Longitudinal Waves in a Plasma Half-Space.

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Summary. — The electric field in a plasma half-space is determined using a normal mode analysis with collisions considered using a BGK particle-conserving collision model. For the collisionless case, the results are shown to be equivalent to the results of Landau who used transform methods. For distances greater than five Debye lengths from the boundary, and for collision frequencies up to $0.01\omega_p$, it is shown that collisions have little effect on the electric field except in the frequency range $0.95\omega_p < \omega < 1.2\omega_p$. Within this frequency band, collisions produce different kinds of effects, depending upon the frequency of the applied field and upon the collision frequency. For most of the frequencies, the field is damped more rapidly with distance. For special frequencies, however, collisions destroy the collisionless damping and reduce the total damping.

1. – Introduction.

The problem of the penetration of an electric field in a plasma half-space was first treated with kinetic theory by LANDAU (1) in the second part of his paper on electrostatic oscillations. LANDAU used transform methods to solve the kinetic equation without collisions. The use of the collisionless Boltzmann (or Vlasov) equation was justified by assuming the applied frequency was sufficiently high that the collisions of electrons with ions and neutrals could be neglected.

In the present paper, Landau's work will be generalized to include collisions via a Bhatnagar, Gross and Krook (BGK) collision model (2). We will employ a

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(1) L. LANDAU: *Journ. Phys.* (*USSR*), **10**, 25 (1946).

(2) P. BHATNAGAR, E. GROSS and M. KROOK: *Phys. Rev.*, **94**, 511 (1954).

normal mode analysis similar to that used by SHURE [3] and modified by CERCIGNANI and PAGANI [4] to include collisions. In Sect. **2** we establish the normal and adjoint modes, determine the normalization coefficients, and obtain the normal mode expansion. The plasma half-space boundary conditions will then be applied to this expansion in Sect. **3**. By setting the collision frequency to zero, this work will be shown in Sect. **4** to reduce to the results of SHURE [5] and to be equivalent to the results of LANDAU [1]. This will demonstrate the equivalence of the normal mode analysis and the transform method for the boundary-value problem. Finally, in Sect. **5**, results are given for asymptotic approximations on integrals obtained for the electric field. By choosing representative values of applied and collision frequencies and of distances from the plasma boundary, the effect of considering collisions on the electric field will be determined.

2. – Normal mode analysis for longitudinal waves.

For our treatment of longitudinal waves, a complete description of the plasma is given by Poisson's equation coupled with the kinetic equation. It is assumed that no magnetic fields are present and that the unperturbed electron velocity distribution is isotropic in velocity space. The frequency of oscillation will be assumed to be of such a magnitude that the ions, with their greater masses, are relatively immobile and form a uniform charge neutralizing background.

We consider propagation in the x-direction normal to the plasma half-space boundary. The linearized kinetic and Poisson equations with the BGK particle-conserving collision model [2] can be written as

$$\frac{\partial f(x,u,t)}{\partial t}+u\,\frac{\partial f(x,u,t)}{\partial x}+\frac{neE(x,t)}{m}\,F'(u)=\omega_c\left[F(u)\int_{-\infty}^{\infty}f(u')\,\mathrm{d}u'-f(x,u,t)\right], \tag{2.1}$$

$$\frac{\partial E(x,t)}{\partial x}=4\pi e\int_{-\infty}^{\infty}f(x,u,t)\,\mathrm{d}u\,, \tag{2.2}$$

where ω_c is the velocity-independent collision frequency, n, m, and e are the electron density, mass and charge, $F(u)$ is the one-dimensional equilibrium electron distribution, and $f(x,u,t)$ is the perturbed distribution.

([3]) F. SHURE: *Plasma Phys.* (*Journ. Nucl. Energy Part C*), **6**, 1 (1964).
([4]) C. CERCIGNANI and C. PAGANI: *Nuovo Cimento*, **40** B, 140 (1965).
([5]) F. SHURE: Ph. D. dissertation, University of Michigan, Ann Arbor, Mich. 1960.

We seek solutions whose time dependence is of the form $\exp[-i\omega t]$, where ω is the frequency of the applied field. Following CERCIGNANI and PAGANI (4), we define

$$Y(x,u,t) = f(x,u,t) - \frac{neF'(u)}{\sigma m} E(x,t)\,, \tag{2.3}$$

where

$$\sigma = -\omega_c + i\omega\,.$$

With this linear combination of $f(x,u,t)$ and $E(x,t)$, eqs. (2.1) and (2.2) become decoupled.

The translational invariance of eqs. (2.1) and (2.2) suggests examining solutions of the form $\exp[\sigma x/\nu]$. These equations can then be written in the form

$$E_\nu = \frac{4\pi e\nu}{\sigma}\int_{-\infty}^{\infty} Y_\nu(u')\,\mathrm{d}u'\,, \tag{2.4}$$

$$(\nu - u)\,Y_\nu(u) = \nu u\gamma(u)\int_{-\infty}^{\infty} Y_\nu(u')\,\mathrm{d}u'\,, \tag{2.5}$$

where

$$\gamma(u) = \frac{\omega_p^2}{\sigma^2}F'(u) - \frac{\omega_c}{\sigma}\frac{F(u)}{u}\,,$$

with

$$\omega_p^2 = \frac{4\pi n e^2}{m}\,.$$

The solutions of these equations can be conveniently classified as follows:

Class I, Discrete spectrum. These modes, denoted by the subscript i, are restricted to solutions of $Y_\nu(u)$ in eq. (2.5) where either ν has a nonvanishing imaginary part, or, if ν is real, then $\gamma(\nu)$ must vanish. Providing that the applied frequency is not zero, $\gamma(\nu)$ will not vanish for finite ν.

The solutions to eqs. (2.4) and (2.5) become

$$E_i = \frac{4\pi e\nu_i}{\sigma}\,. \tag{2.6}$$

$$Y_i(u) = \gamma(u)\frac{\nu_i u}{\nu_i - u}\,, \tag{2.7}$$

where the normalization condition

$$\int_{-\infty}^{\infty} Y_i(u)\,\mathrm{d}u = 1 \tag{2.8}$$

has been chosen. For these relations to be valid, the normalization condition must be satisfied, *i.e.* the quantities ν_i are the eigenvalues of the characteristic function

$$\Lambda(\nu) = 1 + \nu\int_{-\infty}^{\infty} \frac{u\gamma(u)}{u-\nu}\,\mathrm{d}u\,. \tag{2.9}$$

Note that

$$\Lambda^*(\nu) = \Lambda(\nu^*) \qquad \text{and} \qquad \Lambda(\nu) = \Lambda(-\nu)\,.$$

Consequently, if ν_i is a root, then

$$\Lambda(\nu_i) = \Lambda(-\nu_i) = \Lambda(\nu_i^*) = \Lambda(-\nu_i^*) = 0$$

so that $\pm\nu_i$ and $\pm\nu_i^*$ are all roots. The solution to $\Lambda(\nu_i) = 0$ is discussed in Appendix A.

Class II, Continuum spectrum. When ν takes on real values we have, in addition to the discrete modes, a continuum of solutions which we shall denote by the subscript ν. Choosing the normalization condition

$$\int_{-\infty}^{\infty} Y_\nu(u)\,\mathrm{d}u = 1\,, \tag{2.10}$$

eqs. (2.4) and (2.5) can be written

$$E_\nu = \frac{4\pi e\nu}{\sigma} \tag{2.11}$$

and

$$Y_\nu(u) = \nu u\, P\,\frac{\gamma(u)}{\nu-u} + \lambda(\nu)\delta(\nu-u)\,, \tag{2.12}$$

where $\lambda(\nu)$ is an arbitrary function and P indicates the Cauchy principal value.

From eqs. (2.10) and (2.12), $\lambda(\nu)$ can be evaluated as

$$\lambda(\nu) = 1 + \frac{\nu^2\omega_p^2}{\sigma^2} P\int_{-\infty}^{\infty} \frac{F'(u)}{u-\nu}\,\mathrm{d}u - \frac{\nu\omega_c}{\sigma} P\int_{-\infty}^{\infty} \frac{F(u)}{u-\nu}\,\mathrm{d}u\,. \tag{2.13}$$

Class III, Charge free mode. The normal mode solutions for this class are spatially independent. From Poisson's equation, with $\int_{-\infty}^{\infty} f\,\mathrm{d}u = \int_{-\infty}^{\infty} Y\,\mathrm{d}u$ we obtain

$$E_0 = \text{constant} = 1 \text{ (arbitrarily)}\,, \tag{2.14}$$

$$Y_0 = 0\,. \tag{2.15}$$

Following SHURE [3], the adjoint modes can be obtained by similar arguments. The results are summarized below:

$$\begin{Bmatrix} E_i^\dagger \\ E_\nu^\dagger \\ E_0^\dagger \end{Bmatrix} = 0\,, \tag{2.16}$$

$$\begin{Bmatrix} Y_i^\dagger \\ \\ Y_\nu^\dagger \\ \\ Y_0^\dagger \end{Bmatrix} = \begin{Bmatrix} \dfrac{u\nu_i}{\nu_i - u} \\ P\dfrac{\nu u}{\nu - u} + \lambda^\dagger(\nu)\delta(\nu - u) \\ 0 \end{Bmatrix}, \tag{2.17}$$

where normalization for the adjoint modes is given by

$$\int_{-\infty}^{\infty} \gamma(u) \begin{Bmatrix} Y_i^\dagger \\ Y_\nu^\dagger \end{Bmatrix} \mathrm{d}u = 1 \tag{2.18}$$

and where

$$\lambda^\dagger(\nu)\gamma(\nu) = 1 + \nu P\int_{-\infty}^{\infty} \mathrm{d}u\, \frac{u\gamma(u)}{u-\nu} = \lambda(\nu)\,. \tag{2.19}$$

