Dynamics of cascade three-level system interacting with the classical and quantized field

MIHIR RANJAN NATH[1], SURAJIT SEN[1] and GAUTAM GANGOPADHYAY[2]

[1]Department of Physics, Guru Charan College, Silchar 788 004, India
[2]S N Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata 700 098, India
Email: mrnath_95@rediffmail.com; ssen55@yahoo.com; gautam@bose.res.in

Abstract. We study the exact solutions of the cascade three-level atom interacting with a single mode classical and quantized field with different initial conditions of the atom. For the semiclassical model, it is found that if the atom is initially in the middle level, the time-dependent populations of the upper and lower levels are always equal. This dynamical symmetry exhibited by the classical field is spoiled on quantization of the field mode. To reveal this non-classical effect, a Euler matrix formalism is developed to solve the dressed states of the cascade Jaynes–Cummings model (JCM). Possible modification of such an effect on the collapse and revival phenomenon is also discussed by taking the quantized field in a coherent state.

Keywords. Symmetry breaking; three-level JCM; Euler matrix; collapse revival.

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1. Introduction

Over the decades, studies of the population inversion of the two, three and multilevel systems have been proved to be an important tool to understand various fundamental aspects of quantum optics [1,2]. Many interesting coherent phenomena are observed if the number of involved levels exceeds two. In particular, the three-level system exhibits a rich class of coherent phenomena such as two-photon coherence [3], double resonance process [4], three-level super-radiance [5], coherent multistep photo-ionization [6], trilevel echoes [7], STIRAP [8], resonance fluorescence [9], quantum jump [10], quantum zero effect [11] etc. [12–16]. From these studies, it is intuitively clear that the atomic initial conditions of the three-level system can generate diverse quantum optical effects which are not usually displayed by a two-level system [17–20]. The idea of the present investigation is to enunciate the three-level system for various initial conditions while taking the field mode to be either classical or quantized. In this paper the three-level system is modelled by the matrices which are spin-one representation of $SU(2)$ group. A dressed-atom approach is developed where the Euler matrix is used to construct the dressed states. We discuss the time

development of the probabilities both for the semiclassical model and the cascade JCM for various initial conditions and point out the crucial changes. Finally the collapse and revival phenomenon is presented taking the quantized field initially in a coherent state.

The subsequent sections of the paper are organized as follows: To put our treatment in proper perspective, in §2 we have derived the probabilities of three levels taking the field as a classical field. The cascade JCM and its solution in the rotating wave approximation (RWA) is presented in §3. In §4 we have numerically analysed the time-dependent atomic populations and compared with the semiclassical situation by taking the quantized field initially in a number state and in a coherent state. Finally, in conclusion, we highlight the outcome of our paper and make some pertinent remarks.

2. The semiclassical model

The Hamiltonian to describe the semicalssical problem of a cascade three-level system interacting with a single mode classical field is

$$\mathcal{H} = \hbar\omega_0 I_z + \frac{\hbar\omega_1}{\sqrt{2}}(I_+ e^{-i\omega t} + I_- e^{i\omega t}), \tag{1}$$

where Is represent the spin-one representation of $SU(2)$ matrices corresponding to the cascade three-level system with equal energy gaps ($\hbar\omega_0$) between the states, namely,

$$I_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2a}$$

$$I_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2b}$$

$$I_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2c}$$

$\hbar\omega_1$ is the interaction energy between the three-level system with the classical field mode of frequency, ω, in RWA. Let the solution of the Schrödinger equation,

$$i\hbar\left[\frac{\partial\psi}{\partial t}\right] = \mathcal{H}\psi, \tag{3}$$

with Hamiltonian (1) is given by

$$\psi(t) = C_+(t)|+\rangle + C_0(t)|0\rangle + C_-(t)|-\rangle, \tag{4}$$

where $C_+(t)$, $C_0(t)$ and $C_-(t)$ are the time-dependent normalized amplitudes with the eigenfunctions given by

