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Fluctuation of Photoelectrons and Intensity Correlation of Light Beams.

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Summary. — The complexity in writing down expressions for $P(n, T)$, the probability that n electrons are ejected from a photodetector by a fluctuating incident light beam is discussed and the problem of computing coherence functions from a knowledge of the moments of n is solved by recasting the problem in terms of product densities familiar in stochastic point processes. A general method of arriving at $P(n, T)$ from a knowledge of all the intensity correlations for any kind of light beam is studied. For the case of light from a thermal source with usual assumptions on the coherence functions $P(n, T)$ is shown to be the Bose distribution.

1. – Introduction.

Recently stochastic semiclassical methods [1,2] have been applied in the analysis of fluctuation of photoelectrons connected with correlation experiments in light beams. The method involves the statistics of counts in a fast photodetector illuminated by a light beam. The central quantity of interest is $P(n, T)$, the probability frequency function governing the number of counts in a certain time interval $(0, T)$. The Poisson distribution which normally explains the probability of the number of events that are registered in a counter does not give an adequate description of the process. The events do not occur in a Poissonian manner and in fact tend to have a bunching effect characteristic of Bose

(1) L. MANDEL: *Proc. Phys. Soc.*, **81**, 1104 (1963).
(2) L. MANDEL, E. C. G. SUDARSHAN and E. WOLF: *Proc. Phys. Soc.*, **84**, 435 (1964).

particles. Effectively we have here a kind of generalized Poisson process in which the Poisson parameter besides being a *function* of t is itself a random variable subject to certain correlations. In fact it has been recognised that an ensemble average of a simple Poisson distribution [3] may meet the needs of the physical situation. In this note we propose to go into the full intricacies implied in the process of averaging by adopting the methods of stochastic point processes (see for example ref. [4,5]).

We hope that such methods, apart from providing an explicit derivation that leads to the Bose distribution for light beams having Gaussian characteristics, will be useful in other situations involving correlations of electromagnetic fields [6-8].

Section **2** contains a treatment of non-Markovian type of incidence of photons and consequently also the product density description of the process. We next deal with the method of calculation of higher moments of the number of counts and also with an explicit way for arriving at all orders of coherence. The final Section contains the derivation leading to the explicit form of $P(n, T)$ starting from the coherence functions.

2. – Photoelectric counting.

Intensity correlation experiments to determine the coherence properties in light beams are performed by allowing the light to be incident on a fast photoelectric detector. If we treat the problem semiclassically, a stochastic description of the photoelectric counts actuated by the incident radiation field may lead to a probability distribution of n counts in time T, under certain conditions to be described later. But when the incident beam possesses certain statistical features which make themselves manifest through correlations in its intensity $I(t)$ at different time intervals, it is rather difficult to arrive at $P(n, T)$ explicitly. However since such a process can be viewed as a point process defined on the t-space, a description in terms of product densities [9-12] will be very useful.

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(3) E. WOLF and C. L. MEHTA: *Phys. Rev. Lett.*, **13**, 705 (1964).
(4) S. K. SRINIVASAN: *Nuovo Cimento*, **38**, 979 (1965).
(5) S. K. SRINIVASAN and R. VASUDEVAN: *Nuovo Cimento*, **41**, 101 (1966).
(6) R. C. BOURRET: *Nuovo Cimento*, **18**, 347 (1960).
(7) P. ROMAN and E. WOLF: *Nuovo Cimento*, **17**, 462 (1960).
(8) P. ROMAN: *Nuovo Cimento*, **20**, 759 (1961); **22**, 1005 (1961).
(9) A. RAMAKRISHNAN: *Proc. Camb. Phil. Soc.*, **46**, 595 (1950).
(10) A. RAMAKRISHNAN: *Probability and Stochastic Processes*, in *Handbuch der Physik*, vol. **4** (Berlin, 1956).
(11) S. K. SRINIVASAN and K. S. S. IYER: *Nuovo Cimento*, **33**, 273 (Berlin, 1964).
(12) S. K. SRINIVASAN, N. V. KOTESWARA RAO and R. VASUDEVAN: *Nuovo Cimento*, **44** A, 818 (1966).

