c((X, S), G)$$

and

$$H^q(X', j'_! G') = H^q_c((X', S'), G').$$

Since $j'_! G' = (j_! G)'$, we can apply 4).

6) $\theta^q(X, F)$ *is an isomorphism, when* F *is quasiconstructible.* This is obtained from 5), by induction, using Proposition 4.4.4 and the five-lemma.

7) $\theta^q(X, F)$ *is an isomorphism for arbitrary* F. This follows from 6) and Propositions 4.4.5 and 5.2.9. ■

7.6.2. *Corollary.* — *In the situation of Theorem* 7.6.1 *assume that* X *is Hausdorff. Then for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$H^q_c(X, F) \xrightarrow{\sim} H^q_c(X', F').$$

Proof. — From Proposition 5.2.8 and Corollary 5.2.5 it follows that

$$H^q_c(X, F) = \varinjlim H^q(X, j_{\mathcal{U}!}(F|_{\mathcal{U}}))$$

and

$$H^q_c(X', F') = \varinjlim H^q(X, j_{\mathcal{U}'!}(F'|_{\mathcal{U}'})),$$

where the limit is taken over all open subset $\mathcal{U} \subset X$ with compact closure, $\mathcal{U}' = \mathcal{U} \widehat{\otimes} K$, and $j_{\mathcal{U}}$ (resp. $j_{\mathcal{U}'}$) denotes the canonical open immersion $\mathcal{U} \hookrightarrow X$ (resp. $\mathcal{U}' \hookrightarrow X'$). The required statement follows from Theorem 7.6.1. ■

7.7. The Base Change Theorem for Cohomology with Compact Support

7.7.1. *Theorem.* — *Let* $\varphi : Y \to X$ *be a Hausdorff morphism of k-analytic spaces and let* $f : X' \to X$ *be a morphism of analytic spaces over k which give rise to a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

Then for any abelian torsion sheaf F *on* Y *with torsion orders prime to* $\mathrm{char}(\widetilde{k})$ *and any* $q \geqslant 0$ *there is a canonical isomorphism*

$$f^*(R^q \varphi_! F) \xrightarrow{\sim} R^q \varphi'_!(f'^* F).$$

Proof. — The Weak Base Change Theorem 5.3.1 reduces the situation to the case when $X = \mathcal{M}(k)$ and $X' = \mathcal{M}(K)$, where K/k is an extension of algebraically closed fields. In this case our statement follows from Corollary 7.6.2. ■

7.7.2. *Corollary.* — *In the situation of Theorem* 7.7.1 *suppose that* φ *is of finite dimension, and let n be an integer prime to* $\mathrm{char}(\widetilde{k})$. *Then for any* $F'^{\cdot} \in D^-(X', \mathbf{Z}/n\mathbf{Z})$ *and* $G^{\cdot} \in D^-(Y, \mathbf{Z}/n\mathbf{Z})$ *there is a canonical isomorphism*

$$F'^{\cdot} \overset{L}{\otimes} f^*(R\varphi_! G^{\cdot}) \xrightarrow{\sim} R\varphi'_!(\varphi'^* F'^{\cdot} \overset{L}{\otimes} f'^* G^{\cdot}).$$

Proof. — From Theorem 7.7.1 it follows that there is a canonical isomorphism

$$f^*(R\varphi_! G^{\cdot}) \xrightarrow{\sim} R\varphi'_!(f'^* G^{\cdot}).$$

Therefore our statement is obtained by applying Theorem 5.3.9 to $f' : Y' \to X'$. ■

7.7.3. *Corollary (Künneth Formula). — Given a cartesian diagram of Hausdorff morphisms of k-analytic spaces*

$$\begin{array}{ccccc} & & X\times_S Y & & \\ & \swarrow^{g'} & & \searrow^{f'} & \\ X & & \downarrow h & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & S & & \end{array}$$

suppose that f and g are of finite dimensions, and let n be an integer prime to $\mathrm{char}(\widetilde{k})$. *Then for any* $F^\bullet \in D^-(X, \mathbf{Z}/n\mathbf{Z})$ *and* $G^\bullet \in D^-(Y, \mathbf{Z}/n\mathbf{Z})$ *there is a canonical isomorphism*

$$Rf_!\, F^\bullet \overset{\mathbf{L}}{\otimes} Rg_!\, G^\bullet \overset{\sim}{\to} Rh_!(g'^*\, F^\bullet \overset{\mathbf{L}}{\otimes} f'^*\, G^\bullet).$$

