IL NUOVO CIMENTO VOL. LIX B, N. 1 11 Gennaio 1969

A Lagrangian Formulation of the Theory of Random Motion.

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(ricevuto il 22 Marzo 1968; manoscritto revisionato ricevuto il 14 Maggio 1968)

Summary. — The motion of a particle is studied under the action of two types of forces, some slowly varying and others rapidly fluctuating. The slowly varying forces are assumed to be given in every particular problem. The rapidly fluctuating forces, which are unknown, are assumed to have a random character. It is shown that the motion of the particle depends on the random forces only through the diffusion effect that they produce. The theory is statistical in character and only the evolution of a probability density can be determined. A Lagrangian formalism is developed and from it the general equations of motion are derived. These turn out to be formally equivalent to the Schrödinger equation. The generalization to a system of particles is straightforward if it is assumed that the random forces act independently and with like intensity on every elementary particle. The expectation values of the fundamental dynamical variables are obtained. The theory proves to be very similar to, but not fully identical with, nonrelativistic quantum mechanics without spin.

1. – Introduction.

An interpretation of the Schrödinger equation in terms of particle trajectories was first proposed by DE BROGLIE [1] and later developed by BOHM [2], who has since maintained that the possibility of an interpretation of the quantum theory in terms of « hidden variables » cannot be excluded [3]. On

[1] L. DE BROGLIE: *Compt. Rend.*, **183**, 447 (1926); **184**, 273 (1927); **185**, 380 (1927); see also: *Etude critique des bases de l'interprétation actuelle de la Mécanique Ondulatoire* (Paris, 1963); *La Thermodynamique de la particule isolée*, (Paris, 1964).

[2] D. BOHM: *Phys. Rev.*, **85**, 166, 180 (1952).

[3] D. BOHM: *Hidden variables in the quantum theory*, in *Quantum Theory*, edited by D. R. BATES (New York, 1962); D. BOHM and J. BUB: *Rev. Mod. Phys.*, **38**, 453 (1966).

the other hand, a formal analogy between Brownian motion and quantum mechanics had been noticed by FÜRTH (4). Combining these two basic ideas, FÉNYES (5) and WEIZEL (6) showed that the Schrödinger equation can be derived from the hypothesis of a random motion of the material particles. In recent years, a number of alternative derivations have been proposed (7). In general, the parallelism between quantum mechanics and stochastic processes has been studied by several authors (8).

The picture of the atomic systems that results from the hypothesis of random motion is very suggestive (9). Let us take, for instance, the ground state of the hydrogen atom. According to the picture given by the random motion theory, the electron would be a point (or a very small) particle following a very irregular trajectory, similar to that of a particle in Brownian motion. The random motion of the electron would prevent it from falling into the nucleus, but the electrostatic attraction would maintain both close together. In this way, a dynamical equilibrium would exist, similar to the one that prevents a colloidal particle in a liquid from falling to the bottom.

According to this theory, the so-called wave-packets would represent the regions of space in which the probability of finding the particle is greater. If a free particle is known to be in a volume element at some time, the particle may be present in any point of space after a large enough time, due to the random motion. This would represent the « wave-packet spreading » of quantum mechanics. The wave-particle duality should be discarded in this theory; the material systems would consist of classical particles and the radiation of classical waves. The particle appearance of waves and the wave appearance of material particles would be complex phenomena due to quantization of the energy, linear momentum and angular momentum exchange with the material particles which the measuring apparatus consists of. This quantization would be a consequence of the random motion.

To make the comparison between the theory of random motion and quantum mechanics it is necessary that the first be developed in a more general

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(4) R. FÜRTH: *Zeits. Phys.*, **81**, 143 (1933).

(5) I. FÉNYES: *Zeits. Phys.*, **132**, 8 (1952).

(6) W. WEIZEL: *Zeits. Phys.*, **134**, 264 (1953); **135**, 270 (1953); **136**, 582 (1954).

(7) D. KERSHAW: *Phys. Rev.*, **136** B, 1850 (1964); J. C. ARON: *Progr. Theor. Phys.*, **33**, 726 (1965); E. NELSON: *Phys. Rev.*, **150** B, 1079 (1966); L. DE LA PEÑA AUERBACH: *Phys. Lett.*, **24** A, 603 (1967).

(8) G. C. COMISAR: *Phys. Rev.*, **138** B, 1332 (1965); T. MARSHALL: *Proc. Cambridge Phil. Soc.*, **61**, 537 (1965); M. BRAFFORT, M. SURDIN and A. TARONI: *Compt. Rend.*, **261**, 4339 (1965); R. BOURRET: *Phys. Lett.*, **12**, 323 (1964); *Canad. Journ. Phys.*, **43**, 619 (1965); **44**, 2519 (1966); G. DELLA RICCIA and T. HIDA: *Ann. Inst. H. Poincaré*, vol. **4**, n. 1 (1966), p. 31; G. DELLA RICCIA and N. WIENER: *Journ. Math. Phys.*, **7**, 1372 (1966).

