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Relativistic Hydrodynamics with Irreversible Thermodynamics without the Paradox of Infinite Velocity of Heat Conduction.

M. KRANYŠ

Czechoslovak Academy of Sciences, Institute of Plasma Physics - Prague

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Summary. — This paper presents the special relativistic hydrothermodynamics (*i. e.* dynamics of mechanico-thermal processes) in covariant form for a simple viscous fluid, free of the paradox of infinite velocity of heat conduction. The theory was developed by correcting Eckart's theory (1) admitting also infinitely fast heat propagation which is incompatible with Einstein's principle of relativity. The paradox was eliminated in part by replacing the relation between heat flow and temperature — the so-called Fourier's phenomenological transport equation — by a more exact equation due to CATTANEO and VERNOTTE (2,3), and in part by changing in a corresponding manner the energy-momentum tensor. The new energy-momentum tensor contains all terms appearing in Eckart's tensor, as well as terms having in coefficient factors $1/V^2$ where V is the greatest heat velocity in the medium. Finally it is shown that also for diffusion processes one can eliminate the paradox of instantaneous propagation of diffusion flux from the equations describing relativistic diffusion phenomena.

1. – Introduction (*).

When describing heat conduction in a mass medium (or transfer processes in general), it is customary to use Fourier's transport equation (F) in con-

(*) This paper is an extract from the report (4) which was presented as thesis to the Faculty of Mathematics and Physics of Charles University in Prague in April 1964.

(1) C. ECKART: *Phys. Rev.*, **58**, 919 (1940).

(2) C. CATTANEO: *Compt. Rend.*, **247**, 431 (1958).

(3) P. VERNOTTE: *Compt. Rend.*, **246**, 3154 (1958).

(4) M. KRANYŠ: Report No 4/64 IPP Czech. Acad. Sci. Prague, (April 1964).

nection with the equation of heat conservation (*) (1.1) (or, possibly, the equation of conservation of quantity of transfer):

$$\boldsymbol{q} = -K \operatorname{grad} \vartheta\,, \tag{F}$$

$$\gamma m \dot{\vartheta} = -\operatorname{div} \boldsymbol{q}\,, \tag{1.1}$$

so that quantities $\boldsymbol{q}$ and ϑ, *i.e.* heat flow density and temperature are defined by equations

$$\Delta \boldsymbol{q} - \gamma \frac{m}{K} \dot{\boldsymbol{q}} = 0\,, \tag{1.2}$$

$$\Delta \vartheta - \gamma \frac{m}{K} \dot{\vartheta} = 0\,, \tag{1.3}$$

of the parabolic type. But these equations admit the propagation of physical processes at an infinitely large velocity. From the point of view of relativity, this is a paradox — the so-called paradox of infinite velocity of propagation.

CATTANEO (2) and VERNOTTE (3) suggested a generalization of Fourier's phenomenological eq. (F) in the following form

$$\boldsymbol{q} + \varkappa \frac{\partial \boldsymbol{q}}{\partial t} = -K \operatorname{grad} \vartheta\,, \tag{C-V}$$

where $\varkappa$ (**) is a newly introduced macroscopic constant. The connection between $\varkappa$ and the kinetic quantities might be found in (2). The (C-V) transport equation in connection with the equation of conservation (1.1) gives following equations for quantities $\boldsymbol{q}$ and ϑ:

$$\operatorname{rot}\operatorname{rot} \boldsymbol{q} + \Delta \boldsymbol{q} - \varkappa \gamma \frac{m}{K} \ddot{\boldsymbol{q}} - \gamma \frac{m}{K} \dot{\boldsymbol{q}} = 0\,, \tag{1.4}$$

$$\Delta \vartheta - \varkappa \gamma \frac{m}{K} \ddot{\vartheta} - \gamma \frac{m}{K} \dot{\vartheta} = 0\,, \tag{1.5}$$

which are already of the hyperbolic type. This eliminates the paradox of infinite velocity of propagation of heat (or transfer processes); refer also to (5).

(*) Considering an isovolumic precess.

(**) Constant σ introduced in (2) is linked with constant $\varkappa$ used in the present study, by relation $\varkappa = \sigma/K$. $\varkappa$ has here the dimension of time and its order is identical with that of the so-called characteristic molecular time τ_0, *i.e.* the mean time between two successive collisions. For gases close to normal conditions, it is roughly $\varkappa \simeq \tau_0 \simeq (10^{-8} \div 10^{-10})$ s.

(5) P. M. MORSE, H. FESHBACH: *Methods of Theoretical Physics*, Pt. I, Chap. 7,4, p. 865 (New York, 1953).

On the basis of kinetic considerations for the case of a gaseous medium, CATTANEO (2) then stated the grounds of the new equation.

Let us now return to eqs. (1,4) and (1.5) for the vector of heat flow and temperature. The two equations are in the form of a so-called « telegraph equation » which describes pulse propagation in a dispersive medium. This means that various points of the pulse with generally different displacements possess also different translational velocities whose magnitude can reach values only within the interval from 0 to V, the maximum possible velocity of propagation (*) being

$$V = \sqrt{\frac{K}{\varkappa \gamma m}} < \infty . \tag{1.6}$$

Thus only the velocity of propagation of the signal front reaches the value of V, whereas the signal itself increasingly smears and the wave front corresponding to the end of the signal does not propagate at all but remains standing. In the limiting case $\varkappa \to 0$ we obtain $V \to \infty$ (which means that for the old theory the range of velocity dispersion is from zero to infinity), and Cattaneo-Vernotte's (**) eq. (C-V) changes into Fourier's eq. (F).