The orthogonality relations [4] imply the following for the classes of solutions:

Class I, Discrete spectrum.

$$\int_{-\infty}^{\infty} Y_i\, Y_j^\dagger\, \mathrm{d}u = N_i \delta_{ij}\,, \tag{2.20}$$

where

$$N_i = \nu_i^2 \frac{\mathrm{d}\Lambda(\nu)}{\mathrm{d}\nu}\,. \tag{2.21}$$

When the equilibrium distribution is Maxwellian, the normalization coefficient N_i becomes

$$N_i = \nu_i \left[-3 + \nu_i^2 \frac{m}{kT} \left(1 + \frac{\omega_p^2}{\sigma^2}\frac{\omega_c}{\sigma}\right) + \frac{2\omega_c}{\sigma}\left(\frac{1-(\omega_p^2/\sigma^2)\,\nu_i^2(m/kT)}{(\omega_p^2/\sigma^2)\,\nu_i^2(m/kT)+(\omega_c/\sigma)}\right)\right]. \tag{2.22}$$

Class II, Continuum spectrum. The development of the normalization coefficient for the continuum modes differs from that of the discrete modes in that the normalization integral, eq. (2.23) is undefined. Nevertheless, we may infer from the orthogonality relation for the continuum the following symbolic representation:

$$\int_{-\infty}^{\infty} \mathrm{d}u\, Y_\nu\, Y_{\nu'}^\dagger = N(\nu)\delta(\nu-\nu')\,, \tag{2.23}$$

where

$$\frac{1}{N(\nu)} = \frac{1}{2\pi i\nu^2}\left[\frac{1}{\Lambda^-(\nu)} - \frac{1}{\Lambda^+(\nu)}\right] \tag{2.24}$$

and where we define $\Lambda^{\pm}(\nu) = \lim_{\varepsilon\to 0^+} \Lambda(\nu \pm i\varepsilon)$.

With the normal and adjoint modes and the normalization coefficients we are now in a position to expand the functions $Y(x, u)$, $E(x)$ and $f(x, u)$ in terms of the modes, *i.e.*

$$\bar{\psi}(x, u) = \begin{Bmatrix} Y(x, u) \\ E(x) \\ f\ (x, u) \end{Bmatrix} = A_0\psi_0 + \sum_i A_i\psi_i \exp[\sigma x/\nu_i] + \int_{-\infty}^{\infty} A(\nu)\psi_\nu \exp[\sigma x/\nu]\mathrm{d}\nu, \tag{2.25}$$

where f_0, f_i and f_ν are obtained from eq. (2.3), and where the summation $\sum_i$ for the discrete modes is taken over the values of the discrete roots ν_i. The proof that the normal modes form a complete set has been demonstrated by

CERCIGNANI and PAGNANI [4]. The expansion coefficients are obtained from eq. (2.25) to be

$$A(\nu)\exp[\sigma x/\nu] = \frac{\int_{-\infty}^{\infty} Y(x,u)\, Y_\nu^\dagger\, \mathrm{d}u}{N(\nu)} \tag{2.26}$$

and

$$A_i \exp[\sigma x/\nu_i] = \frac{\int_{-\infty}^{\infty} Y(x,u)\, Y_i^\dagger\, \mathrm{d}u}{N_i}\,. \tag{2.27}$$

The term A_0 may not be evaluated in the same manner as A_i and $A(\nu)$ since $Y_0 = Y_0^\dagger = 0$. The evaluation of each of the coefficients including A_0 is carried out in the next Section where the boundary conditions for the plasma half-space are applied.

3. – Application to a plasma half-space.

In this Section we will use the normal modes developed in Sect. 2 to calculate the electric field in a plasma half-space. The required boundary conditions are the following:

1) The electrons are assumed to experience specular reflection at the boundary. This implies

$$f(0,u) = f(0,-u)\,. \tag{3.1}$$

From eq. (2.3), which defines $Y(x,u)$, and eq. (3.1), we obtain

$$Y(0,u) - Y(0,-u) = \frac{-2me}{\sigma m} F'(u) E(0)\,. \tag{3.2}$$

2) As $x \to \infty$ we require that no incoming waves exist and that E remains finite. This is accomplished by specifying that for the continuum mode

$$A(\nu) = 0\,, \qquad \text{when } \nu < 0\,, \tag{3.3}$$

and for the discrete mode

$$\omega[\mathrm{Im}(\nu_i)] - \omega_c[\mathrm{Re}(\nu_i)] < 0\,. \tag{3.4}$$

By including a collision frequency term, we do not require the applied frequency to contain a small imaginary part for the continuum modes as did

SHURE in treating the collisionless case ([5]). For the discrete modes, however, we must assign a small imaginary part to ω when the left side of eq. (3.4) is zero.

3) Across any plane in the plasma parallel to the boundary, the net current flow is zero. Specifically, at the plasma boundary we write

$$J(0)=\int_{-\infty}^{\infty} uf(u,0)\,\mathrm{d}u=0\,. \tag{3.5}$$

Applying the boundary condition 1) to eq. (2.25), we obtain

$$\sum_i A_i[Y_i(u)-Y_i(-u)]+\int_{-\infty}^{\infty} A(\nu)[Y_\nu(u)-Y_\nu(-u)]\,\mathrm{d}\nu=\frac{-2me}{\sigma m}F'(u)E(0)\,. \tag{3.6}$$

Defining $B(\nu)\equiv A(\nu)-A(-\nu)$ and $B_i\equiv A_i-A_{-i}$, and using the symmetry relations $Y_\nu(-u)=Y_{-\nu}(u)$ and $Y_i(-u)=Y_{-i}(u)$, eq. (3.6) becomes

$$\sum_i B_i Y_i(u)+\int_{-\infty}^{\infty} B(\nu')Y_{\nu'}(u)\,\mathrm{d}\nu'=\frac{-2me}{\sigma m}F'(u)E(0)\,. \tag{3.7}$$

From the orthogonality relations, $B(\nu)$ and B_i can be obtained as

$$B(\nu)=\frac{ineE(0)}{\sigma\pi\nu^2 m}\left[\frac{1}{\Lambda^-(\nu)}-\frac{1}{\Lambda^+(\nu)}\right]\left[-\nu^2\int_{-\infty}^{\infty}\frac{F'(u)\,\mathrm{d}u}{u-\nu}+\frac{F'(\nu)}{\gamma(\nu)}\lambda(\nu)\right]. \tag{3.8}$$

$$B_i=2ne\nu_i E(0)\left(1+\frac{\omega_c}{\sigma}\right)\cdot \tag{3.9}$$

$$\cdot\left\{m\nu_i^2\left(1+\frac{\omega_c}{\sigma}\right)\left[\frac{\omega_p^2}{\sigma}\left(1-\nu_i^2\frac{m}{kT}\right)-\omega_c\right]+\frac{\omega_p^2}{\sigma}\nu_i^2 m\left(2-\left(\frac{\omega_p}{\sigma}\right)^2\nu_i^2\frac{m}{kT}\right)+\omega_c kT\right\}^{-1},$$

where, in eq. (3.8) and hereafter, the P is dropped from the Cauchy principal-value integrals.

From boundary condition 2) (the radiation condition)

$$\begin{cases} A(\nu)=B(\nu) & \text{for } \nu>0\,,\\ \quad\;\;\; =0 & \text{for } \nu<0\,. \end{cases} \tag{3.10}$$

In Sect. **2** we found that there are, in general, four values of the discrete root.

Two values are eliminated by the boundary condition, eq. (3.4), and the others are used in the summation $\sum_i A_i$, where $A_i = B_i$, and B_i is given by eq. (3.9).

To obtain A_0, we use boundary condition 3). From eqs. (2.25) and (3.5) we find

$$\int_{-\infty}^{\infty} u f(u, 0)\, \mathrm{d}u = A_0 \int_{-\infty}^{\infty} u f_0\, \mathrm{d}u + \int_0^{\infty} \mathrm{d}\nu A(\nu) \int_{-\infty}^{\infty} u f_\nu\, \mathrm{d}u + \sum_i A_i \int_{-\infty}^{\infty} u f_i\, \mathrm{d}u = 0\,. \tag{3.11}$$

When $F(u)$ is a Maxwellian distribution, the integrals can be evaluated by straightforward manipulation to obtain

$$0 = A_0 \frac{\omega_p^2}{\sigma^2} - \frac{4\pi e}{\sigma}\left(1 + \frac{\omega_c}{\sigma}\right)\left(\int_{-\infty}^{\infty} \nu A(\nu)\, \mathrm{d}\nu + \sum_i A_i \nu_i\right). \tag{3.12}$$

The last can be eliminated in favor of $E(0)$ by recognizing that

$$E(0) = A_0 + \frac{4\pi e}{\sigma} \int_0^{\infty} \nu A(\nu)\, \mathrm{d}\nu + \frac{4\pi e}{\sigma} \sum_i A_i \nu_i\,. \tag{3.13}$$

From eqs. (3.12) and (3.13), A_0 is found to be

$$A_0 = \frac{(1 + \omega_c/\sigma) E(0)}{1 + \omega_c/\sigma + \omega_p^2/\sigma^2}\,. \tag{3.14}$$

The solution for the electric field may now be written as

$$E(x, t) = \exp[-i\omega t] \cdot \\ \cdot \left[\frac{(1 + \omega_c/\sigma) E(0)}{1 + \omega_c/\sigma + \omega_p^2/\sigma^2} + \frac{4\pi e}{\sigma} \sum_i \nu_i A_i \exp[\sigma x/\nu_i] + \frac{4\pi e}{\sigma} \int_0^{\infty} \mathrm{d}\nu\, \nu A(\nu) \exp[\sigma x/\nu]\right]. \tag{3.15}$$

For the collisionless case discussed by SHURE [5], no discrete modes exist for $\omega > \omega_p$. When $\omega_c \neq 0$ there appears to be no simple analytical criterion for the existence of the discrete modes in terms of ω_c. In Appendix A we find that for $\omega_c = 0.01\omega_p$, no discrete modes exist for $\omega > 1.12\omega_p$.