(9) E. NELSON: *Phys. Rev.*, **150** B, 1079 (1966).

form than has been done until now. To make a general formulation of the theory starting from the Lagrangian formalism is the purpose of this paper. The motion of a system of particles is to be studied under the action of a random accelaration plus the acceleration due to some known forces. The only condition imposed on these forces is that they be able to be derived in a classical form from a Lagrangian function.

2. – The hypothesis of the random motion.

The postulates of the theory, besides the usual ones in all classical theories, are the following:

I) *If the force acting on a particle is given by the vectorial function* $\boldsymbol{f}(t)$, *the motion of the particle results in the composition of a random motion and the motion that classical dynamics predicts.*

The position, $\boldsymbol{r}(t)$, of the particle at time t is determined by the equation

$$\ddot{\boldsymbol{r}}(t) = (1/m)\boldsymbol{f}(t) + \boldsymbol{a}(t) , \tag{1}$$

where $\boldsymbol{a}(t)$ is the random acceleration. The force $\boldsymbol{f}(t)$ may depend on the actual position and the actual velocity of the particle.

The above postulate can be considered as a modification of the Newton law of dynamics. However, it is also possible to take the Newton law as valid and assume that the random acceleration is due to a random force $\boldsymbol{F}(t)$, acting on the particle besides the known force $\boldsymbol{f}(t)$. Both hypotheses are formally equivalent, and we will adopt the second in the following. In this way, the theory can be developed within Newtonian mechanics. Then, instead of (1), the following equation must be written

$$\boldsymbol{f}(t) + \boldsymbol{F}(t) = m\ddot{\boldsymbol{r}}(t) , \tag{2}$$

where $\boldsymbol{r}(t)$ gives the actual path of the particle. The function $\boldsymbol{r}(t)$ will be assumed continuous, but not necessarily differentiable, so that $\boldsymbol{F}(t)$ and $\ddot{\boldsymbol{r}}(t)$ will be distributions rather than ordinary functions.

II) *The stochastic parameters of the random process represented by* $\boldsymbol{F}(t)$ *are independent of the state of motion of the particle. There are neither privileged times, nor positions, nor directions in space, in relation to the random motion.*

It would be necessary to specify all the stochastic parameters of the random force in order to develop a complete theory. However, an approximate theory can be developed without such a specification. Indeed, when the force $\boldsymbol{f}(t)$

varies slowly enough, the observable properties of the motion depend on the random force $\boldsymbol{F}(t)$ only through the diffusion effect that it produces.

In fact, let $\boldsymbol{r}$ and $\boldsymbol{v}$ be the position and the velocity of the particle at the moment t_0. The velocity and the position at any other time $t_0 + \Delta t$ will be

$$\boldsymbol{v}(t_0 + \Delta t) = \boldsymbol{v}(t_0) + (1/m) \int_{t_0}^{t_0+\Delta t} \boldsymbol{f}(t)\,\mathrm{d}t + (1/m) \int_{t_0}^{t_0+\Delta t} \boldsymbol{F}(t)\,\mathrm{d}t\,, \tag{3}$$

$$\boldsymbol{r}(t_0 + \Delta t) = \boldsymbol{r}(t_0) + \boldsymbol{v}(t_0)\Delta t + (1/m) \int_{t_0}^{t_0+\Delta t} \mathrm{d}t \int_{t_0}^{t} \boldsymbol{f}(t')\,\mathrm{d}t' + (1/m) \int_{t_0}^{t_0+\Delta t} \mathrm{d}t \int_{t_0}^{t} \boldsymbol{F}(t')\,\mathrm{d}t'\,. \tag{4}$$

Let us now assume that $\boldsymbol{F}(t)$ varies much more quickly than $\boldsymbol{f}(t)$. Then, it will be possible to choose a time interval T, much greater than the characteristic fluctuation time of $\boldsymbol{F}(t)$, during which the force $\boldsymbol{f}(t)$ scarcely varies. Then, from postulate II) it is deduced that

$$(1/\Delta t) \int_{t_0}^{t_0+\Delta t} \boldsymbol{F}(t)\,\mathrm{d}t \approx 0\,,$$

whenever $\Delta t \gg T$.

In these conditions the eqs. (3) and (4) lead to the following equalities (in a statistical sense):

$$\boldsymbol{v}(t_0 + \Delta t) \approx \boldsymbol{v}(t_0) + (1/m)\,\boldsymbol{f}(t_0)\Delta t + o(\Delta t)\,; \tag{5}$$

$$\boldsymbol{r}(t_0 + \Delta t) \approx \boldsymbol{r}(t_0) + \boldsymbol{v}(t_0)\Delta t + \delta\boldsymbol{r} + o(\Delta t)\,. \tag{6}$$

The last integral in (4) gives rise to the random displacement $\delta\boldsymbol{r}$, and $o(\Delta t)$ means « terms of order higher than Δt ». From postulate II) and the central limit theorem it can be deduced that the probability distribution of $\delta\boldsymbol{r}$ approaches a Gaussian when $\Delta t \to \infty$.