We can define the probability that a count occurs in a time interval between t and $t + \mathrm{d}t$ to be $\alpha I(t)\mathrm{d}t$, where α is the sensitivity of the detector, taken to be a constant, and $I(t)$ the intensity of the radiation falling on the detector, is given by

$$I(t) = V^*(t)\,V(t)\,, \tag{2.1}$$

$V(t)$ being the usual analytic signal corresponding to the radiation field. The average number of counts in the time interval $(0, T)$ is given by

$$\bar{n} = \alpha \int_0^T I(t)\,\mathrm{d}t\,. \tag{2.2}$$

If I is a deterministic function of time, the probability of obtaining n counts in time 0 to T obeys the Poisson law

$$P(n, T) = \left[\alpha \int_0^T I(T)\,\mathrm{d}t\right]^n \exp\left[-\alpha \int_0^T I(t)\,\mathrm{d}t\right]^n \Big/ n!\,. \tag{2.3}$$

If however we allow I to be a random parameter (*indepedent* of t) governed by the probability frequency function $p(I)$, $P(n, T)$ is given by

$$P(n, T) = \int P(I)\,\mathrm{d}I \exp\left[-\alpha I T\right] \frac{(\alpha I T)^n}{n!}\,. \tag{2.4}$$

Taking $P(I)$ corresponding to an uncorrelated Gaussian signal in the form

$$P(I) = \frac{1}{I_0} \exp\left[-\frac{I}{I_0}\right], \tag{2.5}$$

we obtain

$$P(n, T) = (1 + \bar{n})^{-1}(1 + \bar{n}^{-1})^{-n}\,, \tag{2.6}$$

leading to a Bose-Einstein distribution with parameter n given by

$$\bar{n} = \alpha I_0 T\,, \tag{2.7}$$

a result discussed in great detail by MEHTA and WOLF (3) who have resorted to a « Poisson transform ».

If however $I(t)$ is a correlated random process, it is not easy to arrive at an expression for $P(n, T)$ similar to (2.4). To realise the magnitude of the difficulty, we wish to draw the attention of the reader to the processes with such non-Markovian features as have been dealt with in connection with the theory of shot noise (4) and Barkhausen noise (5,13-15).

In these papers, an inhomogeneous Poisson process is described by the parameter $\lambda(t)$ which depends on the events that have occurred at previous time intervals (16). This is essentially a non-Markovian process. We assume that $\lambda(t)\mathrm{d}t$ which denotes the probability of occurrence of an event between t and $t+\mathrm{d}t$ is such that the occurrence of an event at t_i increases the probability of occurrence of an event at a later time t by $b\exp[-a(t-t_i)]$. Thus given that an event has occurred between t_1 and $t_1+\mathrm{d}t_1$ with probability $\lambda_1\mathrm{d}t_1$ the probability that the next event occurs between t_2 and $t_2+\mathrm{d}t_2$ is given by

$$(2.8)\qquad p(t_1|t_2)\,\mathrm{d}t_2 = \\ = \exp\left[-\int_{t_1}^{t_2}\{\lambda_1+b\exp[-a(t'-t_1)]+(\lambda_1-\lambda_0)\exp[-a(t'-t_1)]\}\,\mathrm{d}t'\right]\cdot \\ \cdot\left\{\lambda_0+b\exp\left[-a\sum_i(t_2-t_i)\right]\right\}\mathrm{d}t_2\,,$$

where λ_0 denotes the value assumed by λ at $t=0$. (2.8) clearly shows the difficulty in calculating $P(n, T, t)$ the probability of obtaining n counts of photoelectrons in time $(t, t+T)$. Let us divide the interval $(t, t+T)$ into $T/\Delta T$ short intervals labelled such that $t+i\Delta T=t_i$ $(i=0, 1, 2, ..., T/\Delta T)$. Then the probability of obtaining n counts in the time interval $(t, t+T)$ is given by