Another difference between the theories of heat conduction based respectively on (1.5) and (1.3), lies in that the partial differential eq. (1.3) possesses but a single initial condition while the corresponding generalized eq. (1.5) must have two initial conditions

$$[\vartheta(\boldsymbol{r}, t)]_{t=0} = g(\boldsymbol{r}) , \tag{1.7}$$

$$\left[\frac{\partial \vartheta(\boldsymbol{r}, t)}{\partial t}\right]_{t=0} = G(\boldsymbol{r}) \; (^{***}), \tag{1.8}$$

where g and G are the given continuous functions of position. We note that according to Cattaneo-Vernotte's theory and in contradistinction to Fourier's theory heat pulses can also be excited by the initial change in temperature alone, as indicated by condition (1.8). In the vast majority of practical applications, it will, on the other hand, be possible to simply put $G(\boldsymbol{r}) = 0$. Similarly, the boundary conditions for the solution of eq. (1.5) will have somewhat different form than those for eq. (1.3). Thus *e.g.* the so-called Newton's law of cooling (for details refer to Sect. **8**) will be

$$- K \nu_j \partial^j \vartheta = h(\vartheta - \theta) + h\varkappa \frac{\partial}{\partial t} (\vartheta - \theta) \; (^{***}), \tag{1.9}$$

(*) By its order, it corresponds to the velocity of sound in the gas under given conditions.

(**) The order of author's names is alphabetical.

(***) In case that the surface of the body does not move relatively to its surroundings and the boundary surface does not change its shape with time.

where ν_j $(j = 1, 2, 3)$ is the outer unit normal to the boundary surface, h the coefficient of surface heat transfer, θ the temperature of the surroundings.

An attempt to formulate the energy-momentum tensor of a fluid with heat conduction was made by VAN DANTZIG as early as 1939 [6]. But so far as we know, the first systematic relativistic theory of flowing fluid which takes into consideration the effects of heat conduction, was formulated in 1940 by ECKART in [1]. The basis of this Eckart's relativistic thermodynamics of flowing continuum is just the relativistic form of Fourier's law (F). The essential insufficiency of Eckart's formulation is that — although in covariant form — it admits an infinitely fast heat propagation which is incompatible with the relativistic theory because the primary postulate of relativity is that the maximum velocity of propagation of physical processes cannot exceed the velocity of light in vacuum [*]. It is just Fourier's law (F) or the equation of heat conduction (1.3) following therefrom, that does not respect the important consequence (as well as the postulate) of Einstein's principle of relativity, because according to (1.3), the front of the heat wave propagates with an infinite velocity. But if we put the relativistic form of Cattaneo-Vernotte's eq. (C-V) instead of the relativistic form of Fourier's law (F) as the basis of the relativistic thermodynamics of flowing fluids, we eliminate the paradox of infinite velocity of heat propagation from Eckart's theory. This will be the subject of our discussion in the Sections to follow.

The general relativistic theory, with which Stueckelberg works (see [7,8]), is practically identical with that of Eckart if the gravitational field in the former is neglected.

The author is also acquainted with a study by Pham Mau Quan [9] on the relativistic theory of fluids with heat conduction. The theory presented in that study, is again a version of that of Eckart's. The theories of Eckart and of Pham Mau Quan can be generalized to include the case of relativistic magnetohydrodynamics — refer, *e.g.* to HUGHES [10], or PHAM MAU QUAN [11]; and as the studies of Stueckelberg and of Pham Mau Quan indicate, general relativistic versions of such theories are also possible.

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(6) D. VAN DANTZIG: *Physica* **6**, 673 (1939) (see p. 688).

(*) This means that in the arbitrary pulse representing the real physical phenomenon whose parts propagate at various velocities, *none* of the parts of the pulse can propagate faster than the light in vacuum.

(7) E. C. G. STUECKELBERG and G. WANDERS: *Helv. Phys. Acta*, **26**, 307 (1953).

(8) E. C. G. STUECKELBERG: *Helv. Phys. Acta*, **35**, 568 (1962).

(9) PHAM MAU QUAN: *Ann. di Mat. pura appl.*, Ser IV, **38**, 121, (1955).

(10) W. F. HUGHES: *Proc. Cambr. Phil. Soc.*, **57**, 4, 878 (1961).

(11) PHAM MAU QUAN: *Journ. Rational Mech. Anal.* **5**, 473 (1956).

2. – Geometry of space-time. Resolution of space-time and world tensors (*).

We shall use Galilean cordinate systems

$$x^\alpha = (ct, x, y, z)\,, \qquad x_\alpha = (-ct, x, y, z)\,, \qquad \alpha = 0, 1, 2, 3, \tag{2.1}$$

for which the metric tensor is

$$-g_{00} = g_{11} = g_{22} = g_{33} = 1\,, \quad g_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta\,, \quad g^{\alpha\beta} = g_{\alpha\beta}\,. \tag{2.2}$$

The world interval ds is directly identified with the « proper time »:

$$\mathrm{d}s^2 \equiv -g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = c^2\,\mathrm{d}t^2\left(1-\frac{w^2}{c^2}\right), \qquad w^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2\,. \tag{2.3}$$

Furthermore, we define the four-velocity of mass as

$$u^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \left(\frac{c}{\sqrt{c^2-w^2}}, \frac{\boldsymbol{w}}{\sqrt{c^2-w^2}}\right), \qquad u_\alpha \equiv \frac{\mathrm{d}x_\alpha}{\mathrm{d}s}\,. \tag{2.4}$$

Then it holds that

$$u^\alpha u_\alpha = -1\,, \tag{2.5}$$

i.e. u^α is a dimensionless unit timelike vector for which

$$u^\alpha\,\partial_\beta u_\alpha = 0\,, \qquad u_\alpha\,\partial_\beta u^\alpha = 0\,. \tag{2.6}$$

For partial derivatives we use the symbols:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \qquad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right). \tag{2.7}$$

The proper local frame in which mass medium is at rest ($w_x = w_y = w_z = 0$), we denote by K_0. In this case

$$u^\alpha = (1, 0, 0, 0)\,, \qquad u_\alpha = (-1, 0, 0, 0)\,. \tag{2.8}$$

In a proper local co-ordinate system the velocity four-vector u^α, forming a tangent unit vector to the world line, determines the local axis of proper time.

(*) We have adopted almost ewerywhere Eckart's terminology and notation from paper (1).