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{5a}$$

$$|0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{5b}$$

$$|-\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5c}$$

We now proceed to calculate the probability amplitudes of the three states. Substituting eq. (4) into eq. (3) and equating the coefficients of $|+\rangle$, $|0\rangle$ and $|-\rangle$ from both sides we obtain

$$i\dot{C}_+(t) = \omega_0 C_+(t) + \frac{\omega_1}{\sqrt{2}} \exp(-i\omega t) C_0(t), \tag{6a}$$

$$i\hbar \dot{C}_0(t) = \frac{\omega_1}{\sqrt{2}} \exp(i\omega t) C_+(t) + \frac{\omega_1}{\sqrt{2}} \exp(-i\omega t) C_-(t), \tag{6b}$$

$$i\hbar \dot{C}_-(t) = \frac{\omega_1}{\sqrt{2}} \exp(i\omega t) C_0(t) - \omega_0 C_-(t), \tag{6c}$$

where the dot represents the derivative with respect to time.

Let the solutions of eqs (6a)–(6c) are of the following form:

$$C_+(t) = A_+ \exp(is_+ t), \tag{7a}$$

$$C_0(t) = A_0 \exp(is_0 t), \tag{7b}$$

$$C_-(t) = A_- \exp(is_- t), \tag{7c}$$

where As are the time-independent constants to be determined. Plucking back eqs (7a)–(7c) in eqs (6a)–(6c) we obtain

$$(s_0 - \omega + \omega_0)A_+ + \frac{1}{\sqrt{2}}\omega_1 A_0 = 0, \tag{8a}$$

$$s_0 A_0 + \frac{1}{\sqrt{2}}\omega(A_+ + A_-) = 0, \tag{8b}$$

$$(s_0 + \omega - \omega_0)A_- + \frac{1}{\sqrt{2}}\omega_1 A_0 = 0. \tag{8c}$$

In deriving eqs (8a)–(8c), the time independence of the amplitudes A_+, A_- and A_0 are ensured by invoking the conditions $s_+ = s_0 - \omega$ and $s_- = s_0 + \omega$. The solution of (8) readily yields

$$s_0 = 0 \tag{9a}$$

$$s_0 = \pm\sqrt{[(\omega - \omega_0)^2 + \omega_1^2]}(\equiv \pm\Omega) \tag{9b}$$

and we have three values of s_+ and s_-, namely,

$$s_+^1 = -\omega, \quad s_+^2 = \Omega - \omega, \quad s_+^3 = -\Omega - \omega \tag{10a}$$

$$s_-^1 = \omega, \quad s_-^2 = \Omega + \omega, \quad s_-^3 = -\Omega + \omega. \tag{10b}$$

Using (10), eqs (7) can be written as

$$C_+(t) = A_+^1 \exp[-i\omega t] + A_+^2 \exp[i(\Omega - \omega)t] + A_+^3 \exp[i(-\Omega - \omega)t] \tag{11a}$$

$$C_0(t) = A_0^1 + A_0^2 \exp(i\Omega t) + A_0^3 \exp(-i\Omega t) \tag{11b}$$

$$C_-(t) = A_-^1 \exp(i\omega t) + A_-^2 \exp[i(\Omega + \omega)t] + A_-^3 \exp[i(-\Omega + \omega)t]. \tag{11c}$$

where As are the constants to be calculated from the following initial conditions:

Case I: Let us consider at $t = 0$, the atom is in the lower level, i.e., $C_+(0) = 0$, $C_0(0) = 0$, $C_-(0) = 1$. Using eqs (6) and (11), the time-dependent probabilities of the three levels are given by

$$|C_+(t)|^2 = \frac{\omega_1^4}{\Omega^4} \sin^4 \Omega t/2, \tag{12a}$$

$$|C_0(t)|^2 = \frac{\omega_1^2}{2\Omega^4}[4(\omega - \omega_0)^2 \sin^4 \Omega t/2 + \Omega^2 \sin^2 \Omega t], \tag{12b}$$

$$|C_-(t)|^2 = \frac{1}{\Omega^4}[(\omega_1^2 \sin^2 \Omega t/2 + \Omega^2 \cos \Omega t)^2 + (\omega - \omega_0)^2 \Omega^2 \sin^2 \Omega t]. \tag{12c}$$

