Wall and Collision Effects in Plasma Capacitors.

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Summary. — The problem of finding the electric field within a plasma-filled plate condenser, upon which an alternating e.m.f. is imposed, is considered. The adopted model implies the solution of the Boltzmann-Vlasov system of equations with a single relaxation time collision term, matched by suitable boundary conditions at the walls of the condenser. The plasma is assumed to be completely ionized and the frequency large enough to consider negligible the ion motions. In order to solve the Boltzmann-Vlasov system the method of separating the variables is used. Firstly a general mathematical theory of such solutions is developed, then applications to the plasma capacitor are considered. Both diffusing and reflecting walls are considered. In the limiting cases of large and small separation of the plates the effective permittivity is evaluated.

1. – Introduction and basic equations.

The problem of finding the electric field within a plasma-filled plate condenser upon which an e.m.f. is imposed has been studied extensively, both in the frame of a classical continuum treatment [1-3] and with the methods of kinetic theory [4,5]. Our researches aim to extend the previous results, with particular regard to that obtained in [4,5], when a more accurate model and more realistic boundary conditions are taken into account.

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(1) D. WEISSGLAS: *Journ. Nucl. Energy*, Part *C*, **4**, 329 (1962).
(2) P. VANDEPLAS and R. GOULD: *Physica*, **28**, 357 (1962).
(3) A. MESSIAEN and P. VANDEPLAS: *Physica*, **30**, 303 (1964).
(4) F. SHURE: *Journ. Nucl. Energy*, Part *C*, **6**, 1 (1964).
(5) R. AAMODT and K. CASE: *Ann. Phys.*, **21**, 289 (1963).

The model which we adopted implies the solution of the Boltzmann-Vlasov system of equations with a single relaxation time collision term, matched by suitable boundary conditions at the walls of the condenser. The plasma is assumed to be completely ionized and the frequency of the potential difference which is applied to the plates is assumed to be large enough that we can consider the ions at rest because of their inertia. According to this assumption, our equations do not conserve the momentum and the energy, but only the number of particles; consequently we are prevented from considering the plasma oscillations; our attention is principally devoted to the study of the thin plasma sheath adherent to the walls, for which a kinetic treatment is required.

The coupled one-dimensional Vlasov and Poisson equations for electrons are

$$\frac{\partial}{\partial t} f(x, v, t) + v \frac{\partial}{\partial x} f(x, v, t) - v \frac{n_0 e}{RTm} E(x, t) = \frac{1}{\theta}\left[\int_{-\infty}^{+\infty} F(v) f(x, v, t)\, \mathrm{d}v - f(x, v, t)\right], \tag{1.1}$$

$$\frac{\partial}{\partial x} E(x, t) = 4\pi e \int_{-\infty}^{+\infty} F(v) f(x, v, t)\, \mathrm{d}v\,. \tag{1.2}$$

The original Vlasov equation has been linearized about its equilibrium solution

$$\mathscr{F}(x, v, t) = F(v)[n_0 + f(x, v, t)]\,, \tag{1.3}$$

where

$$\mathscr{F}(x, v, t) = \int_{-\infty}^{+\infty} \mathscr{F}(x, \mathbf{v}, t)\, \mathrm{d}v_y\, \mathrm{d}v_z$$

is the equilibrium electron density and $F(v)$ is the dimensional Maxwellian

$$F(v) = \frac{1}{\sqrt{2\pi RT}} \exp[- v^2/2RT]\,. \tag{1.4}$$

θ has the meaning of the collision mean free time, e and m are the charge and the mass of electron.

It will be convenient to introduce the nondimensional variables

$$\frac{x}{\lambda_D}, \qquad t\omega_p\,, \qquad \frac{v}{\lambda_D\,\omega_p}\,,$$

and the nondimensional parameter $\zeta = 1/\omega_p \theta$ where the Debye lenght λ_D and the plasma frequency ω_p are given by

$$\lambda_D^2 = \frac{RTm}{4\pi n_0 e^2}, \qquad \omega_p^2 = \frac{4\pi n_0 e^2}{m}. \tag{1.5}$$

Now, rewriting the equations without changing the names of the variables, and putting $E/4\pi e\lambda_D$ in place of E gives

$$\frac{\partial}{\partial t} f(x, v, t) + v \frac{\partial}{\partial x} f(x, v, t) - vE(x, t) = \zeta \left[\int_{-\infty}^{+\infty} F(v) f(x, v, t)\, dv - f(x, v, t) \right], \tag{1.6}$$

$$\frac{\partial}{\partial t} E(x, t) = \int_{-\infty}^{+\infty} F(v) f(x, v, t)\, dv, \tag{1.7}$$

where now

$$F(v) = \frac{1}{\sqrt{2\pi}} \exp[-v^2/2].$$

We are looking for solutions of eqs. (1.6) and (1.7) with the time dependence $\exp[-i\omega t]$, ω being the frequency of the impressed field measured in ω_p units. The solution will describe the situation resulting from an applied voltage when the equilibrium is re-established.

2. – Elementary solutions of eqs. (1.6), (1.7).

It is useful to consider the combination

$$Y(x, v, t) = f(x, v, t) - \frac{v}{\sigma} E(x, t), \tag{2.1}$$

where $\sigma = \zeta - i\omega$; the symmetry of equations now suggests looking for solutions of the form

$$\begin{pmatrix} Y(x, v, t) \\ E(x, t) \end{pmatrix} = \begin{pmatrix} Y_\nu(v) \\ E_\nu \end{pmatrix} \exp[-i\omega t] \exp\left[-\sigma \frac{x}{\nu}\right]. \tag{2.2}$$

Equations (1.6) and (1.7) become

$$(2.3)\qquad \left(1-\frac{v}{\nu}\right) Y_\nu(v) = \frac{\sigma\zeta - v^2}{\sigma^2} \int_{-\infty}^{+\infty} F(v)\, Y_\nu(v)\, \mathrm{d}v\,,$$

$$(2.4)\qquad E_\nu = -\frac{\nu}{\sigma} \int_{-\infty}^{+\infty} F(v)\, Y_\nu(v)\, \mathrm{d}v\,.$$

Choosing a convenient normalization of $Y_\nu(v)$, *i.e.*

$$(2.5)\qquad \int_{-\infty}^{+\infty} F(v)\; Y_\nu(v)\, \mathrm{d}v = 1$$

we obtain three different types of elementary solutions.

CLASS I. *Discrete spectrum.* – If $\operatorname{Im}\nu \neq 0$, we can easily solve eqs. (2.3) ... by writing

$$(2.6)\qquad Y_{\nu_i}(v) = \frac{\sigma\zeta - v^2}{\sigma^2}\, \frac{\nu_i}{\nu_i - v}\,,$$

$$(2.7)\qquad E_{\nu_i} = -\nu_i/\sigma,$$

provided that the ν_i are roots of the equation

$$(2.8)\qquad \Lambda(\nu) = \frac{\sigma^2 - \nu^2}{\sigma^2} + \nu\, \frac{\sigma\zeta - \nu^2}{\sigma^2} \int_{-\infty}^{+\infty} \frac{F(v)}{v - \nu}\, \mathrm{d}v = 0\,.$$

The characteristic function $\Lambda(\nu)$ is analytic in the complex ν-plane with a cut along the entire real axis and asymptotically

$$(2.9)\qquad \Lambda(\nu) \underset{|\nu|\to+\infty}{\simeq} \frac{1 - i\omega\sigma}{\sigma^2} + O\left(\frac{1}{\nu^2}\right)\,.$$

Clearly if ν_i is a root of eq. (2.8), so is $-\nu_i$ and $\pm\nu_i^*$. Defining $\nu_{-i} = -\nu_i$, we assume for convenience that the ν_i can be labeled so that $\operatorname{Re}\nu_i > 0$, $i > 0$ and if $\nu_{i+1}^* \neq \nu_i$, then $|\operatorname{Re}\nu_i| < |\operatorname{Re}\nu_{i+1}|$. In general we have $2N$ zeros (which, for simplicity in notations, we will consider to be simple). About these zeros some results are listed in Appendix A.