Any world vector P^α can be resolved into a longitudinal (*i.e.* time) component $\overset{\parallel}{P}{}^\alpha$ and a transversal (*i.e.* space) component $\overset{\perp}{P}{}^\alpha$. Thus we have

$$P^\alpha \equiv \overset{\parallel}{P}{}^\alpha + \overset{\perp}{P}{}^\alpha = u^\alpha \overset{\parallel}{P} + S^\alpha_\beta P^\beta \,, \tag{2.9}$$

where

$$\overset{\parallel}{P} = - u_\beta P^\beta \tag{2.10}$$

and

$$S^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta \tag{2.11}$$

is projection operator to local proper space. From this definition it follows that

$$u_\alpha S^\alpha_\beta = 0 \,, \qquad u^\beta S^\alpha_\beta = 0 \,, \qquad S^\alpha_\beta S^\beta_\gamma = S^\alpha_\gamma \,. \tag{2.12}$$

Equally we can invariantly resolve differential operators:

$$\partial^\alpha \equiv \overset{\parallel}{\partial}{}^\alpha + \overset{\perp}{\partial}{}^\alpha = - u^\alpha D + S^\alpha_\beta \partial^\beta \,, \tag{2.13}$$

where symbol D designates

$$D \equiv u_\beta \partial^\beta = \frac{\mathrm{d}}{\mathrm{d}s} \qquad \text{or} \qquad D = \frac{1}{\sqrt{c^2 - w^2}} \left(\frac{\partial}{\partial t} + \boldsymbol{w} \nabla \right) . \tag{2.14}$$

Taking into account the latter definition and eq. (2.6) we obtain

$$u^\alpha D u_\alpha = 0 \,, \qquad u_\alpha D u^\alpha = 0 \,. \tag{2.15}$$

Later we shall also use the following relations

$$\Box \equiv \partial_\alpha \partial^\alpha = - D^2 + S^\beta_\gamma \partial_\beta \partial^\gamma + (D u_\gamma) \partial^\gamma \,, \tag{2.16}$$

$$\partial_\alpha [S^\alpha_\beta \partial^\beta] = \Box + D^2 + (\partial_\alpha u^\alpha) D \,, \tag{2.17}$$

$$\partial^\alpha D - D \partial^\alpha = (\partial^\alpha u_\beta) \partial^\beta \,. \tag{2.18}$$

In the last formula one must assume that the differential operator will always be applied only to functions $\Phi(x)^\alpha$ of such properties that $\partial^\alpha \partial^\beta \Phi = \partial^\beta \partial^\alpha \Phi$ holds.

3. – Description of the motion of matter.

Moving matter will be decribed in the same way as by ECKART [1] using a four-vector of the mass flow

$$m^\alpha = mu^\alpha\,, \qquad m^\alpha m_\alpha = -m^2\,, \tag{3.1}$$

where m is the invariant proper mass density of the continuum.

The law of conservation of matter is expressed by equation

$$\partial_\alpha m^\alpha = 0\,, \tag{3.2}$$

which on introduction of the invariant specific volume

$$v = \frac{1}{m} \tag{3.3}$$

may be written as follows

$$mDv = \partial_\alpha u^\alpha\,. \tag{3.4}$$

The interpretation is as follows: $\partial_\alpha u^\alpha$ is a measure of expansion or compression of the specific proper three-volume v. Thus for a solid body is $v = \text{const}$ and hence also $u^\alpha = \text{const}$ for all mass elements of the body.

4. – Construction of the energy-momentum tensor in relativistic hydrothermodynamics.

In a mass continuum, there occur next to purely mechanical, reversible processes, also thermal processes in the broader sense, *i.e.* irreversible processes such as heat conduction, viscous processes, diffusion phenomena, chemical processes, etc.

In the Section to follow we shall discuss only such an isotropic continuum (concretely, a fluid) in which only two irreversible processes — heat conduction and viscous processes — take place in addition to mechanical events. Such a restriction results from the fact that our considerations are only concerned with a homogeneous medium forming a single pure substance so that diffusion is absent and neither chemical processes nor phase transitions occur according to our assumption.

Since our dynamic study of the fluid will involve both thermal and mechanical aspects, we shall from time to time — for the sake of clarity — resort to the classical term of continuum dynamics (or hydrodynamics) with irreversible thermodynamics, although we are of the opinion that the term hydrothermodynamics is more consistent inasmuch as the word « dynamics » pertains to mechanical as well as thermal changes of the event being studied, which are, strictly speaking, inseparable. Let us now formulate the relativistic hydrothermodynamics in the sense just stated.

We ask for the basic equation of relativistic hydrothermodynamics to have the well-known form of four-divergence of the so-called energy-momentum tensor $W^{\alpha\beta}$

$$\partial_\alpha W^{\alpha\beta} = 0 \,, \tag{4.1}$$

which is symmetrical

$$W^{\alpha\beta} = W^{\beta\alpha} \,. \tag{4.1'}$$

Equations (4.1) and (4.1′) which represent the basic postulate of the theory, express both the conservation of energy-momentum (eq. (4.1)) and the conservation of moment of energy-momentum (eq. (4.1′)) of the fluid in which irreversible processes are taking place. Tensor $W^{\alpha\beta}$ can be resolved into proper components (see ([1])) which again form tensors

$$\overline{w} \;= W^{\alpha\beta} u_\alpha u_\beta \,, \tag{4.2}$$

$$w^\alpha \;= - S^\alpha_\beta W^{\beta\gamma} u_\gamma \,, \tag{4.3}$$

$$w^{\alpha\beta} = S^\alpha_\gamma S^\beta_\delta W^{\gamma\delta} \,, \tag{4.4}$$

hence it holds that

$$W^{\alpha\beta} = \overline{w} u^\alpha u^\beta + w^\alpha u^\beta + w^\beta u^\alpha + w^{\alpha\beta} \,. \tag{4.5}$$

Let us now discuss an important question, namely the physical interpretation of the various tensors (4.2), (4.3), (4.4) and consequently, also of (4.5). According to Eckart the invariant $\overline{w}$ is

$$\overline{w} = mc^2 + m\varepsilon \,. \tag{4.6}$$

Here mc^2 is the invariant density of the internal energy of matter at rest and $m\varepsilon$ is the density of the internal thermal energy.

Furthermore, tensor $w^{\alpha\beta}$ is the elastic-stress tensor which satisfies condition

$$u_\alpha w^{\alpha\beta} = 0 \,, \tag{4.7}$$

as follows from (4.4) and (2.12).