The approximate equalities (5) and (6), and the form of the probability distribution of $\delta\boldsymbol{r}$ when $\Delta t \to \infty$, are not enough to develop a rigorous theory. However, if the equalities (5) and (6) were exact and the distribution function of $\delta\boldsymbol{r}$ were Gaussian for any Δt, the random motion would be completely determined as a Wiener process. Starting from the hypothesis that the random motion is a Wiener process, the Schrödinger equation has been derived [6,7]. In this paper a new derivation will be proposed, starting also from the theory of Wiener processes. However, the preceding analysis shows that the Wiener processes have no special physical meaning. They represent only an approximation, valid to substitute any random motion when the known force $\boldsymbol{f}(t)$ varies much more slowly than the random force $\boldsymbol{F}(t)$, whenever the latter fulfils the conditions of postulate II).

It is interesting to point out that the Wiener processes are not suitable for relativistic generalization, due to the fact that infinite velocities are involved in them. Nevertheless, if the Wiener process is only considered as an approximation of a random motion with finite velocity, the relativistic generalization may be possible. It seems also that the relativistic formulation might be easier starting from the Lagrangian formulation in a form similar to the one developed in this paper for the nonrelativistic case.

3. – Equations of motion.

Let us take a particle in a given force field. The theory that we are developing, being classical, rests upon the hypothesis that the particle has always a definite position and a definite velocity. The problem of the error with which these quantities can be measured will be considered as a purely experimental one. From the theoretical point of view, the basic problem is to determine the position and the velocity of the particle at any time t from data taken at time t_0. As a consequence of the assumed random force, the future motion of the particle cannot be accurately determined, even if it were possible to measure simultaneously the position and the velocity of the particle at one time. Only probability distributions of the quantities can be predicted. In this way, the theory is statistical in character like quantum mechanics.

It would be possible to develop a statistical theory that could predict the evolution of the probability density in phase space. However, we intend to develop a more restricted theory dealing only with the probability distribution in configuration space. This is more adapted to experimental situations in which only the position of a particle can be measured directly.

Let us consider a particle under the action of the random force (without any other force), whose velocity be zero at time t_0, and whose position is $\boldsymbol{r}_0$ at the same time. As a consequence of the theory of Wiener processes, the probability distribution of the position at any later time would be given by a function $\varrho(\boldsymbol{r}, t)$ fulfilling the equation

$$\partial\varrho/\partial t = D\nabla^2\varrho \,. \tag{7}$$

If eq. (7) is integrated with the initial condition

$$\varrho(\boldsymbol{r}, t_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \,,$$

where $\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ is the Dirac's « delta », a Gaussian distribution is obtained.

To study the motion of a particle under the combined action of given forces and the rapidly fluctuating random force, eq. (7) could be substituted by

the more general Fokker-Planck equation (9). In this paper, however, another procedure will be used starting from the Lagrangian formulation. To do this, the actual problem of the motion of a single particle will be replaced by the fictitious problem of the motion of a statistical ensemble of independent particles. Every one of the particles in the ensemble should follow one of the possible paths of the actual particle. Both problems are formally equivalent if the probability distribution of the position of the actual particle is the same function of $\boldsymbol{r}$ and t as the density of particles of the statistical ensemble. The statistical ensemble can be considered a fluid with density $\varrho(\boldsymbol{r}, t)$, whose motion results from the superposition of a classical motion due to the known forces, plus a diffusion effect due to the random forces.

In order to obtain the equations of motion of the statistical fluid it is convenient to start with the equation of motion of a classical fluid in Lagrangian formulation. Let us assume that the classical motion (no random force) of the actual particle would be determined by the Lagrangian function

$$L[\boldsymbol{r}_0(t), \dot{\boldsymbol{r}}_0(t), t] .$$

Then, the motion of a classical statistical fluid could be obtained from the Hamilton principle

$$\int_{t_1}^{t_2} \mathrm{d}t \int \mathscr{L}_0 \, \mathrm{d}v = \text{extremum} , \tag{8}$$

where

$$\mathscr{L}_0 = \varrho(\boldsymbol{r}, t) L[\boldsymbol{r}, \boldsymbol{v}(\boldsymbol{r}, t), t] . \tag{9}$$

The variations of $\mathscr{L}_0$ in (8) correspond to arbitrary variations of $\boldsymbol{v}$, ϱ being related to $\boldsymbol{v}$ by the continuity equation

$$\dot{\varrho} + \operatorname{div}(\varrho \boldsymbol{v}) = 0 . \tag{10}$$

To prove that the motion of the classical fluid is determined by the Lagrangian density $\mathscr{L}_0$, we choose $\varrho(\boldsymbol{r}, t)$ in the form of a Dirac's « delta »

$$\varrho(\boldsymbol{r}, t) = \delta\big(\boldsymbol{r} - \boldsymbol{r}_0(t)\big) . \tag{11}$$

From (10) and (11), it follows that

$$\boldsymbol{v}(\boldsymbol{r}, t) = \dot{\boldsymbol{r}}_0(t) . \tag{12}$$

By substituting (11) and (12) into (9) and performing the volume integral in

(8) it results that the variational equation

$$\int_{t_1}^{t_2} L[\boldsymbol{r}_0(t), \dot{\boldsymbol{r}}_0(t), t] \, \mathrm{d}t = \text{extremum}$$

gives the motion of every material point of the fluid, which completes the proof.

