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An Analytic Expression for Closed-Loop Output Behavior under Multiloop PID Control

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Abstract−An analytic expression is derived for closed-loop output behavior under a multiloop PID control. Based on the analytic expression obtained, optimization problems are formulated to assess 1) best achievable quadratic performance using multiloop PID control, 2) best achievable quadratic performance on key process variables while maintaining reasonable performance on other less critical process variables, 3) achievable performance improvement with decouplers, and 4) effects of loop pairing on achievable performance. It is shown through a simulated example that individual loop performance as well as the overall multiloop PID control performance can be assessed by using the proposed method.

Key words: Analytic Expression, Multiloop PID Control, Performance Assessment, Time-Series Analysis

INTRODUCTION

Control loop performance monitoring technology has received increasing attention over the past decade in both academia and industry. In this technology, minimum variance control has played a very important role as a benchmark performance measure in various feedback/feedforward control systems [Harris, 1989; Desborough and Harris, 1992, 1993; Harris et al., 1996; Huang et al., 1997; Ko and Edgar, 2000, 2001a, b]. Detailed review and survey of literature in this area can be found in Qin [1998] and Harris and Seppala [2001]. However, virtually no progress has been reported in the literature for multiloop feedback control systems despite the fact that it is one of the commonly used control structures in process industries.

In multiloop control systems difficulties arise in controlling process variables due to process interactions. When severe process interactions exist, even the best-tuned multiloop controller may not provide satisfactory control performance. Hence the assessment of best achievable performance in a multiloop control system has significant practical importance in control system redesign and performance monitoring. The necessity of assessing the best achievable multiloop control performance has also been reported in the literature [Kozub, 1996; Qin, 1998].

When the control performance is measured by the l_∞-norm of the output signal, Sourlas et al. [1994] have developed a methodology to quantify the best performance achievable by low order decentralized controllers such as PI and PID controllers. However, as noted by Stanfelj et al. [1993], the most commonly used measure of performance is the variance of key process variables due to its direct relationship to process performance and profit.

In this paper, an analytic expression is derived for closed-loop output behavior under multiloop PID control. Based on the analytic expression derived, an optimization approach is proposed for the assessment of best achievable performance in multiloop PID control system when the control performance is measured by output variance. Also discussed are several extensions of the proposed method to some important issues in multiloop PID control system.

PROBLEM FOURMULATOIN

Consider the following multivariable system with n inputs and n outputs represented by a linear time-invariant process with additive disturbance at the output:

$$y_t = H(q^{-1})u_t + N(q^{-1})a_t \tag{1}$$

where $H(q^{-1})$ and $N(q^{-1})$ are the process and the disturbance transfer function matrices, respectively; y_t, u_t and a_t are process output, input and white noise vectors of appropriate dimensions. The discrete multiloop PID controller transfer function $C(q^{-1})$ has the following diagonal form:

$$C(q^{-1}) = \mathrm{diag}\{c_1(q^{-1}), c_2(q^{-1}), \ldots, c_n(q^{-1})\} \tag{2}$$

where $c_i(q^{-1}) = \dfrac{k_{1,i}+k_{2,i}q^{-1}+k_{3,i}q^{-2}}{1-q^{-1}}$, (i=1, 2, …, n), and $k_{1,i}$, $k_{2,i}$, $k_{3,i}$ are PID controller tuning parameters of the i^{th} loop. The quadratic performance measure of the output variable to be minimized is

$$J = E(y_t^T Q y_t) \tag{3}$$

where Q is a positive-definite output weighting matrix.

The closed-loop output is then given by $y_t = G_{CL}(q^{-1})\cdot a_t$, where $G_{CL}(q^{-1}) \equiv [I+H(q^{-1})C(q^{-1})]^{-1}N(q^{-1})$ is the closed-loop transfer function matrix from a_t to y_t. One way of obtaining closed-loop impulse response and the corresponding quadratic cost function is to perform long division on each element of the matrix transfer function $G_{CL}(q^{-1})$ after solving for the inversion of matrix polynomial. This method becomes non-trivial and requires numerous polynomial manipulations and greater computation time as the number of loops increases.