In case that the elastic continuum is a fluid the stress tensor $w^{\alpha\beta}$ can be resolved into the viscous component $P^{\alpha\beta}$ and the pressure component $pS^{\alpha\beta}$ ($\bar{\lambda}$ is the coefficient of viscosity):

$$w^{\alpha\beta} = - P^{\alpha\beta} + pS^{\alpha\beta} \,, \qquad P^{\alpha\beta} = \bar{\lambda} c \{ S^{\alpha\gamma} S^{\beta\delta} [\partial_\delta u_\gamma + \partial_\gamma u_\delta] - \tfrac{2}{3} S^{\alpha\beta} S^{\gamma\delta} \partial_\delta u_\gamma \} \,. \tag{4.8}$$

Particular attention must be devoted to the construction of the vector defined by eq. (4.3). In [1], ECKART puts $w^\alpha = q^\alpha/c$ where q^α denotes the heat flow defined by the covariantly transcribed Fourier's equation. Such a procedure leads, however, to a theory suffering — as explained in the Introduction — by the paradox of heat velocity greater than the velocity of light. In order to get rid of this paradox, we must give to vector w^α a somewhat different physical interpretation. First of all, we shall assume that heat propagation in a mass medium is governed by Cattaneo-Vernotte's theory which — in contradistinction to Fourier's theory — does not imply the paradox of instantaneous propagation of heat. Cattaneo-Vernotte's eq. (C-V) in covariant form is written as follows

$$q^\alpha + \varkappa c D q^\alpha = -K S^\alpha_\beta \partial^\beta \vartheta \,, \tag{4.9}$$

which is immediately clear if we recall that (4.9) is but a transcription of the (C-V) equation valid for proper local rest system K^0 (2.8), into an arbitrary co-ordinate system. Thus the classical three-vector of heat $\boldsymbol{q}$ goes over into spacelike four-vector q^α, $\partial/\partial t$ goes over to cD (2.14), and gradient ∇ goes over into transversal derivative $\overset{\perp}{\partial}{}^\alpha = S^\alpha_\beta \partial^\beta$.

Another possible covariant transcription of (C-V) equation is

$$q^\alpha + \varkappa c D q^\alpha = -K S^\alpha_\beta \{\partial^\beta \vartheta + \vartheta D u^\beta\} \,. \tag{4.10}$$

In this case, the temperature gradient $\nabla\vartheta$ on the right-hand side of the (C-V) equation has been transcribed to a somewhat more extended covariant form used by ECKART. Either of the alternative covariant transcriptions of the (C-V) equation might be used in the discussion to follow, but we shall prefer the simpler eq. (4.9).

First, we shall determine the longitudinal and transversal components of the four-vector of heat flow q^α in an arbitrary local co-ordinate system. According to (2.15)

$$q^\alpha = u^\alpha \overset{\parallel}{q} + \overset{\perp}{q}{}^\alpha \,, \tag{4.11}$$

$$\overset{\parallel}{q} = -u_\beta q^\beta \,, \tag{4.12}$$

$$\overset{\perp}{q}{}^\alpha = S^\alpha_\beta q^\beta \,. \tag{4.13}$$

Introducing in (4.12) for q^β from (4.9), we get with respect to (2.12)

$$\overset{\parallel}{q} = \varkappa u_\beta c D q^\beta \,, \tag{4.14}$$

which introduced back into (4.11) gives

$$\overset{\perp}{q}{}^\alpha = q^\alpha - \varkappa u^\alpha u_\beta c D q^\beta \,. \tag{4.15}$$

Naturally, it holds that

$$u_\alpha \overset{\perp}{q}{}^\alpha = 0 \,. \tag{4.16}$$

The construction of vector w^α will be carried out as follows: Let us put

$$w^\alpha = \frac{\overset{\perp}{q}{}^\alpha}{c} \qquad \text{or} \qquad w^\alpha = \frac{q^\alpha}{c} - \varkappa u^\alpha u_\beta D q^\beta \tag{4.18}$$

where $\overset{\perp}{q}{}^\alpha$ is the transversal component of the four-vector of heat flow relative to the world line in the sense of (2.9); *i.e.* a quantity defined by eq. (4.15). According to definition (4.3) and because of (2.12), vector w^α must also satisfy condition

$$u_\alpha w^\alpha = 0 \,. \tag{4.19}$$

But this condition is compatible with the construction of four-vector w^α done in accordance with (4.18), as directly follows from eq. (4.16). For the transition $\varkappa \to 0$ eq. (4.18) also goes over into the corresponding Eckart's equation.

In contradistinction to Eckart's theory from which it follows that there is no invariant density of heat different from zero, *i.e.* that it holds that $\overset{\parallel}{q} \equiv -u_\alpha q^\alpha = 0$ — refer to (1) eq. (25), — the more exact theory of heat conduction due to Cattaneo-Vernotte, leads to the existence of nonzero invariant density of heat defined by eq. (4.14). The complete energy-momentum tensor $W^{\alpha\beta}$ for an elastic continuum with heat conduction according to (4.5), (4.6) and (4.18) is obtained in the form

$$W^{\alpha\beta} = (mc^2 + m\varepsilon) u^\alpha u^\beta + w^{\alpha\beta} + \frac{1}{c}\,(\overset{\perp}{q}{}^\alpha u^\beta + \overset{\perp}{q}{}^\beta u^\alpha) \,. \tag{4.20}$$

5. – Relativistic formulation of the first law of nonequilibrium thermodynamics (or hydrothermodynamics).