Case II: If we choose the atom initially in the middle level, i.e., $C_+(0) = 0$, $C_0(0) = 1$, $C_-(0) = 0$, the corresponding probabilities of the levels are given by

$$|C_+(t)|^2 = \frac{\omega_1^2}{2\Omega^4}[4(\omega - \omega_0)^2 \sin^4 \Omega t/2 + \Omega^2 \sin^2 \Omega t] = |C_-(t)|^2, \tag{13a}$$

$$|C_0(t)|^2 = \frac{4(\omega - \omega_0)^4}{\Omega^4} \sin^4 \Omega t/2 + \frac{4(\omega - \omega_0)^2}{\Omega^2} \sin^2 \Omega t/2 \cos \Omega t + \cos^2 \Omega t. \tag{13b}$$

Here we note that, unlike the previous case, the probabilities of the upper and lower levels are equal.

Case III: When the atom is initially in the upper level, i.e., $C_+(0) = 1$, $C_0(0) = 0$, $C_-(0) = 0$, we obtain the following occupation probabilities in the three levels:

$$|C_+(t)|^2 = \frac{1}{\Omega^4}[(\omega_1^2 \sin^2 \Omega t/2 + \Omega^2 \cos \Omega t)^2 + (\omega - \omega_0)^2 \Omega^2 \sin^2 \Omega t], \qquad (14a)$$

$$|C_0(t)|^2 = \frac{\omega_1^2}{2\Omega^4}[4(\omega - \omega_0)^2 \sin^4 \Omega t/2 + \Omega^2 \sin^2 \Omega t], \qquad (14b)$$

$$|C_-(t)|^2 = \frac{\omega_1^4}{\Omega^4} \sin^4 \Omega t/2. \qquad (14c)$$

We note that the probability of the middle level for Case III is precisely identical to that of Case I while those of the upper and lower levels are interchanged.

3. Cascade Jaynes–Cummings model

Here we consider the cascade three-level system interacting with a single mode quantized field. The cascade JCM system in the rotating wave approximation [17,18] is described by the Hamiltonian

$$H = \hbar\omega(a^\dagger a + I_z) + (\Delta I_z + g\hbar(I_+ a + I_- a^\dagger)), \qquad (15)$$

where $a^\dagger$ and a are the creation and annihilation operators, g the coupling constant and $\Delta = \hbar(w_0 - w)$ the detuning frequency. It is easy to check that both diagonal and interaction parts of the Hamiltonian commute with each other. The eigenfunction of this Hamiltonian is given by

$$|\psi_n(t)\rangle = \sum_{n=0}^{\infty} [C_-^{n+1}(t)|n+1,-\rangle + C_0^n(t)|n,0\rangle + C_+^{n-1}(t)|n-1,+\rangle]. \qquad (16)$$

We note that the Hamiltonian couples the atom-field states $|n-1,+\rangle$, $|n,0\rangle$ and $|n+1,-\rangle$, where n represents the number of photons of the field. The interaction part of the Hamiltonian (15) can also be written in the matrix form

$$H = \begin{bmatrix} -\Delta & g\hbar\sqrt{n+1} & 0 \\ g\hbar\sqrt{n+1} & 0 & g\hbar\sqrt{n} \\ 0 & g\hbar\sqrt{n} & \Delta \end{bmatrix}. \qquad (17)$$

At resonance ($\Delta = 0$), the eigenvalues of the Hamiltonian are given by $\lambda_+ = g\hbar\sqrt{2n+1}$, $\lambda_0 = 0$ and $\lambda_- = -g\hbar\sqrt{2n+1}$ with the corresponding dressed eigenstates

$$\begin{bmatrix} |n,1\rangle \\ |n,2\rangle \\ |n,3\rangle \end{bmatrix} = T \begin{bmatrix} |n+1,-\rangle \\ |n,0\rangle \\ |n-1,+\rangle \end{bmatrix}. \qquad (18)$$

In eq. (18), the dressed states are constructed by rotating the bare states with the Euler matrix T parametrized as