Another way of obtaining the quadratic cost function in Eq. (3) is to calculate the complex contour integral given below.

$$J = E(y_t^T Q y_t) = \mathrm{trace}[Q\cdot E(y_t y_t^T)]$$

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‡This paper is dedicated to Professor Hyun-Ku Rhee on the occasion of his retirement from Seoul National University.

$$=\mathrm{trace}\left[Q\cdot\frac{1}{2\pi i}\oint G_{CL}(z)\Sigma_a G_{CL}^T(z^{-1})\frac{dz}{z}\right]$$
$$=\mathrm{trace}\left[\left(\frac{1}{2\pi i}\oint G_{CL}^T(z^{-1})QG_{CL}(z)\frac{dz}{z}\right)\cdot\Sigma_a\right] \tag{4}$$

where $\oint$ denotes a counter-clockwise integral along the unit circle in the complex plane; Σ_a represents the covariance matrix of the noise vector a_t. Practically speaking, the evaluation of the cost function J should be carried out numerically; for example, with the scheme suggested by Åström [1970]. This method also has a drawback in that it requires matrix polynomial inversion. Moreover, it does not give the closed-loop impulse response.

In the next section, an alternative and more convenient way of obtaining the closed-loop impulse response and the corresponding quadratic cost function is developed as an explicit function of multiloop PID tuning parameters.

ANALYTIC EXPRESSION FOR THE CLOSED-LOOP OUTPUT BEHAVIOR

Consider the multivariable system in Eq. (1) that is rewritten by using the impulse response forms as follows:

$$y_t=\sum_{i=1}^{m}H_i q^{-i}u_t+\sum_{i=0}^{\infty}N_i q^{-i}a_t \tag{5}$$

where H_i and N_i are the impulse response coefficients of the process and the disturbances, respectively, and m is the largest number of time intervals for an output to reach a steady-state threshold. When the set point is held at zero, the output from the multiloop PID controller becomes

$$u_t=-\frac{1}{1-q^{-1}}[\mathrm{diag}(k_{1,1},k_{1,2},\ldots,k_{1,n})+\mathrm{diag}(k_{2,1},k_{2,2},\ldots,k_{2,n})q^{-1}$$
$$+\mathrm{diag}(k_{3,1},k_{3,2},\ldots,k_{3,n})q^{-2}]\cdot y_t \tag{6}$$

Substituting Eq. (6) into (5) results in the following relation:

$$y_t=-\sum_{i=1}^{m}S_i\cdot[\mathrm{diag}(k_{1,1},k_{1,2},\ldots,k_{1,n})+\mathrm{diag}(k_{2,1},k_{2,2},\ldots,k_{2,n})q^{-1}$$
$$+\mathrm{diag}(k_{3,1},k_{3,2},\ldots,k_{3,n})q^{-2}]\cdot q^{-i}y_t+\sum_{i=0}^{\infty}N_i q^{-i}a_t \tag{7}$$

where S_i(i=1, 2, …, m) represents the step response coefficients of the process model. If we define $S_{i,j}$ as the matrix that has the same *j*th column with S_i and has zeros elsewhere, Eq. (7) can be rewritten as:

$$y_t=-\sum_{j=1}^{n}\left[\sum_{i=1}^{m}S_{i,j}(k_{1,j}+k_{2,j}q^{-1}+k_{3,j}q^{-2})\cdot q^{-i}y_t\right]+\sum_{i=0}^{\infty}N_i q^{-i}a_t \tag{8}$$

To obtain an output expression under a multiloop PID control, we first compute the closed-loop output behavior for a single load disturbance driven by a white noise a_o, which is introduced to the system at t=0. Then, we apply the principle of superposition to obtain a general expression for the closed-loop output behavior.