4. – Equivalence of results.

For the limit $\omega_c \to 0$, eq. (3.15) reduces to the expression obtained by SHURE [5] if $\omega > \omega_p$ (no discrete modes). SHURE did not show the equivalence of his result, which was obtained by a normal mode analysis, to that of LAN-

DAU [1], who used Fourier transforms. In this Section, the equivalence of the results will be established and the restriction $\omega > \omega_p$ will be removed.

Using Landau's notation, we rewrite our results in the form

$$E(x,t) = \exp[-i\omega t][E(\infty) + E_1(x)]\,, \tag{4.1}$$

where $E(\infty)$ is precisely what LANDAU found:

$$E(\infty) = \lim_{x\to\infty} E(x) = A_0 = \frac{E(0)}{1-\omega_p^2/\omega^2}\,. \tag{4.2}$$

The other term of eq. (4.1) is seen from eq. (3.15) to be

$$E_1(x) = \frac{E(0)}{i\pi}\int_0^\infty \exp\left[\frac{i\omega}{\nu}x\right]\frac{1}{\nu}\left(\frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)}\right)\mathrm{d}\nu + \sum_i \frac{2E(0)}{3-\nu_i^2(m/kT)(1-\omega_p^2/\omega^2)}\exp[i\omega x/\nu_i]\,. \tag{4.3}$$

LANDAU found for $E_1(x)$

$$E_1(x) = \frac{iE(0)}{\pi\varepsilon}\int_{-\infty}^{\infty}\frac{K_0 - K_k}{k(1-K_k)}\exp[ikx]\,\mathrm{d}k\,, \tag{4.4}$$

where

$$K_k = \frac{\omega_p^2}{\omega}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{uF'(u)}{ku-\omega}\,, \qquad \operatorname{Im}\omega > 0\,, \tag{4.5}$$

$$K_0 = \frac{\omega_p^2}{\omega^2} = 1-\varepsilon\,. \tag{4.6}$$

We write eq. (4.4) in the form

$$E_1(x) = \frac{iE(0)}{\pi\varepsilon}\cdot \left[\int_{-\infty}^{0}\frac{(K_0-1)+(1-K^-)}{k(1-K^-)}\exp[ikx]\,\mathrm{d}k + \int_0^\infty\frac{(K_0-1)+(1-K^+)}{k(1-K^+)}\exp[ikx]\,\mathrm{d}k\right], \tag{4.7}$$

where

$$K_k = K^+ \qquad \text{for } k>0\,,$$
$$\quad = K^- \qquad \text{for } k<0\,.$$

With the change of variable $\nu = \omega/k$, it is easily shown that

$$\Lambda^{\pm}(\nu) = 1 - K^{\pm}(\nu) . \tag{4.8}$$

Equation (4.7) can be rewritten in terms of $\Lambda^{\pm}(\nu)$ as

$$E_1(x) = \frac{iE(0)}{\pi\Lambda(\infty)} \cdot \left[\int_{-\infty}^{0} \frac{-\Lambda(\infty) + \Lambda^{-}(\nu)}{\nu\Lambda^{-}(\nu)} \exp[i\omega x/\nu]\, d\nu + \int_{0}^{\infty} \frac{-\Lambda(\infty) + \Lambda^{+}(\nu)}{\nu\Lambda^{+}(\nu)} \exp[(i\omega/\nu)x]\, d\nu\right], \tag{4.9}$$

where

$$\Lambda(\infty) = 1 - \frac{\omega_p^2}{\omega^2} = 1 - K_0 = \varepsilon .$$

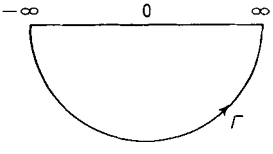

Fig. 1. – Contour of integration in the complex ν-plane.

Consider the first integral by examining the closed contour shown in Fig. 1. If we allow $\omega < \omega_p$ as well as $\omega > \omega_p$, then we must admit the presence of a pole in the lower half ν-plane. We label this pole $\nu = \nu_i$. From Fig. 1 we write

$$\oint_c = \sum_i 2\pi i \text{ residue (poles at } \nu_i) = \int_\Gamma + \int_\infty^0 + \int_0^{-\infty} . \tag{4.10}$$

We obtain for the residue

$$R = 2\pi i \sum_i \text{residue} = \sum_i \frac{-2E(0)}{\Lambda(\infty)} \left[\frac{-\Lambda(\infty) + \Lambda^{-}(\nu_i)}{\nu_i[\Lambda^{-}(\nu_i)]'}\right] \exp[i\omega x/\nu_i] , \tag{4.11}$$

where the prime indicates differentiation with respect to ν at $\nu = \nu_i$. We evaluate the expression $[\Lambda^{-}(\nu_i)]'$ to be

$$[\Lambda^{-}(\nu_i)]' = \nu_i \frac{m}{kT}\left(1 - \frac{\omega_p^2}{\omega^2}\right) - \frac{3}{\nu_i} . \tag{4.12}$$

The expression for the residue becomes

$$R = \sum_i \frac{-2E(0)}{3 - \nu_i^2(m/kT)(1 - \omega_p^2/\omega^2)} \exp[i\omega x/\nu_i] . \tag{4.13}$$

As $\nu \to \pm\infty$ the integral designated $\int_\Gamma \to 0$; therefore, eq. (4.9) may be written as

$$E_1(x) = \frac{iE(0)}{\pi\Lambda(\infty)} \int_0^\infty \mathrm{d}\nu \left[\frac{\Lambda(\infty) - \Lambda^-(\nu)}{\nu\Lambda^-(\nu)} + \frac{\Lambda^+(\nu) - \Lambda(\infty)}{\nu\Lambda^+(\nu)}\right] \exp[i\omega x/\nu] + \\ + \sum_i \frac{2E(0)}{3 - \nu_i^2(m/kT)(1 - \omega_p^2/\omega^2)} \exp[i\omega x/\nu_i]\,. \tag{4.14}$$

This expression for $E_1(x)$ reduces to eq. (4.3) and demonstrates the equivalence of the two methods for all frequencies.

5. – Effect of collisions on the electric field.

The complete expression for the electric field, eq. (3.15), consists of contributions from the charge free, discrete, and continuum modes. If we consider large distances (greater than several Debye lengths) from the plasma boundary we may approximate the integral which results from the continuum mode. The electric field may then be expressed in the form

$$[E(x)]_{\text{large } x} = E(\infty) + E_{\text{mode}}(x) + E_{\text{S.D.}}(x) + E_{\text{pole}}(x)\,. \tag{5.1}$$

We shall proceed to define and discuss individually each term of eq. (5.1).

The field at infinity, $E(\infty)$, shown in Fig. 2, results from the charge free mode. The effect of collisions is greatest at the plasma frequency where the magnitude of $E(\infty)$ reaches its maximum value. For the collisionless case, $E(\infty)$ is infinite at $\omega = \omega_p$, while for $\omega_c = 0.01\,\omega_p$, $E(\infty) \simeq 100\ E(0)$.

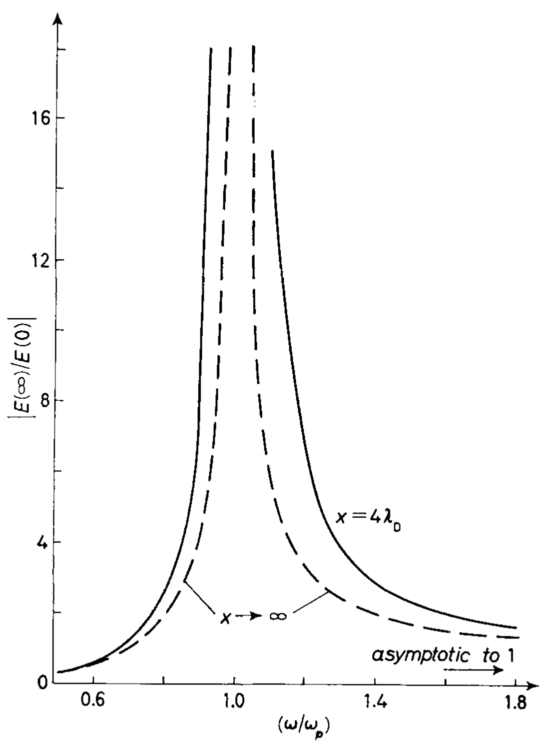

Fig. 2. – Electric field *vs.* applied frequency for $\omega_c = 0.01\omega_p$.

