Sādhanā Vol. 29, Part 4, August 2004, pp. 381–387. © Printed in India

Stokes flow past a swarm of porous circular cylinders with Happel and Kuwabara boundary conditions

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MS received 28 February 2003; revised 27 November 2003

Abstract. The problem of creeping flow past a swarm of porous circular cylinders with Happel and Kuwabara boundary conditions is investigated. The Brinkman equation for the flow inside the porous cylinder and the Stokes equation outside the porous cylinder in their stream function formulations are used. The force experienced by each porous circular cylinder in a cell is evaluated. Explicit expressions of stream functions are obtained for both the inside and outside flow fields. The earlier results reported by Happel and Kuwabara for flow past a solid cylinder in Happel's and Kuwabara's cell model, have been deduced. Analytical expressions for the velocity components, pressure, vorticity and stress- tensor are also obtained.

Keywords. Particle-in-cell model; modified Bessel functions; stress; vorticity; Brinkman equation.

1. Introduction

Flow through a swarm of porous particles arises in many important processes such as flow in sand beds, in petroleum reservoir rocks, in aloxite materials, in flow sedimentation etc. (Qin & Kaloni 1993). These problems can be easily solved by using the cell model technique. In this technique, it is assumed that each particle is surrounded by a fluid envelope (or cell) and all the disturbances due to each particle are confined to the envelope (Happel & Brenner 1983). The fluid envelope is assumed to contain the same volumetric proportion of fluid to solid as exists in the entire assemblage.

Happel (1959) and Kuwabara (1959) proposed a cell model in which two concentric cylinders serve as the model for fluid moving through an assemblage of circular cylinders. These authors solved the problem when the inner cylinder is solid with respective boundary conditions on the cell surface. The Happel model assumes uniform velocity condition and no tangential stress at the cell surface, whereas, the Kuwabara model assumes vanishing of vorticity in place of no tangential stress. An analytical study of the steady incompressible flow past a circular cylinder embedded in a porous medium based on the Brinkman model has been reported by Pop & Cheng (1992).

In the present work, we shall extend the idea of Datta & Deo (2002) for flow past a swarm of porous circular cylinders with Happel and Kuwabara boundary conditions. The Brinkman equation (Brinkman 1947) for the flow inside the porous cylinder and the Stokes equation outside the porous cylinder in their stream function formulations are used. As boundary conditions, continuity of velocity and surface stresses at the porous cylinders are employed. On the cell surface, Happel and Kuwabara boundary conditions are used. The force experienced by each porous circular cylinder in a cell is evaluated. The earlier results reported by Happel (1959) and Kuwabara (1959), for flow past a solid cylinder in their cell models, have been deduced.

2. Mathematical formulations

The governing equations for the steady slow motion of fluids relative to assemblage of porous circular cylinders must be written for two regions: For the outside region of porous cylinders, namely (1), we assume that flow are governed by Stokes equation (Happel & Brenner 1983)

$$\mu^{(1)}\nabla^2\mathbf{v}^{(1)} - \text{grad}\ p^{(1)} = 0, \quad \text{div}\ \mathbf{v}^{(1)} = 0. \tag{1}$$

For the inside region (2), occupied by the porous cylinder, we use the Brinkman's equation (Zlatanovski 1999)

$$\nabla^2\ \mathbf{v}^{2)} - K^2\mathbf{v}^{(2)} = (1/\mu^{(2)})\ \text{grad}\ p^{(2)}, \quad \text{div}\ \mathbf{v}^{(2)} = 0. \tag{2}$$

Here, $K^2 = \beta/k$ with $\beta = \mu^{(1)}/\mu^{(2)}$, $\mu^{(1)}$ the viscosity of fluid, $\mu^{(2)}$ denotes the effective viscosity of porous medium, k being the permeability of porous medium. The viscosity coefficients $\mu^{(1)}$ and $\mu^{(2)}$ are, in general, different. For high porosity material, it is assumed that $\mu^{(1)} = \mu^{(2)}$. Also, the velocity vector and pressure in the two regions are denoted by $\mathbf{v}^{(i)}$ and $p^{(i)}$, $i = 1, 2$. It may be noted that here $p^{(1)}$ is not the total but the thermodynamic pressure alone.