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}, \qquad (19)$$

where

$$\alpha_{11} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi,$$
$$\alpha_{12} = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi,$$
$$\alpha_{13} = \sin\psi\sin\theta,$$
$$\alpha_{21} = -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi,$$
$$\alpha_{22} = -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi,$$
$$\alpha_{23} = \cos\psi\sin\theta,$$
$$\alpha_{31} = \sin\theta\sin\phi,$$
$$\alpha_{32} = -\sin\theta\cos\phi,$$
$$\alpha_{33} = \cos\theta.$$

The evaluation of its various elements is presented in the appendix and here we quote the results as follows:

$$\alpha_{11} = \sqrt{\frac{n+1}{4n+2}}, \quad \alpha_{12} = \frac{1}{\sqrt{2}}, \quad \alpha_{13} = \sqrt{\frac{n}{4n+2}},$$

$$\alpha_{21} = -\sqrt{\frac{n}{2n+1}}, \quad \alpha_{22} = 0, \quad \alpha_{23} = \sqrt{\frac{n+1}{2n+1}},$$

$$\alpha_{31} = \sqrt{\frac{n+1}{4n+2}}, \quad \alpha_{32} = -\frac{1}{\sqrt{2}}, \quad \alpha_{33} = \sqrt{\frac{n}{4n+2}}. \tag{20}$$

The time-dependent probability amplitudes of the three levels are given by

$$\begin{bmatrix} C_-^{n+1}(t) \\ C_0^n(t) \\ C_+^{n-1}(t) \end{bmatrix} = T^{-1} \begin{bmatrix} e^{-i\Omega_n t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\Omega_n t} \end{bmatrix} T \begin{bmatrix} C_-^{n+1}(0) \\ C_0^n(0) \\ C_+^{n-1}(0) \end{bmatrix}, \tag{21}$$

where $\Omega_n = g\sqrt{2n+1}$. In the following we consider different initial condition of the atom with the quantized field in a number state $|n\rangle$.

Case IV: Here we consider that the atom is initially polarized in the lower level and the combined atom-field state is $|n+1,-\rangle$, i.e., $C_+^{n-1}(0) = 0$, $C_0^n(0) = 0$, $C_-^{n+1}(0) = 1$. Using eqs (20) and (21) the time-dependent atomic population of the three levels are given by

$$\left|C_+^{n-1}(t)\right|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^4 \Omega_n t/2, \tag{22a}$$

$$\left|C_0^n(t)\right|^2 = \frac{(n+1)}{(2n+1)} \sin^2 \Omega_n t, \tag{22b}$$

$$\left|C_-^{n+1}(t)\right|^2 = 1 - 4\left[\frac{n(n+1)}{(2n+1)^2} + \frac{(n+1)^2}{(2n+1)^2} \cos^2 \Omega_n t/2\right] \sin^2 \Omega_n t/2. \tag{22c}$$

Case V: At $t = 0$ when the atom is in the middle level and the combined atom-field state is $|n,0\rangle$, i.e., $C_+^{n-1}(0) = 0$, $C_0^n(0) = 1$, $C_-^{n+1}(0) = 0$, we find

$$\left|C_+^{n-1}(t)\right|^2 = \frac{n}{(2n+1)} \sin^2 \Omega_n t, \tag{23a}$$

$$|C_0^n(t)|^2 = \cos^2 \Omega_n t \tag{23b}$$

$$\left|C_-^{n+1}(t)\right|^2 = \frac{(n+1)}{(2n+1)} \sin^2 \Omega_n t. \tag{23c}$$

Case VI: $C_+^{n-1}(0) = 1$, $C_0^n(0) = 0$, $C_-^{n+1}(0) = 0$.

If the atom is initially in the upper level and the atom-field state is $|n-1,+\rangle$, i.e., $C_+^{n-1}(0) = 1$, $C_0^n(0) = 0$, $C_-^{n+1}(0) = 0$ we obtain the following probabilities:

$$\left|C_+^{n-1}(t)\right|^2 = 1 - 4\left[\frac{n(n+1)}{(2n+1)^2} + \frac{n^2}{(2n+1)^2}\cos^2 \Omega_n t/2\right] \sin^2 \Omega_n t/2, \tag{24a}$$

$$|C_0^n(t)|^2 = \frac{n}{(2n+1)} \sin^2 \Omega_n t, \tag{24b}$$

$$\left|C_-^{n+1}(t)\right|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^4 \Omega_n t/2. \tag{24c}$$

Finally we note that, at resonance, for large value of *n* the probabilities of Case IV, V and VI are identical to Case I, II and III, respectively indicating the validity of the correspondence principle.