ω_c/ω_p	$\lvert E(\infty)/E(0)\rvert$
0	∞
0.01	100

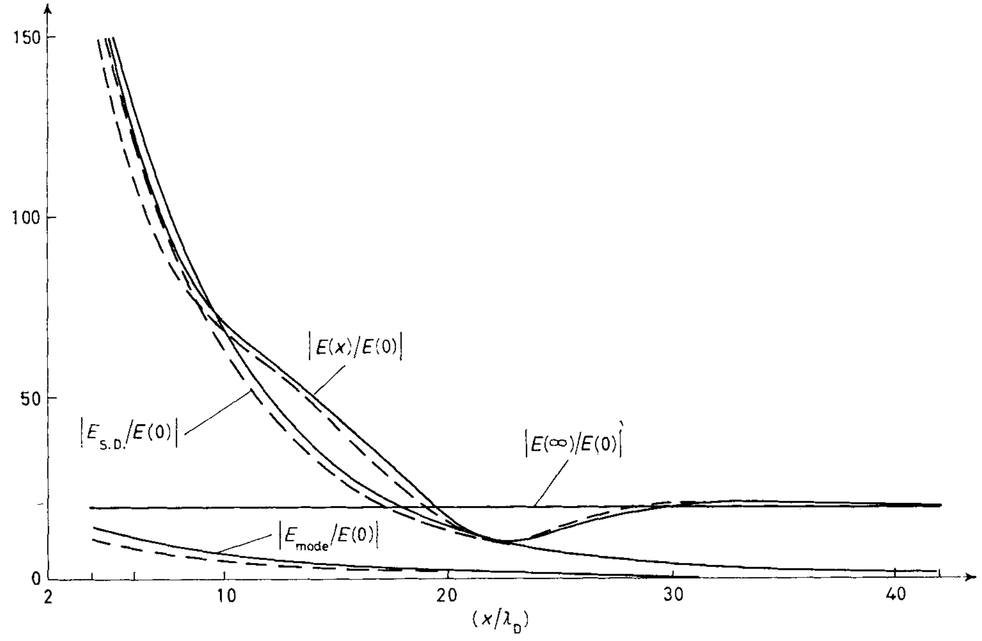

Fig. 3. – Electric field *vs.* distance for $\omega = 0.975\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

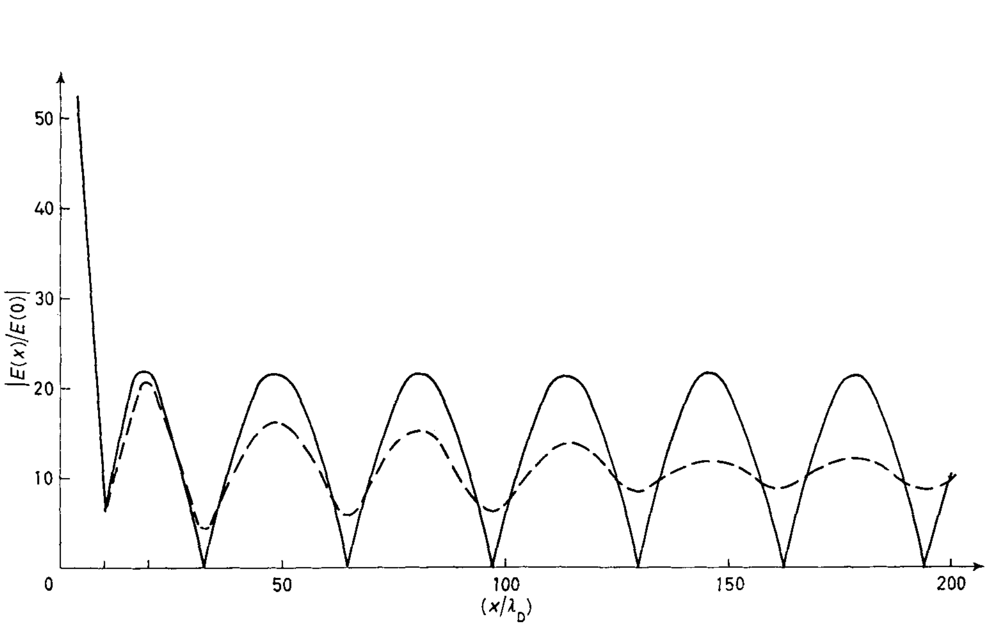

Fig. 4. – Total electric field *vs.* distance for $\omega = 1.05\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

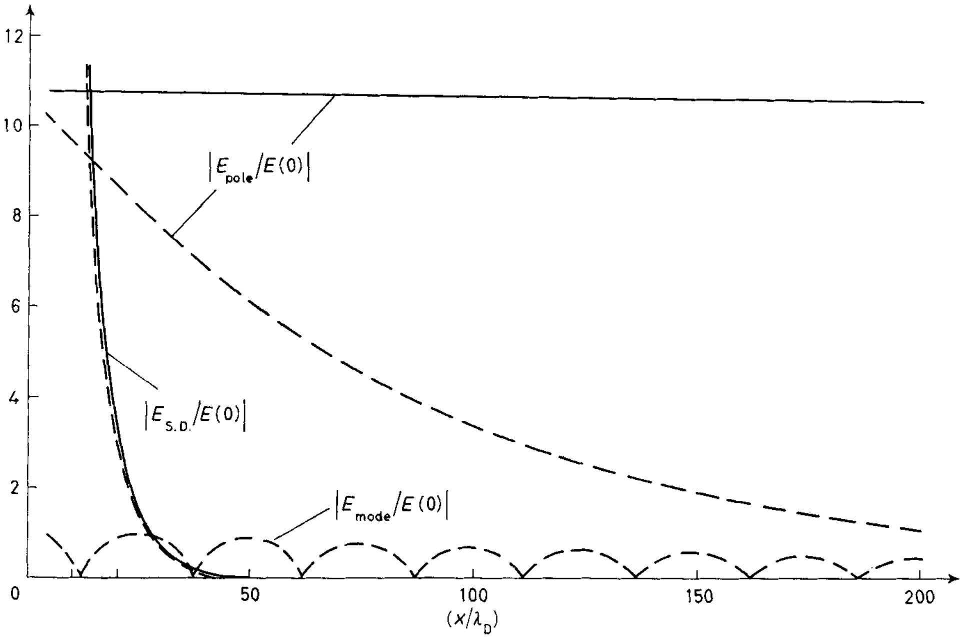

Fig. 5. – Electric field components *vs.* distance for $\omega = 1.05\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

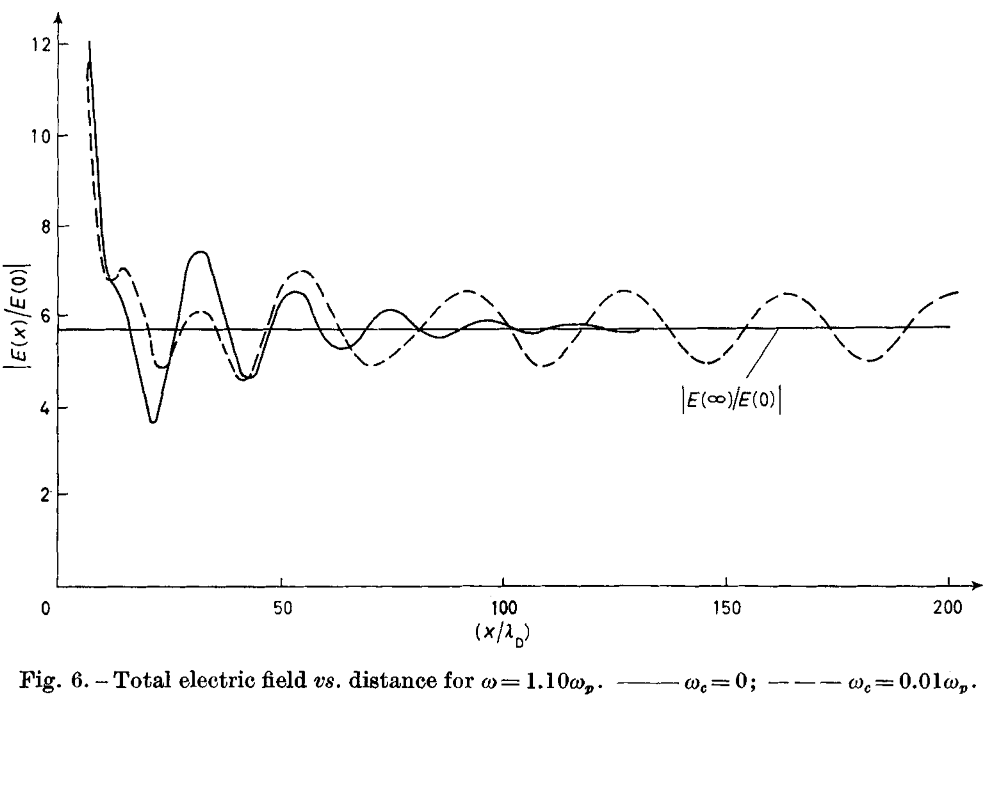

Fig. 6. – Total electric field *vs.* distance for $\omega = 1.10\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