The stream function formulation of the above equations (1) and (2) in plane polar coordinates (r, θ) is reduced to solve the following differential equations respectively:

$$\nabla^4\psi^{(1)} = \nabla^2(\nabla^2\ \psi^{(1)}) = 0, \tag{3}$$

and

$$\nabla^4\ \psi^{(2)} - K^2\ \nabla^2\psi^{(2)} = \nabla^2(\nabla^2 - K^2)\psi^{(2)} = 0, \tag{4}$$

where the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{5}$$

The range of r and θ in the above equations (3) and (4) are given below as:

$$0 < r < \infty, \quad 0 \le \theta < 2\pi \tag{6}$$

Further, the velocity components $(v_r^{(i)},\ v_\theta^{(i)})$ and tangential and normal stresses can be expressed (Langlois 1964) respectively as

$$v_r^{(i)} = (1/r)(\partial\psi^{(i)}/\partial\theta, \qquad v_\theta^{(i)} = -(\partial\psi^{(i)}/\partial r), \tag{7}$$

$$T_{r\theta}^{(i)} = \mu^{(i)} \left[\frac{1}{r^2} \frac{\partial^2 \psi^{(i)}}{\partial \theta^2} + \frac{1}{r} \frac{\partial \psi^{(i)}}{\partial r} - \frac{\partial^2 \psi^{(i)}}{\partial r^2} \right], \tag{8}$$

$$T_{rr}^{(i)} = -p^{(i)} + \frac{2\mu^{(i)}}{r} \left(\frac{\partial^2 \psi^{(i)}}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \psi^{(i)}}{\partial \theta} \right), \quad i = 1, 2. \tag{9}$$

Also, the pressure in both regions may be obtained by integrating the following relations (Pop & Cheng 1992) respectively:

$$\frac{\partial p^{(i)}}{\partial r} = \mu^{(i)} \left(\nabla^2 v_r^{(i)} - \frac{v_r^{(i)}}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta^{(i)}}{\partial \theta} - \delta_{2i} K^2 v_r^{(i)} \right) \tag{10}$$

and

$$\frac{1}{r} \frac{\partial p^{(i)}}{\partial \theta} = \mu^{(i)} \left(\nabla^2 v_\theta^{(i)} - \frac{v_\theta^{(i)}}{r^2} + \frac{2}{r^2} \frac{\partial v_r^{(i)}}{\partial \theta} - \delta_{2i} K^2 v_\theta^{(i)} \right), \quad i = 1, 2, \tag{11}$$

where $\delta_{2i} = 0$ when $i = 1$ and $\delta_{2i} = 1$ when $i = 2$.

3. Solution of the problem with Happel boundary conditions

In the mathematical model, we assume that all the porous circular cylinders have the same radius and are randomly and homogeneously distributed parallel to each other. Let us suppose that flow with uniform velocity U is perpendicular to each stationary porous circular cylinder. The porous medium is assumed to be homogeneous and isotropic. We take the model to consist of a hypothetical circular cylinder of radius b, termed cell surface, enclosing and concentric with the porous circular cylinder. The radius of the cell surface can be determined by the assumption that it contains the same volumetric proportion of fluid to porous cylinder as exists in the entire assemblage. Therefore, if a is the radius of each porous cylinder, then the radius b of the cell surface is given by

$$\pi a^2 / \pi b^2 = \gamma, \tag{12}$$

where γ is the volume fraction of porous cylinders in the fluid medium.

The boundary conditions of our model may be expressed as below.

3.1 *On the porous cylinder* $(r = a)$

The continuity of velocity and surface stresses across the surface of porous cylinder implies that

$$v_r^{(1)}(a, \theta) = v_r^{(2)}(a, \theta), \tag{13}$$

$$v_\theta^{(1)}(a, \theta) = v_\theta^{(2)}(a, \theta), \tag{14}$$

$$T_{r\theta}^{(1)}(a, \theta) = T_{r\theta}^{(2)}(a, \theta), \tag{15}$$

$$T_{rr}^{(1)}(a, \theta) = T_{rr}^{(2)}(a, \theta). \tag{16}$$

3.2 *On the cell surface* ($r = b$)*, Happel conditions*

The continuity of normal component of velocity implies:

$$v_r^{(1)} = U \cos\theta \quad \text{for} \quad r = b. \tag{17}$$

The vanishing of tangential stress implies:

$$\frac{1}{r^2}\frac{\partial^2 \psi^{(1)}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial r} - \frac{\partial^2 \psi^{(1)}}{\partial r^2} = 0, \quad \text{for } r = b. \tag{18}$$

A suitable solution of Stokes equation (3) as (Happel & Brenner 1983) can be written in the form

$$\psi^{(1)}(r,\theta) = \left\{A^{(1)}r + B^{(1)}r^3 + C^{(1)}(1/r) + D^{(1)}r\ln(r/a)\right\}\sin\theta. \tag{19}$$

A particular solution of Brinkman's equation (4) (Pop & Cheng 1992) may be expressed as

$$\psi^{(2)}(r,\theta) = \left(A^{(2)}r + B^{(2)}(1/r) + C^{(2)}I_1(Kr) + D^{(2)}\mathcal{K}_1(Kr)\right)\sin\theta. \tag{20}$$

Here, $I_1(Kr)$ and $\mathcal{K}_1(Kr)$ are the modified Bessel functions of order one of the first and second kinds respectively.