Proof. — Applying Theorem 5.3.9 to the morphism f and Corollary 7.7.2, one has

$$Rf_!\, F^\bullet \overset{\mathbf{L}}{\otimes} Rg_!\, G^\bullet \overset{\sim}{\to} Rf_!(F^\bullet \overset{\mathbf{L}}{\otimes} f^*(Rg_!\, G^\bullet)) \overset{\sim}{\to} Rf_!\, Rg'_!(g'^*\, F^\bullet \overset{\mathbf{L}}{\otimes} f'^*\, G^\bullet)$$
$$\overset{\sim}{\to} Rh_!(g'^*\, F^\bullet \overset{\mathbf{L}}{\otimes} f'^*\, G^\bullet). \quad \blacksquare$$

7.8. The Smooth Base Change Theorem

We say that a morphism of k-analytic spaces $\varphi : Y \to X$ is *almost smooth* if, for any point $x \in X$, there exist an open Hausdorff neighborhood $\mathcal{U}$ of x and affinoid domains $V_1, \ldots, V_n \subset \mathcal{U}$ such that $V_1 \cup \ldots \cup V_n$ is a neighborhood of x and all the induced morphisms $\varphi^{-1}(V_i) \to V_i$ are smooth. Or course, smooth morphisms are almost smooth. (And we believe that the converse implication is also true.)

7.8.1. *Theorem. — Let* $f : X' \to X$ *be an almost smooth morphism of k-analytic spaces, and let* $\varphi : Y \to X$ *be a morphism of analytic spaces over k, which give rise to a cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

Then for any $(\mathbf{Z}/n\mathbf{Z})_Y$*-module* F, *where n is an integer prime to* $\mathrm{char}(\widetilde{k})$, *and for any* $q \geqslant 0$ *there is a canonical isomorphism*

$$f^*(R^q\, \varphi_*\, F) \overset{\sim}{\to} R^q\, \varphi'_*(f'^*\, F).$$

Proof. — First of all, the reasoning from the beginning of the proof of Theorem 7.3.1 reduces the situation to the case when f is a smooth morphism of good k-analytic spaces.

Furthermore, since the statement is local with respect to X' and is evidently true when f is étale, it suffices to assume that f is of pure dimension one.

Let x' be a point of X', and set $x = \varphi(x')$. One has

$$(f^*(R^q \varphi_* F))_{x'} = (R^q \varphi_* F)_x = \varinjlim H^q(Y \times_X U, F),$$

where the limit is taken over all étale morphisms $U \to X$ with a fixed point $u \in U$ over x and with a fixed embedding of fields $\mathscr{H}(u) \hookrightarrow \mathscr{H}(x)^s$ over $\mathscr{H}(x)$. Similarly, one has

$$(R^q \varphi'_*(f'^* F))_{x'} = \varinjlim H^q(Y' \times_{X'} W, f'^* F),$$

where the limit is taken over all étale morphisms $W \to X'$ with a fixed point $w \in W$ over x' and with a fixed embedding of fields $\mathscr{H}(w) \hookrightarrow \mathscr{H}(x')^s$ over $\mathscr{H}(x')$. Therefore, it suffices to show that for any étale morphism $(W, w) \to (X', x')$ there exist étale morphisms $g : U \to X$ and $h : U'' \to U' = W \times_X U$ such that the point w is contained in the image of U'' and for any $q \geqslant 0$ one has

$$\operatorname{Im}(H^q(V', f'^* F) \to H^q(V'', f'^* F)) \subset \operatorname{Im}(H^q(V, F) \to H^q(V'', f'^* F)),$$

where $V = Y \times_X U$, $V' = Y \times_X U'$ and $V'' = Y \times_X U''$. We can shrink W and assume that $W \to X$ is a separated smooth morphism of pure dimension one.