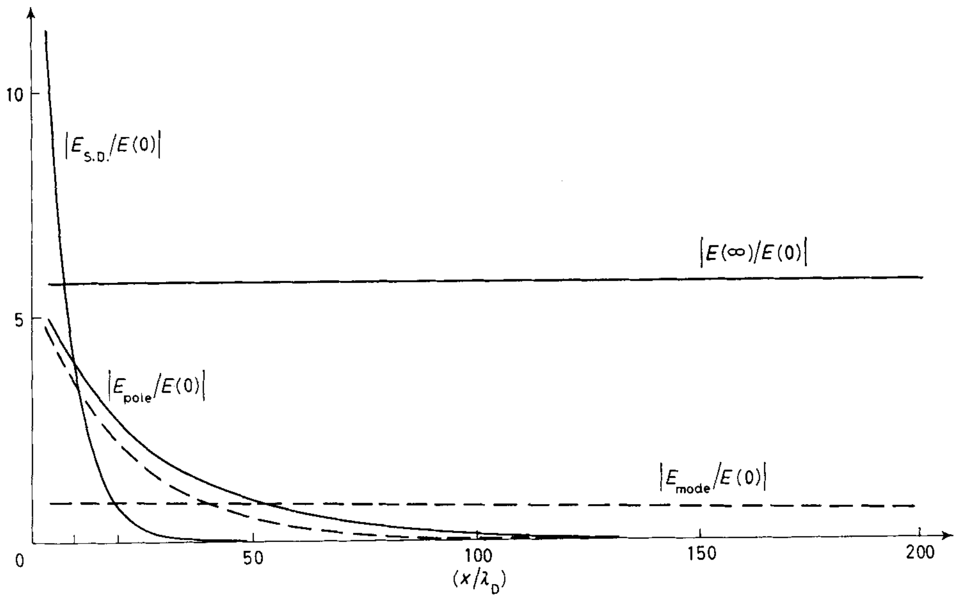

Fig. 7. – Electric field components *vs.* distance for $\omega = 1.10\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

Except for the frequency ratio range of approximately $1 < \omega/\omega_p < 1.15$, $E(\infty)$ is the dominant field component after a distance of roughly $30\lambda_D$. (See Fig. 3-9). The Debye length λ_D is defined by $\lambda_D = kT/4\pi ne^2$.

The field due to the discrete modes, E_{mode}, exists for values of applied frequency up to a certain cut-off frequency. Using the argument principle to examine the complex $\Lambda(\nu)$-plane for encirclements of the origin, we found that

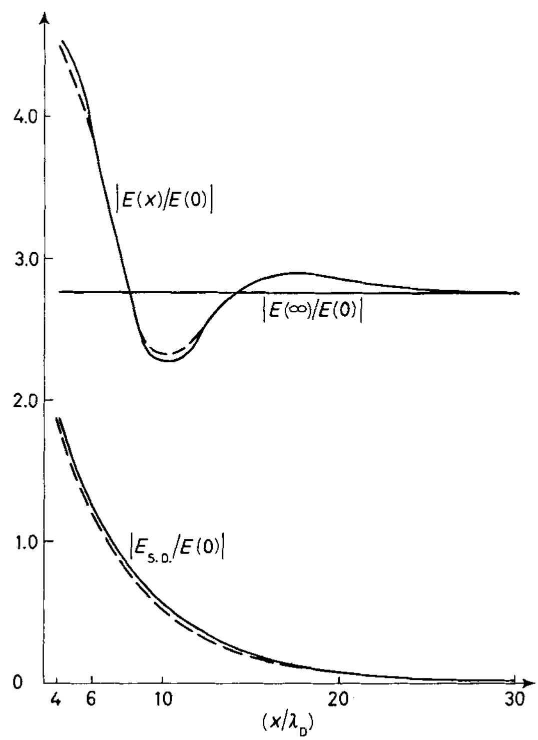

Fig. 8. – Electric field *vs.* distance for $\omega = 1.25\omega_p$. —— $\omega_c = 0$; – – – $\omega_c = 0.01\omega_p$.

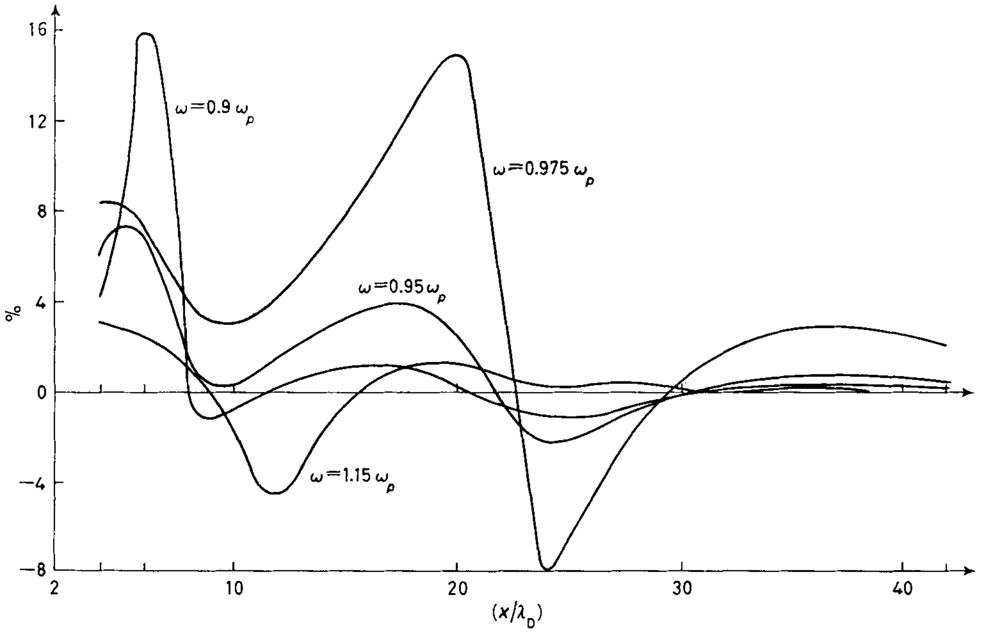

Fig. 9. – Percent change in the electric field due to collisions *vs.* distance for $\omega_c = 0.01\omega_p$.

for $\omega_c = 0.01\omega_p$ the discrete modes do not exist for $\omega > 1.12\omega_p$ as compared to $\omega > \omega_p$ for the collisionless case. (See Appendix A.) For the collisionless case E_{mode} is the least significant component for all frequencies up to $0.975\omega_p$. When collisions are considered, however, this component may be significant at frequencies above the plasma frequency. Although the magnitude of E_{mode} for small distances is small relative to the other components, it decreases very slowly with distance as shown in Fig. 7. For $\omega = 1.10\omega_p$ and distances greater than $100\lambda_D$, E_{mode} is the most significant spatially dependent field contribution.

The field component labeled $E_{\text{S.D.}}(x)$ results from the evaluation of the integral of the continuum mode by the method of steepest descent as shown in Appendix B; the result is

$$(5.2)\qquad E_{\text{S.D.}}(x) = \frac{-2\omega_p^2 E(0)}{\sqrt{3}\sigma^2(1+\omega_p^2/\sigma^2+\omega_c/\sigma)^2}\cdot \left[\frac{\omega_c}{\sigma}-\left(1+\frac{\omega_c}{\sigma}\right)\left(\frac{m}{kT}\right)^{\frac{1}{3}}(\omega^2+\omega_c^2)^{\frac{1}{3}}\left(x^{\frac{1}{3}}\exp\left[-i\left(\tfrac{2}{3}\right)\theta\right]\right)\right]\cdot \exp\left[\frac{3}{2}\left(\frac{m}{kT}\right)^{\frac{1}{3}}x^{\frac{2}{3}}(\omega^2+\omega_c^2)^{\frac{1}{3}}\exp\left[i\left(\pi-\frac{2}{3}\theta\right)\right]\right],$$

where $\theta = \operatorname{tg}^{-1}(\omega/\omega_c)$.

This component exists for the entire range of ω, and, except for frequencies in the vicinity of the plasma frequency, it constitutes the major field component which is spatially dependent. The field $E_{\text{S.D.}}(x)$ is the only field component in which the collisions reduce the magnitude of the field for every frequency and distance considered. The behavior of the real part of the exponent of $E_{\text{S.D.}}(x)$ is plotted as a function of ω/ω_p for values of ω_c/ω_p in Fig. 10. This exponential factor is labeled « damping factor » and may be considered as the Landau damping for the collisionless case. As the collision frequency increases, this « damping factor » decreases in value. The particles undergo increased collisions, thereby reducing the opportunity of an exchange of energy between the incoming wave and the plasma particles which are moving with the phase velocity of the wave (collisionless damping).

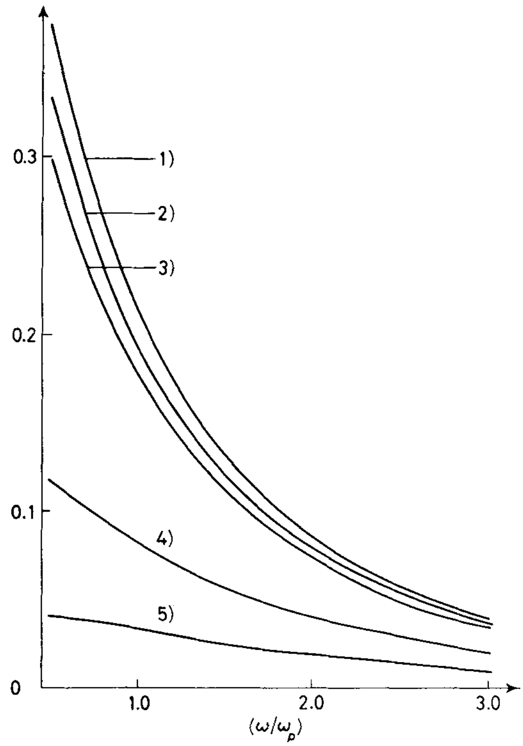

Fig. 10. – Damping factor *vs.* applied frequency for $x = 3\lambda_{\text{D}}$: 1) $\omega_c = 0$; 2) $\omega_c = 0.05\omega_p$; 3) $\omega_c = 0.1\omega_p$; 4) $\omega_c = 0.5\omega_p$; 5) $\omega_c = \omega_p$.