4. Numerical results

To explore the physical content, we now proceed to analyse the probabilities of the semi-classical model and the cascade JCM numerically.

For the classical field at resonance, the time evolution of the probabilities $|C_+(t)|^2$ (solid line), $|C_0(t)|^2$ (dashed line) and $|C_-(t)|^2$ (dotted line) corresponding to Case I, II and III, respectively are shown in figure 1. We note that for the cases with atom initially in the lower and upper level, which are displayed in figure 1a and 1c respectively, the probabilities $|C_+(t)|^2$ and $|C_-(t)|^2$ can attain a maximum value equal to unity while $|C_0(t)|^2$ cannot. If we compare these two figures, the time-dependent populations of the lower and upper levels are different by a phase lag corresponding to the initial condition of population. This clearly shows that the probabilities oscillate between the levels $|+\rangle$ and $|-\rangle$ alternatively at a Rabi frequency of $\nu_I = \omega_1/2\pi$. On the contrary, the plot of Case II where the atom is initially in the middle level depicted in figure 1b shows that the system oscillates with a Rabi frequency of $\nu_{II} = \omega_1/\pi$ such that the probabilities of $|+\rangle$ and $|-\rangle$ states are always equal. When the atom is initially in the middle level, the exactly sinusoidal resonant field interacts with the atom in such a way that the upper and lower levels are dynamically treated on an equal footing. This dynamically symmetrical distribution of population between the upper and lower levels is possible because of the classical field.

For quantized field we consider the time evolution of the probabilities in two different situations of initial condition of the field: (a) when the field is in a number state and (b) when the field is in a coherent state.
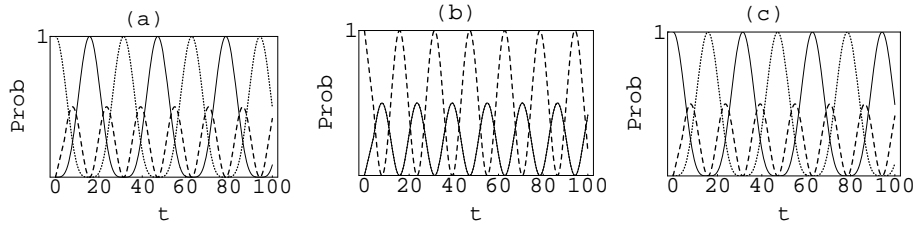
Figure 1. The time evolution of the probabilities of the semiclassical model corresponding to Cases I, II and III. The symmetric pattern of evolution is evident from figures 1a and 1c which are in opposite phase.
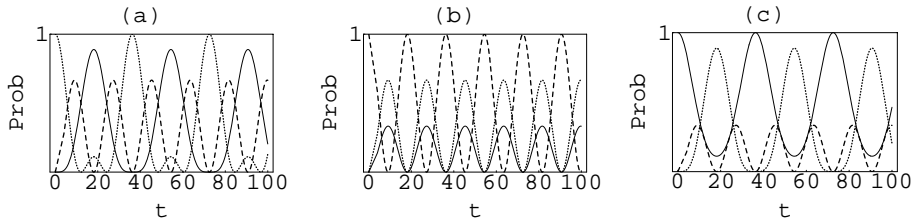

Figure 2. The time evolution of the probabilities of the cascade JCM corresponding to Cases IV, V and VI. Figures 2a and 2c depict that the symmetry exhibited by the semiclassical model is spoiled on quantization of the field mode.

