IL NUOVO CIMENTO VOL. LVI B, N. 1 11 Luglio 1968

Gravitational Instability of a Plasma.

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(ricevuto il 9 Agosto 1967)

Summary. — Using a modified Chew-Goldberger-Low equation (C.G.L. equation), the problem of gravitational instability of a plasma has been studied. The dispersion relation has been obtained for the case of wave propagation along the magnetic field. It is found that Jeans' criterion still determines the gravitational instability.

1. – Introduction.

An extensive study of the problem of gravitational instability of an infinite homogeneous medium, as carried out by various authors under varying conditions, has been given by CHANDRASEKHAR (1).

Recently GLIDDON (2) has extended this problem to the case of an anisotropic plasma by using the C.G.L. (3) equation. The C.G.L. equation is a strong tool for the study of macroscopic motion of a collisionless plasma in a strong magnetic field. But this is applicable only for small Larmor radii and low frequencies. Several authors (4,5) have demonstrated the importance of the finite Larmor radius (F.L.R.) of the ions on the plasma instabilities. It is, therefore, of some interest to study the F.L.R. effect of the ions on the problem of anisotropic plasma.

(1) S. CHANDRASEKHAR: *Hydrodynamic and Hydromagnetic Stability* (Oxford, 1961).

(2) J. E. C. GLIDDON: *Astrophys. Journ.*, **145**, 583 (1966).

(3) G. F. CHEW, M. L. GOLDBERGER and F. E. LOW: *Proc. Roy. Soc.* A **236**, 112 (1956).

(4) K. V. ROBERTS and J. B. TAYLOR: *Phys. Rev. Lett.*, **8**, 197 (1962).

(5) M. N. ROSENBLUTH, N. A. KRALL and N. ROSTOKER: *Suppl. Nucl. Fusion*, Part 1, 143 (1962).

Using the modified C.G.L. equation, as given by THOMPSON [6], in which the pressure tensor is corrected by the F.L.R. effects, the problem of gravitational instability of a plasma has been discussed here. In this note the discussion has been carried out for the case of wave propagation along the direction of magnetic field. The discussion of the case of wave propagation transverse to the magnetic field has already been discussed by the author [7] in a separate attempt.

It has been shown there that F.L.R. does not have any influence on the Jeans instability.

2. – Perturbation equations and the dispersion relation.

The relevant linearized perturbation equations are:

$$\frac{\partial}{\partial t}\delta\varrho + \varrho\nabla\cdot\boldsymbol{u} = 0\,, \tag{1}$$

$$\varrho\frac{\partial\boldsymbol{u}}{\partial t} = -\nabla\cdot\delta P + \frac{1}{4\pi}(\nabla\times\delta\boldsymbol{B})\times\boldsymbol{B} - \varrho\nabla\,\delta V\,, \tag{2}$$

$$\frac{\partial}{\partial t}\delta\boldsymbol{B} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{B_0}{4\pi\varrho\Omega c}\nabla\times[(\nabla\times\delta\boldsymbol{B})\times\boldsymbol{B}]\,, \tag{3}$$

$$\frac{\delta p_{\parallel}}{p_{\parallel}} = \frac{3\,\delta\varrho}{\varrho} - 2\,\frac{\delta B}{B}\,, \tag{4}$$

$$\frac{\delta p_{\perp}}{p_{\perp}} = \frac{\delta\varrho}{\varrho} + \frac{\delta B}{B}\,, \tag{5}$$

$$\nabla^2\delta V = 4\pi G\,\delta\varrho\,. \tag{6}$$

In eqs. (1)-(6), $\delta\varrho$, $\delta\boldsymbol{B}$, δV are respectively the perturbations in density ϱ, magnetic field $\boldsymbol{B}$ and the gravitational potential V. δV satisfies Poisson's equation (6). Here $\boldsymbol{u}$ is the velocity vector. We take here $\boldsymbol{B}$ to be along the z-axis, *i.e.* $\boldsymbol{B} = (0, 0, B_0)$.

Equation (3) governs the perturbations in the magnetic field. It has been obtained by including the F.L.R. effect of the ion and, therefore, using the generalized Ohm's law in the form

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B} = \frac{B_0}{\varrho\Omega c}\left(\frac{\boldsymbol{j}\times\boldsymbol{B}}{c}\right), \tag{7}$$

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[6] W. B. THOMPSON: *An Introduction to Plasma Physics* (London, 1962), p. 225.