The next step is to modify the Lagrangian density $\mathscr{L}_0$ in order to obtain a new Lagrangian density $\mathscr{L}$, from which the equations of motion of the statistical fluid can be obtained, taking into account both the known and the random forces. Actually, the diffusion effect due to the random force is not suitable for a Lagrangian formulation. In fact, the process of diffusion is not a motion of the statistical ensemble as a continuous fluid, but an exchange of particles between different volume elements. However, if we assume that only the density of the fluid at every point is physically meaningful, we can substitute the diffusion by an equivalent continuous motion that carries fluid from the regions of greater density to the regions of lower one. This motion may be suitable for a Lagrangian formulation.

The actual motion of the fluid is the composition of two independent motions, a « classical » one, given by eq. (8), and a « diffusionlike » one, given by eq. (7). If we are able to obtain the diffusionlike motion from a variational condition similar to the Hamilton principle, our problem will be solved. In fact, let us assume that we find a condition of the form

$$\int_{t_1}^{t_2} \mathrm{d}t \int (\tfrac{1}{2} m \varrho \boldsymbol{v}^2 + \mathscr{U}) \, \mathrm{d}v = \text{extremum} ,$$

such that it is equivalent to eq. (7). Then, assuming that the classical and the diffusionlike motions are independent, the Lagrangian density

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{U}$$

should determine the composite motion.

In order to obtain a Lagrangian formulation for the diffusionlike motion alone, we must take into account that eq. (7) is a statistical equation, related to our information about the position of the particle rather than to an objective reality. Then, it must be a consequence of the condition of maximum rate of entropy increase, which is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (-\varrho \ln \varrho) \, \mathrm{d}v = \text{maximum} . \tag{13}$$

The increase of entropy is constrained by two additional conditions. The first one, which follows from the definition of probability, is the normalization of ϱ, which is

$$\int \varrho \, \mathrm{d}v = 1 \, .$$

The second condition is the conservation of energy, which follows from postulate II). According to this postulate, the random force is independent of the state of motion of the particle, so that the mean energy is conserved. However, when we go from the actual problem of the motion of a particle to the fictitious one of the motion of a fluid, a kinetic energy term appears in the form

$$\tfrac{1}{2} m \int \varrho \boldsymbol{v}^2 \, \mathrm{d}v \, .$$

In order that energy be conserved, we must add to the kinetic energy a kind of potential energy $\mathscr{U}$ such that, whenever eq. (7) is fulfilled,

$$\tfrac{1}{2} m \int \varrho \boldsymbol{v}^2 \, \mathrm{d}v + \int \mathscr{U} \, \mathrm{d}v = 0 \, . \tag{14}$$

The term $\mathscr{U}$ must be a function of ϱ and $\boldsymbol{\nabla}\varrho$, but it cannot be a function of $\boldsymbol{v}$ or $\dot{\varrho}$, which are related to time evolution. In order to obtain $\mathscr{U}$, we compare (7) with (10) and we obtain, after an integration,

$$\boldsymbol{v} = -D\boldsymbol{\nabla}\varrho/\varrho \, .$$

Actually, any vector $\boldsymbol{w}$ such that

$$\operatorname{div}(\varrho \boldsymbol{w}) = 0$$

may be added to $\boldsymbol{v}$ as a result of the integration. As we are considering the simplest motion, we will put $\boldsymbol{w} = 0$. In this way, the term $\mathscr{U}$ is given by

$$\mathscr{U} = -\tfrac{1}{2} m D^2 (\operatorname{grad} \ln \varrho)^2 \varrho \, , \tag{15}$$

because this function fulfils the equality (14) whenever the motion is given by (7).

It is easy to prove that conditions (13), (14) and (15) lead to eq. (7). In making the derivation, eq. (10) and the normalization of ϱ must be taken into account, from which eq. (13) is transformed into

$$\int \boldsymbol{v} \cdot \operatorname{grad} \varrho \, \mathrm{d}v = \text{minimum} \, .$$

Given the function ϱ, this condition, together with eq. (14), leads to a function $\boldsymbol{v}$ which, through eq. (10), gives eq. (7). In this way, we have transformed the diffusion eq. (7) into the variational condition (13) constrained by (14). Obviously other variational formulations can be found whish lead also to eq. (7); for example,

$$\int (\boldsymbol{v} + D \operatorname{grand} \ln \varrho)^2 \varrho \mathrm{d}v = \text{minimum} .$$

However, these alternative conditions, not being a consequence of general principles—as eqs. (13) and (14) are—might not hold in cases other than the pure diffusion represented by eq. (7).