CLASS II. *Continuous spectrum.* – In addition to discrete solutions we have the continuum of solutions corresponding to real values of ν

$$Y_\nu(v) = \frac{\sigma\zeta - v^2}{\sigma^2}\left[P\frac{\nu}{\nu - v}\right] + \lambda(\nu)\,\delta(\nu - v)\,, \tag{2.10}$$

$$E_\nu = -\nu/\sigma\,. \tag{2.11}$$

From eq. (2.5) one finds

$$F(\nu)\,\lambda(\nu) = \tfrac{1}{2}\left(\Lambda^+(\nu) + \Lambda^-(\nu)\right), \tag{2.12}$$

where $\Lambda^\pm(\nu)$ are the boundary values of $\Lambda(\nu)$ as ν approaches the real axis from above and below, respectively.

It should be noted that eqs. (1.6) and (1.7) are satisfied also by the spatially independent solutions

$$f_0(v, t) = \frac{v}{\sigma} A_0 \exp[-i\omega t]\,, \tag{2.13}$$

$$E_0(t) = A_0 \exp[-i\omega t]\,, \tag{2.14}$$

where A_0 is an arbitrary constant.

Now, in order to represent our functions $Y(x, v, t)$ and $E(x, t)$ as superposition of elementary solutions, we must prove the completeness theorem. This will be made in Sect. **5**; in the next Section we study the solution in a particular and instructive case.

3. – The time-independent case.

As it is shown in Appendix A, if $\omega = 0$, eq. (2.8) has only two roots, exceptionally real, $\nu = \pm\zeta$. One will note that this value of ω is out of the range that we are considering and the assumption of neglecting the movement of ions is not justified in this case. However, some interesting results have been obtained, which can give some information about the actual situation.

For future calculations it is now more convenient to consider, in place of (2.1), the following combination

$$\psi(x, v) = f(x, v) + V(x)\,, \tag{3.1}$$

where $V(x)$ is the electric potential

$$E(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}\,. \tag{3.2}$$

Our equations are now

$$v\,\frac{\partial\psi(x,v)}{\partial x}=\zeta\left[\int_{-\infty}^{+\infty}F(v)\,\psi(x,v)\,\mathrm{d}v-\psi(x,v)\right],\tag{3.3}$$

$$\frac{\mathrm{d}^2V(x)}{\mathrm{d}x^2}=V(x)-\int_{-\infty}^{+\infty}F(v)\,\psi(x,v)\,\mathrm{d}v\,.\tag{3.4}$$

Looking for solutions of the form

$$\begin{pmatrix}\psi(x,v)\\ V(x)\end{pmatrix}=\begin{pmatrix}\psi_\nu(v)\\ V_\nu\end{pmatrix}\exp[-\zeta x/\nu]\,,\tag{3.5}$$

eqs. (3.3), (3.4) become

$$\left(1-\frac{v}{\nu}\right)\psi_\nu(v)=\int_{-\infty}^{+\infty}F(v)\,\psi_\nu(v)\,\mathrm{d}v\,,\tag{3.6}$$

$$(1-\zeta^2/\nu^2)\,V_\nu=\int_{-\infty}^{+\infty}F(v)\,\psi_\nu(v)\,\mathrm{d}v\,.\tag{3.7}$$

Equation (3.3) is just the one considered by CERCIGNANI (6) in studying shear flow problems; its solution can then be written

$$\Psi(x,v)=A_0+A_1(v-\zeta x)+\int_{-\infty}^{+\infty}A(\nu)\,\exp[-\zeta x/\nu]\,\Psi_\nu(v)\,\mathrm{d}\nu\,,\tag{3.8}$$

where

$$\Psi_\nu(v)=P\,\frac{\nu}{\nu-v}+p(\nu)\,\delta(\nu-v)\tag{3.9}$$

and, in order to satisfy the normalization condition (2.5), we have

$$p(\nu)=\sqrt{2\pi}\left\{\exp[\nu^2/2]-2\left(\nu/\sqrt{2}\right)\int_0^{\nu/\sqrt{2}}\exp[t^2]\,\mathrm{d}t\right\}.\tag{3.10}$$

(6) C. CERCIGNANI: *Ann. Phys.*, **20**, 219 (1962).

Equation (3.7) is immediately solved and the general integral of eq. (3.4) can be written

$$(3.11)\qquad V(x)=A_0-A_1\zeta x+A_+\exp[x]+A_-\exp[-x]+\int_{-\infty}^{+\infty}A(\nu)\exp[-\zeta x/\nu]\,V_\nu\,d\nu\,,$$

where

$$(3.12)\qquad V_\nu=P\frac{\nu^2}{\nu^2-\zeta^2}\,.$$

From eqs. (3.1), (3.2), using the representations (3.8), (3.11), we have

$$(3.13)\qquad f(x,v)=A_1v-A_+\exp[x]-A_-\exp[-x]+\int_{-\infty}^{+\infty}A(\nu)f_\nu(v)\exp[-\zeta x/\nu]d\nu,$$

$$(3.14)\qquad E(x)=A_1\zeta-A_+\exp[x]+A_-\exp[-x]+\int_{-\infty}^{+\infty}A(\nu)E_\nu\exp[-\zeta x/\nu]d\nu\,,$$

where, from eqs. (3.9), (3.12),

$$(3.15)\qquad f_\nu(v)=P\frac{\nu v-\zeta^2}{\nu^2-\zeta^2}\,\frac{\nu}{\nu-v}+p(\nu)\delta(\nu-v)\,,$$

$$(3.16)\qquad E_\nu=P\frac{\nu\zeta}{\nu^2-\zeta^2}\,.$$

4. – Some time-independent problems.

The general solution found in the previous Section allows us to give immediately the solution of the following problem: the plasma is confined between two parallel plates at the same temperature (equal to the plasma temperature). The walls are assumed to diffuse the electrons according to a Maxwellian distribution function. Then the distribution will be Maxwellian everywhere and we shall have

$$(4.1)\qquad f(x)=-A_+e^x-A_-\exp[-x]\,,$$

$$(4.2)\qquad E(x)=-A_+e^x+A_-\exp[-x]\,,$$

$$(4.3)\qquad V(x)=\quad A_0+A_+e^x+A_-\exp[-x]\,.$$

In order to determine completely the problem we must give the potential difference between the walls (or, better, the excess of electrons with respect to ions); moreover we must choose the reference system: thus the constants A_+ and A_- can be determined.

With respect to the walls just considered, the plasma behaves as a dielectric (there is no current); moreover the collisions have no influence (the result does not depend on the parameter ζ).

On the other hand, if we assume that one of the walls is a sink of electrons and the other a source (so that the charge is constant in time, as is required by the stationary conditions), then the simplest (even if little realistic) boundary condition is of type

$$f(x, v) = kv \qquad \text{for } x = -\frac{d}{2}\,\mathrm{sgn}\, v\,, \tag{4.4}$$

where k represents the number of charges created at one wall and destroyed at the other for unit time and unit area.

Then the solution (global neutrality is assumed; if not, one must add the solution of the previous problem) is merely

$$f(x, v) = kv\,, \qquad E = k\zeta\,. \tag{4.5}$$

The plasma behaves now as a conducting medium: the electric current is constant (and equal to k) and ζ measures the plasma conductivity.

One must note that, while the first solution (eqs. (4.1) and (4.2)) does not depend on ζ, the second (eqs. (4.5)) varies with ζ, and, for $\zeta \to 0$ (no collisions), the plasma becomes a perfect conductor. Finally one verifies that the condition of specular reflection at the walls implies a solution of the first kind too.

As a third example, we consider the following problem: the plasma fills a half-space bounded by a wall which diffuses the electrons according to a Maxwellian distribution; but no condition of conservation of the number of electrons at the wall is now imposed.