By the covariant form of the first law of hydrothermodynamics we denote a scalar equation describing the projection of vector $\partial_\alpha W^{\alpha\beta}$ on the direction of the world line

$$u_\beta \partial_\alpha W^{\alpha\beta} = 0 \,. \tag{5.1}$$

By using eqs. (4.20, 16, 7), (3.2), (2.5, 14) and identity

$$D(u_\beta \overset{\perp}{q}{}^\beta) \equiv u_\beta D\overset{\perp}{q}{}^\beta + \overset{\perp}{q}{}^\beta D u_\beta = 0$$

we get for an elastic continuum

$$mD\varepsilon + \frac{1}{c}\,\partial_\alpha \overset{\perp}{q}{}^\alpha + \frac{1}{c}\overset{\perp}{q}{}^\alpha Du_\alpha + w^{\alpha\beta}\partial_\alpha u_\beta = 0 \tag{5.2}$$

or, in view of (4.8), (2.12, 13), (3.4) for a viscous fluid

$$-\frac{1}{mc}\{\partial_\alpha \overset{\perp}{q}{}^\alpha + \overset{\perp}{q}{}^\alpha Du_\alpha\} = D\varepsilon + pDv - \frac{1}{m}P^{\alpha\beta}\partial_\alpha u_\beta\,. \tag{5.3}$$

The next equation

$$mcDq = -(\partial_\alpha \overset{\perp}{q}{}^\alpha + \overset{\perp}{q}{}^\alpha Du_\alpha) \tag{5.4}$$

defines quantity q as the « specific heat » received by the unit mass from the surroundings. Through the use of eq. (5.4), the first law of thermodynamics, *i.e.* eq. (5.3) may be written in the following form:

$$Dq = D\varepsilon + pDv - vP^{\alpha\beta}\partial_\alpha u_\beta\,, \tag{5.5}$$

which in the nonrelativistic approximation $w/c \ll 1$, where it holds that $cD \to \mathrm{d}/\mathrm{d}$ (see (2.14)), goes over into the well-known form of the first law of thermodynamics for the viscous fluid. The total heat content Q in some three-dimensional space volume V inside of which the fluid is at rest at a given instant is then

$$Q = \int_V mq\,\mathrm{d}V \;(^*). \tag{5.6}$$

The right-hand side of eq. (5.4) contains, in addition to the usual four-divergence, also the term $-\overset{\perp}{q}{}^\alpha Du_\alpha$, expressing evidently some supply of heat arising from the accelerated flow of heat in the fluid. The term has no analog in the classical theory. It drops out for a uniformly flowing fluid.

6. – The relativistic equation for heat conduction in a continuum.

To illustrate the procedure of obtaining from the just formulated theory, the equations of heat conduction, and to make sure that our theory really

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(*) Integral quantities will also be derived similarly from other specific quantities. Thus *e.g.* for the total internal energy, we write $E = \int_V m\varepsilon\,\mathrm{d}V$, for entropy $\mathscr{S} = \int_V m\eta\,\mathrm{d}V$, etc.

leads in a special case to correct equations of heat conduction, let us derive the equation of heat conduction directly, at least for the case of an incoherent gas at mechanical equilibrium, *i.e.* behaving from the standpoint of macroscopic motion as a solid body in which thermal changes produce no dilatation. It then holds that

$$u^\alpha = \text{const} . \tag{6.1}$$

The first law of thermodynamics (5.2) then has the form

$$mD\varepsilon + \frac{1}{c}\,\partial_\alpha \overset{\perp}{q}{}^\alpha = 0 . \tag{6.2}$$

Assume that the internal energy (thermal) of an incoherent fluid explicitly depends on v and ϑ only, *i.e.* that it holds that $\varepsilon = \varepsilon(v, \vartheta)$, and then introduce the concept of specific heat at constant volume (in the rest system)

$$\gamma = \left(\frac{\partial \varepsilon}{\partial \vartheta}\right)_{v=\text{const}} . \tag{6.3}$$

A quantity thus defined might also depend on temperature; but assuming that this is not the case, it holds that

$$\varepsilon = \gamma\vartheta . \tag{6.4}$$

Or, eq. (6.2) has the form

$$-\,m\gamma c D\vartheta = \partial_\alpha \overset{\perp}{q}{}^\alpha . \tag{6.5}$$

Combining now eq. (6.5) with the equation of heat flow (4.9), we obtain the required covariant equation of heat conduction. Let us carry out operation ∂_α on eq. (4.13) and take account of (6.1) and (4.9), (2.12, 18); noting the interchangeability of operators $\partial_\alpha D = D\partial_\alpha$, we obtain

$$\partial_\alpha \overset{\perp}{q}{}^\alpha = -\,K\partial_\alpha[S^\alpha_\beta \partial^\beta \vartheta] - \varkappa c D \partial_\alpha [S^\alpha_\beta q^\beta] , \tag{6.6}$$

which, in view of (2.17) and (6.1), and of (4.13) gives

$$\partial_\alpha \overset{\perp}{q}{}^\alpha = -\,K\{\Box + D^2\}\vartheta - \varkappa c D\{\partial_\alpha \overset{\perp}{q}{}^\alpha\} . \tag{6.7}$$

And this equation, in connection with (6.5), finally gives the promised covariant form of the equation describing heat conduction according to CATTANEO and VERNOTTE:

$$m\gamma c D\vartheta = K\{\Box + D^2\}\,\vartheta - \varkappa m\gamma c^2 D^2 \vartheta . \tag{6.8}$$

To make sure of it right away, let us observe the process, of heat conduction from the proper local system K^0. Then eq. (6.8) changes in fact into eq. (1.5).

Finally, in any arbitrary inertial co-ordinate system, the relativistic covariant formulation of the equation of heat conduction (6.8) may be rewritten incovariantly with the aid of (2.18) as follows

$$(6.9)\quad \Box\vartheta-\left(\frac{m\gamma\varkappa}{K}-\frac{1}{c^2}\right)\frac{1}{1-w^2/c^2}\cdot\frac{\mathrm{d}^2\vartheta}{\mathrm{d}t^2}-\frac{m\gamma}{K}\frac{1}{\sqrt{1-w^2/c^2}}\cdot\frac{\mathrm{d}\vartheta}{\mathrm{d}t}=0\qquad\left(\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{w}\cdot\nabla\right).$$

For $w\to 0$, we again obtain (1.5).

A general equation of heat conduction in a flowing fluid could be obtained from the first law (5.3) if a single equation containing neither q^α nor ε could be arrived at with the aid of eqs. (4.9) and $\varepsilon=\varepsilon(\vartheta, m)$.

7. – Relativistic formulation of the second law of nonequilibrium thermodynamics (or hydrothermodynamics), and determination of entropy production (case of a simple fluid).

