IL NUOVO CIMENTO
RIVISTA INTERNAZIONALE
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE
E DEL COMITATO NAZIONALE PER L'ENERGIA NUCLEARE

VOL. LX B, N. 2 *Serie decima* 11 Aprile 1969

# Exact Fields of Charge and Mass Distributions in General Relativity.

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(ricevuto il 3 Gennaio 1968)

**Summary.** — Presented in this paper are exact solutions to Einstein's field equations generated via static spherically symmetric mass and charge distributions. In the limit in which the mass density vanishes but the charge density does not, the mass seen by an observer at infinity does not vanish and the metric exterior to the source can be the Ressner-Nordstrom or Schwarzschild solution depending upon the charge distribution. This mass, generated by the energy density of the electromagnetic field, cannot be set equal to zero for a body of finite size. If the charge is concentrated in a thin shell and the charge and mass as seen by an observer at infinity are set equal to those of an electron, the radius of the shell is half the classical electron radius. Finally, we exhibit a class of solutions in which the red-shift (from a point in the body to infinity) is maximum at the surface rather than at the center.

## 1. – Introduction.

In a recent paper (¹), exact fields of a number of charge and mass distributions were found and the self-energy (¹) of these distributions was discussed. Because of their complexity, these solutions are difficult to work with. In this

(¹) C. F. KYLE and A. W. MARTIN: *Nuovo Cimento*, **50** A, 583 (1967).

paper, we present some especially simple charge and mass distributions and find exact solutions to Einstein's field equations corresponding to these distributions. The metric coefficients are sufficiently simple so that all quantities of interest (such as self-energy) can be computed exactly; no power-series expansions are necessary.

The mass and charge distributions treated here consist of one or more thin shells. The method used to treat such shells is closely related to that used elsewhere (2).

## 2. – Charged shell.

To treat the problem of a thin spherically symmetric charged shell of radius $r_0$, it is convenient to write the metric in the form

$$ds^2 = A^2\,dt^2 + B^2\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\,. \tag{1}$$

This form of the metric has the advantage that the radial co-ordinate $r$ is related to the area of a sphere of constant radius $r$ via area $= 4\pi r^2$. An especially simple way to write Einstein's field equations in orthonormal frames is

$$8\pi r^2 T^{00} = [r(1 - B^{-2})]_r\,, \tag{2a}$$

$$8\pi A B T^{33} = [(rB)^{-1}(rA)_r]_r + (A/r^2 B)\,, \tag{2b}$$

$$A^2 B^2 (1 + 8\pi r^2 T^{11}) = (rA^2)_r\,. \tag{2c}$$

If we consider only radial electric fields generated by a spherically symmetric charge distribution, we obtain only one nontrivial Maxwell equation (3)

$$(r^2\varepsilon)_r = 4r^2\sigma B\,, \tag{3}$$

where $\sigma$ is the charge density and $\varepsilon$ is the radial electric field in orthonormal frames. Inside the shell $\varepsilon$ vanishes, while outside the shell we obtain

$$\varepsilon = q/r^2\,, \tag{4}$$

where the charge $q$ is related to the charge density $\sigma$ via

$$q = 4\pi\int_{r_0^-}^{r_0^+} \sigma r^2 B\,dr\,. \tag{5}$$

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(2) J. M. COHEN: *Relativity Theory and Astrophysics*, edited by J. EHLERS (New York, 1967); D. BRILL and J. COHEN: *Phys. Rev.*, **143**, 1011 (1966); J. COHEN and D. BRILL: *Nuovo Cimento*, **56** B, 209 (1968).

(3) J. M. COHEN: *Phys. Rev.*, **148**, 1264 (1966); B. HOFFMAN: *Quart. Journ. Math.*, **3**, 226 (1932).

Here $r_0^+$ and $r_0^-$ denote the limit $r \to r_0$ taken from above and below respectively.