If we integrate eqs. (13) and (14) with respect to time, we obtain

$$(16) \qquad \begin{cases} \int [\varrho(\boldsymbol{r}, t_1) \ln \varrho(\boldsymbol{r}, t_1) - \varrho(\boldsymbol{r}, t_2) \ln \varrho(\boldsymbol{r}, t_2)] \mathrm{d}v = \text{maximum} , \\ \dfrac{1}{2} m \displaystyle\int_{t_1}^{t_2} \mathrm{d}t \int [\boldsymbol{v}^2 - D^2 (\operatorname{grad} \ln \varrho)^2] \varrho \, \mathrm{d}v = 0 . \end{cases}$$

These conditions are a generalization of eqs. (13) and (14) and therefore they imply eq. (7). Now, from a formal point of view, the following two variational problems are equivalent:

a) the first integral of eq. (16) be extremal whilst the second be a constant,

b) the second integral of eq. (16) be extremal whilst the first be a constant.

Problem *a*) is related to the case in which we known (with some uncertainty, maybe) the position of the particle at time t_1 and we are asking for the probability distribution of the position at any time $t_2 > t_1$. Problem *b*) is related to the case in which we know the positions of the particle at times t_1 and t_2 and we are asking for the probability distribution at intermediate times t ($t_1 < t < t_2$). Actually, the difference between these two cases is only subjective, the objective reality being the same in both cases, that is, a particle moving under the action of a random force. This is consistent with the fact that the general variational statement is formally identical in both problems *a*) and *b*).

Luckily, the variational problem *b*) is of the form of the Hamilton principle if we assume that the functions $\varrho(\boldsymbol{r}, t_1)$ and $\varrho(\boldsymbol{r}, t_2)$—and therefore the first integral of eqs. (16)—are given. However, if we use the variational formulation *b*), we no longer can calculate the function ϱ at any time from the knowl-

edge of this function at one time (as in problem *a*) or through eq. (7)) but we need the function ϱ at two different times. This is necessary because any problem of motion in Newtonian mechanics needs two conditions to be determined (position at two times or position and velocity at one time) and, on the other hand, we must deal with the diffusionlike motion of the fictitious fluid as if it were a Newtonian motion.

When the diffusion is formally considered as a Newtonian motion of a fluid, some properties appear which do not seem to be inherent to the theory of random motion. For instance, as we will see later, the equation of motion of the fluid turns out to be the equation of Schrödinger, from which—as is well known—the Heisenberg uncertainty relations follow. In this way, the theory that we are developing seems too restrictive, because all problems are excluded in which both the position and the velocity of the particle are known with great accuracy at a time. Clearly, the uncertainty relations do not follow from the postulates I) and II), as was stated at the beginning of this Section. Actually, the restriction imposed by the uncertainty relations does not matter in practice. In fact, any practical measurement of the position and the velocity of the particle must be made through some interaction with a measuring apparatus. If the interaction forces are slowly varying, the position and the velocity measured will be averages over time intervals large in comparison with the fluctuation time of the random force. Then some uncertainties will appear which well be, at least, of the order

$$\Delta x \approx (D\,\Delta t)^{\frac{1}{2}}\,, \qquad \Delta v \approx \Delta x/\Delta t = (D/\Delta t)^{\frac{1}{2}}\,;$$

from which the uncertainty relations follow:

$$\Delta x\,\Delta v \approx \hbar/2m\,; \qquad \hbar = 2mD\,.$$

On the other hand, if the interaction forces were rapidly varying, the uncertainty relations should not appear, but the theory which we are developing would not hold at all, because eqs. (5) and (6) should not be fulfilled.

Having obtained the equations of the diffusionlike motion in the form

$$\frac{1}{2}m\int_{t_1}^{t_2}\mathrm{d}t\int[\boldsymbol{v}^2 - D^2(\operatorname{grad}\ln\varrho)^2]\,\varrho\,\mathrm{d}v = \text{extremum}\,,$$

it is enough to substitute the Lagrangian density (9) instead of the kinetic energy term $\frac{1}{2}m\boldsymbol{v}^2\varrho$ in it, in order to obtain the equation of motion under the action of both the random force and another given force. The correctness of this statement rests upon the hypothesis that the random and the known

forces act on the particle with total independence. This may not be the case if the known forces are velocity-dependent, because the motion produced by the random force may lead to forces of the known type very different from those calculated from the average motion of the particle. This possibility will not be studied in this paper, and the total independence of the forces will be assumed in the following. In this way, the general equations of motion can be obtained from the Hamiltonian principle (8) with the Lagrangian density

$$\mathscr{L} = \varrho[L(\boldsymbol{r}, \boldsymbol{v}, t) - \tfrac{1}{2} mD^2(\operatorname{grad} \ln \varrho)^2] .$$