$$(2.9)\qquad \overline{P(n,T,t)}=\varepsilon\left\{\lim_{\Delta T\to 0}\sum_{r_1=0}^{T/\Delta T}\sum_{r_2=0}^{T/\Delta T}\dots\sum_{r_n=0}^{T/\Delta T}\alpha^n I(t_{r_1})I(t_{r_2})\dots I(t_{r_n})(\Delta T)^n\cdot\right. \\ \left.\cdot\left\{\prod_{i=0}^{T/\Delta T}(1-\alpha I(t_i)\,\Delta T)\right\}\left\{\prod_{j=1}^{n}(1-\alpha I(t_{r_j})\,\Delta T)\right\}\right\},$$

where the expectation value is over all possible values of $I(t_i)$. (2.9) is a very complicated limit and cannot be evaluated explicitly. We can circumvent the difficulty by resorting to a product density approach as has been done in ref. (4,5). If $f_1(t_1)$, $f_2(t_1t_2)$, $f_3(t_1r_2t_3)$ ... are the product densities of events on the t-axis, then it is easy to see that

$$(2.10)\qquad \begin{cases} f_1(t_1) = \alpha\varepsilon\{I(t_1)\}\,, \\ f_2(t_1t_2) = \alpha^2\varepsilon\{I(t_1)I(t_2)\}\,, \\ f_3(t_1t_2t_3) = \alpha^3\varepsilon\{I(t_1)I(t_2)I(t_3)\}\ \dots\ \text{etc.}\,, \end{cases}$$

(13) P. MAZZETTI: *Nuovo Cimento*, **25**, 1322 (1962).

(14) P. MAZZETTI: *Nuovo Cimento*, **31**, 38 (1964).

(15) S. K. SRINIVASAN: *On a class of non-Markovian processes*, IIT preprint, 1962.

(16) Such processes have been subsequently termed « doubly stochastic processes » by BARTLETT. See *Journ. Roy. Statist. Soc.*, **25** B, 264 (1963).

where the expectation is to be evaluated with the help of the joint probability frequency function of the analytic signals at different times. At first sight it might appear that the problem is simpler than the space-charge-limited shot noise. However the complexity of the present problem lies in the continuous random nature of $I(t)$. In fact the problem would be intractable if we did not assume a Gaussian character of the thermal source combined with some equally elegant choice of the coherence functions.

The mean square number of counts in the interval $(0, T)$ is given by

$$\overline{n}^2 = \int_0^T f(t)\,\mathrm{d}t + \int_0^T\!\!\int_0^T f(tt')\,\mathrm{d}t\,\mathrm{d}t' \,, \tag{2.11}$$

where $I(t)$ is usually expressed in terms of the complex amplitude $V(t)$ of the light beam:

$$I(t) = V^*(t)\,V(t) \,. \tag{2.12}$$

If we assume further that the Fourier components of $V(t)$ are distributed according to a Gaussian law, then $\pi(V, t)$ the p.f.f. of V can be calculated. $\pi(V, t)$ is given by

$$\pi(V, t) = A \exp\left[-\int V^*(t + \tau/2)\,V(t - \tau/2)\,\mathrm{d}\tau\right] , \tag{2.13}$$

which in turn yields

$$\overline{I(t_1)I(t_2)} = I(t_1)I(t_2) + |\Gamma(t_1 - t_2)|^2 \,, \tag{2.14}$$

where Γ is called the coherence function. We can substitute (2.14) into (2.11) and we find that there is a deviation from the simple Poisson law due to the second term of the right-hand side of (2.14) which is usually attributed to the wave interference effect of the incident beam.

The above result can be readily extended to the higher moments of the counts by calculating higher-order correlations of the intensity. The third moment is given by

$$\overline{n^3} = \alpha\int \overline{I(t)}\,\mathrm{d}t + 3\alpha^2 \int\!\!\int \overline{I(t_1)I(t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 + \alpha^3\int\!\!\int\!\!\int \overline{I(t_1)I(t_2)I(t_3)}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3 \,. \tag{2.15}$$