The stress-energy tensor (generated by the electic field, the mass density $\rho$ of the shell, and the elastic stress $t^{22} = t^{33}$ necessary to support the shell) has the nonvanishing components

$$T^{00} = \rho + (\varepsilon^2/8\pi)\,, \qquad T^{11} = -\,\varepsilon^2/8\pi\,, \qquad T^{22} = T^{33} = t^{33} + (\varepsilon^2/8\pi)\,. \tag{6}$$

In the limit of a shell of vanishing thickness, the mass density and elastic stress can be expressed in terms of a delta « function », *i.e.*

$$\rho = K\delta(r - r_0)\,, \qquad t^{22} = t^{33} = S\delta(r - r_0)\,, \tag{7}$$

where the delta « function » is normalized to satisfy

$$1 = 4\pi \int_{r_0^-}^{r_0^+} \delta(r - r_0) r^2 \,\mathrm{d}r\,. \tag{8}$$

For the field equations (2) with the stress-energy tensor (6) we have the solution

$$A^2 = B^{-2} = 1 - (2mr - q^2)r^{-2}\,, \qquad \varepsilon = q/r^2 \qquad \text{for } r > r_0\,, \tag{9}$$

and

$$A^2 = 1 - (2mr_0 - q^2)r_0^{-2}\,, \qquad B^2 = 1\,, \qquad \varepsilon = 0 \qquad \text{for } r < r_0\,. \tag{10}$$

The solution (9) is the well-known Reisner-Nordstrom solution while the solution (10) is just flat space.

All that remains to be done is to find the connection between the mass and to find the elastic stress supporting the shell. The former relation is found by integrating eq. ($2a$) across the shell and using eq. (8):

$$m = K + (q^2/2r_0)\,. \tag{11}$$

If we assume that the mass density is positive definite then $K$ must also be positive definite. It is interesting to note that even when the mass density vanishes ($K = 0$), the mass does not. Thus an observer at infinity sees a mass even if no « real » mass is present. It is the energy density of the electric field which generates this mass. It is often stated that the effect of the electric charge on the metric dies off faster at large distances than the effect of the mass, since the $q^2$ term in the metric (9) falls off faster than the $m$ term. But relation (11) shows that the electric field also contributes to the mass via its energy density. Thus the effect of the charge does not fall off faster than that of a

« real » mass (nonelectromagnetic mass): the energy density of the electric field makes a contribution to the mass seen by an observer at infinity which can be as important as the contribution from any « real » mass. In the extreme case $K = 0$, the energy density of the electric field makes the only contribution to the Schwarzschild mass. *If the Schwarzchild mass and charge are set equal to the values for an electron (with $K = 0$), the radius of the body becomes $r_0 = q^2/2m$, half the classical electron radius.* This result came directly out of the field equations with the gravitational interaction taken into account automatically via the field equations.

We define the gravitational self-energy $\delta m$ via [4]

$$\delta m = m - m_0 , \tag{12}$$

where $m$ is the Schwarzchild mass and $m_0$ is the proper volume integral of the energy density, *i.e.*

$$m_0 = 4\pi \int_0^\infty T^{00} r^2 B \, \mathrm{d}r \tag{13}$$

for the spherically symmetric distribution considered here while

$$m = 4\pi \int_0^\infty T^{00} r^2 \, \mathrm{d}r . \tag{14}$$

Thus the gravitational self-energy $\delta m$ is given by

$$\delta m = 4\pi \int_0^\infty r^2 \, \mathrm{d}r \, T^{00}(1 - B) , \tag{15}$$

which is negative since the integrand is negative definite ($B \geqslant 1$). Integration yields the exact result

$$\delta m = r_0(1 - A_0) + (|q|/2) \ln \left[(A_0 + |q| r_0^{-1} - m|q|^{-1})/(1 - m|q|^{-1})\right] . \tag{16}$$

When $K = 0$ this expression (16) takes the form

$$\delta m = m\left(1 - (|q|/2m) \ln \left[(1 + m|q|^{-1})(1 - m|q|^{-1})^{-1}\right]\right) , \tag{17}$$

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[4] ADLER, BAZIN and SCHIFFER: *Introduction to General Relativity* (New York, 1965).

which becomes $\delta m \simeq -mm^2/3q^2$ when the charge and mass of an electron are substituted into (17) since $|q| \gg m$. Note that $m \gg |\delta m|$ for this charge distribution and charge mass ratio.