Condition (10) can be taken into account by the method of Lagrange multipliers. So, Hamilton's principle must be applied to the function

$$\mathscr{L} = \varrho[L(\boldsymbol{r}, \boldsymbol{v}, t) - \tfrac{1}{2} mD^2(\operatorname{grad} \ln \varrho)^2] + S[\dot{\varrho} + \operatorname{div}(\varrho \boldsymbol{v})] ,$$

where S is a function of $\boldsymbol{r}$ and t, at first unknown. After substitution of this function in eq. (8), it follows that

$$\int_{t_1}^{t_2} \mathrm{d}t \int \varrho[(\boldsymbol{r}, \boldsymbol{v}, t) - \tfrac{1}{2} mD^2(\operatorname{grad} \ln \varrho)^2 - \dot{S} - \boldsymbol{v} \cdot \nabla S] \, \mathrm{d}v = \text{extremum} ,$$

where the last two terms come from an integration by parts. The variations of ϱ and $\boldsymbol{v}$ are now independent. For a variation of $\boldsymbol{v}$, this equation leads to

$$\partial L / \partial \boldsymbol{v} = \boldsymbol{p} = \nabla S , \tag{17}$$

where the partial derivation means the gradient with respect to the components of $\boldsymbol{v}$. Hence results that the first and the last terms of the integral can be combined to give the classical Hamiltonian function. In this way, the vectorial function $\boldsymbol{v}$ can be taken out of the integral and the variational problem is written

$$\int_{t_1}^{t_2} \mathrm{d}t \int \varrho[-H(\boldsymbol{r}, \nabla S, t) - \tfrac{1}{2} mD^2(\operatorname{grad} \ln \varrho)^2 - \dot{S}] \, \mathrm{d}v = \text{extremum} . \tag{18}$$

The integral must be stationary with respect to arbitrary variations of ϱ, whereas S is related to ϱ by the equalities (10) and (17).

It is easy to prove that this relation between S and ϱ is the same that is obtained by equating to zero the variations of (18) corresponding to arbitrary

variations of S. To do this, we must take into account the Hamilton equation

$$\boldsymbol{v} = \frac{\partial H(\boldsymbol{r}, \nabla s, t)}{\partial(\nabla s)} . \tag{19}$$

In this way, the equations of motion of the statistical fluid can be obtained from the single variational problem (18), taking ϱ and S as independent.

It is useful to combine both real functions ϱ and S, in the single complex function

$$\Psi = \sqrt{\varrho}\, \exp[iS/h]; \qquad h = 2mD. \tag{20}$$

In terms of this function it is possible to write, instead of (18),

$$\int_{t_1}^{t_2} \mathrm{d}t \int \Psi^* [i\hbar\partial/\partial t - H(\boldsymbol{r}, -i\hbar\nabla, t)]\Psi \,\mathrm{d}v = \text{extremum}\,, \tag{21}$$

where Ψ^* is the complex conjugate of Ψ and the usual rules of operator algebra are assumed. The proof of the equivalence between (18) and (21) is easy if it is assumed that $H(\boldsymbol{r}, \boldsymbol{p}, t)$ is quadratic in $\boldsymbol{p}$, which is the case in all problems of practical interest. In (21) it is indifferent to vary the function Ψ or its complex conjugate. By equating to zero the variation corresponding to Ψ^*, the equation of motion of the statistical ensemble results to be given by

$$ih\frac{\partial \Psi}{\partial t} = H(\boldsymbol{r}, -i\hbar\nabla, t)\Psi\,, \tag{22}$$

which is identical to the Schrödinger equation.

The derivation of the equation of motion for a system of N particles is straightforward. We need only assume now that the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ are in the configuration space of $3N$ dimensions. In order to obtain, in this case also, the Schrödinger equation of N particles, it is necessary to make the following hypotheses:

III) *The random motion of every particle is independent of that of the others.*

IV) *The product of the mass m of every particle times its diffusion parameter D equals $h/2$.*

These postulates complete the many-particle theory.

As in quantum mechanics, it is easy to prove that the equation of a system of particles can be separated into an equation of the relative motion plus an

equation for the motion of the centre of mass. This result is interesting because it shows that to obtain the Schrödinger equation for every body, it is not necessary to assume that a random force acts on the body as a whole, but only on a number of elementary particles, which we may assume all material bodies consist of.

4. – Expectation values.

In order to test the theory with experiments, it is necessary to calculate the expectation values of the fundamental quantities in terms of either the functions ϱ and S or the functions Ψ and Ψ^*. Actually, the main interest of the theory of random motion is due to its formal analogy with quantum mechanics and, to see whether the analogy is complete, it is also necessary to know how the expectation values must be calculated. In order to make the development clearer we will work with a single particle. The generalization to a system of particles is straightforward.