When the white noise a_o is introduced at t=0, the future closed-loop outputs over a finite-horizon p are related to the tuning parameters in the following way.

$$\begin{bmatrix}y_o\\y_1\\\vdots\\y_p\end{bmatrix}=-\sum_{j=1}^{n}(G_jk_{1,j}+\tilde{G}_jk_{2,j}+\tilde{\tilde{G}}_jk_{3,j})\cdot\begin{bmatrix}y_o\\y_1\\\vdots\\y_p\end{bmatrix}+\begin{bmatrix}N_o\\N_1\\\vdots\\N_p\end{bmatrix}\cdot a_o \tag{9}$$

where

$$G_j=\begin{bmatrix}0&\cdots&&&\\S_{1,j}&0&&\mathbf{0}&\\S_{2,j}&S_{1,j}&0&&\\\vdots&\vdots&\ddots&\ddots&\\S_{p,j}&S_{p-1,j}&\cdots&S_{1,j}&0\end{bmatrix},\quad \tilde{G}_j=\begin{bmatrix}0&\cdots&&&&\\0&0&&&\mathbf{0}&\\S_{1,j}&0&0&&&\\S_{2,j}&S_{1,j}&0&0&&\\\vdots&\vdots&\ddots&\ddots&\ddots&\\S_{p-1,j}&S_{p-2,j}&\cdots&S_{1,j}&0&0\end{bmatrix},$$

and

$$\tilde{\tilde{G}}_j=\begin{bmatrix}0&\cdots&&&&&\\0&0&&&&\mathbf{0}&\\0&0&0&&&&\\S_{1,j}&0&0&0&&&\\S_{2,j}&S_{1,j}&0&0&0&&\\\vdots&\vdots&\ddots&\ddots&\ddots&\ddots&\\S_{p-2,j}&S_{p-3,j}&\cdots&S_{1,j}&0&0&0\end{bmatrix}.$$

Eq. (9) can be rewritten in a compact form as:

$$\bar{y}=\left[I+\sum_{j=1}^{n}(G_jk_{1,j}+\tilde{G}_jk_{2,j}+\tilde{\tilde{G}}_jk_{3,j})\right]^{-1}\cdot\bar{N}a_o \tag{10}$$

where $\bar{y}=\begin{bmatrix}y_o\\y_1\\\vdots\\y_p\end{bmatrix}$, and $\bar{N}=\begin{bmatrix}N_o\\N_1\\\vdots\\N_p\end{bmatrix}$.

The relation in Eq. (10) gives an explicit expression for the closed-loop output behavior when a single load disturbance is introduced to the system. When the load disturbance is generated by a series of white noise occurring at every sampling instant, the closed-loop output y_t at sampling time t can be obtained, by the principle of superposition, as

$$y_t=\sum_{i=0}^{p}Y_i a_{t-i} \tag{11a}$$

where

$$\begin{bmatrix}Y_o\\Y_1\\\vdots\\Y_p\end{bmatrix}=\left[I+\sum_{j=1}^{n}(G_jk_{1,j}+\tilde{G}_jk_{2,j}+\tilde{\tilde{G}}_jk_{3,j})\right]^{-1}\cdot\bar{N} \tag{11b}$$

In deriving Eq. (11), it is assumed that the closed-loop output returns to its set point within the given horizon p, and hence Y_i=0 (i>p). For unstable or excessively sluggish systems, the expression in Eq. (11) will only give a truncated output response up to lag p. It is also worth noting that the inverse operation in Eq. (11b) is greatly simplified, since the matrix being inverted has a special lower triangular structure.

The quadratic performance measure defined in Eq. (3) can now be evaluated by substituting the output expression in Eq. (11) into Eq. (3), and it is given by

$$
\begin{aligned}
J &= E(y_t^T Q y_t)\\
&= \mathrm{trace}\left[\left(\sum_{i=0}^{p} Y_i^T Q Y_i\right)\cdot \Sigma_a\right]\\
&= \mathrm{trace}\left[\overline{N}^T\left\{I+\sum_{j=1}^{n}(G_j^T k_{1,j}+\tilde{G}_j^T k_{2,j}+\tilde{\tilde{G}}_j^T k_{3,j})\right\}^{-1}\overline{Q}\right.\\
&\quad \left.\times\left\{I+\sum_{j=1}^{n}(G_j k_{1,j}+\tilde{G}_j k_{2,j}+\tilde{\tilde{G}}_j k_{3,j})\right\}^{-1}\overline{N}\cdot\Sigma_a\right] \qquad (12)
\end{aligned}
$$