Finally we consider the field component labeled E_{pole}. This contribution results from encountering a pole when deforming the contour of integration to the path of steepest descent. In Appendix B it is shown that for large x/λ_{D}

$$(5.3) \qquad E_{\text{pole}} = \frac{-E(0)\omega_p^2}{\omega_2^2}\left[\frac{(\sigma+\omega_c)(\omega_1^2/\omega_2^2)+\omega_c}{\omega_1^2\omega_p^2/\omega_2^2\sigma+\omega_c}\right]\exp\left[\frac{\sigma x(\omega_2^2+i\sigma^2\sqrt{(\pi/2)}K_3)^{\frac{1}{2}}}{i\omega_p\lambda_{\text{D}}\omega_1}\right],$$

where

$$\omega_1^2 = 3\omega_p^2 + \sigma\omega_c\,,$$

$$\omega_2^2 = \omega_p^2 + \sigma^2 + \omega_c\sigma\,,$$

$$K_3 = i\,\frac{\omega_1}{\omega_2}\left[\frac{\omega_p^2\omega_1^2}{\sigma^2\omega_2^2} - \frac{\omega_c}{\sigma}\right]\exp\left[\frac{\omega_1^2}{2\omega_2^2}\right].$$

By examining the arguments of the pole and saddle point the pole was encountered within the frequency range of $\omega_p < \omega < 1.2\omega_p$. The contribution E_{pole} is most significant at approximately $\omega = 1.05\omega_p$, since at this frequency E_{pole} decreases only very slightly with distance. The ratio $|E_{\text{pole}}/E(0)|$ decreases from 10.75 at $4\lambda_D$ to 10.50 at $290\lambda_D$. For $\omega > 1.05\omega_p$, E_{pole} becomes less significant until it vanishes when the argument of the pole lies on the path of steepest descent at approximately $\omega \simeq 1.2\omega_p$.

For distances greater than $10\lambda_D$ it is evident from Fig. 2-9 that the effect of collisions is greatest when the applied frequency is near the plasma frequency. In particular, from Fig. 9 we find that when $\omega \geqslant 1.15\omega_p$ and $x > 4\lambda_D$, the value of the field with collisions is within 5% of that without collisions. When $\omega < 0.95\omega_p$ and $x > 10\lambda_D$ the field with and without collisions is within 4%.

The effect of including collisions for $0.95 < \omega/\omega_p < 1.00$ is one of enhancing the damping of the field. For $\omega = 0.975\omega_p$ we find that for $x > 30\lambda_D$, the effect of collisions is to decrease the field by a constant value 1.9%. For frequencies slightly greater than the plasma frequency, this is not at all the case as seen from Fig. 4 for $\omega = 1.05\omega_p$. Here we find the field magnitude for $\omega_c = 0$ undergoing large oscillations with distance. The behavior of the field components shown in Fig. 5 indicates that the oscillations are primarily caused by the sum of E_{pole} and $E(\infty)$, since their magnitudes are nearly equal for all x to $200\lambda_D$. Hence the inclusion of collisions results in the field approaching the constant value $E(\infty)$ at a much closer distance to the plasma boundary than predicted in the collisionless case.

The results are quite different for the frequency $\omega = 1.10\omega_p$ plotted in Fig. 6 and 7. The field E_{pole} is damped much more rapidly than for $\omega = 1.05\omega_p$, but when $\omega = 0.01\omega_c$ the damping of E_{mode} is very slight. This causes the field to continue to oscillate significantly with distance. Since the discrete mode vanishes at this frequency for $\omega_c = 0$, we find the unusual condition of the collisionless case being damped out more rapidly than when collisions are included. The collisions tend to reinforce the waves rather than destroy them for this particular combination of collision and applied frequencies.

6. – Discussion.

The longitudinal electric field in a plasma half-space was calculated by use of the normal mode technique. Collisions between electrons and neutrals were accounted for by use of a particle-conserving BGK collision model. The equivalence of the normal mode approach and the transform method used by LANDAU was established for the collisonless case. For distances greater than five Debye lengths from the boundary, and for collision frequencies up to $0.01\omega_p$, it was shown that collisions have little affect on the electric field except in the frequency range $0.95\omega_p < \omega < 1.2\omega_p$. Within this frequency band, several different kinds of effects are possible. The field may be damped much more rapidly with distance, or for special frequencies which depend on the collision frequency the collisions may destroy the collisionless damping and thereby reduce the total damping. The behavior of the field is highly dependent upon the applied frequency and the collision frequency, particularly when the former is in the vicinity of the plasma frequency.

It should be noted that the collision model which is used only conserves particle number. The results obtained are intended to provide a qualitative rather than a quantitative indication of the behavior of the electric field.

* * *

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APPENDIX A

Discussion of the discrete mode.

In formulating a solution for the electric field using the BGK collision model, we noted that the discrete modes do not exist above a certain frequency. This cut-off frequency and the value of the discrete root will be the topics covered in this Section.

In Sect. **2** we found that the ν_i are the eigenvalues of the characteristic function $\Lambda(\nu)$. Equation (2.9) can be written as

$$\Lambda(\nu) = 1 + \frac{\nu^2\omega_p^2}{\sigma^2}\int_{-\infty}^{\infty}\frac{F'(u)}{(u-\nu)}\,\mathrm{d}u - \frac{\nu\omega_c}{\sigma}\int_{-\infty}^{\infty}\frac{F(u)}{u-\nu}\,\mathrm{d}u\,. \tag{A.1}$$

To investigate the number of zeros of $\Lambda(\nu)$ in the upper half-plane, we employ the argument principle (6). Consider the contour C in the ν-plane shown in Fig. 11 and follow $\Lambda(\nu)$ as ν traverses this contour. The change of argument of $\Lambda(\nu)$ will be equal to 2π times the number of zeros of $\Lambda(\nu)$ inside C. For the semicircle 1-2-3, ν is sufficiently large that $\Lambda(\nu)$ assumes its asymptotic behavior.

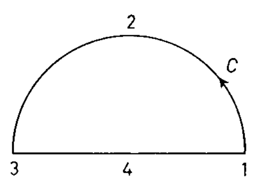

Fig. 11. – Contour of integration in the ν-plane.

(6) E. COPSON: *Theory of Functions of a Complex Variable* (Oxford, 1935).

Since the whole semicircle (1-2-3) in the ν-plane maps into a single point in the $\Lambda(\nu)$-plane, it is the only necessary to examine the real ν-axis (3-4-1) in using the argument principle to find encirclements of the origin in the $\Lambda(\nu)$-plane. If an encirclement is to occur, there must be at least one value of real ν, call it $\bar{\nu}$, that satisfies $\operatorname{Im}\Lambda(\bar{\nu})=0$ and $\operatorname{Re}\Lambda(\bar{\nu})<0$. The procedure therefore will be to find the roots of $\operatorname{Im}\Lambda(\bar{\nu})=0$ and examine the behavior of $\operatorname{Re}\Lambda(\bar{\nu})$. To complete the problem, values of $\bar{\nu}$ near the roots will be investigated to ensure that complete encirclement was made. By following this procedure, we prevent the occurrence of mappings such as shown in Fig. 12 where the imaginary axis was crossed but no encirclement occurred.

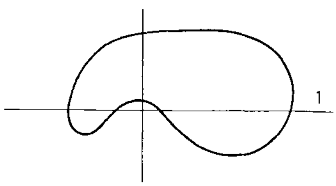

Fig. 12. – Real values of ν mapped in the complex $\Lambda(\nu)$-plane without encirclement of the origin.

To examine the real ν-axis along 3-4-1 of Fig. 11, we consider $F(u)$ as a Maxwellian distribution and define

$$Z(\alpha)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{e^{-\xi^2}}{\xi-\alpha},$$

where

$$\xi=\left(\frac{m}{2kT}\right)^{\frac{1}{2}}u,$$

$$\alpha=\frac{\nu}{\sqrt{2}u_e}=\frac{\nu}{\sqrt{2}}\left(\frac{m}{kT}\right)^{\frac{1}{2}}.$$

Equation (A.1) can then be written as

$$\Lambda(\alpha)=1-\frac{2\omega_p^2}{\sigma^2}\alpha^2-\left(\frac{\omega_c}{\sigma}\alpha+\frac{2\omega_p^2}{\sigma^2}\alpha^3\right)Z(\alpha)\,. \tag{A.2}$$

For the collisionless case we find that to satisfy $\operatorname{Im}\Lambda(\alpha)=0$ and $\operatorname{Re}\Lambda(\alpha)<0$, we require $\alpha\to\infty$. Then, from eq. (A.2) we find $\omega<\omega_p$. There are no zeros of $\Lambda(\nu)$ when $\omega>\omega_p$.

For the case $\omega_c\neq 0$, we find $\operatorname{Im}\Lambda(\infty)\neq 0$ and the analysis becomes considerably more complex. For the purpose of numerical calculation, we choose $\omega_c/\omega_p=10^{-2}$. This ratio, which is reasonable for a weakly ionized plasma, should be sufficiently high for demonstrating the effect of $\omega_c\neq 0$. By using this parameter we find both $\operatorname{Re}\Lambda(\infty)$ and $\operatorname{Im}\Lambda(\infty)$ are zero for $\omega=1.12\,\omega_p$ and $\alpha=-3.025$. When $\omega>1.12\,\omega_p$, there are no zeros of $\Lambda(\nu)$ and there are no discrete modes.

In Fig. 13 three values of ω are plotted: $\omega=\omega_p$, $\omega=\sqrt{2}\omega_p$, and $\omega=1.12\,\omega_p$. The first two show typical mappings of $\Lambda(\bar{\nu})$ for cases of encirclement and nonencirclement of the origin, and $\omega=1.12\,\omega_p$ is the marginal case of the mapping passing through the origin.