The elastic stress in the shell is obtained by integrating eq. (2*b*) across the shell yielding

$$(18)\qquad 8\pi S\int_{r_0^-}^{r_0^+} B\delta(r-r_0)\,\mathrm{d}r = B^{-1}(r_0^{-1}+A_rA^{-1})\Big|_{r_0^-}^{r_0^+} = (2r_0A_0)^{-1}[(1-A_0)^2-q^2r_0^{-2}]\,.$$

*The integral on the left of eq.* (18) *is not well defined* (*since the integral of the step function times a delta « function » is not well defined*) *and can only be determined if additional information is given.* This additional information can be obtained by integrating (2*a*) across the shell

$$(19)\qquad 4\pi r_0K\int_{r_0^-}^{r_0^+}\delta(r-r_0)B\,\mathrm{d}r = -B^{-1}\Big|_{r_0^-}^{r_0^+} = 1-A_0\,.$$

Substitution of this relation (19) into eq. (18) yields

$$(20)\qquad S=(K/4)[A_0^{-1}-1-q^2r_0^{-2}A_0^{-1}(1-A_0)^{-1}]\,.$$

When $e^2\to 0$, this relation (14) reduces to

$$(21)\qquad S=(m/4)(A_0^{-1}-1)\,,$$

in agreement with previous results [2]. In the limit $K\to 0$, eq. (13) reduces to eq. (8) and eq. (20) or eq. (18) yields $S=-q^2/4r_0$. While if $|q|=m$, $S$ vanishes; the gravitational attraction just balances the electric repulsion. This is not hard to show if one notes that $A_0^2=(1-mr_0^{-1})^2$ in this case.

When this latter case $|q|=m$ is considered via isotropic co-ordinates, strange results are often obtained. The co-ordinate transformation $r=\varrho\Psi^2$ yields the isotropic form of the metric (1), (9):

$$(22)\qquad \mathrm{d}s^2=-V^2dt^2+\Psi^4(\mathrm{d}\varrho^2+\varrho^2\mathrm{d}\theta^2+\varrho^2\sin^2\theta\,\mathrm{d}\varphi^2)\,,$$

where

$$(23)\qquad \Psi^2=[1+(m/2\varrho)]^2-(q^2/4\varrho^2)\,,\qquad V=\big(1-[(m^2-q^2)/4\varrho^2]\big)\Psi^{-2}\,.$$

When $|q|=m$, these metric coefficients become

$$(24)\qquad \Psi^2=1+(m/\varrho)\,,\qquad V=\Psi^{-2}\,.$$

A sphere of constant radius $\varrho$ has the area $4\pi(\varrho\Psi^2)^2 = 4\pi r^2$. Thus, if we measure the « radius » of a shell via the area of the shell through the relation area $= 4\pi r_0^2$, we find that the « radius » of the shell is related to $\varrho_0$ via (for $m = |q|$)

$$r_0 = \varrho_0 + m . \tag{25}$$

Thus when $\varrho_0$ approaches zero, the shell radius $r_0$ approaches $m$, a nonzero value. Consequently, the surface area of a charged body with $m = |q|$ and $\varrho_0 \to 0$ does not vanish, and thus the body is not a point charge in the usual sense. However, it is often called a « point charge », since it exhibits many of the properties of a point charge (5).

For this case the expression for the gravitational self-energy (18) is

$$\delta m = (m/2) \ln (1 - m r_0^{-1}) , \tag{26}$$

which becomes large as $r_0$ approaches $m$. Thus the gravitational potential « renormalizes » the mass and keeps it finite as pointed out previously by others (6).

## 3. – Schwarzschild metric.

In this Section we consider two concentric spherical charged shells with charges $q, -q$ and radii $r_1, r_2$ respectively. For this case the metric takes the form (1) with the electric field and metric coefficients being

$$\varepsilon = 0 , \qquad A^2 = B^{-2} = 1 - 2mr^{-1} , \qquad \text{for } r_2 < r , \tag{27}$$

$$\begin{cases} \varepsilon = q/r^2 , \\ A^2 = (1 - 2mr_2^{-1})\big(1 - q^2 r^{-2}(r r_1^{-1} - 1)\big)\big(1 - q^2 r_2^{-2}(r^2 r_1^{-1} - 1)\big)^{-1} , \\ B^{-2} = 1 - q^2 r^{-2}(r r_1^{-1} - 1) , \qquad \text{for } r_1 < r < r_2 . \end{cases} \tag{28}$$

$$\begin{cases} \varepsilon = 0 , \\ A^2 = (1 - 2mr_2^{-1})\big(1 - q^2 r_2^{-2}(r_2 r_1^{-1} - 1)\big)^{-1} , \qquad \text{for } r < r_1 . \\ B = 1 , \end{cases} \tag{29}$$

(5) R. Arnowitt, S. Deser and C. Misner: *Ann. of Phys.*, **33**, 88 (1965).
(6) R. Arnowitt, S. Deser and C. Misner: *Phys. Rev. Lett.*, **1**, 375 (1960).