The the boundary condition is

$$f(x, v) = \delta n \qquad \text{for } x = 0 \text{ and } v > 0\,, \tag{4.6}$$

where δn measures the excess (or defect) of electrons with respect to ions at the wall and must be determined from the total number of electrons. Taking into account the conditions at infinity, assuming $V(0) = 0$ and denoting by k, as previously, the constant electric current, we have

$$\delta n = A_0 + A_1 v + \int_0^\infty A(\nu)\Psi_\nu(v)\,\mathrm{d}\nu\,, \qquad v>0\,, \tag{4.7}$$

$$0 = A_0 + A_- + \int_0^\infty A(\nu)\,V_\nu\,\mathrm{d}\nu\,, \tag{4.8}$$

$$k = A_1\,. \tag{4.9}$$

Solving eq. (4.7) by the well-known technique (6) we obtain

$$A(\nu) = -k\,\frac{1}{X^-(\nu)[p(\nu)+\pi i\nu]}\,, \tag{4.10}$$

$$A_0 = \delta n - k\lambda\sqrt{\pi/2}\,, \tag{4.11}$$

where $\lambda = 1.1466$ is the slip coefficient (7), while

$$X(z) = z^{-1}\exp\left[\frac{1}{\pi}\int_0^\infty \mathrm{tg}^{-1}\left[\frac{\pi t}{p(t)}\right]\frac{\mathrm{d}t}{t-z}\right]. \tag{4.12}$$

From eq. (4.8) we can calculate A_-; using the identity (8)

$$\frac{1}{X(z)} = z + \frac{1}{\pi}\int_0^\infty \mathrm{tg}^{-1}\left[\frac{\pi t}{p(t)}\right]\mathrm{d}t - \int_0^\infty \frac{t\,\mathrm{d}t}{(t-z)X^-(t)[p(t)+\pi i t]} \tag{4.13}$$

we have

$$A_- = -\delta n - k\left[\frac{1}{X(\zeta)} + \frac{1}{X(-\zeta)}\right], \tag{4.14}$$

where $X(z) = \frac{1}{2}[X^+(z) + X^-(z)]$ when z is real and positive.
Thus, with simple manipulations, we can write

$$E(x) = k\zeta - \left[\delta n + k\left(\frac{1}{X(-\zeta)} + \frac{X(-\zeta)}{F(\zeta)\,p(\zeta)}\right)\right]\exp[-x] - k\zeta\int_0^\infty \frac{\nu X(-\nu)}{\nu^2-\zeta^2}\,\frac{\exp[-\zeta x/\nu]}{[p(\nu)]^2+\pi^2\nu^2}\,\frac{\mathrm{d}\nu}{F(\nu)}\,. \tag{4.15}$$

(7) S. ALBERTONI, C. CERCIGNANI and L. GOTUSSO: *Phys. Fluid.*, **6**, 993 (1963).

(8) C. CERCIGNANI: *The Kramers problem for a not completely diffusing wall*, in *Journ. Math. Anal. Appl.*, in press.

It is interesting now to note that, while the current is constant, the electric field shows a spatial transient; the proportionality between E and j, as found in the Chapman-Enskog theory, is atisfied only far from the wall; the thickness of the sheath where strong deviations from Ohm's law arise is of the order of $\max(1, \zeta)$ in the chosen units.

If $\zeta \to 0$, the integral in eq. (4.15) diverges logarithmically, and, since $X(0) = 1$, we have

$$E(x) = -(\delta n + 2k) \exp[-x] . \tag{4.16}$$

The plasma becomes a perfect conductor only a few Debye lenghts far from the wall.

Other problems should be interesting to study, like those involving temperature exchanges between the plasma and the walls; but then a more realistic model than that here used should be assumed in order to take into account the conservation of energy properly.

5. – The general case.

Let us now turn to consider the case when $\omega \neq 0$. As we have already noted, we must prove the completeness theorem for elementary solutions. It is an easy matter to prove, according to well-known methods [5,6], the following theorems:

Theorem I. – « Eigenfunctions » corresponding to different eigenvalues (both of continuous and discrete spectrum) are orthogonal on the whole real axis with respect to the weight function

$$\varrho(v) = \frac{\sigma^2}{\sigma\zeta - v^2} vF(v) .$$

In fact, one easily finds

$$\int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2} vF(v)\, Y_{\nu_i}(\nu v)\, Y_{\nu_j}(v)\, \mathrm{d}v = \delta_{ij} C_{\nu_i} , \tag{5.1}$$

$$\int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2} vF(v)\, Y_\nu(v)\, Y_{\nu'}(v)\, \mathrm{d}v = \delta(\nu - \nu')\, C(\nu) , \tag{5.2}$$

where

$$C_{\nu_i} = -\nu_i^2 \frac{dA(\nu)}{d\nu}\bigg|_{\nu_i} = \nu_i\left[-3 + \nu_i^2\frac{1-i\omega\sigma}{\sigma^2} + \frac{2\sigma\zeta}{\sigma\zeta-\nu_i^2}\left(1-\frac{\nu_i^2}{\sigma^2}\right)\right], \tag{5.3}$$

$$C(\nu) = -2\pi i\nu^2\left[\frac{1}{A^+(\nu)} - \frac{1}{A^-(\nu)}\right]^{-1} = \frac{\sigma\zeta-\nu^2}{\sigma^2}\,\nu F(\nu)\{[p(\nu)]^2 + \pi^2\nu^2\}. \tag{5.4}$$

In eq. (5.4) we have put

$$p(\nu) = \frac{\sigma^2}{\sigma\zeta-\nu^2}\,\lambda(\nu). \tag{5.5}$$

For $\omega = 0$, the function $p(\nu)$ now introduced reduces to that defined in eq. (3.10).

In addition to eqs. (5.1), (5.2) we have also

$$\int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2}\, vF(v)\, Y_{\nu_i}(v)\, Y_\nu(v)\, dv = 0. \tag{5.6}$$

Theorem II. - The set $\{Y_\nu(v)\}$, $(-\infty<\nu<+\infty)$ complemented with $\{Y_{\nu_i}(v)\}$, is complete on the real axis for the functions $Y(v)$ satisfying a Hölder condition in every finite interval of the real axis and such that

$$\int_{-\infty}^{+\infty} F(v)\, Y(v)\, dv < \infty. \tag{5.7}$$

One must show that $Y(v)$ can be expressed in the form

$$Y(v) = \sum_{\substack{i=-N\\ i\neq 0}}^{N} A_{\nu_i} Y_{\nu_i}(v) + \int_{-\infty}^{+\infty} A(\nu)\, Y_\nu(v)\, d\nu. \tag{5.8}$$

Let us begin by showing that expansion (5.8), if possible, is unique. In fact, using eq. (5.1) and subsequently eq. (5.2), gives

$$A_{\nu_i} C_{\nu_i} = \int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2}\, vF(v)\, Y(v)\, Y_{\nu_i}(v)\, dv, \tag{5.9}$$

$$A(\nu)\, C(\nu) = \int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2}\, vF(v)\, Y(v)\, Y_\nu(v)\, dv, \tag{5.10}$$

$A(\nu)$ and A_{ν_i} are thus found.

Now, in order to show that Theorem II is true, one must prove that the expansion does exist. We construct an actual solution of eq. (5.8).

All the integrals appearing converge, by condition (5.7) and by the property of $Y(v)$ to be Hölderian. The solution is given by eqs. (5.9) and (5.10) and $\nu A(\nu)$ results as the jump through the real axis of the following function of the complex variable z:

$$N(z) = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2}\, vF(v)\,[Y(v) - \sum_{\substack{-N \\ i\neq 0}}^{+N} A_{\nu_i} Y_{\nu_i}(v)]\,\mathrm{d}v}{\displaystyle\int_{-\infty}^{+\infty} \frac{vF(v)}{v - z}\,\mathrm{d}v}\,. \tag{5.11}$$

It is an easy matter to use Theorem II to construct the Green function of eqs. (1.6) and (1.7) for an infinite medium.