The value of the discrete root may be determined by solving $\Lambda(\zeta) = 0$, where $\zeta = \alpha + i\beta$. First we examine the collisionless case. From a discussion of the properties of $\Lambda(\nu)$ in Sect. **2**, we found there are four possible roots of

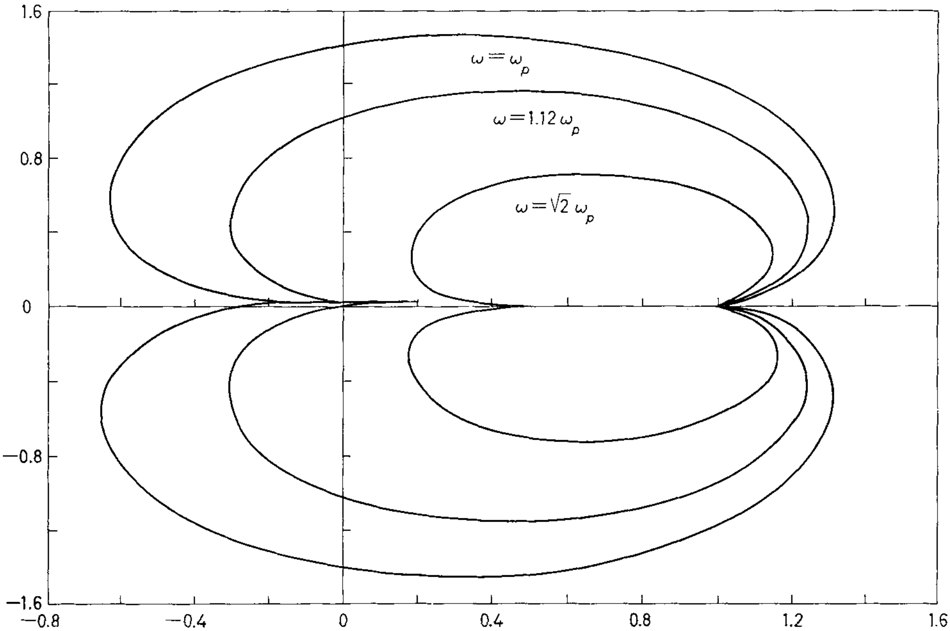

Fig. 13. – Complex $\Lambda(\bar{\nu})$-plane for $\omega_c = 0.01\omega_p$.

$\Lambda(\nu) = 0$. But from the first part of this Appendix, we found there is only one possible root in the upper half-plane; therefore, it must lie on the imaginary axis. With $\omega_c = 0$, $\zeta = i\beta$, $\alpha = 0$, and we find

$$\Lambda(i\beta) = \operatorname{Re}\Lambda(i\beta) = 0\,.$$

This yields

$$\text{(A.3)} \qquad \frac{\omega}{\omega_p} = \sqrt{2}\beta\,[1 - \pi^{\frac{1}{2}}\beta e^{\beta^2}(1 - \operatorname{erf}\beta)]^{\frac{1}{2}}\,.$$

This relation gives the value of the root as a function of the dimensionless frequency.

When we set $\omega_c = 0.01\omega_p$, the solution to $\Lambda(\nu) = 0$ was found with the aid of an IBM 360/50 computer. The computer results are plotted in Fig. 14. Observe that the value of the real part of the discrete root below ω_p is small, and the magnitudes of the real and imaginary parts of the root for $\omega \approx \omega_p$ are roughly equal. The curve terminates on the real ν_i axis at the cut-off frequency of $1.12\omega_p$. For the collisionless case, as $\omega \to \omega_p$ the root is entirely imaginary and approaches infinity. For $\omega_c = 0.01\omega_p$ and $\omega_p < \omega < 1.12\omega_p$,

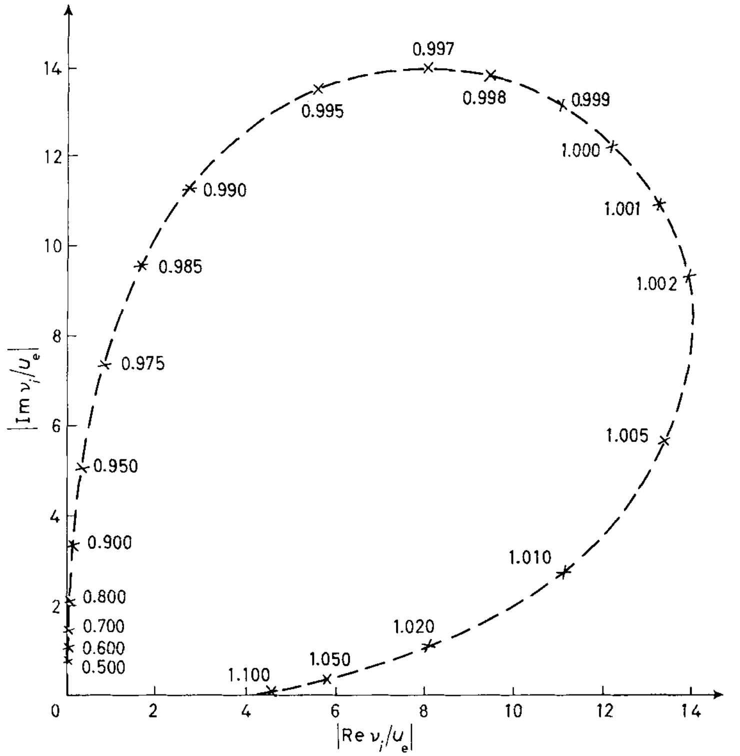

Fig. 14. – Complex discrete root normalized to the electron thermal speed as a function of applied frequency for $\omega_c = 0.01\omega_p$. All points indicate applied frequency.

the magnitude decreases for both the real and imaginary parts of the root. However, the real part is now greater than the imaginary part. For $\omega = 1.10\,\omega_p$ the sign of one of the roots changed to satisfy the boundary condition that the field vanishes at infinity.

Appendix B

Asymptotic approximations.

By assuming distances to be greater than several Debye lengths, we can evaluate the integrals of the electric field expression. The method of steepest descents [7] will be applied to the integrals, and then any poles crossed in deforming the contour to the path of steepest descent will be investigated.

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(7) E. Copson: *Asymptotic Expansions* (Cambridge, 1965).

The term of the electric field solution that is in integral form can be re-written using the Plemelj formulas [8] as

$$(\text{B.1})\qquad E_{\text{S.D.}}(x) = \frac{-2\omega_p^2 E(0)}{\sigma^2}\int_0^\infty \mathrm{d}\nu \exp[\sigma x/\nu]\left(\frac{\nu\gamma(\nu)}{\Lambda^-(\nu)\Lambda^+(\nu)}\right)\cdot$$
$$\cdot\left[-\nu^2\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{F'(u)}{u-\nu}+\frac{F'(\nu)}{\gamma(\nu)}\left(1+\frac{\nu^2\omega_p^2}{\sigma^2}\int_{-\infty}^{\infty}\frac{F'(u)}{u-\nu}\,\mathrm{d}u-\frac{\nu\omega_c}{\sigma}\int_{-\infty}^{\infty}\frac{F(u)}{u-\nu}\,\mathrm{d}u\right)\right].$$

If we consider $F(\nu)$ as Maxwellian and use the definition of $\gamma(\nu)$, we obtain an expression in the form

$$(\text{B.2})\qquad E_{\text{S.D.}}(x) = C\int_0^\infty \mathrm{d}\nu\, G(\nu)\exp[xg(\nu)]\,,$$

where $G(\nu)$ is the nonexponential function of the complex variable ν, C is a constant in ν, and

$$(\text{B.3})\qquad g(\nu) = \frac{\sigma}{\nu} - \frac{m\nu^2}{2kTx}\,.$$

Applying the method of steepest descent, we find the saddle point at

$$(\text{B.4})\qquad \nu_0 = (\omega^2+\omega_c^2)^{\frac{1}{6}}\left(\frac{kT}{m}x\right)^{\frac{1}{3}}\exp\left[\frac{i}{3}(-\theta+2n\pi)\right],$$

where

$$\theta = \operatorname{tg}^{-1}(\omega/\omega_c)\,,\qquad n = 0, 1, 2\,.$$

Possible values for the arguments of the three roots of ν_0 are shown in the hatched areas of Fig. 15. To deform the original contour (0 to ∞) to the regions indicated by $n=1$ and $n=2$ the imaginary axis must be crossed. Since the integrand has an essential singularity at these crossings, the integral fails to converge; hence, the only possible choice for the root is the one defined by $n=0$.

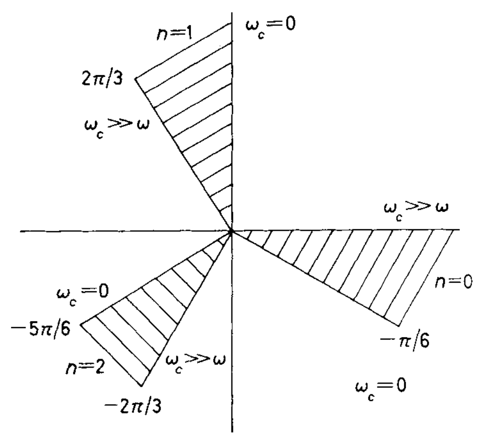

Fig. 15. – Possible arguments of ν_0 in the complex ν-plane.