where $\overline{Q}$ is the (p+1)-block diagonal matrix of Q. The quadratic cost function is now expressed as an explicit function of multiloop PID tuning parameters in Eq. (12), and it provides a more convenient way of evaluating the quadratic cost function over the previous approaches, especially when the number of loops involved in the system increases. For n×n processes, there are 3n adjustable tuning parameters for the multiloop PID controllers. By minimizing the quadratic cost function with respect to these tuning parameters, the best achievable multiloop PID control performance can be found.

EXTENSIONS

The multiloop PID control performance assessment procedure proposed in the previous section is capable of providing performance benchmarks for each individual output variable as well as the overall performance bound. It is demonstrated in this section that the proposed method can also be readily extended to some important performance assessment issues such as (1) plant-wide variability analysis, (2) achievable performance improvement with decouplers, and (3) effects of controller pairing on achievable performance.

1. Plant-wide Variability Analysis

As mentioned by Qin [1998], it is desirable to transfer variability from key process output variables to other output variables that are less critical in process operations in multiloop control systems. In this case, one important question is “What is the lowest achievable variance of the key process variable under the constraint that the variances of other less critical variables are bounded by some prescribed values?” The answer to this question can be obtained by solving an optimization problem where the cost function is the variance of key process variable and the optimization is subject to variance inequality constraints on other less critical process variables.

2. Achievable Performance Improvement with Decouplers

Multiloop control systems often exhibit adverse loop interactions and cause the achievable multiloop control performance to be unacceptable. When severe loop interactions occur, one might want to implement decouplers to compensate for the effect of loop interactions brought about by cross-coupling of the process variables. The calculation of the potential performance improvement with decouplers is helpful in designing control systems, and it can be quantified as demonstrated in the example section below.

3. Effects of Loop Pairing on Achievable Performance

When the best achievable multiloop control performance is unacceptably poor with a given pairing of controlled and manipulated variables, one can consider different loop pairings. A more effective controller pairing could result in a big improvement in achievable multiloop control performance. The extent of such improvement in achievable multiloop control performance can also be assessed by using the process transfer function matrix for a new controller pairing.

AN EXAMPLE

Consider the following 2×2 multivariable process used by Huang et al. [1997].

$$
H(q^{-1})=\begin{bmatrix}\dfrac{q^{-1}}{1-0.4q^{-1}} & \dfrac{K_{12}q^{-2}}{1-0.1q^{-1}}\\[2ex] \dfrac{0.3q^{-1}}{1-0.1q^{-1}} & \dfrac{q^{-2}}{1-0.8q^{-1}}\end{bmatrix} \qquad (13)
$$

We assume that the process transfer function above is known from previous identification experiments. The disturbance sequence entering at the process output was generated from the disturbance model below:

$$
N(q^{-1})=\begin{pmatrix}1 & -0.6\\ 0.5 & 1\end{pmatrix}\cdot\begin{pmatrix}1-q^{-1} & 0\\ 0 & 1-q^{-1}\end{pmatrix}^{-1} a_t \qquad (14)
$$

where a_t is a two-dimensional normally distributed white noise with Σ_a=I. A multiloop PID controller was implemented to reject the disturbance, and its settings are given by

$$
\tilde{C}(q^{-1})=\mathrm{diag}\left(\frac{1-0.9q^{-1}+0.1q^{-2}}{1-q^{-1}},\frac{0.21-0.2q^{-1}+0.01q^{-2}}{1-q^{-1}}\right)
$$

Closed-loop simulations were performed with this multiloop controller, and two thousand multivariate observations were collected from the simulation for the performance assessment of the installed multiloop PID controller. Multivariate time-series analysis was then carried out on the closed-loop outputs, and a disturbance model $\hat{N}(q^{-1})$ was identified from the following relation.

$$
\hat{N}(q^{-1})=[I+H(q^{-1})\tilde{C}(q^{-1})]\cdot G_{CL}(q^{-1})
$$

The quadratic performance measure in this example is $J=E(y_t^T y_t)$, *i.e.*, equal weighting on each output variable.