Theorem III. – Let α and β be two real numbers ($\alpha < \beta$); the set $\{Y_\nu(v)\}$, ($\alpha \leqslant \nu \leqslant \beta$) is complete for all the $Y(v)$ which satisfy a Hölder condition in the open interval $\alpha < v < \beta$ and are bounded by $A\,|v-\alpha|^{-\gamma}$ or $B\,|v-\beta|^{-\gamma}$ (with $\gamma < 1$) at the endpoints. In the limiting cases $\alpha = -\infty$ or $\beta = +\infty$ the theorem also holds, provided that the set $\{Y_\nu(v)\}$ is complemented with those of the eigenfunctions belonging to the discrete spectrum for which $\operatorname{Re}\nu_i < 0$ or $\operatorname{Re}\nu_i > 0$, respectively.

The case of the semiaxis $(0, \infty)$ is the most important one for applications, as we shall see; we restrict ourselves to the consideration of this case only.

To prove the theorem, we have to show that the singular integral equation

$$\frac{\sigma^2}{(\sigma\zeta - v^2)}\, Y(v) - \sum_{i=1}^{N} A_{\nu_i} \frac{\nu_i}{\nu_i - v} = P\int_0^\infty A(\nu)\, \frac{\nu\,\mathrm{d}\nu}{\nu - v} + A(v)\, p(v) \tag{5.12}$$

has a unique solution when $Y(v)$ is subject to the restrictions quoted in the theorem.

Let us consider the following function of the complex variable z:

$$X(z) = z^{-N} \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{G(v)}{v - z}\,\mathrm{d}v\right], \tag{5.13}$$

where

$$G(v) = \lg\frac{p(v) + \pi i v}{p(v) - \pi i v} \tag{5.14}$$

and the logarithm is such that $G(v) \to 0$ when $v \to \infty$; with this choice, $G(v) \to -2\pi i N$, when $v \to 0$. $X(z)$ is easily seen to be bounded when $z \to 0$ and have a z^{-N}-behavior when $z \to \infty$.

Let us now define a function $N(z)$ with the following relation (where $X^{\pm}(v) = \lim_{\varepsilon \to 0} X(v \pm i\varepsilon)$):

$$N(z) = \frac{1}{X(z)} \frac{1}{2\pi i} \int_0^\infty \frac{v X^-(v) y(v)}{[p(v) - \pi i v](v - z)} \, dv ; \tag{5.15}$$

here

$$y(v) = \frac{\sigma^2}{\sigma\zeta - v^2} Y(v) - \sum_{i=1}^{N} A_{\nu_i} \frac{\nu_i}{\nu_i - v} . \tag{5.16}$$

The integral in eq. (5.15) converges for every z not belonging to the positive real semiaxis; thus, $N(z)$ is an analytic function in the complex plane with a cut along the positive real semiaxis; further, $N(z)$ is bounded when $z \to 0$, and $N(z) \sim z^{-1}$ when $z \to \infty$, provided that the following N conditions are satisfied:

$$\int_0^\infty v^l \frac{v X^-(v) y(v)}{p(v) - \pi i v} \, dv = 0 \qquad [l = 0, 1, \dots, N-1] . \tag{5.17}$$

Writing eq. (5.17) explicitly gives

$$\int_0^\infty v^l \frac{\sigma^2}{\sigma\zeta - v^2} \frac{v X^-(v) Y(v)}{p(v) - \pi i v} \, dv = \sum_{i=1}^{N} A_{\nu_i} \nu_i \int_0^\infty v^l \frac{v X^-(v) \, dv}{[p(v) - \pi i v](\nu_i - v)} . \tag{5.18}$$

Using the representation of $X(z)$ given by (B.2), eqs. (5.18) take the form

$$\int_0^\infty v^l \frac{\sigma^2}{\sigma\zeta - v^2} \frac{v X^-(v) Y(v)}{p(v) - \pi i v} \, dv = -\sum_{i=1}^{N} A_{\nu_i} \nu_i^{l+1} X(\nu_i) . \tag{5.19}$$

Equations (5.19) constitute a linear system with determinant equal to

$$(-)^N \prod_{i=1}^{N} \nu_i X(\nu) \prod_{l>j=1}^{N} (\nu_l - \nu_j) \tag{5.20}$$

that is clearly nonzero. Thus, by using Cramer's rule, the A_{ν_i} are uniquely determined.

Then, if one defines

$$\nu A(\nu) = N^+ - N^- , \tag{5.21}$$

he can conclude

$$N(z) = \frac{1}{2\pi i} \int_0^\infty \frac{\nu A(\nu)}{\nu - z} \, d\nu \, . \tag{5.22}$$

Using the Plemely formulas for $\log X(z)$ gives

$$X^+ = X^- \frac{p(v) + \pi i v}{p(v) - \pi i v} \, . \tag{5.23}$$

Analogously from (5.15)

$$N^+ X^+ - N^- X^- = \frac{v X^-(v)\, y(v)}{p(v) - \pi i v} \, . \tag{5.24}$$

Inserting into this equation N^+, N^- as calculated by the Plemely formulas from eq. (5.22), and X^+ as given by eq. (5.23), gives eq. (5.12).

Conversely, if A_{ν_i} and $A(\nu)$ are constants and a function respectively such that eq. (5.12) is satisfied, let $N(z)$ be defined by eq. (5.22); using Plemely formulas and straightforward algebra gives eq. (5.24). Then $N(z)$ $X(z)$ must be given by eq. (5.15); letting z go to infinity in eq. (5.15) gives eqs. (5.17). Thus the solution given by eqs. (5.15) and (5.17) is the only solution of eq. (5.12).

If there are no discrete eigenfunctions, the previous procedure must be slightly modified. In fact it is sufficient to consider the same function $X(z)$ as defined in eq. (5.13) with $N = 0$; $X(z)$ has now a constant behavior when $z \to \infty$. The same definition (5.15) for $N(z)$ holds where $y(v) = Y(v)\big(\sigma^2/(\sigma\zeta - v^2)\big)$. Now $N(z) \sim z^{-1}$ without requesting any additional condition to be satisfied. Then defining $A(v)$ as in eq. (5.21), the demonstration proceeds as previously shown.

6. – Applications to the plasma capacitor.

To illustrate the usefulness of the theorems derived in the preceding Section, we consider now a specific problem. As we have already assumed in Sect. **4**, the plasma is confined between two plane parallel walls: as boundary conditions we suppose:

A) first case: the walls are specularly reflecting,

B) second case: the walls diffuse the electrons according to a Maxwellian distribution.

We note that these two cases were not distinguishable for $\omega = 0$.

In order to take the limit $L \to \infty$ later, we assume the reference system sketched in Fig. 1.

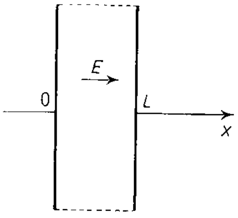

Fig. 1. – Geometry of the capacitor and reference system.

A) Reflecting walls. – This is the problem treated by WEISSGLAS [1] and by SHURE [4] without taking into account collisions.

However, it is to be noted that when treating such problems, the above theorems would not be strictly required: a Fourier expansion would be sufficient.

The then boundary conditions are now

$$Y(v, 0) - Y(-v, 0) = -\frac{2v}{\sigma} E(0) , \tag{6.1}$$

$$Y(v, L) - Y(-v, L) = -\frac{2v}{\sigma} E(L) . \tag{6.2}$$

Applying eqs. (6.1) and (6.2) to an expansion such eq. (5.8) we obtain

$$-\frac{2v}{\sigma} E(0) = \sum_{\substack{i=-N \\ i\neq 0}}^{+N} B_{\nu_i} Y_{\nu_i}(v) + \int_{-\infty}^{+\infty} B(\nu)\, Y_\nu(v)\, d\nu , \tag{6.3}$$

$$-\frac{2v}{\sigma} E(L) = \sum_{\substack{i=-N \\ i\neq 0}}^{+N} B'_{\nu_i} Y_{\nu_i}(v) + \int_{-\infty}^{+\infty} B'(\nu)\, Y_\nu(v)\, d\nu , \tag{6.4}$$

where we have put

$$B(\nu) = A(\nu) - A(-\nu) , \tag{6.5}$$

$$B'(\nu) = A(\nu) \exp[-\sigma L/\nu] - A(-\nu) \exp[\sigma L/\nu] , \tag{6.6}$$

and similarly for B_{ν_i} and B'_{ν_i}.