The statistical meaning of the function ϱ in the developed theory is such that the expectation value for the position of the particle must be calculated by means of the integral

$$\langle \boldsymbol{r} \rangle = \int \boldsymbol{r} \varrho \, \mathrm{d}v = \int \boldsymbol{r} \Psi^* \Psi \, \mathrm{d}v \,. \tag{23}$$

Generally, the expectation value of any function of $\boldsymbol{r}$ can be calculated by means of the integral

$$\langle f(\boldsymbol{r}) \rangle = \int f(\boldsymbol{r}) \varrho \, \mathrm{d}v = \int f(\boldsymbol{r}) \Psi^* \Psi \, \mathrm{d}v \,. \tag{24}$$

The expectation value of the velocity can be obtained from the integral (19), which gives the velocity of the statistical fluid in every point of space. So,

$$\langle \boldsymbol{v} \rangle = \int \varrho \, \frac{\partial H(\boldsymbol{r}, \boldsymbol{\nabla} S, t)}{\partial (\boldsymbol{\nabla} S)} \, \mathrm{d}v \,. \tag{25}$$

From this equality, taking into account (10), it is easy to show that

$$\langle \boldsymbol{v} \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{r} \rangle \,,$$

as it should, the derivation being a linear operation. The expectation value of $\boldsymbol{v}$ can also be obtained from the function ψ, which gives

$$\langle \boldsymbol{v} \rangle = \int \Psi^* \, \frac{\partial H(\boldsymbol{r}, -i\hbar \boldsymbol{\nabla}, t)}{\partial (-i\hbar \boldsymbol{\nabla})} \, \Psi \, \mathrm{d}v \,. \tag{26}$$

In this equality, H is considered a linear operator, function of the variables $\boldsymbol{r}$ and t and of the differential operator $\boldsymbol{\nabla}$, which acts on the function ψ. The proof of the equivalence of (26) and (25) is easy, taking into account that H is quadratic in $\boldsymbol{p}$. In fact, the equality (26), after the substitution of Ψ by the expression (20) leads to

$$\langle \boldsymbol{v} \rangle = \int \varrho \frac{\partial H(\boldsymbol{r}, \boldsymbol{\nabla} S, t)}{\partial (\boldsymbol{\nabla} S)} \, \mathrm{d}v - \int (i\hbar/2m) \boldsymbol{\nabla} \varrho \, \mathrm{d}v \, ,$$

where $1/2m$ is the coefficient of the term quadratic in $\boldsymbol{p}$. The first integral is identical with (25), and the last is zero if it is assumed that ϱ decreases rapidly enough at infinity.

The expression under the integral in (21) can be considered formally as a Lagrangian density. However, this function is complex and the imaginary terms were introduced rather artificially in going from (18) to (21). This shows that we must take as a Lagrangian density only the real part, which is identical with the expression under the integral (18). From this Lagrangian density the following Hamiltonian density is obtained for the statistical fluid:

$$\mathscr{H} = \varrho [H(\boldsymbol{r}, \boldsymbol{\nabla} S, t) + (\hbar^2/2m)(\mathrm{grad} \ln \varrho)^2] \, . \tag{27}$$

By integrating (27), the expectation value of the Hamiltonian function is obtained. In terms of ψ, the expectation value is

$$\langle H \rangle = \int \Psi^* H(\boldsymbol{r}, -i\hbar \boldsymbol{\nabla}, t) \Psi \, \mathrm{d}v \, , \tag{28}$$

which is equal to the integral of (27).

Now, it is necessary to obtain the expectation values of the linear and angular momenta. Equation (17) leads to the following expression for the density of linear momentum

$$\boldsymbol{\pi} = \varrho \boldsymbol{p} = \varrho \boldsymbol{\nabla} S \, ,$$

whence results the expectation value

$$\langle \boldsymbol{p} \rangle = \int \varrho \boldsymbol{\nabla} S \, \mathrm{d}v \, .$$

In terms of Ψ, this integral can be written as

$$\langle \boldsymbol{p} \rangle = \int \Psi^* (-i\hbar \boldsymbol{\nabla}) \Psi \, \mathrm{d}v \, . \tag{29}$$

The real part of this integral equals (29) and the imaginary part is zero. Simi-

larly, the expectation value of the angular momentum is given by

$$\langle \boldsymbol{l} \rangle = \int \boldsymbol{r} \cdot \boldsymbol{\pi} \, \mathrm{d}v = \int \Psi^* [\boldsymbol{r} \cdot (-i\hbar \nabla)] \Psi \, \mathrm{d}v \,. \tag{30}$$

The equalities (24) to (30), generalized for a system of particles, can be considered condensed in the following rule:

A quantity defined in classical mechanics by a function $F(q_k, p_k, t)$ of a system of orthogonal co-ordinates q_k and conjugated momenta p_k, quadratic at most in the momenta, has the operator $F(q_k, -i\hbar \partial/\partial q_k, t)$ associated with it. The expectation value of the quantity is given by the integral

$$\int \Psi^* F\left(q, -i\hbar \frac{\partial}{\partial q_k}, t\right) \Psi \, \mathrm{d}v \,. \tag{31}$$

5. – Discussion.