According to the nonequilibrium thermodynamics, a state function $\mathscr{S}$ called the entropy of the system (see ([12]), Chap. III, § 1) can be introduced in any macroscopic system. In the Sections to follow, we shall make use of the term specific entropy, *i.e.* entropy of unit mass in a rest system η, and of the concept of four-flow of entropy η^α which will evidently depend, in a hitherto not determined manner, on the four-vector of heat flow and on the reciprocal value of temperature.

The mathematical formulation of the law of entropy of nonequilibrium thermodynamics in a covariant form is

$$(7.1)\qquad mcD\eta=\partial_\alpha\eta^\alpha+\sigma\,,$$

$$(7.2)\qquad \sigma\geqslant 0\,.$$

Equation (7.1) is a general form of the so-called specific entropy balance equation. The symbol σ denoting a source term is the entropy production.

Our discussion will be restricted to a so-called simple viscous fluid (see ([1])) characterized by the following: specific entropy is an explicit function of only the local internal energy ε and volume v, *i.e.*

$$(7.3)\qquad \eta=\eta(\varepsilon, v)$$

([12]) S. R. DeGroot, P. Mazur: *Nonequilibrium Thermodynamics.* (Amsterdam, 1962).

and not of the co-ordinates or gradients ε or v. Assumption (7.3) is required in order to express the total differential of the specific entropy η with the aid of Gibbs equation in the form (the second law)

$$\vartheta D\eta = D\varepsilon + pDv\,. \tag{7.4}$$

In nonequilibrium thermodynamics this formula is assumed to remain valid for each element of mass along the world line even outside thermostatic equilibrium. The validity of formula (7.4) and of the corresponding nonequilibrium thermodynamics as well, is of course restricted to states not too far from the equilibrium states. For detailed information on the range of validity of equation (7.4), refer, *e.g.* to ([12]) Chap IX, or ([13]), p. 11, p. 220. Nevertheless, it can be said that for the most usual cases of heat propagation, the application of (7.4) is justified, because the relative change of temperature $\Delta\vartheta/\vartheta$ on the mean free path of molecules is always slight for a mass medium. The accuracy of transport equations for heat, be it Fourier's or Cattaneo-Vernotte's, is dependent on virtually the same limitations, and it roughly holds that it is the greater, the higher the temperature of the system and the smaller the differences in temperature, because then even the thermal conductivity approaches a constant, which is an assumption accepted herein.

Equation (5.3) in connection with Gibbs' equation (7.4) gives

$$mD\eta = -\frac{1}{c}\,\partial_\alpha\left(\frac{\overset{\perp}{q}{}^\alpha}{\vartheta}\right) - \frac{1}{c}\,\frac{1}{\vartheta^2}\,\overset{\perp}{q}{}^\alpha\{\partial_\alpha\vartheta + \vartheta Du_\alpha\} + \frac{1}{\vartheta}\,P^{\alpha\beta}\partial_\alpha u_\beta\,, \tag{7.5}$$

which is but an explicitly formulated equation of entropy balance (7.1). A comparison of eqs. (7.1) and (7.5) yields two important relations

$$\eta^\alpha = \frac{\overset{\perp}{q}{}^\alpha}{\vartheta} \qquad \text{or} \qquad \eta^\alpha = \frac{q^\alpha}{\vartheta} - \frac{\varkappa}{\vartheta}\,u^\alpha u_\beta c D q^\beta\,, \tag{7.6}$$

$$\sigma = -\frac{1}{\vartheta^2}\,\overset{\perp}{q}{}^\alpha\{\partial_\alpha\vartheta + \vartheta Du_\alpha\} + \frac{c}{\vartheta}\,P^{\alpha\beta}\partial_\alpha u_\beta\,. \tag{7.7}$$

Equation (7.6) defines the four-vector of entropy flow and eq. (7.7) gives an explicit expression for entropy production in the simple viscous fluid.

The expression for σ might be written in yet another form

$$\sigma = \frac{1}{\vartheta^2}\left[\frac{1}{K}\,\{\overset{\perp}{q}{}^\alpha\overset{\perp}{q}_\alpha + \varkappa\overset{\perp}{q}{}^\alpha c(Dq_\alpha)^\perp\} - \ell\,\overset{\perp}{q}{}^\alpha Du_\alpha\right] + \frac{c}{\vartheta}\,P^{\alpha\beta}\partial_\alpha u_\beta\,. \tag{7.8}$$

If we use instead of eq. (4.9), eq. (4.10) the last term in the bracket will fall out.

([13]) S. R. De Groot: *Thermodynamics of Irreversible Processes*. (Amsterdam, 1962).

There are two irreversible processes that in our case are responsible for the entropy production inside a mass element: first, heat conduction with its corresponding first term on the right-hand side of eq. (7.8), and second, irreversible processes associated with the dissipation of energy in viscous flow; the latter effect is represented by the second term on the right-hand side of eq. (7.8).

That the invariant rest value, and hence also the value given by eq. (7.8), of the entropy production σ always satisfies formula (7.2), is well known from the kinetic theory as Boltzmann's H-theorem ((12), p. 171).

Through the use of eq. (5.5), eq. (7.4) may be rewritten also in another form as follows:

$$D\eta = \frac{Dq + vP^{\alpha\beta}\,\partial_\alpha u_\beta}{\vartheta}\,. \tag{7.9}$$

Finally we shall give several notes about mathematical determinateness of the theory. Mechanical and thermal behaviour of the simple fluid are completely described by thirteen quantities $m=1/v$, u^α, p, ε, q^α, ϑ, η for the determination of which we need thirteen equations with the necessary number of boundary and initial conditions. These are eqs. (2.5), (3.2), (4.1), (4.9), (7.4), the equation of internal energy $\varepsilon=\varepsilon(m,\vartheta)$ and the equation of state $p=p(m,\vartheta)$. This system of equations must be completed by condition $\sigma\geqslant 0$ (see eq.(7.2)) which selects from the possible solutions of the system only solutions that have physical meaning. Thus the system of equations for a simple fluid in question constitutes axiomatic theorems of the theory, from which all other laws of the phenomenological theory can be deduced. Other thermodynamical functions that are used in thermostatics can also be introduced in our theory, and their corresponding balance equations can be derived (Details see (14)).

8. – Boundary and initial conditions.