From eqs. (6.5), (6.6), taking into account that, for symmetry reasons, $E(0) = E(L)$, we have

$$A(\nu) = 2i\omega E(0) \frac{\nu^2}{\sigma\zeta - \nu^2} \frac{\exp[(\sigma/\nu)(L/2)]}{2 \cosh\big((\sigma/\nu)(L/2)\big)} \frac{1}{C(\nu)} \tag{6.7}$$

and

$$A_{\nu_i} = 2i\omega E(0) \frac{\nu_i^2}{\sigma\zeta - \nu_i^2} \frac{\exp[(\sigma/\nu_i)(L/2)}{2 \cosh\big((\sigma/\nu_i)(L/2)\big)} \frac{1}{C_{\nu_i}} . \tag{6.8}$$

The electric field can now be written as follows

$$E(x) = A_0 + \sum_{\substack{i=-N \\ i\neq 0}}^{N} A_{\nu_i} E_{\nu_i} \exp\left[-\frac{\sigma x}{\nu_i}\right] + \int_{-\infty}^{+\infty} A(\nu) E_\nu \exp[-\sigma x/\nu]\, \mathrm{d}\nu\,, \tag{6.9}$$

where A_{ν_i} and $A(\nu)$ are given by eqs. (6.7), (6.8) and E_{ν_i} and E_ν by eqs. (2.7), (2.11).

The integral in eq. (6.9) can be easily calculated by the method of residues (see Appendix C). One has

$$E(x) = E(0)\left\{\frac{\cosh[\sqrt{\sigma/\zeta}\,(L/2 - x)]}{\cosh(\sqrt{\sigma/\zeta}\,(L/2))} - \right. \tag{6.10}$$

$$\left. - \frac{2\omega}{\sigma} \sum_{n=1,3,5\ldots} \frac{1}{\pi n} \frac{1}{1+\sigma\zeta(\pi n/\sigma L)^2}\left[\frac{\exp[i(\pi n x/L)) - 1}{\Lambda(i(\sigma L/\pi n))} - \frac{\exp[-i(\pi n x/L) - 1]}{\Lambda(-i(\sigma L/\pi n))}\right]\right\}.$$

For $\zeta \to 0$, by taking into account that $\Lambda^+(\nu) = \Lambda^-(-\nu)$, eq. (6.10) reduces to Shure's result ([4]).

From Ohm's law we can calculate the impedance of the capacitor:

$$Z = \frac{1}{J}\int_0^L E(x)\, \mathrm{d}x\,, \tag{6.11}$$

where J is the displacement current at the plates; introducing the capacitance C_0 of the condenser gives

$$J = i\omega L C_0 E(0)\,. \tag{6.12}$$

Performing the integration indicated in eq. (6.11) gives

$$Z = \frac{1}{i\omega C_0}\left\{\frac{2}{L}\sqrt{\frac{\zeta}{\sigma}}\,\mathrm{tgh}\left(\sqrt{\frac{\sigma}{\zeta}}\,\frac{L}{2}\right) + \frac{2\omega}{\sigma}\sum_{n=1,3,5\ldots}\frac{1}{\pi n}\frac{1}{1+\sigma\zeta(\pi n/\sigma L)^2}\cdot\right. \tag{6.13}$$

$$\left.\cdot\left[\left(1 - \frac{2i}{\pi n}\right)\Big/\Lambda\left(i\frac{\sigma L}{\pi n}\right) - \left(1 + \frac{2i}{\pi n}\right)\Big/\Lambda\left(-i\frac{\sigma L}{\pi n}\right)\right]\right\}.$$

One can easily note that, owing to the fact that $\zeta \neq 0$, the impedance Z is always a complex number; *i.e.*, the collisions prevent the capacitor from becoming a largely dissipative device at certain frequencies ω_n.

The effect of collisions is also pointed out if one considers the actual distribution function $f(x, v)$.

As it was noted by SHURE ([4]), if $\zeta = 0$, the expression of $f(x, v)$ is singular at the velocities $v = \omega L/\pi n$ with which an electron may make a round trip in an odd number of periods. As is physically reasonable, these singularities vanish if the collisions are taken into account.

From eq. (6.13) we can calculate the complex dielectric permittivity ε

$$\varepsilon = \frac{1}{i\omega C_0 Z}\,. \tag{6.14}$$

It may be interesting to consider the limit $L \to \infty$; we have

$$\varepsilon = \left(1 - \frac{1}{i\omega\sigma}\right)\left[1 + \frac{2}{L}\sqrt{\frac{\zeta}{\sigma}}\,\frac{1 - i\omega\sigma}{i\omega\sigma}\right] + o(L^{-1})\,. \tag{6.15}$$

Taking in eq. (6.15) only the constant term and separating the real from the imaginary part gives the expressions for the dielectric permittivity and plasma conductivity; one obtains the familiar result given by GINZBURG (([9]): see particularly Chap. II).

Conversely, if one considers the limit $L \to 0$, one has the simple result

$$\varepsilon = 1 + \frac{L^2}{12} + \dots\,. \tag{6.16}$$

Now ε is a constant with respect to ω and ζ and is real, *i.e.*, the plasma conductivity is zero in this approximation.

B) Diffusing walls. We consider now the same problem in so far treated, but with more realistic boundary conditions; the walls are assumed to diffuse the electrons according to a Maxwellian distribution. Thus we have now

$$Y(v, 0) = \quad C - \frac{E(0)}{\sigma}\,v\,, \qquad v > 0\,, \tag{6.17}$$

$$Y(v, L) = -C - \frac{E(L)}{\sigma}\,v\,, \qquad v < 0\,. \tag{6.18}$$

where C is a costant proportional to the current at $X = 0$.

The condition that the electron current is conserved at the walls gives

$$C = -\sqrt{2\pi}\int_{-\infty}^{0} vF(v)\,f(0, v)\,dv\,. \tag{6.19}$$

([9]) V. L. GINZBURG: *Propagation of Electromagnetic Waves in Plasma* (Amsterdam, 1961).

Then, by taking into account Theorem III, we can write the general integral of our problem

$$(6.20)\qquad Y(v,x)=\sum_{\substack{i=-N\\ i\neq 0}}^{+N} A_{\nu_i}\exp\left[-\frac{\sigma x}{\nu_i}+\frac{\sigma L}{\nu_i}H(-\nu_i)\right]Y_{\nu_i}(v)+ \\ +\int_{-\infty}^{+\infty} A(\nu)\exp\left[-\frac{\sigma x}{\nu}+\frac{\sigma L}{\nu}H(-\nu)\right]Y_\nu(v)\,d\nu\,,$$

where we have put $H(-\nu_i)$ in place of $H(-\operatorname{Re}\nu_i)$.

We consider now the condition (6.17). We have the equation, valid only for $v>0$

$$(6.21)\qquad C-\frac{E(0)}{\sigma}v=\sum_{\substack{i=-N\\ i\neq 0}}^{-N} A_{\nu_i}\exp\left[\frac{\sigma L}{\nu_i}H(-\nu_i)\right]Y_{\nu_i}(v)+ \\ +\int_{-\infty}^{+\infty} A(\nu)\exp\left[\frac{\sigma L}{\nu}H(-\nu)\right]Y_\nu(v)\,d\nu\,.$$

As one can easily note by considering the transformation $v\to -v$, $x\to L-x$, the constants A_{ν_i} and the function $A(\nu)$ are subject to the antisymmetry conditions

$$(6.22)\qquad A_{\nu_i}=-A_{-\nu_i}\,,$$

$$(6.23)\qquad A(\nu)=-A(-\nu)\,.$$

The procedure that we shall use consists of reducing the singular integral eq. (6.21) to a Fredholm equation to which the classic iterative method of Neumann-Liouville can be applied.