Fig. 1 shows results of multiloop PID performance benchmark estimates (circled symbol) along with theoretical ones (solid line) as a function of the process model parameter K_{12}. In this plot, the multivariable minimum variance performance bounds are also drawn for comparison. As the interaction increases (i.e., as K_{12} increases), Fig. 1 shows that the achievable multiloop PID performance deteriorates gradually, while that of multivariable minimum variance control remains almost unchanged due to its perfect interaction compensation.

The closed-loop performance using the installed multiloop PID

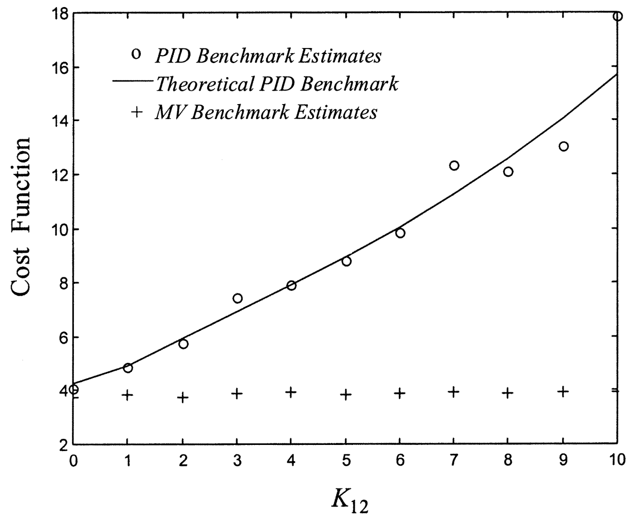

Fig. 1. Achievable performance bounds with increased process interactions.

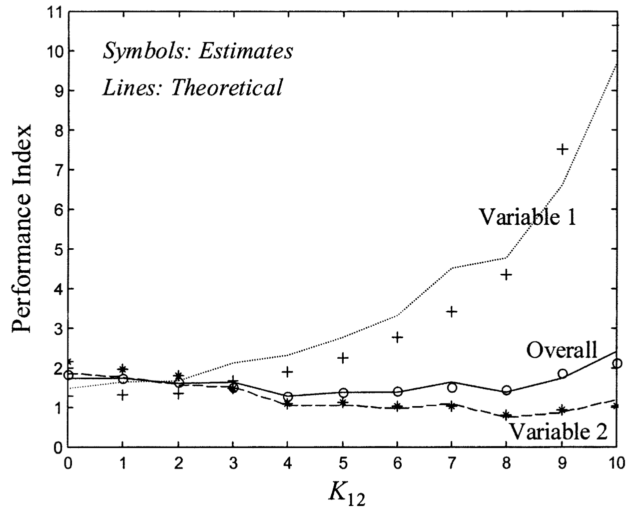

Fig. 2. Performance assessment of the installed multiloop PID controller.

controller is shown in Fig. 2 in terms of overall and individual performance indices. The performance indices in this case are defined as the ratio of the output variance (or quadratic cost function) of the installed multiloop PID controller to the multiloop PID benchmark performance. It is seen from this figure that the output variance of variable 1 deteriorates with the increased interaction, while that of variable 2 remains relatively constant as K_{12} increases.

1. Plant-wide Variability Analysis

Assume that we are mainly interested in the variance reduction of variable 2 while keeping the variance of variable 1 less than or equal to 3.0. We wish to obtain the best achievable performance in variable 2 under the condition $E(y_1^2)\leq 3$. This task is easily handled by choosing the cost function as $E(y_2^2)$ and adding the inequality constraint $E(y_1^2)\leq 3$ to the optimization problem. When $K_{12}=5$, the constrained optimization resulted in an improvement in achievable output variance of variable 2 from 7.30 to 5.98, while the variance of variable 1 increased from 1.64 to 3.0.