(a) For the cascade JCM, the probabilities of Case IV, V and VI are plotted in figure 2 when the field is in a number state with $n = 1$ and $g = 0.1$. In figure 2a we note that for Case IV, i.e., when the atom is initially in the lower level, the Rabi frequency of oscillation is $v_I^n = \Omega_n/2\pi$. However, unlike Case I of the semiclassical model, the probabilities $|C_+^{n-1}(t)|^2$ never become unity. On the other hand, figure 2b illustrates the probabilities of Case V, i.e., when the atom is initially in the middle level, where the system oscillates with a Rabi frequency $v_{II}^n = \Omega_n/\pi$ and once again, in contrast with the corresponding semiclassical situation in Case II, the probabilities of the upper and lower level are not equal. The probabilities of Case VI, i.e., when the atom is initially in the upper level, depicted in figure 2c shows that although it possesses the same Rabi frequency v_I^n, the pattern of oscillation is not out of phase of Case IV. To compare with one can look back the semiclassical interaction where we have shown that in Case I the pattern of oscillation of upper (lower) level population is precisely identical to the lower (upper) level population of Case III.

To understand the implications of such dynamical symmetry breaking qualitatively, various bounds on the probabilities are given (see table 1).

We note that for the semiclassical model, the symmetric evolution of the probabilities results in identical bounds for Cases I and III as shown in table 1. On quantization of the field mode, the bounds corresponding to Cases IV and VI are no longer similar although those for Cases II and V remain the same. At resonance, for large values of n, eqs (22)–(24) of Cases IV, V and VI are precisely identical to eqs (12)–(14) of Case I, II and III, respectively and we recover the same bounds of the semiclassical model.

(b) Finally, we consider that the atom is interacting with the quantized field mode in a coherent state. The coherently averaged probabilities of Cases IV, V and VI are given by

Table 1.

Case	Semiclassical model	Case	Cascade JCM
I	$0 \leq \left\|C_-(t)\right\|^2 \leq 1,$ $0 \leq \left\|C_0(t)\right\|^2 < 1,$ $0 \leq \left\|C_+(t)\right\|^2 \leq 1$	IV	$0 \leq \left\|C_-^n(t)\right\|^2 \leq 1,$ $0 \leq \left\|C_0^n(t)\right\|^2 < 1,$ $0 \leq \left\|C_+^n(t)\right\|^2 < 1$
II	$0 \leq \left\|C_-(t)\right\|^2 < 1,$ $0 \leq \left\|C_0(t)\right\|^2 \leq 1,$ $0 \leq \left\|C_+(t)\right\|^2 < 1$	V	$0 \leq \left\|C_-^n(t)\right\|^2 < 1,$ $0 \leq \left\|C_0^n(t)\right\|^2 \leq 1,$ $0 \leq \left\|C_+^n(t)\right\|^2 < 1$
III	$0 \leq \left\|C_-(t)\right\|^2 \leq 1,$ $0 \leq \left\|C_0(t)\right\|^2 < 1,$ $0 \leq \left\|C_+(t)\right\|^2 \leq 1$	VI	$0 \leq \left\|C_-^n(t)\right\|^2 < 1,$ $0 \leq \left\|C_0(t)^n\right\|^2 < 1,$ $0 < \left\|C_-^n(t)\right\|^2 \leq 1$

$$\langle P_+(t) \rangle = \sum_n P_n \left| C_+^{n-1}(t) \right|^2, \tag{25a}$$

$$\langle P_0(t) \rangle = \sum_n P_n \left| C_0^n(t) \right|^2, \tag{25b}$$

$$\langle P_-(t) \rangle = \sum_n P_n \left| C_-^{n+1}(t) \right|^2, \tag{25c}$$

where $P_n = \exp[-\bar{n}]\bar{n}^n/n!$ be the Poisson distribution function and $\bar{n}$ be mean photon number. For all numerical purpose we choose $g = 0.1$. We have studied extensively for various values of $\bar{n}$. Figures are given only for $\bar{n} = 50$. Figures 3–5 display the numerical plots of (25) where the collapse and revival of the Rabi oscillation is clearly evident. For low $\bar{n}$ and when the atom is in the middle level the symmetrical values of population of upper and lower levels are not observed until $\bar{n}$ is very high as given in the figures. However, even if $\bar{n} = 50$, the numerical values of the time dependent populations of the upper and lower level are not exactly equal although very close and becomes exactly equal in the limit $\bar{n} \to \infty$. We further note that, if the atom is initially polarized either in the upper or in the lower level, it exhibits similar population oscillation, which is different from the case if it is initially polarized in the middle level. The reproduction of this result analogous to the semiclassical model shows the proximity of the coherent state with large $\bar{n}$ to the classical field.