The boundary conditions must be satisfied at all times on a (two-dimensional) surface forming the boundary between the fluid and the surroundings; from the standpoint of world geometry, this means that the boundary conditions must be fulfilled on the sheet of the world tube (*i.e.* on a three-dimensional surface (*)) associated with the fluid. The boundary condition of a thermodynamic problem states — according to Newton's cooling law — that the normal heat flow on the boundary between the fluid and its surroundings, is propor-

(14) M. KRANYŠ: Preprint IPP-14, Czec. Acad. Sci. Prague, (Januar 1965).

(*) *i.e.* two space and one time dimensions.

tional to the difference of temperature ϑ of the surface, and that of the surroundings θ:

$$q^{\alpha}\nu_{\alpha} = h(\vartheta - \theta) \quad (^{*}) , \tag{8.1}$$

where ν^{α} is the unit normal to the sheet of world tube ($\nu_{\alpha}\nu^{\alpha} = +1$) so that it evidently holds that $u^{\alpha}\nu_{\alpha} = 0$, and h is the invariant constant called coefficient of surface heat transfer. Combining the latter equation whith (4.9), we obtain after rearrangement the required equation in the explicit form. In case $u^{\alpha} = \text{const}$, it is also $\nu_{\alpha} = \text{const}$, *i.e.* the boundary area between the fluid and its surroundings does not change with time, and we get

$$-K\nu_{\alpha}\partial^{\alpha}\vartheta = h(\vartheta - \theta) + \varkappa hcD(\vartheta - \theta) . \tag{8.2}$$

In a rest system when $\nu_{\beta} = (0, \nu_j)$ holds, the above equation reduces to eq. (1.9). In case when the fluid boundary is thermally insulated, the above condition holds true with $h = 0$. And in case the fluid boundary is permanently maintained at the temperature θ of the surroundings, the above condition holds true with $h = \infty$ or $\vartheta = \theta$.

The initial conditions of the thermodynamics problem (1.7, 8) which must be fulfilled at the initial time $x_0 = \text{const}$ (in covariant writing $u^{\alpha}x_{\alpha} = \text{const}$) within the entire volume of the fluid in covariant transcription are

$$[\vartheta]_{x_0=\text{const}} = g , \qquad [cD\vartheta]_{x_0=\text{const}} = G . \tag{8.3}$$

Here g and G are given functions of space co-ordinates of points lying inside the fluid, or functions specified on the initial normal cross-section of the world tube of the respective fluid.

Naturally, all the boundary conditions go over into a well-known form in Fourier's approximation $\varkappa \to 0$. In the classical, *i.e.* nonrelativistic approximation when $w^2/c^2 \ll 1$, expression cD goes over to $\partial/\partial t + \boldsymbol{w}\cdot\nabla$ in all equations.

In a hydrothermodynamical problem, the boundary and initial conditions for mechanical quantities retain their form known from hydrodynamics. Thus the boundary condition for a viscous fluid represents the fact that on the boundary of a three-dimensional space filled with the fluid, the fluid velocity u^{α} must be equal to the velocity of the confining wall U^{α}, namely

$$u^{\alpha} = U^{\alpha} . \tag{8.4}$$

At the same time, the pressure in the fluid on the boundary must be equal to

(*) The discussion is restricted to cases for which $Du^{\alpha} = 0$.

the pressure of the confining medium P. Thus

$$p = P \,. \tag{8.5}$$

The initial conditions consist in that the initial density $\overset{0}{m}$, or pressure $\overset{0}{p}$ and flow velocity $\overset{0}{u}{}^{\alpha}$ must be known within the entire (three-dimensional) space occupied by the fluid.

9. – Conclusion.

The paper presents a formulation of special relativistic hydrothermodynamics (*i.e.* dynamics of mechanico-thermal processes) of a simple viscous fluid, free of the paradox of heat propagation at a velocity exceeding the velocity of light.

The theory was developed by correcting Eckart's (1) theory in the sense to satisfy the requirement of finite heat velocity. Although covariant, Eckart's theory admits also infinitely fast heat propagation which is incompatible with Einstein's principle of relativity. The paradox was eliminated in part by substituting a more exact relation, Cattaneo-Vernotte's eq. (4.9) for the so-called Fourier's transport equation describing the relation between heat flow and temperature, and in part by introducing into the energy-momentum tensor only the transversal part (4.15) of the heat flow four-vector as defined by the C-V equation, instead of the original Eckart's four-vector of heat flow. The new energy-momentum tensor (4.20) thus obtained forms the basis of the entire theory.

The equation of heat conduction (6.8) derived with the aid of the theory founded on the corrected energy-momentum tensor, clearly indicates that the velocity of heat propagation is in fact limited by a velocity $V < c$, accordingly to Cattaneo-Vernotte's theory of heat conduction.

At the same time, it was necessary to change the definition of entropy flow (7.6). All other changes followed automatically.

In Fourier's approximation, *i.e.* $V \to \infty$ or $\varkappa \to 0$, all newly defined quantities as well as equations go over into the corresponding quantities and equations of Eckart's theory.

So far as the modification carried out on the energy-momentum tensor is concerned, we believe that we have brought not too great a change in Eckart's statement ((1), p. 919): « The correct form of the energy-momentum tensor is still a matter of discussion ».

We have seen that hydrothermodynamics admits formulations that are in accord with the principle of special relativity. Further extension to the theory of general relativity offers no special difficulties (see *e.g.* (8-11-9)). An extension of the hydrothermodynamics of simple fluids to the magnetohydrothermodynamics of simple fluids is similarly clear ([10,11]).

Eckart's relativistic theory resting on Fourier's phenomenological equation, does not exclude Onsager's reciprocal relations in case the heat is conducted in an anisotropic medium or is accompanied by diffusion (for mixture of substances). On the contrary, our theory based on Cattaneo-Vernotte's equation, *i. e.* on the equation in which the heat flow is no longer proportional to the temperature gradient only, is of that kind that it does not satisfy Onsager's reciprocal relations (see *e.g.* ([13]) p. 224), at least in their present form. Thus Onsanger's relations are valid solely on the assumption that the phenomenological transport law is in the form of Fourier's equation, *i.e.* the presence of the paradox of instantaneous transport velocity is tacitly implied.