2. Achievable Performance Improvement with Decouplers

Again consider the case where $K_{12}=5$, which is the case with considerable process interactions. It can be seen from the inspection of the process model in Eq. (13) that the effect of (1, 2) element of process transfer function can be completely eliminated using an exact decoupler $I_{12}=-5(q^{-1}-0.4q^{-2})/(1-0.1q^{-1})$, while the effect of (2, 1) element is best eliminated by using a decoupler $I_{21}=-0.03\,(1-0.8q^{-1})/(1-0.1q^{-1})$. The causal decoupler I_{21} cannot remove the cross-coupling term entirely, but it results in a term $0.3q^{-1}$ in the (2,1) element of the process transfer function matrix. The effective process transfer function with the decouplers above becomes

$$\tilde{H}(q^{-1})=\begin{bmatrix} H_{11}+H_{12}I_{21} & H_{12}+H_{11}I_{12} \\ H_{21}+H_{22}I_{21} & H_{22}+H_{21}I_{12} \end{bmatrix}=\begin{bmatrix} H_{11}+H_{12}I_{21} & 0 \\ 0.3q^{-1} & H_{22}+H_{21}I_{12} \end{bmatrix}$$

where H_{ij} is the (i, j) element of the process transfer function matrix. Achievable performance improvement with decouplers can then be obtained by replacing the process transfer function with $\tilde{H}(q^{-1})$ and optimizing the corresponding cost function. Table 1 shows the potential improvement in both overall and individual performance measures by incorporating decouplers into the multiloop control system.

3. Effects of Loop-pairing on Achievable Performance

Consider the following discretized process transfer function (based on a continuous model used by Gagnepain and Seborg [1982]):

$$H(q^{-1})=\begin{bmatrix} \dfrac{-0.190q^{-2}}{1-0.905q^{-1}} & \dfrac{0.948q^{-2}}{1-0.368q^{-1}} \\ \dfrac{0.948q^{-2}}{1-0.368q^{-1}} & \dfrac{0.190q^{-2}}{1-0.905q^{-1}} \end{bmatrix}$$

Table 1. Potential performance improvement with decouplers

	Quadratic performance measure	Output variance of variable 1	Output variance of variable 2
Without decouplers	8.94	1.64	7.30
With decouplers (%reduction)	7.13 (20.2%)	1.55 (5.5%)	5.58 (23.6%)

Table 2. Effects of different controller pairing

Controller pairing	Overall performance measure	Performance measure of variable 1	Performance measure of variable 2
1-1/2-2	14.57	7.30	7.27
1-2/2-1	6.78	3.26	3.52

With this process model, the relative gain array analysis indicates that 1-1/2-2 pairing is favorable with the relative gain λ_{11}= 0.64. However, the diagonal time constants in the process transfer function matrix are significantly larger than off-diagonal time constants. Thus there exists a conflict between steady-state and dynamic considerations.

Let us assume that the stochastic load disturbance model given in Eq. (14) was identified. We want to assess the best achievable performance for each controller pairing with the current disturbance characteristics. Table 2 shows the effects of different controller pairing on achievable performance. From this table, we can see that 1-2/2-1 pairing is potentially advantageous over 1-1/2-2 pairing, and the variability in the process outputs can be reduced significantly by employing the 1-2/2-1 configuration. This result coincides with that of Gagnepain and Seborg [1982], where deterministic step changes were used in evaluating the performance of each control configuration.

CONCLUSIONS

In this paper, an optimization approach for the estimation of best achievable quadratic performance under a multiloop PID controller was suggested. To evaluate the quadratic cost function, an analytic expression was derived for the closed-loop impulse response under a multiloop PID control. The proposed performance assessment procedure utilizes the knowledge on a process model and finds the best achievable performance under a multiloop PID control scheme. The proposed method was then extended to some important performance assessment issues such as (1) plant-wide variability analysis, (2) achievable performance improvement with decouplers, and (3) effects of controller pairing on achievable performance. A simulated example was employed to quantify the effects of process interactions on achievable multiloop PID control performance and the changes in individual loop performance.

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