By Corollary 7.3.6, there exist separated étale morphisms $g : U \to X$ and $h : U'' \to U' = W \times_X U$

$$\begin{array}{ccc} W & \longrightarrow & X \\ \uparrow & & \uparrow{\scriptstyle g} \\ U' & \xrightarrow{\psi} & U \\ \uparrow{\scriptstyle h} & \nearrow{\scriptstyle \chi} & \\ U'' & & \end{array}$$

such that the point w is contained in the image of U'' and there is a factorization

$$\begin{array}{ccc} R\chi_!(\mu_{n,U''}) & \longrightarrow & R\psi_!(\mu_{n,U'}) \\ & {\scriptstyle t}\searrow \quad \nearrow & \\ & (\mathbf{Z}/n\mathbf{Z})_U[-2] & \end{array}$$

where t is induced by the trace mapping. Consider the fibre product of the previous diagram with Y over X

$$\begin{array}{ccc} Y \times_X W & \longrightarrow & Y \\ \uparrow & & \uparrow{\scriptstyle g'} \\ V' & \xrightarrow{\psi'} & V \\ \uparrow{\scriptstyle h'} & \nearrow{\scriptstyle \chi'} & \\ V'' & & \end{array}$$

We claim that for any $q \geqslant 0$ there is a factorization

$$\begin{array}{ccc} H^q(V', f'^* F) & \longrightarrow & H^q(V'', f'^* F) \\ & \searrow \quad \nearrow & \\ & H^q(V, F) & \end{array}$$

where the homomorphism $H^q(V, F) \to H^q(V'', f'^* F)$ is the canonical one.

Indeed, by the Base Change Theorem for Cohomology with Compact Support 7.7.1, there is a factorization

$$\begin{array}{ccc} R\chi'_!(\mu_{n, V''}) & \longrightarrow & R\psi'_!(\mu_{n, V'}) \\ & {\scriptstyle t'}\searrow \quad \nearrow & \\ & (\mathbf{Z}/n\mathbf{Z})_V[-2] & \end{array}$$

where t' is induced by the trace mapping. By Theorem 7.4.1, we have

$$R\psi'_*(\psi'^*(F|_V)) \tilde{\to} \underline{\mathcal{H}om}(R\psi'_!(\mu_{n, V'}), F|_V[-2]),$$
$$R\chi'_*(\chi'^*(F|_V)) \tilde{\to} \underline{\mathcal{H}om}(R\chi'_!(\mu_{n, V''}), F|_V[-2]).$$

Since $\underline{\mathcal{H}om}((\mathbf{Z}/n\mathbf{Z})_V[-2], F|_V[-2]) \tilde{\to} F|_V$, then there is a commutative diagram

$$\begin{array}{ccc} R\psi'_*(\psi'^*(F|_V)) & \longrightarrow & R\chi'_*(\chi'^*(F|_V)) \\ & \searrow \quad \nearrow & \\ & F|_V & \end{array}$$

where the morphism $F|_V \to R\chi'_*(\chi'^*(F|_V))$ is the canonical one.

We now apply to the latter commutative diagram the derived functor $R\Gamma_V$, where $\Gamma_V : \mathbf{S}(V, \mathbf{Z}/n\mathbf{Z}) \to \mathcal{A}b$ is the functor of global sections on V. Since $\Gamma_V \circ \psi'_* = \Gamma_{V'}$ and $\Gamma_V \circ \chi'_* = \Gamma_{V''}$, we get a commutative diagram

$$\begin{array}{ccc} R\Gamma_{V'}(\psi'^*(F|_V)) & \longrightarrow & R\Gamma_{V''}(\chi'^*(F|_V)) \\ & \searrow \quad \nearrow & \\ & R\Gamma_V(F|_V) & \end{array}$$

which gives, for any $q \geqslant 0$, a commutative diagram

$$\begin{array}{ccc} H^q(V', f'^* F) & \longrightarrow & H^q(V'', f'^* F) \\ & \searrow \quad \nearrow & \\ & H^q(V, F) & \end{array}$$

in which the homomorphism $H^q(V, F) \to H^q(V'', f'^* F)$ is the canonical one. The theorem is proved. ■

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