The equation of Schrödinger and the general rule for calculating the expectation values of the quantities, derived in the preceding Section, seem identical with the postulates of nonrelativistic quantum mechanics (without spin). However, the parallelism is not complete for two reasons, which we must analyse:

a) Not all the solutions of the Schrödinger equation valid according to quantum mechanics, are acceptable in the theory of random motion.

b) In quantum mechanics it is assumed that the expectation value of any function of q_i and p_i is given by the integral (31), but in the theory developed here, it is valid only for polynomials in p_k of degree two or less.

Let us analyse the first point. In quantum mechanics, some bound states of a particle are associated with functions Ψ that have nodal surfaces. For instance, the electron of a hydrogen atom in the state $2s$ is associated with a function which has a spherical nodal surface. To interpret this fact in keeping with the theory of random motion, we should assume that the electron is sometimes in the internal region and sometimes in the external one without ever crossing the nodal surface, which is absurd. This shows that some solutions of (22) are not acceptable in the theory of random motion.

As it is well known, every solution of (22) can be expressed as a linear combination of complex particular solutions which have a constant modulus and a phase proportional to the time. If a solution is obtained by combining only eigenfunctions of the discrete spectra, this solution represents also a bound state. In the theory of random motion the only acceptable functions are those which can be obtained by continuous evolution from the function

$$\varrho(\boldsymbol{r}, t_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \,.$$

This function represents a particle localized in the point $\boldsymbol{r}_0$ at time t_0. It is always possible to assume that the position of the particle was measured at a time t_0, however remote. The evolution of ϱ from that time on, would be given by the equation of Fokker-Planck, generalization of (7). It is not easy to see whether by considering only solutions of this type it is possible to explain the experimentally observed transitions between discrete levels. So, no definite conclusion can be obtained in relation to this point without a more careful analysis.

Let us discuss now the second point. In quantum mechanics the solutions that correspond to a definite energy fulfil the condition

$$\langle H^2 \rangle = \langle H \rangle^2 ,$$

and these functions are solutions of the time-independent Schrödinger equation. However, in the theory that we are developing, it is not possible to define the expectation value of the square of H, because this value should depend on the actual fluctuation of the random force and not only on the diffusion parameter.

In conclusion, the theory of random motion, resting upon the four postulates stated above, is not fully equivalent to nonrelativistic quantum mechanics. However, without a more careful analysis, it is not possible to decide whether, after all, the theory could interpret all the known experimental facts in the low-energy domain, as nonrelativistic quantum mechanics does.

RIASSUNTO (*)

Si studia il moto di una particella sotto l'azione di due tipi di forze, le une che variano lentamente e le altre che fluttuano rapidamente. Si suppone che le forze lentamente varianti siano date in ogni particolare problema. Si suppone che le forze fluttuanti rapidamente, che sono sconosciute, abbiano un carattere casuale. Si dimostra che il moto della particella dipende dalle forze casuali solo attraverso l'effetto di diffusione che esse producono. La teoria è di carattere statistico e si può determinare solo l'evoluzione di una densità di probabilità. Si sviluppa un formalismo lagrangiano e da esso si deducono le equazioni generali del moto. Questi risultati sono formalmente equivalenti all'equazione di Schrödinger. La generalizzazione ad un sistema di particelle è immediata se si suppone che le forze casuali agiscano indipendentemente e con uguale intensità su ogni particella elementare. Si ottengono i valori di attesa delle variabili dinamiche fondamentali. La teoria si dimostra molto simile, ma non completamente identica, alla meccanica quantistica non relativistica senza spin.

(*) *Traduzione a cura della Redazione.*

Лагранжианная формулировка теории случайного движения.

Резюме (*). — Исследуется движение частицы под действием сил двух типов, одних медленно меняющихся и других быстро флуктуирующих. Предполагается, что медленно меняющиеся силы заданы в каждом частном случае. Также предполагается, что быстро флуктуирующие силы, которые неизвестны, имеют случайный характер. Показывается, что движение частицы зависит от случайных сил только через эффект диффузии, который они же обуславливают. По своему характеру теория является статистической, и единственно возможно определить изменение плотности вероятности. Развивается лагранжианный формализм, и из него выводятся общие уравнения движения. Оказывается, что уравнения формально эквивалентны уравнению Шредингера. Обобщение для системы частиц производится непосредственно, если предполагается, что случайные силы действуют независимо и с одинаковой интенсивностью на каждую элементарную частицу. Получаются ожидаемые величины для основных динамических переменных. Показывается, что теория очень похожа, но полностью неидентична, на нерелятивистскую квантовую механику без спина.

(*) *Переведено редакцией.*