This can be done in two different ways, the one particularly useful in the case $L\ll 1$, the other in the opposite case.

For simplicity in calculations we suppose that there are no discrete eigenfunctions (for example, this is true if ω is sufficiently large).

Let us begin by writing eq. (6.21) in the form

$$(6.24)\qquad y(v)-\int_{-\infty}^{0} A(\nu)\exp\left[\frac{\sigma L}{\nu}\right]\frac{\nu}{\nu-v}\,d\nu=\int_{0}^{\infty} A(\nu)\frac{\nu}{\nu-v}\,d\nu+p(v)A(v)\,,$$

where we have put

$$(6.25)\qquad y(v)=\frac{\sigma^2}{\sigma\zeta-v^2}\left(C-\frac{E(0)}{\sigma}v\right).$$

Applying to eq. (6.24) the inverse of the operator appearing in the right-hand side gives

$$A(v) = \frac{p(v)}{[p(v)]^2 + \pi^2 v^2}\left\{y(v) - \int_{-\infty}^{0} A(\nu)\exp\left[\frac{\sigma L}{\nu}\right]\frac{\nu}{\nu - v}\,d\nu\right\} - \tag{6.26}$$

$$- \frac{1}{X^-(v)}\frac{1}{p(v) + \pi i v} P\int_0^\infty \frac{tX^-(t)\left\{y(t) - \int_{-\infty}^{0} A(\nu)\exp[\sigma L/\nu]\left(\nu/(\nu - t)\right)d\nu\right\}}{[p(t) - \pi i t](t - v)}\,dt\,.$$

Performing the integrations in eq. (6.26) by using the identity (B.3) gives

$$A(v) = -\frac{1}{X^-(v)}\frac{1}{p(v) + \pi i v}\left\{a(v)X(\sqrt{\sigma\zeta}) + b(v)X(-\sqrt{\sigma\zeta}) + \int_{-\infty}^{0}\frac{A(\nu)\nu X(\nu)}{v - \nu}\exp\left[\frac{\sigma L}{\nu}\right]d\nu\right\}; \tag{6.27}$$

here

$$a(v) = \frac{1}{2\sqrt{\sigma\zeta}}\left(\frac{\sigma^2 C - \sigma E(0)v}{v - \sqrt{\sigma\zeta}} + \sigma E(0)\right), \tag{6.28}$$

$$b(v) = \frac{1}{2\sqrt{\sigma\zeta}}\left(\frac{-\sigma^2 C + \sigma E(0)v}{v + \sqrt{\sigma\zeta}} - \sigma E(0)\right). \tag{6.29}$$

In such a way the problem has been reduced to solving the integral eq. (6.27). It is an easy matter, using a well-known technique [8,10,11], to prove the convergence of the Neumann-Liouville series of eq. (6.27); also one can show that the larger is L the more rapid is the convergence.

We are now interested in the limit $L \to \infty$. Then the first term of the series gives (the terms neglected are exponentially decreasing with L)

$$A(v) = -\frac{a(v)X(\sqrt{\sigma\zeta}) + b(v)X(-\sqrt{\sigma\zeta})}{X^-(v)[p(v) + \pi i v]}\,. \tag{6.30}$$

([10]) C. CERCIGNANI: *Plane Couette flow according to the method of elementary solutions*, in *Journ. Math. Anal. Appl.*, in press.

([11]) C. CERCIGNANI: *Plane Poiseuille flow according to the method of elementary solutions*, in *Journ. Math. Anal. Appl.*, in press.

The electric field can now be written as follows:

$$(6.31)\qquad E(x) = E(0) + \frac{1}{\sigma}\int_0^\infty \nu A(\nu)\left(1 + \exp\left[-\frac{\sigma L}{\nu}\right]\right)d\nu - \frac{1}{\sigma}\int_0^\infty \nu A(\nu)\cdot \cdot\left(\exp\left[-\frac{\sigma x}{\nu}\right] + \exp\left[\frac{\sigma x}{\nu} - \frac{\sigma L}{\nu}\right]\right)d\nu\,,$$

where $A(\nu)$ is solution of eq. (6.27), if eq. (6.23) is taken into account.

In order to have the impedance of the capacitor, we must calculate

$$(6.32)\qquad \int_0^L E(x)\,dx = E(0)L + \frac{L}{\sigma}\int_0^\infty \nu A(\nu)\left(1 + \exp\left[-\frac{\sigma L}{\nu}\right]\right)d\nu - -\frac{2}{\sigma^2}\int_0^\infty \nu^2 A(\nu)\left(1 - \exp\left[-\frac{\sigma L}{\nu}\right]\right)d\nu\,.$$

In the approximation in which eq. (6.30) is valid, eq. (6.32) becomes

$$(6.33)\qquad \int_0^L E(x)\,dx = E(0)L + \frac{L}{\sigma}\int_0^\infty \nu A(\nu)\,d\nu - \frac{2}{\sigma^2}\int_0^\infty \nu^2 A(\nu)\,d\nu\,.$$

Performing the integrations by using the identity (B.4) gives

$$(6.34)\qquad \int_0^\infty v\;A(v)\,dv = aX(\sqrt{\sigma\zeta}) + bX(-\sqrt{\sigma\zeta}) - (a+b)\,,$$

$$(6.35)\qquad \int_0^\infty v^2 A(v)\,dv = \sqrt{\sigma\zeta}\left[aX(\sqrt{\sigma\zeta}) - bX(-\sqrt{\sigma\zeta}) - (a-b)\right],$$

here

$$(6.36)\qquad a = \frac{\sigma E(0)}{2} - \frac{\sigma^2 C}{2\sqrt{\sigma\zeta}}\,,$$

$$(6.37)\qquad b = \frac{\sigma E(0)}{2} + \frac{\sigma^2 C}{2\sqrt{\sigma\zeta}}\,.$$

We must now express the constant C through the only datum of the problem: the electric field at the walls. From eq. (6.19) we have

$$-\frac{C}{\sqrt{2\pi}}=\int_{-\infty}^{0} vF(v)\left[Y(0,v)+\frac{E(0)}{\sigma}v\right]dv\,. \tag{6.38}$$

Substituting in eq. (6.38) the expression of $Y(0,v)$ and exchanging the order of integration gives

$$-\frac{C}{\sqrt{2\pi}}=\int_0^{\infty} A(\nu)\,d\nu\int_{-\infty}^{0} vF(v)[Y_\nu(v)-Y_\nu(-v)]\,dv+\frac{E(0)}{2\sigma}\,. \tag{6.39}$$

Now, by performing the first integral, we have

$$-\frac{C}{\sqrt{2\pi}}=\frac{1-i\omega\sigma}{\sigma^2}\int_0^{\infty}\nu A(\nu)\,d\nu+\frac{E(0)}{2\sigma}\,. \tag{6.39 bis}$$

If now use eq. (6.30), by taking into account (6.34), we have finally

$$-C=\frac{E(0)}{\sigma}\,\frac{i\omega\sigma/(1-i\omega\sigma)+\frac{1}{2}[X(\sqrt{\sigma\zeta})+X(-\sqrt{\sigma\zeta})]}{(1/(2\sqrt{\sigma\zeta}))[X(-\sqrt{\sigma\zeta})-X(\sqrt{\sigma\zeta})]}\,. \tag{6.40}$$

Thus we have the current of electrons entering the wall. One can observe that, if $\zeta\to 0$ (*),

$$C=\frac{E(0)}{i\omega}X(0)\frac{1-X(0)}{X'(0)}\,. \tag{6.41}$$

The preceding expression coincides with that found in ([5]).