When the field is quantized, population oscillation depends on the occupation number, n, of the field state, for example, $\cos\left(g\sqrt{2n+1}t\right)$. For a statistical distribution of field state, the spontaneous factor 1 plays a dominant role when n is low. For an initial number state of the field when n is slightly higher than 1, the upper and lower levels of the atom are not treated dynamically on an equal footing even when the atom is initially in the middle level. This fine graining of the quantized distribution of photons over the number states $\{|n\rangle\}$ generates a complex interference between individual Rabi oscillation corresponding to each n and plays a role until n is very large compared to 1 and effectively acts as a classical field and thereby the semiclassical situation is satisfied. Note that for an initial vacuum field, i.e., $n = 0$ for the number state and $\bar{n} = 0$ for the coherent state, with the atom initially in the middle level, it cannot go to the upper level at all and the population will oscillate between the lower and middle levels with Rabi frequency Ω_0. This asymmetry is still present when the field is in a coherent state with a Poissonian photon distribution with low average photon number, $\bar{n}$, which is generally not symmetric around $\bar{n}$. A Poisson
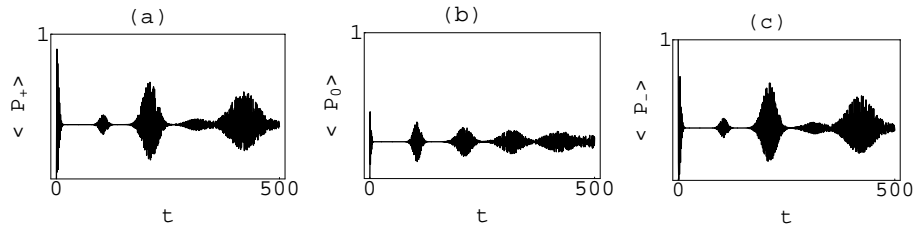
Figure 3. The collapse and revival are shown for Case IV when the atom is initially in the upper level. The time-dependent profiles of the upper and lower level populations are similar in figures 3a and 3c.
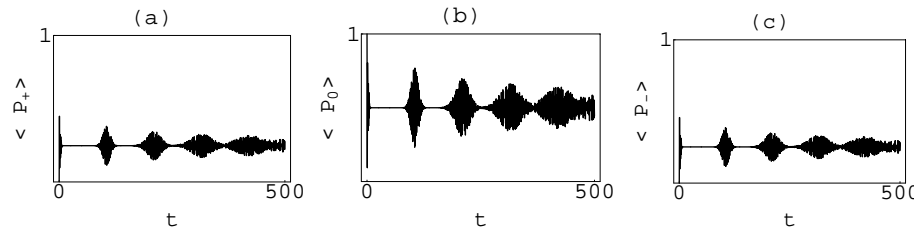

Figure 4. The collapse and revival for Case V are displayed when the atom is initially in the middle level. The time-dependent patterns of the upper and lower level populations are similar in figures 4a and 4c as in the semiclassical cases of figure 1b.
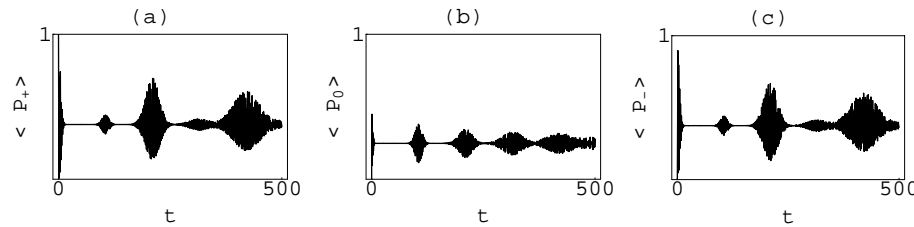

Figure 5. The collapse and revival for Case VI are shown when the atom is initially polarized in the lower level. Similar to figures 1a and 1c of the semiclassical model, figures 3 and 5 are closely alike.

distribution is almost symmetric, a Gaussian, around an $\bar{n}$ if $\bar{n}$ is very large which is the case of a classical field. In that situation the upper and lower levels of the atom are treated dynamically on an equal footing and maintains the symmetrical distribution of population in upper and lower levels.