Here the Schwarzchild mass $m$ is given by

$$m = (q^2/2r_1)(1 - r_1 r_2^{-1}) \tag{30}$$

and $r_1$ is the radius of the inner shell. The mass $m$ seen by an observer at infinity is generated by the electric field between the two charged shells. However, *this observer cannot tell that the mass is due to two charges rather than to a « real » mass.*

## 4. – Extended charge distribution.

Another exact solution is

$$\varepsilon = q(r)/r^2\,, \qquad A^2 = B^{-2} = 1 - (8\pi\mu/3)r^2\,, \qquad \text{for } r < r_0\,, \tag{31}$$

with the solution for $r_0 < r$ being the same as that in eq. (9). Here $\mu$ is the constant related to the mass $m$ and total charge $q$ via the relation

$$m = (4\pi/3)\mu r_0^3 + (q^2/2r_0)$$

and $q(r)$ is an arbitrary function satisfying the conditions $q(r_0) = q$ and $q(r) \sim r^n$ with $n \leqslant 3$ as $r$ approaches zero. This latter condition ensures that the charge density remains finite at the origin. The nonvanishing components of the stress-energy tensor are

$$\begin{cases} T^{00} = \mu = \rho + \big(q^2(r)/8\pi r^4\big)\,, \\ T^{11} = -\mu = p_1 - \big(q^2(r)/8\pi r^4\big)\,, \\ T^{22} = T^{33} = -\mu = p_3 + \big(q^2(r)/8\pi r^4\big)\,, \end{cases} \tag{32}$$

where $\rho$, $p_1$, $p_3$ are determined once $\mu$ and $q(r)$ are given. *This solution has the surprising property that the red-shift (from a point in the sphere to infinity) is maximum at the surface rather than the center.*

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For helpful discussion, we are indebted to Drs. R. GAUTREAU and A. FINZI. This work was completed while the the first author held an NAS-NRC Research Associateship at the Institute for Space Studies, Goddard Space Flight Center, NASA, New York.

## RIASSUNTO (*)

Si presentano le soluzioni esatte delle equazioni di campo di Einstein generate tramite distribuzioni statiche e sfericamente simmetriche della massa e delle cariche. Al limite in cui la densità di massa, ma non la densità di carica, si annulla, la massa vista da un osservatore all'infinito non si annulla e può verificarsi che la metrica esterna alla sorgente sia la soluzione di Ressner-Nordstrom o di Schwarzschild dipendente dalla distribuzione di carica. Questa massa, generata dalla densità di energia del campo elettromagnetico, non può porsi uguale a zero per un corpo di dimensioni finite. Se la carica è concentrata in un guscio sottile e la carica e la massa, viste da un osservatore all'infinito, si pongono uguali a quelle di un elettrone, si trova che il raggio del guscio è la metà del raggio classico dell'elettrone. Infine si espone una classe di soluzioni in cui lo spostamento verso il rosso (da un punto del corpo fino all'infinito) è massimo alla superficie anzichè che al centro.

(*) *Traduzione a cura della Redazione.*

## Точные поля для распределений заряда и массы в общей теории относительности.

**Резюме** (*). — В этой статье сообщаются точные решения уравнений поля Эйнштейна, образованных посредством статических сферически симметричных распределений массы и заряда. В пределе, в котором плотность массы обращается в нуль, а плотность заряда не обращается в нуль, масса, которую видит наблюдатель на бесконечности, не обращается в нуль, и метрическая наружность к источнику может быть решениемРесснера-Нордстрёма или Шварцшильда, которое зависит от распределения заряда. Эта масса, образованная плотностью энергии электромагнитного поля, не может быть равной нулю для тела бесконечных размеров. Если заряд сконцентрирован в тонкой оболочке, то заряд и масса, которые видит наблюдатель на бесконечности, равны заряду и массе электрона, а радиус оболочки равен половине классического радиуса электрона. В заключение, мы показываем класс решений, для которых красное смещение (от точки в теле до бесконечности) является максимальным на поверхности, а не в центре.

(*) *Переведено редакцией.*