We are now able to give an explicit expression for the impedance Z of the capacitor and for the complex dielectric permittivity ε.

After some calculations we obtain

$$\varepsilon=\frac{1-i\omega\sigma}{-i\omega\sigma}\cdot\left\{1-\frac{2}{\sigma L}\left[2+\frac{1-2i\omega\sigma}{i\omega\sigma}\,\frac{X(\sqrt{\sigma\zeta})+X(-\sqrt{\sigma\zeta})}{2}\right]\frac{2\sqrt{\sigma\zeta}}{X(\sqrt{\sigma\zeta})-X(-\sqrt{\sigma\zeta})}\right\}\,. \tag{6.42}$$

(*) Since $(\partial/\partial\zeta)[X(z,\zeta,\omega)]<\infty$ we can write

$$\frac{\partial}{\partial\sqrt{\sigma\zeta}}[X(z,\zeta,\omega)]=\frac{\partial\zeta}{\partial\sqrt{\sigma\zeta}}\,\frac{\partial}{\partial\zeta}[X(z,\zeta,\omega)]\xrightarrow[\zeta\to 0]{}0\,.$$

As we can see, the constant term in eq. (6.42) is just the same as in eq. (6.15). which, in turn is the same as one finds when treating the full-range problem, As is physically reasonable, this term does not depend on any boundary condition.

Let us now turn to consider the case $L \ll 1$. For this purpose let us begin by writing eq. (6.21) in the following form:

$$C \operatorname{sgn} v - \frac{E(0)}{\sigma} v - \operatorname{sgn} v \int_0^\infty A(\nu)\left(1-\exp\left[-\frac{\sigma L}{\nu}\right]\right) Y_\nu(-|v|)\,\mathrm{d}\nu = \int_{-\infty}^{+\infty} A(\nu)\, Y_\nu(v)\mathrm{d}\nu \tag{6.43}$$

which is valid also for $v<0$.

Applying to eq. (6.43) the inverse of the operator appearing in right-hand side gives

$$\begin{aligned} A(\nu)\,C(\nu) = {} & C\int_{-\infty}^{+\infty} \frac{\sigma^2}{\sigma\zeta - v^2}\,|v|\,F(v)\,Y_\nu(v)\,\mathrm{d}v - \\ & -\frac{E(0)}{\sigma}\int_{-\infty}^{+\infty}\frac{\sigma^2}{\sigma\zeta - v^2}\,v^2 F(v)\,Y_\nu(v)\,\mathrm{d}v - \\ & -\int_0^\infty A_\mu\left(1-\exp\left[-\frac{\sigma L}{\mu}\right]\right)\mathrm{d}\mu \int_{-\infty}^{+\infty}\frac{\sigma^2}{\sigma\zeta - v^2}\,|v|\,F(v)\,Y_\mu(-|v|)\,Y_\nu(v)\,\mathrm{d}v\,. \end{aligned} \tag{6.44}$$

Performing the integrations in eq. (6.44) and rearranging some terms gives

$$\begin{aligned} A(\nu)\,C(\nu) = {} & 2CF(\nu)q(-\nu) + \nu\frac{-i\omega}{\sigma\zeta-\nu^2}\,(\sigma C - E(0)\,\nu) + \\ & + \frac{i\omega}{\sigma}\,\nu\int_0^\infty \frac{\mu A_\mu}{\mu+\nu}\left(1-\exp\left[-\frac{\sigma L}{\mu}\right]\right)\mathrm{d}\mu - \frac{\nu}{\sigma^2}\int_0^\infty \mu A_\mu\left(1-\exp\left[-\frac{\sigma L}{\mu}\right]\right) S(\mu,\nu)\,\mathrm{d}\mu\,, \end{aligned} \tag{6.45}$$

where we have defined

$$F(\nu)q(\nu) = \int_0^\infty \frac{vF(v)}{v-\nu}\,\mathrm{d}v\,, \tag{6.46}$$

$$S(\mu,\nu) = \frac{2}{\mu+\nu}\int_0^\infty \frac{vF(v)\left[\sigma\zeta(\mu+\nu+v)+\mu\nu v\right]}{(v+\mu)(v+\nu)}\,\mathrm{d}v\,. \tag{6.47}$$

Note that the following relation holds

$$F(\nu)p(\nu) = F(\nu)[q(\nu) + q(-\nu)] + \frac{\sigma^2 - \sigma\zeta}{\sigma\zeta - \nu^2}\,. \tag{6.48}$$

In such a way the problem has been reduced to solving the integral eq. (6.45). Again one could easily prove the convergence of the Neumann-Liouville series of eq. (6.45); also one could show that, the smaller is L, the more rapid is the convergence.

Since we are inteıested in the limit $L \to 0$, we take only the first term of the series; thus we have

$$A(\nu)\,C(\nu) = 2CF(\nu)q(-\nu) + \nu\,\frac{-i\omega}{\sigma\zeta - \nu^2}\,(\sigma C - E(0)\nu)\,. \tag{6.49}$$

We wish now to calculate the permittivity ε. Then we write eq. (6.32) in the approximation in which eq. (6.49) is valid

$$\int_0^L E(x)\,\mathrm{d}x = E(0)L + \frac{L^3\sigma}{6}\int_0^\infty \frac{A(\nu)}{\nu}\,\mathrm{d}\nu\,. \tag{6.50}$$

Now, in order to calculate C, we take into account eq. (6.39bis). If we substitute, in place of $A(\nu)$, the expression (6.49), we easily find $C = 0$.

In order to realize the fact that, also for $L \ll 1$ (as for $L \ll 1$) the first approximation for the permittivity ε does not depend on the chosen boundary conditions, we check solutions of eq. (1.6) by an iterative method based on the following scheme:

$$v\,\frac{\partial f^{(n)}(x, v)}{\partial x} = vE^{(n-1)}(x) - \sigma f^{(n-1)}(x, v) + \zeta\int_{-\infty}^{+\infty} F(v)\,f^{(n-1)}(x, v)\,\mathrm{d}v\,, \tag{6.51}$$

$$\frac{\partial E^{(n)}(x)}{\partial x} = \int_{-\infty}^{+\infty} F(v)\,f^{(n-1)}(x, v)\,\mathrm{d}v\,. \tag{6.52}$$

Equations (6.51) and (6.52) are accompanied by the following boundary conditions

$$x = 0\,, \qquad f(0, v) = (1-\alpha)f(0, -v) + \alpha C\,, \qquad v > 0, \tag{6.53}$$

$$x = L\,, \qquad f(L, v) = (1-\alpha)f(L, -v) - \alpha C\,, \qquad v < 0, \tag{6.54}$$

where α is a weight-constant ranging from 0 (conditions of specular reflexion) to 1 (conditions of Maxwellian diffusion). The condition that the electron current is conserved at the walls gives, for C, eq. (6.19) again.

We start with the zero solution $f^{(0)}(x, v) = 0$, $E^{(0)}(x) = 0$. For $n = 1$ we have simply $f^{(1)}(x, v) = 0$, $E^{(1)}(x) = E(0)$. For $n = 2$ we have, after easy calculations

$$f^{(2)}(x, v) = E(0)\left[x - \frac{L}{2} - \frac{\alpha}{2-\alpha} L \operatorname{sgn} v\right], \tag{6.55}$$

while $E^{(2)}(x) = E(0)$. Then, from $f^{(2)}(x, v)$, we have

$$E^{(3)}(x) = E(0)[1 + \tfrac{1}{2}x(x - L)]. \tag{6.56}$$

This expression is already independent of α; if we calculate ε, we have just eq. (6.56).

7. – Concluding remarks.

The system of Boltzmann-Vlasov has been treated by the method of the elementary solutions for different boundary conditions.

Some time-independent problems have been explicitly solved. One could usefully extend this first part by taking into account the ion motion, that is important in the limit of very small frequencies of the applied field.