5. Conclusion

We conclude by recapitulating the essential content of our investigation. At the outset we have sculpted the semiclassical model by choosing the spin-one representation of $SU(2)$ group and have calculated the transition probabilities of the three levels. It is shown that at resonance, if the atom is initially polarized in the lower or in the upper level, the various atomic populations oscillate quite differently when it is initially populated in the middle level. When the atom is initially populated in the middle level, the classical field interacts in such a way that the populations of the upper and lower levels are always equal. This

dynamically symmetrical populations of the upper and lower levels are destroyed due to the quantization of the field. To show this quantum behavior, a dressed-atom approach is presented to solve the cascade JCM. Finally we discuss the restoration of the symmetry taking the quantized field in a coherent state with large average photon number, the closest state to the classical state. Although the collapse and revival and some other non-classical features are well-studied in the context of two-level systems, the above dynamical breaking of symmetry due to the quantization of the field has no two-level analog. We hope that this dynamical behavior in the cascade three-level system should show its signature on the time-dependent profile of the second-order coherence of the quantized field which will be discussed elsewhere. The dressed-atom approach developed here may also find its application in the V- and Λ-type three-level systems where the nature of the symmetry should be different from the cascade system.

Appendix

At resonance, the interaction part of the Hamiltonian of the three-level system is given by

$$H_{\text{int}} = \begin{bmatrix} 0 & g\sqrt{n+1} & 0 \\ g\sqrt{n+1} & 0 & g\sqrt{n} \\ 0 & g\sqrt{n} & 0 \end{bmatrix}, \tag{A1}$$

where the eigenvalues are $\lambda_+ = g\sqrt{2n+1}$, $\lambda_0 = 0$ and $\lambda_- = -g\sqrt{2n+1}$. The Euler matrix T, diagonalizes the Hamiltonian as $H_D = TH_{\text{int}}T^{-1}$, is given by eq. (19). Using the trick $(H_{\text{int}} - \lambda_j I)\{X_j\} = 0$, where $\{X_j\}$ is the column matrix of T^{-1}, corresponding to the eigenvalue λ_+ we have

$$\begin{bmatrix} -g\sqrt{2n+1} & g\sqrt{n+1} & 0 \\ g\sqrt{n+1} & -g\sqrt{2n+1} & g\sqrt{n} \\ 0 & g\sqrt{n} & -g\sqrt{2n+1} \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{bmatrix} = 0. \tag{A2}$$

These linear equations readily yield

$$\alpha_{12} = \frac{\sqrt{2n+1}}{\sqrt{n}}\alpha_{13}, \quad \alpha_{12} = \frac{\sqrt{2n+1}}{\sqrt{n+1}}\alpha_{11} \quad \text{and} \quad \alpha_{11} = \frac{\sqrt{n+1}}{\sqrt{n}}\alpha_{13}. \tag{A3}$$

Using the normalization condition

$$\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1, \tag{A4}$$

we get $\alpha_{11} = \sqrt{(n+1)/(4n+2)}$, $\alpha_{12} = \sqrt{(2n+1)/(4n+2)} = 1/\sqrt{2}$ and $\alpha_{13} = \sqrt{n/(4n+2)}$. Similarly, corresponding to the eigenvalues λ_0 and λ_- we can obtain other elements of T, namely,

$$\alpha_{21} = -\sqrt{\frac{n}{2n+1}}, \quad \alpha_{22} = 0 \quad \text{and} \quad \alpha_{23} = \sqrt{\frac{n+1}{2n+1}}, \tag{A5}$$

$$\alpha_{31} = \sqrt{\frac{n+1}{4n+2}}, \quad \alpha_{32} = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \alpha_{33} = \sqrt{\frac{n}{4n+2}}. \tag{A6}$$

One can now easily read off the Euler's angles

$$\sin\theta = \sqrt{\frac{3n+2}{4n+2}}, \quad \sin\phi = \sqrt{\frac{n+1}{3n+2}} \quad \text{and} \quad \sin\psi = \sqrt{\frac{n}{3n+2}}. \tag{A7}$$

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