[7] P. K. BHATIA: *Effect of finite Larmor radius on the gravitational instability of a plasma*, *Zeit. f. Astrophys.* (1968) in press.

instead of Ohm's law used in the C.G.L. equation

$$E + \frac{1}{c}\, u \times B = 0 \,. \tag{8}$$

In (7), we are neglecting the electron inertia term and we are also neglecting the effect of the F.L.R. of the ion on the electron pressure. In eq. (7) E denotes the electric field and $\Omega = eB_0/Mc$ is the Larmor frequency of the ion, M being the mass of the ion.

The pressure tensor P can be written, taking into account the F.L.R. effect,

$$P = p_\perp(1 - \hat{n}\hat{n}) + p_\parallel \hat{n}\hat{n} + P_1 = P^* + P_1 \,. \tag{9}$$

In eq. (9), $p_\perp$ and $p_\parallel$ are the components of pressure perpendicular and parallel, respectively, to the direction $\hat{n}$ of the magnetic field and P_1 is the first-order correction term to the pressure tensor due to the F.I.R. effect. The components of P_1, for the magnetic field along the z-axis, are given by MACMAHON [8] as

$$\left\{\begin{aligned}
&P_{1xx} = -P_{1yy} = -\frac{P_{i\perp}}{2\Omega}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right), \\
&P_{1zz} = 0 \,, \\
&P_{1xy} = P_{1yx} = \frac{P_{i\perp}}{2\Omega}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right), \\
&P_{1yz} = P_{1zy} = \frac{1}{\Omega}\left\{P_{i\perp}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - 2P_{i\parallel}\frac{\partial u_x}{\partial z}\right\}, \\
&P_{1zx} = P_{1xz} = \frac{1}{\Omega}\left\{P_{i\perp}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) - 2P_{i\parallel}\frac{\partial u_y}{\partial z}\right\},
\end{aligned}\right. \tag{10}$$

where $P_{i\perp}$ and $P_{i\parallel}$ are, respectively the normal and parallel components of the zeroth-order pressure tensor of the ions.

Equations (4) and (5) govern the perturbations in $p_\parallel$ and $p_\perp$.

Let us assume that the perturbations of all the quantities are of the form

$$\exp\,[ikz - i\omega t] \,. \tag{11}$$

Then

$$u = \frac{\partial \xi}{\partial t} = -i\omega \xi \,. \tag{12}$$

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[8] A. MACMAHON: *Phys. Fluids*, **8**, 1840 (1965).

For the perturbations of the form (11), the component of eq. (3) can be written as

$$ikB_0u_x + i\omega\,\delta B_x - \frac{B_0^2k^2}{4\pi\varrho\Omega c}\,\delta B_y = 0\,, \tag{13}$$

$$ikB_0u_y + \frac{B_0^2k^2}{4\pi\varrho\Omega c}\,\delta B_x + i\omega\,\delta B_y = 0\,, \tag{14}$$

$$\delta B_z = 0\,. \tag{15}$$

The values of $\nabla\cdot\delta p^*$ and $\nabla\,\delta V$ have been obtained by GLIDDON for the wave vector of the form $(k_x,\ k_y,\ k_z)$. We can, therefore, obtain their values for the present problem by putting $k_x = k_y = 0,\ k_z = k$ in his calculations. We, therefore, have

$$\nabla\cdot\delta P^* = [-(p_\parallel - p_\perp)k^2\xi_x,\ -(p_\parallel - p_\perp)k^2\xi_y,\ 3k^2\xi_z p_\parallel]\,, \tag{16}$$

$$\nabla\,\delta V = -\frac{4\pi G\varrho}{k^2}\,(k\xi_z)(0,0,k)\,. \tag{17}$$

Evaluating $\nabla\cdot\delta P_1$ from (10) and (11) we have

$$\nabla\cdot\delta P_1 = \left[\frac{i\omega k^2\xi_y}{\Omega}\,(P_{i\perp} - 2P_{i\parallel}),\ -\frac{i\omega k^2\xi_x}{\Omega}\,(P_{i\perp} - 2P_{i\parallel}),\ 0\right]. \tag{18}$$