For time-dependent (but stationary) problems it has been shown how to handle the elementary solutions method in order to ensure a rapid convergence of the approximated solution. The method is applied to the evaluation of the complex dielectric permittivity of a plasma capacitor for extreme values of the wall distance. The results, when first-order corrections are neglected, are in accordance with the elementary theory of the dielectric permittivity given by GINZBURG (9).

A possible extension of the above results could be that of taking into account a three-dimensional model of the plasma so that momentum and energy are conserved; thus the plasma oscillations effects could be studied.

Moreover a rather different collision term could be used, as *e.g.*, a simplified Fokker-Planck model, which allows us to treat the grazing collisions, which are neglected in the B.G.K. model.

* * *

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Appendix A

Let us consider eq. (2.8), $\Lambda(\nu)=0$. If $\nu^2=\sigma\zeta$ this equation gives $\sigma^2=\nu^2$, *i.e.*, $\omega=0$ and $\nu=\pm\zeta$. If $\nu^2\neq\sigma\zeta$, we can write eq. (2.8) in the form

$$\int_{-\infty}^{+\infty} F(v)\frac{\nu}{\nu-v}\,dv=\frac{\sigma^2-\nu^2}{\sigma\zeta-\nu^2}\,. \tag{A.1}$$

If now we look for the solutions of (A.1) for which $\omega=0$, we must satisfy the following equation

$$\int_{-\infty}^{+\infty} F(v)\frac{\nu}{\nu-v}\,dv=1\,, \tag{A.2}$$

which is an impossible one. Thus we can conclude that, for $\omega=0$, the equation $\Lambda(\nu)=0$ is satisfied only by $\nu=\pm\zeta$.

If, on the contrary, $\zeta=0$, eq. (A.1) becomes

$$\int_{-\infty}^{\infty} F(v)\frac{\nu}{\nu-v}\,dv=1+\frac{\omega^2}{\nu^2}\,. \tag{A.3}$$

Equation (A.3) is well known ([4]); it is easy to show that, when $\omega>1$ there are no solutions; when $\omega<1$, there are two: $\nu_\pm=\pm i\nu_0$, with ν_0 real and positive.

When $\omega\neq0$ and $\zeta\neq0$ eq. (A.1) is difficult to treat and no general result has been got. However, it seems a plausible thing to suppose that, for ω sufficiently large, no roots of eq. (A.1) exist.

Appendix B

We derive here very briefly some useful identities for the function $X(z)$, which have been used in the main text.

From eq. (5.23) we have

$$X^+(v)-X^-(v)=2\pi i\,\frac{vX^-(v)}{p(v)-\pi i v}\,. \tag{B.1}$$

Thus, the Plemely formulas applied to $X(z)$ give

$$X(z) = \int_0^\infty \frac{vX^-(v)}{p(v) - \pi i v} \frac{dv}{(v-z)^N} \tag{B.2}$$

according to the fact that $X(z)$ has a z^{-N} behavior for $z \to \infty$. If $N = 0$, and $X(z) \to 1$ for $z \to \infty$, the Plemely formulas give

$$X(z) = 1 + \int_0^\infty \frac{vX^-(v)}{p(v) - \pi i v} \frac{dv}{v - z}. \tag{B.3}$$

Now, we restrict ourselves to the case $N = 0$.

Let us consider the function $\varphi(z) = 1/X(z) - 1$, to which the Plemely formulas can be applied. Since, from (B.1), one has

$$\frac{1}{X^+(v)} - \frac{1}{X^-(v)} = \frac{1}{X^-(v)} \left[\frac{X^+(v)}{X^-(v)} - 1\right] = -\frac{1}{X^-(v)} \frac{2\pi i v}{p(v) + \pi i v},$$

we have

$$\frac{1}{X(z)} = 1 - \int_0^\infty \frac{v}{(v-z)X^-(v)} \frac{dv}{p(v) + \pi i v}. \tag{B.4}$$

Finally, if we consider the function

$$W(z) = \frac{X(z)X(-z)}{\Lambda(z)},$$

we have, for $z \to \infty$, $W(z) \to 1/\Lambda(\infty)$. Since

$$W^+ - W^- = X(-v) \left[\frac{X^+}{\Lambda^+} - \frac{X^-}{\Lambda^-}\right] = 0,$$

we have, taking into account Liouville's theorem,

$$X(z)X(-z) = \frac{\Lambda(z)}{\Lambda(\infty)}. \tag{B.5}$$

APPENDIX C

We have to calculate the integral

$$I = \int_{-\infty}^{+\infty} A(\nu) E_\nu \exp[-\sigma x/\nu] \, d\nu \,, \tag{C.1}$$

where $A(\nu)$ is given by eq. (6.7), $E_\nu = -\nu/\sigma$. Taking into account eq. (5.4) we have

$$I = \frac{i\omega}{\sigma} E(0) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\nu}{\sigma\zeta - \nu^2} \left[\frac{1}{\Lambda^+} - \frac{1}{\Lambda^-}\right] \frac{\exp[(\sigma/\nu)(L/2 - x)]}{\cosh\left((\sigma/\nu)(L/2)\right)} \, d\nu \,. \tag{C.2}$$

Let us consider

$$\int_{C_1} \frac{\nu}{\sigma\zeta - \nu^2} \frac{d\nu}{\Lambda(\nu)} \frac{\exp[(\sigma/\nu)(L/2 - x)]}{\cosh\left((\sigma/\nu)(L/2)\right)} - \int_{C_2} \frac{\nu}{\sigma\zeta - \nu^2} \frac{d\nu}{\Lambda(\nu)} \frac{\exp[(\sigma/\nu)(L/2 - x)]}{\cosh\left((\sigma/\nu)(L/2)\right)} \,, \tag{C.3}$$

where the integration paths C_1 and C_2 (represented in Fig. 2) are such to encircle all the singularities of the integrands. These singularities lie at points

$$\nu = \pm \sqrt{\sigma\zeta} \,,$$

$$\nu = \pm i \frac{\sigma L}{\pi n} \,,$$

$$\nu = \nu_i \qquad [n = 1, 3, 5, \ldots] \,,$$

where ν_i are the roots of the equation $\Lambda(\nu) = 0$.

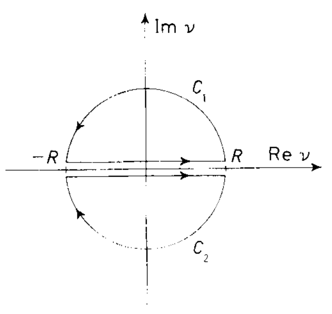

Fig. 2. – Integration path for the evaluation of the integral (C.1).

Denoting by S the sum of residues of the poles and taking the limit $R \to \infty$, we have

$$I = \frac{i\omega}{\sigma} E(0) S - \text{constant} \,, \tag{C.4}$$

where the constant represents the contribution to the integrals (C.3) from infinity. These constant combines with A_0 in expression (6.9). The residue contribution of the poles $\nu = \nu_i$ indentically cancel the sum in expression (6.9).

Thus, requiring that $E(x) = E(0)$ for $x = 0$, we have just eq. (6.10).

RIASSUNTO

Si considera il problema di valutare il campo elettrico che si stabilisce tra le armature di un condensatore a facce piane parallele, pieno di plasma, quando una f.e.m. alternata viene applicata alle facce. Il modello adottato richiede la soluzione del sistema di Boltzmann-Vlasov con termine di collisione a rilassamento unico, accompagnato da opportune condizioni al contorno. Si suppone che il plasma sia completamente ionizzato e la frequenza della f.e.m. abbastanza grande cosicché si possa trascurare il moto degli ioni. Per risolvere le equazioni di Boltzmann-Vlasov si adotta il metodo della separazione delle variabili. Innanzitutto si sviluppa una teoria matematica generale delle soluzioni elementari; quindi se ne considerano le applicazioni al condensatore a plasma. Si prendono in considerazione entrambi i casi di pareti diffondenti e riflettenti. Nei casi limite in cui la distanza tra le armature è molto grande o molto piccola si valuta la permittività efficace.