With the proper path of integration chosen, we may evaluate the

[8] N. Muskhelishvili: *Singular Integral Equations* (Gronigen, 1953).

integral using the Debye formula (7):

$$E_{\text{S.D.}}(x) = C\exp[xg(\nu_0)]G(\nu_0)\left[\frac{2\pi}{-xg''(\nu_0)}\right]^{\frac{1}{2}}. \tag{B.5}$$

To find $G(\nu_0)$ we note that as $x \to \infty$, $\gamma_0 \to \infty$. We therefore make the following approximations:

$$\Lambda^+(\nu_0) = \Lambda^-(\nu_0) = 1 + \frac{\omega_p^2}{\sigma^2} + \frac{\omega_c}{\sigma}, \tag{B.6}$$

$$\int_{-\infty}^{\infty} \frac{F'(u)}{u-\nu}\,\mathrm{d}u = \int_{-\infty}^{\infty} \mathrm{d}u\,\frac{F(u)}{(u-\nu)^2} \simeq \frac{1}{\nu^2}, \tag{B.7}$$

$$\int_{-\infty}^{\infty} \frac{F(u)}{u-\nu}\,\mathrm{d}u \simeq -\frac{1}{\nu}. \tag{B.8}$$

With these relations, eq. (B.1) becomes

$$\begin{aligned} E_{\text{S.D.}}(x) = {} & \frac{-2\omega_p^2 E(0)}{\sqrt{3}\,\sigma^2(1+\omega_p^2/\sigma^2+\omega_c/\sigma)^2}\cdot \\ & \cdot\left[\frac{\omega_c}{\sigma} - \left(1+\frac{\omega_c}{\sigma}\right)\left(\frac{m}{kT}\right)^{\frac{1}{3}}(\omega^2+\omega_c^2)^{\frac{1}{3}}\left(x^{\frac{2}{3}}\exp\left[-i\frac{2}{3}\theta\right]\right)\right]\cdot \\ & \left[\exp\left[\frac{3}{2}\left(\frac{m}{kT}\right)^{\frac{1}{3}} x^{\frac{2}{3}}(\omega^2+\omega_c^2)^{\frac{1}{3}}\exp\left[i\pi - i\frac{2}{3}\theta\right]\right]\right]. \end{aligned} \tag{B.9}$$

One possibility that has been overlooked in this development is that a pole could have been encountered when deforming the contour to the path of steepest descent. If such a root exists, we know it must be near the real axis in the lower half ν-plane, so we examine the roots of $\Lambda^-(\nu) = 0$. Assuming the phase velocity of the wave is large compared to the thermal velocity, we find

$$\begin{aligned} \Lambda^-(\nu) = 1 + \left(\frac{\omega_p}{\sigma}\right)^2 + \frac{\omega_c}{\sigma} + \frac{1}{\nu^2}\left(\frac{kT}{m}\right)\left(3\frac{\omega_p^2}{\sigma^2} + \frac{\omega_c}{\sigma}\right) + \\ + \pi i\nu\left[\frac{\nu^2}{\sigma^2\lambda_D^2} + \frac{\omega_c}{\sigma}\right]\left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}}\exp\left[-\frac{m\nu^2}{2kT}\right]. \end{aligned} \tag{B.10}$$

To find the root ν_1 that causes the above expression to vanish, we assume the last term in the above expression is less than the combination of the remaining terms. An appropriate solution for ν_1 results:

$$\nu_1 \simeq \left[\frac{(kT/m)(-\omega_c/\sigma - 3(\omega_p^2/\sigma^2))}{1+\omega_p^2/\sigma^2+\omega_c/\sigma}\right]^{\frac{1}{2}}. \tag{B.11}$$

This relation is then used in the last term of eq. (B.10), and again we solve that equation for ν_1. The result is

$$\nu_1 = \left[\frac{-(kT/m)\omega_1^2}{\omega_2^2 + (\pi/2)^{\frac{1}{2}}(\omega_1/\omega_2)\big((\omega_p^2\omega_1^2/\omega_2^2) - \omega_c\sigma\big)\exp[\omega_1^2/2\omega_2^2]}\right]^{\frac{1}{2}}, \tag{B.12}$$

where

$$\omega_1^2 = 3\omega_p^2 + \sigma\omega_c\,, \qquad \omega_2^2 = \omega_p^2 + \sigma^2 + \omega_c\sigma\,, \qquad \sigma = -\omega_c + i\omega\,.$$

If the argument of ν_1 lies within an angle from 0 to the argument of ν_0 from the path of steepest descent, then the pole ν_1 will provide an additional contribution to the field. Since eq. (B.12) is too cumbersome for an analytical expression for the argument, a computer solution was obtained. The results show the arguments of ν_0 and ν_1 to be equal at $\omega = 1.21\,\omega_p$ for no collisions and $\omega = 1.20\,\omega_p$ for $\omega_c = 0.01\,\omega_p$.

We have now established the existence of a pole which will contribute to the asymptotic field when the applied frequency is approximately $\omega_p < \omega \lesssim 1.2\,\omega_p$. To evaluate this field contribution we use the theory of residues.

$$\int_0^\infty (\;) + \int_\infty^{-(i\pi/6)\infty} (\;) + 2\pi i \text{ (residue at } \nu_1) + \int_{-(i\pi/6)\infty}^{0} (\;) = 0\,, \tag{B.13}$$

where

$$(\;) = \exp[\sigma x/\nu]A(\nu)E_\nu\,\mathrm{d}\nu\,.$$

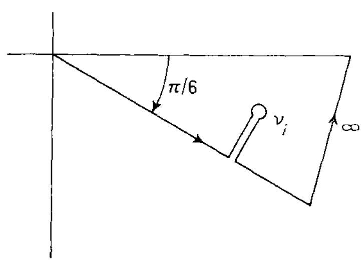

Fig. 16. – Contour of integration in the complex ν-plane.

Figure 16 indicates the contour of these integrals. Since the second term of eq. (B.13) vanishes by Jordan's lemma we obtain

$$\int_0^\infty (\;) = \int_0^{-(i\pi/6)\infty} - 2\pi i \text{ (residue at } \nu_1)\,. \tag{B.14}$$

To calculate the residue we again assume large real ν, such that the integral of eq. (3.15) may be evaluated. The field resulting from the residue we shall call E_{pole}. It takes the form

$$E_{\text{pole}} = \frac{-E(0)\omega_p^2}{\omega_2^2}\left[\frac{(\sigma^2+\omega_c\sigma)(\omega_1^2/\omega_2^2)+\omega_c\sigma}{\omega_1^2\omega_p^2/\omega_2^2+\omega_c\sigma}\right]\exp\left[\frac{\sigma x\big(\omega_2^2 + i\sigma^2\sqrt{(\pi/2)}K_3\big)^{\frac{1}{2}}}{i\omega_p\lambda_D\omega_1}\right], \tag{B.15}$$

where

$$K_3 = i\,\frac{\omega_1}{\omega_2}\left[\frac{\omega_p^2\omega_1^2}{\sigma^2\omega_2^2} - \frac{\omega_c}{\sigma}\right]\exp\left[\frac{\omega_1^2}{2\omega_2^2}\right].$$

For the collisionless case, eq. (B.15) reduces to

$$E_{\text{pole}} = \frac{E(0)}{-\varepsilon} \exp\left[\frac{x}{\lambda_D}\left[i\,\frac{\omega^2}{\omega_p^2}\sqrt{\frac{\varepsilon}{3}} - \frac{1.5}{\varepsilon^2}\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p}{\omega}\right)^3 \exp\left[-\frac{1.5\,\omega_p^2}{\varepsilon\omega^2}\right]\right]\right], \tag{B.16}$$

which was obtained by LANDAU.

RIASSUNTO (*)

Si determina il campo elettrico nel semispazio di un plasma usando un'analisi del modo normale con collisioni studiate tramite un modello di collisione che conserva le particelle di BGK. nel caso in cui non si hanno collisioni si dimostra che i risultati sono equivalenti a quelli di Landau che ha usato metodi di trasformazione. Per distanze maggiori di cinque lunghezze di Debye dal bordo, e per frequenze di collisione sino a $0.01\omega_p$, si dimostra che le collisioni hanno scarso effetto sul campo elettrico tranne che nell'intervallo di frequenze $0.95\omega_p < \omega < 1.2\omega_p$. Entro questa banda di frequenze, le collisioni producono diverse specie di effetti, dipendenti dalla frequenza del campo applicato e dalla frequenza delle collisioni. Per la maggior parte delle frequenze il campo è smorzato più rapidamente con le distanze. Per speciali frequenze, però, le collisioni distruggono lo smorzamento senza collisioni e riducono lo smorzamento totale.

(*) *Traduzione a cura della Redazione.*

Продольные волны в полубесконечной плазме.

Резюме (*). — Определяется электрическое поле в полубесконечной плазме, используя анализ нормальных мод с соударениями, которые рассматриваются с помощью модели BGK для соударений, которые сохраняют число частиц. Показывается, что в бесстолкновительном случае результаты эквивалентны результатам Ландау, который использовал трансформационные методы. Отмечается, что для расстояний больше, чем пять дебаевских длин от границы и для частот соударений вплоть до $0.01\omega_p$, столкновения имеют малое влияние на электрическое поле, за исключением области частот $0.95\omega_p < \omega < 1.2\omega_p$. Внутри этой области частот соударения вызывают различные типы эффектов, зависящих от частоты приложенного поля и от частоты соударений. Для большинства частот, поле затухает более быстро с расстоянием. Однако, для характерных частот, соударения уничтожают бесстолкновительное затухание и уменьшают полное затухание.

(*) *Переведено редакцией.*