With the help of (16)-(18), we can obtain the components of eq. (2). These three equations together with eqs. (13)-(15) can be written in the following determinantal form:

$$\begin{Vmatrix} \varrho\omega^2 + (p_\parallel - p_\perp)k^2\,, & \dfrac{(2P_{i\parallel} - P_{i\perp})}{\Omega}\,i\omega k^2, & 0 & , \dfrac{ikB_0}{4\pi} & , & 0 \\ -\dfrac{2P_{i\parallel} - P_{i\perp}}{\Omega}\,i\omega k^2, & \varrho\omega^2 + (p_\parallel - p_\perp)k^2, & 0 & , 0 & , & \dfrac{ikB_0}{4\pi} \\ 0 \quad , & 0 & , \varrho\omega^2 - 3k^2p_\parallel + 4\pi G\varrho^2, & 0 & , & 0 \\ B_0k\omega \quad , & 0 & , 0 & , i\omega & , & \dfrac{-B_0^2k^2}{4\pi\varrho\Omega c} \\ 0 \quad , & B_0k\omega & , 0 & , \dfrac{B_0^2k^2}{4\pi\varrho\Omega c}, & & i\omega \end{Vmatrix} \begin{Vmatrix} \xi_x \\ \xi_y \\ \xi_z \\ \delta B_x \\ \delta B_y \end{Vmatrix} = 0\,.$$

This leads to the dispersion relation

$$\{\varrho\omega^2 - 3k^2 p_\parallel + 4\pi G\varrho^2\}[\omega^6 - \omega^4\{\Omega_c^2 + 2A + 2\Omega_B^2 + \Omega_L^2\} + \omega^2\{A^2 + 2A\Omega_c^2 + 2A\Omega_B^2 + (\Omega_L\Omega_c - \Omega_B^2)^2\} - A^2\Omega_c^2] = 0 , \tag{20}$$

where

$$\begin{cases} A = \left(\dfrac{p_\perp - p_\parallel}{\varrho}\right)k^2 , & \Omega_B^2 = \dfrac{B_0^2 k^2}{4\pi\varrho} , \\ \Omega_L = \left(\dfrac{2P_{i\parallel} - P_{i\perp}}{\Omega\varrho}\right)k^2 , & \Omega_c^2 = \dfrac{B_0^2 k^2}{4\pi\varrho\Omega c} . \end{cases} \tag{21}$$

The first factor of (20) gives

$$\frac{\omega^2}{k^2} = \frac{3p_\parallel}{\varrho} - \frac{4\pi G\varrho}{k^2} . \tag{22}$$

This shows that we have a mode of instability when $\omega^2 < 0$, *i.e.* for instability, the wave number k is given by

$$k < \left(\frac{4\pi G\varrho}{3p_\parallel/\varrho}\right)^{\frac{1}{2}} . \tag{23}$$

This critical wave number is the same that as obtained by JEANS for determining the gravitational instability.

The second factor of (20) shows that there are three modes of wave propagation. If ω_1^2, ω_2^2, ω_3^2 are the three modes, then

$$\omega_1^2\omega_2^2\omega_3^2 = + ive .$$

This means that either all the roots are positive real or there is a positive real root and a pair of complex roots. A close look at the second factor of (20) shows that there are no negative roots because a change of sign in the value of ω^2 $(= X)$ into $(-X)$ yields an equation in X in which all the coefficients are negative.

When all the roots are positive real, all the three modes are stable. For the complex roots, we have either a stable mode or an over-stable mode.

We thus find that the gravitational instability of the plasma considered is determined by (23) *i.e.* by Jeans' criterion.

We thus conclude that Jeans' criterion still determines the gravitational instability of a plasma when we include the F.L.R. effects in the case of an anisotropic plasma and take longitudinal wave propagation.

RIASSUNTO (*)

Per mezzo dell'equazione di Chew-Goldberger-Low modificata si è studiato il problema dell'instabilità gravitazionale di un plasma. Si è ottenuta la relazione di dispersione per il caso della propagazione di un'onda lungo il campo magnetico. Si trova che il criterio di Jeans determina anche l'instabilità gravitazionale.

(*) *Traduzione a cura della Redazione.*

Гравитационная нестабильность плазмы.

Резюме (*). — Используя модифицированное уравнение Чу-Гольдбергера-Лоу (C.G.L. уравнение), была исследована проблема гравитационной нестабильности плазмы. Было получено дисперсионное соотношение для случая распространения волны вдоль магнитного поля. Обнаружено, что критерий Джинса попрежнему определяет гравитационную нестабильность.

(*) *Переведено редакцией.*