For thermal light, the third-order correlation is given by

$$\begin{aligned}\overline{I(t_1)I(t_2)I(t_3)} = {}& \overline{I(t_1)}\,\overline{I(t_2)}\,\overline{I(t_3)} + \overline{I(t_1)}\,|\Gamma(t_2 - t_3)|^2 + \\ &+ \overline{I}(t_2)\,|\Gamma(t_1 - t_3)|^2 + \overline{I(t_3)}\,|\Gamma(t_1 - t_2)|^2 + [\Gamma(t_1 - t_2)\Gamma(t_2 - t_3)\Gamma(t_3 - t_1) + \text{c.c.}] \,,\end{aligned} \tag{2.16}$$

an equation which shows that the third-order correlation involves also the phase of the coherence function. Thus if we are in possession of the first three moments of the total number of counts, then (2.11) and (2.14) to (2.16) will determine the coherence function completely. If as in the case of the experiments of HANBURY-BROWN and TWISS ([17]) the incident intensity $\alpha I(t)$ is the input of a linear filter whose output $S(t)$ is given by

$$S(t) = \alpha \int_0^t I(t') b(t-t') \, dt' , \tag{2.17}$$

where $b(t)$ is the response of a filter, then we can obtain the correlation of the output (see ref. ([5])):

$$\overline{S(t_1) S(t_2)} = \int_0^{t_1}\int_0^{t_2} b(t_1 - t') b(t_2 - t') f_2(t' t'') \, dt' \, dt'' + \int_0^{(\min t_1 t_2)} b(t_1 - t') b(t_2 - t') f_1(t') \, dt' . \tag{2.18}$$

In fact the power spectrum of the output which is directly obtainable by experiments can be calculated by the formulas given in ref. ([5]).

3. – Higher moments and the distribution of the counts.

To obtain the general moment of the number of counts, it is convenient to define the product density generating functional ([18]) by

$$L(u) = \varepsilon \left\{ \exp \left[\int u(t) I(t) \, dt \right] \right\} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int \dots \int f_m(t_1 t_2 \dots t_m) u(t_1) u(t_2) \dots u(t_m) \, dt_1 \, dt_2 \dots dt_m , \tag{3.1}$$

which in turn can be related to a different type of cluster functions g_i that are related to the product densities f_i by

$$\begin{cases} f_1(t_1) = g_1(t_1) , \\ f_2(t_1 t_2) = g_1(t_1) g_1(t_2) + g_2(t_1 t_2) , \\ f_3(t_1 t_2 t_3) = g_1(t_1) g_1(t_2) g_1(t_3) + 3 \{ g_1(t_1) g_2(t_2 t_3) \}_{\text{sym}} + g_3(t_1 t_2 t_3) \dots \text{etc.} , \end{cases} \tag{3.2}$$

([17]) R. HANBURY-BROWN and R. Q. TWISS: *Phil. Mag.*, **45**, 663 (1954); *Proc. Roy. Soc.* **242** A, 300 (1957); **243** A, 291 (1957).

([18]) P. I. KUZYESTSOV, R. L. STRATONOVICH and V. I. TIKHONOV: *Nonlinear Transformations of Stochastic Processes* (London, 1965), Chapter I, Sect. **6**.

where g_i can be called the *actual correlation functions since these* functions give a measure of the deviation from the Poisson law. It is interesting to note that $g_i(t_1 t_2 \dots t_i)$ are identical with the coherence functions $\Gamma^i(t_1 t_2 \dots t_i)$ defined by

$$\Gamma^i(t_1 t_2 \dots t_i) = \overline{V^*(t_1) V^*(t_2) \dots V^*(t_i) V(t_1) V(t_2) \dots V(t_i)} \tag{3.3}$$

and it is easy to verify the identity

$$L(u) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m!} \int g_m(t_1 t_2 \dots t_m) u(t_1) u(t_2) \dots u(t_m)\, \mathrm{d}t_1\, \mathrm{d}t_2 \dots \mathrm{d}t_m\right], \tag{3.4}$$

which shows that once we are in possession of all the moments of the n, then we *can readily obtain the magnitude and phase of all the coherence functions.* If, for a beam, coherence functions of all orders less than or equal to l determine all higher-order coherence functions, then correspondingly a knowledge of the first $l+1$ moments of n will be sufficient to determine all the coherence functions completely.