Further more, the modified Bessel functions $\mathcal{K}_1(Kr)$ are singular at $r = 0$. The same is true for the term multiplied by $B^{(2)}$. Thus, the regular solution inside the circular cylinder takes the form

$$\psi^{(2)}(r,\theta) = \left(A^{(2)}r + C^{(2)}I_1(Kr)\right)\sin\theta, \tag{21}$$

The arbitrary constants $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$, $A^{(2)}$ and $C^{(2)}$ in (19) and (21) can be determined from the boundary conditions (13))–(18). Here, for sake of simplicity, we give below these constants for the particular case when $\beta = 1$, i.e., $\mu^{(1)} = \mu^{(2)}$. Thus,

$$A^{(1)} = U - D^{(1)}\ln l, \tag{22}$$

$$B^{(1)} = -K^2U[KaI_1(Ka) - 2I_2(Ka)]/\Delta_1, \tag{23}$$

$$C^{(1)} = -a^4l^4B^{(1)}, \tag{24}$$

$$D^{(1)} = 2K^2a^2U[Ka(1+l^4)I_1(Ka) - 4I_2(Ka)]/\Delta_1, \tag{25}$$

$$A^{(2)} = (1-l^4)a^2B^{(1)} - D^{(1)}\ln l + U - (1/a)C^{(2)}I_1(Ka), \tag{26}$$

$$C^{(2)} = -4Ka^2U(1-l^4)/\Delta_1, \tag{27}$$

where,

$$\Delta_1 = \left[\begin{array}{r}(1-l^4)\left\{K^3a^3I_1(Ka) - 2(K^2a^2+4)I_2(Ka)\right\} + 2(K^2a^2\ln l + 4)\\ \left\{Ka(1+l^4)I_1(Ka) - 4I_2(Ka)\right\}\end{array}\right], \tag{28}$$

and $l = b/a$ is a dimensionless parameter.

Also, we have used above the following recurrence relations (Abramowitz & Stegun 1970):

$$aI_n'(Ka) - nI_n(Ka) = KaI_{n+1}(Ka), \tag{29}$$

$$aI_n'(Ka) + nI_n(Ka) = KaI_{n-1}(Ka), \tag{30}$$

where primes denote differentiations with respect to a.

Further, the drag experienced by the porous cylinder in a cell may be calculated by using the formula

$$F = \int_0^{2\pi} \big(T_{rr}^{(1)} \cos\theta - T_{r\theta}^{(1)} \sin\theta\big) r \mathrm{d}\theta. \tag{31}$$

Since,

$$T_{rr}^{(1)} = -4\mu^{(1)} \big(B^{(1)}r - (D^{(1)}/r) + (C^{(1)}/r^3)\big) \cos\theta, \tag{32}$$

$$T_{r\theta}^{(1)} = -4\mu^{(1)} \big(B^{(1)}r + (C^{(1)}/r^3)\big) \sin\theta, \tag{33}$$

inserting (32) and (33) in (31) and integrating, we get

$$\begin{aligned} F &= 4\pi\mu^{(1)} D^{(1)} \\ &= 8\pi\mu^{(1)} UK^2a^2[Ka(1 + l^4)I_1(Ka) - 4I_2(Ka)]/\Delta_1, \end{aligned} \tag{34}$$

where Δ_1 is given by (28).

Now, when permeability $k \to o$, i.e. $K \to \infty$, i.e. porous cylinder behaves like a solid cylinder, then (34) reduces to,

$$F = 8\pi\mu^{(1)} U/[(2\ln l - 1) + 2/(1 + l^4)], \tag{35}$$

which is a known result reported earlier by Happel (1959) for flow past a solid cylinder.

4. Kuwabara boundary conditions

Kuwabara (1959) assumes that on the cell surface vorticity ω vanishes instead of no shearing stress. In this case, we take the five boundary conditions (13)–(17) to be the same as in the previous case but in place of the sixth boundary condition (18), Kuwabara boundary condition is used. Thus, vanishing of vorticity on the cell surface implies that

$$\nabla^2\psi^{(1)} = 0 \quad \text{on} \quad r = b. \tag{36}$$

Solving (13)–(17) with (36) we get the values of unknown constants $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$, $A^{(2)}$ and $C^{(2)}$ in (19) and (21) as

$$A^{(1)} = U - (1/a^2)C^{(1)}l^{-2} - D^{(1)}\big(\ln l - (1/4)\big), \tag{37}$$