* * *

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Appendix

Notes on the relativistic theory of diffusion.

Another typical transport phenomenon (besides thermal conduction) is ordinary diffusion usually described by an equation of parabolic type and this equation is also loaded with the paradox of instantaneous propagation of diffusion flow. Also in this case we can reach the delimitation of propagation velocity in a similar way as we did in the case of heat conduction.

Let us consider the phenomenon of diffusion in an isotropic nonreacting mixture of N components (which are numbered by index $A = 1, 2, \dots N$) in the absence of external forces.

The law of conservation of mass can be written

$$\partial_\alpha(\underset{A}{u^\alpha}\,\underset{A}{m}) = 0\,, \qquad A = 1, 2, \dots N, \tag{A.1}$$

where $\underset{A}{m}$ is the mass density and $\underset{A}{u^\alpha}$ the four-velocity of component A. After sumation over all components we obtain

$$\partial_\alpha(u^\alpha m) = 0\,, \tag{A.2}$$

where m is the total mass density and u^α the barycentric velocity, which are defined by

$$m = \sum_{A=1}^{N} \underset{A}{m}\,, \qquad mu^\alpha = \sum_{A=1}^{N} \underset{A}{u^\alpha}\,\underset{A}{m} \qquad \text{or} \qquad u^\alpha = \sum_{A=1}^{N} \underset{A}{c}\,\underset{A}{u^\alpha}\,. \tag{A.3}$$

Here $\underset{A}{c} = \underset{A}{m}/m$ is the concentration of the component A. The diffusion flow of a substance A with respect to the barycentric motion is

$$\underset{A}{p^\alpha} = c\underset{A}{m}(\underset{A}{u^\alpha} - u^\alpha)\,. \tag{A.4}$$

If we put the expression for $\underset{A}{u^\alpha}\,\underset{A}{m}$ from (A.4) into eq. (A.1) we obtain

$$\frac{1}{c}\,\partial_\alpha \underset{A}{p^\alpha} = -\,\partial_\alpha(\underset{A}{m}u^\alpha)\,. \tag{A.5}$$

With $\underset{A}{m} = m\underset{A}{c}$ and with eq. (A.2) we obtain eq. (A.5) in the form

$$\frac{1}{c}\,\partial_\alpha \underset{A}{p^\alpha} = -\,mD\underset{A}{c}\,. \tag{A.6}$$

Now we use instead of Fick's relativistic transport equation (see STUECKELBERG and WANDERS ([7])) the equation

$$\underset{A}{p^\alpha} + \underset{A}{\varkappa}cD\underset{A}{p^\alpha} = -\,\underset{A}{\lambda}S^\alpha_\beta[\,\partial^\beta \underset{A}{m} + \underset{A}{m}Du^\beta]\,, \tag{A.7}$$

where $\underset{A}{\lambda}$ is the diffusion coefficient (of component A) in the rest frame, and $\underset{A}{\varkappa}$ is the new coefficient (*) which is connected with $\underset{A}{\lambda}$ and maximal diffusion velocity V_A in the rest frame by the relation

$$\underset{A}{\varkappa} = \frac{\underset{A}{\lambda}}{V_A^2}\ (^{**})\,. \tag{A.8}$$

Now we restrict to the most simple case of ordinary diffusion without conduction of heat, and cross-effects. Let as now suppose that $\vartheta = \text{const}$, and simultaneously $q^\alpha = 0$ and $u^\alpha = \text{const}$ (*i.e.* $m = \text{const}$) and that we have only a binary system ($N = 2$) with $\underset{1}{c} \ll 1$ and $\underset{2}{c} \simeq 1$ (see ([12]), p. 254).

Then eq. (A.7) after application of ∂^α and with the use of (2.17) and (2.18) is

$$\partial_\alpha \underset{1}{p^\alpha} + \underset{1}{\varkappa}cD(\partial_\alpha \underset{1}{p^\alpha}) = -\,m\underset{1}{\lambda}\,\partial_\alpha[S^\alpha_\beta\,\partial^\beta \underset{1}{c}] = -\,m\underset{1}{\lambda}[\Box + D^2]\underset{1}{c}\,. \tag{A.9}$$

(*) We may call $\underset{A}{\varkappa}$ the velocity cut-off coefficient for diffusion.

(**) For (a gas) $\underset{A}{\lambda} \simeq 10^{-1}\ \text{cm}^2\cdot\text{s}^{-1}$, $V_A \simeq 3.10^4\ \text{cm}\cdot\text{s}^{-1}$ (velocity of sound) is $\underset{A}{\varkappa} \simeq 10^{-10}\text{s}$. This is again $\underset{A}{\varkappa} \simeq \tau_0$ (see footnote in Sect. 1).

Finally using eq. (A.6) we obtain

$$cD\underset{1}{c} = \underset{1}{\lambda}\{\Box + D^2\}\underset{1}{c} - \varkappa c^2 D^2 \underset{1}{c} \,. \tag{A.10}$$

This is the diffusion equation in covariant form free of the paradox of instantaneous diffusion propagation.

RIASSUNTO (*)

Si presenta l'idrotermodinamica relativistica speciale (cioè la dinamica dei processi termomeccanici) in forma covariante per un fluido viscoso semplice, priva del paradosso della velocità infinita della conduzione del calore. Si è sviluppata la teoria correggendo quella di ECKART [1] che ammette anche una propagazione del calore infinitamente veloce, il che è incompatibile col principio della relatività di Einstein. Il paradosso è stato eliminato, in parte sostituendo la relazione fra flusso di calore e temperatura — la cosiddetta equazione fenomenologica del trasporto di Fourier — con un'equazione più esatta dovuta a CATTANEO e VERNOTTE [2,3] ed in parte modificando in modo corrispondente il tensore energia-impulso. Il nuovo tensore energia-impulso contiene tutti i termini che compaiono nel tensore di Eckart, ed anche termini che hanno a coefficiente fattori $1/V^2$ dove V è la massima velocità del calore nel mezzo. Infine si mostra che anche nei processi di diffusione si può eliminare il paradosso della propagazione istantanea del flusso di diffusione dalle equazioni che descrivono fenomeni di diffusione relativistici.

(*) *Traduzione a cura della Redazione.*