It is customary (see references ([1-3])) to guess the probability $P(n, T)$ from comparing the second moment of n:

$$\overline{n^2} = \alpha \bar{I} T + \alpha^2 (\bar{I} T)^2 + \alpha^2 \iint \Gamma(t_1 - t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2 . \tag{3.5}$$

If we assume that $\Gamma(t_1 - t_2)$ is given by $\bar{I}^2$, we obtain

$$\overline{n^2} = \bar{n} + 2\bar{n}^2 , \tag{3.6}$$

which is exactly the second moment of the distribution

$$P(n, T) = (1 + \bar{n})^{-1} (1 + \bar{n}^{-1})^{-n} , \tag{3.7}$$

which in turn describes the distribution of Bose particles with $\bar{n} = \alpha \bar{I} T$. Howver this does not ensure that the higher moments agree with those corresponding to a Bose distribution. This can be shown rigorously if we observe that the probability-generating function of $P(n, T)$ defined by

$$\sum_n P(n, T) z^n = h(z, T) \tag{3.8}$$

can be obtained from $L(u)$:

$$L(z - 1) = h(z, T) . \tag{3.9}$$

If we can make some reasonable assumption for the coherence functions, then (3.4) can be used to obtain $h(z, T)$ explicitly. Let us assume that the intensities

$I(t)$ are distributed according to the Gaussian law as in the case of thermal light so that all orders of coherence functions can be expressed in terms of the coherence function of order 2. Thus we may postulate that

$$(3.10)\qquad \Gamma^m(t_1 t_2 \dots t_m) = (m-1)!\,\Gamma^2_2(t_1-t_2)\Gamma^2(t_2-t_3)\dots\Gamma^2(t_m-t_{m-1})\,.$$

If in addition we assume that $\Gamma^2(t_1-t_2)=\alpha^2\bar{I}\bar{I}$ then we can write $L(u)$ explicitly if we identify g_i with Γ^i. Using (3.10) and performing the summation we obtain

$$(3.11)\qquad L[u] = \left[1-\alpha\bar{I}\int_0^T u(t)\,\mathrm{d}t\right]^{-1},$$

from which we find

$$(3.12)\qquad h(z, T) = [1+\alpha\bar{I}T-\alpha\bar{I}zT]^{-1}\,,$$

a result that identifies $P(n, T)$ with the boson distribution (3.7).

Our demonstration leading to the Bose-Einstein distribution shows that we can always arrive at $P(n, T)$ starting with the experimentally observed coherence functions. In particular the Bose-Einstein distribution can be confirmed if the coherence functions of a few more orders are available.

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RIASSUNTO (*)

Si discute la complessità strutturale delle espressioni di $P(n, t)$, la probabilità che n elettroni siano emessi da un fotorivelatore per azione di un fascio di luce incidente fluttuante, e si risolve il problema di calcolare le funzioni di coerenza dati gli impulsi di n, riformulandolo in funzione di densità di prodotti familiari nei processi puntuali stocastici. Si studia un metodo generale per giungere a $P(n, T)$ dalla conoscenza di tutte le correlazioni di intensità di ogni specie di fasci di luce. Nel caso di luce emessa da una sorgente termica, si dimostra con le usuali ipotesi sulle funzioni di coerenza, che $P(n, T)$ è la distribuzione di Bose.

(*) *Traduzione a cura della Redazione.*

Флуктуации фотоэлектронов и корреляции интенсивности пучков света.

Резюме (*). — Отмечая сложность в написании выражений $P(n, T)$, вероятности того, что n электронов испускаются фотодетектором, благодаря флуктуирующему падающему пучку света, решается проблема вычисления когерентных функций, зная моменты n, путем пересчитывания проблемы в выражениях, произведений плотностей, обычных для стохастических точечных процесспв. В деталях рассматривается общий метод получения $P(n, T)$ из знания всех корреляций интенсивностей для любого сорта светового пучка. Показано, что для случая теплового свечения, при обычных предположениях о когерентных функциях, $P(n, T)$ представляет Бозе-распределение.

(*) *Переведено редакцией.*