$$B^{(1)} = -(1/4a^2l^2)D^{(1)}, \tag{38}$$

$$C^{(1)} = (a^2/4l^2)[Kal^2I_2'(Ka)C^{(2)} + D^{(1)}], \tag{39}$$

$$D^{(1)} = -4UK^2a^2l^2I_1(Ka)/\Delta_2, \tag{40}$$

$$C^{(2)} = 8al^2(1 - l^2)U/\Delta_2, \tag{41}$$

$$A^{(2)} = (1/K^2a^4)[D^{(1)}a^2(4 + l^{-2}) - 4C^{(1)} - 2Ka^2I_2(Ka)C^{(2)}], \tag{42}$$

where,

$$\Delta_2 = \begin{bmatrix} \left\{4l^2\{K^2a^2(l^2-1)-4l^2\} - K^2a^2\{(4\ln l - 1)l^4+1\}\right\} I_1(Ka) \\ -2Ka(l^2-1)\{Ka(1+l^2)I_1(Ka)+(l^2-2)I_2(Ka)\} \end{bmatrix}, \tag{43}$$

and $l = b/a$ is a dimensionless parameter.

Further, the drag experienced by the porous cylinder in a cell can be evaluated and comes out as

$$\begin{aligned} X &= 4\pi\mu^{(1)}D^{(1)} \\ &= -16\pi\mu^{(1)}UK^2a^2l^4I_1(Ka)/\Delta_2, \end{aligned} \tag{44}$$

where Δ_2 is given by (43).

Now, when permeability $k \to 0$, i.e., $K \to \infty$, i.e., the porous cylinder behaves like a solid cylinder, then (44) reduces to

$$X = 4\pi\mu^{(1)}U/[P + l^{-2} - (1/4)l^{-4}], \tag{45}$$

where,

$$P = \ln l - (3/4). \tag{46}$$

which is a known result reported earlier by Kuwabara (1959) for flow past a solid cylinder.

5. Velocity components, pressure, vorticity and stress tensor

Substituting the values of $\psi^{(1)}(r,\theta)$ from (19) and $\psi^{(2)}(r,\theta)$ from (21) into (7), we obtain the following expressions for the velocity components for the outside region (1) and inside region (2) of porous cylinder respectively as

$$v_r^{(1)} = \left[A^{(1)} + B^{(1)}r^2 + C^{(1)}(1/r^2) + D^{(1)}\ln(r/a)\right]\cos\theta, \tag{47}$$

$$v_\theta^{(1)} = \left[A^{(1)} + 3B^{(1)}r^2 - C^{(1)}(1/r^2) + D^{(1)}\{1+\ln(r/a)\}\right]\sin\theta, \tag{48}$$

$$v_r^{(2)} = \left[A^{(2)} + C^{(2)}(I_1(Kr)/r\right]\cos\theta, \tag{49}$$

$$v_\theta^{(2)} = -\left[A^{(2)} + C^{(2)}I_1'(Kr)\right]\sin\theta, \tag{50}$$

where the values of constants $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$, $A^{(2)}$ and $C^{(2)}$ are given by (22)–(27) for the case of the Happel boundary conditions and by (37)–(42) for the case of the Kuwabara boundary conditions.

Using the above expression (47)–(50) of velocity components in (10) and (11) and integrating the resulting equations, we get

$$p^{(1)} = 2\mu^{(1)}\left[4B^{(1)}r - (D^{(1)}/r)\right]\cos\theta, \tag{51}$$

$$p^{(2)} = -\mu^{(2)}K^2A^{(2)}r\cos\theta, \tag{52}$$

where $p^{(1)}$ and $p^{(2)}$ are the pressures in the outside and inside regions respectively.

The vorticity $\omega^{(i)}$ in both the regions can be expressed in terms of velocity components as

$$\omega^{(i)} = \frac{\partial v_\theta^{(i)}}{\partial r} - \frac{1}{r}\frac{\partial v_r^{(i)}}{\partial \theta} + \frac{v_\theta^{(i)}}{r}$$
$$= -\nabla^2 \psi^{(i)}, \quad i = 1, 2. \tag{53}$$

Thus,

$$\omega^{(1)} = -2\left[4B^{(1)} r + D^{(1)}(1/r)\right] \sin\theta, \tag{54}$$
$$\omega^{(2)} = -K^2 C^{(2)} I_1(Kr) \sin\theta. \tag{55}$$

Components of the stress tensor for the outside region (1) are given by (32) and (33). Components for the inside region (2) can be expressed by using (8) and (9) as given below,

$$T_{r\theta}^{(2)} = -\mu^{(2)} K C^{(2)} I_2'(Kr) \sin\theta, \tag{56}$$
$$T_{rr}^{(2)} = \mu^{(2)} K \left[K r A^{(2)} + (2C^{(2)}/r) I_2(Kr)\right] \cos\theta. \tag{57}$$

The author wishes to express his sincere thanks to Professor Sunil Datta, formerly of the Lucknow University, Lucknow for his valuable suggestions during the work. The author also thanks the referees for their suggestions which led to much improvement in the presentation of the paper.

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