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Propagation of Rayleigh surface waves with small wavelengths in nonlocal visco-elastic solids

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Abstract. This paper investigates Rayleigh waves, propagating on the surface of a visco-elastic solid under the linear theory of nonlocal elasticity. Dispersion relations are obtained. It is observed that the waves are dispersive in nature for small wavelengths. Numerical calculations and discussions presented in this paper lead us to some important conclusions.

Keywords. Continuum approach; propagation of waves; nonlocal solids; frequency equation; dispersion; attenuation exponents.

1. Introduction

In the linear theory of classical elasticity, it is observed that Rayleigh waves propagating on the surface of a semi-infinite isotropic elastic space are non-dispersive in nature (Love 1944). It is well-confirmed by experiment that the atomic theory of lattices predicts otherwise. Maradudin *et al* (1971) considered the problem from the view point of lattice dynamics for cubic crystals. Their investigations in this matter confirm the existence of dispersive character in such waves. However, so far we know, there exists no systematic study of continuum theory which may lead to similar conclusions in all such problems. A continuum approach to such problems has special advantages due to many facts as stated by Eringen (1973) in his research paper. Eringen (1973) investigated Rayleigh surface waves with small wavelengths under the nonlocal theory of elasticity. From the conclusions made by Eringen (1973), we observe that Rayleigh waves are definitely dispersive in nature while the rate of amplitude attenuations of waves remain the same as in classical elasticity. Again, the effect of internal friction on the propagation of plane waves in an elastic medium may also be considered owing to the fact that dissipation accompanies vibrations in solid media due to the conversion of elastic energy to heat energy (Ewing *et al* 1957). Several mathematical models have been used by many authors (Ewing *et al* 1957; Hunter 1960) to accomodate the energy dissipation in vibrating solids where it is observed that internal friction produces attenuation and dispersion and hence the effect of the visco-elastic nature of the material medium in the process of wave propagation cannot be neglected. The visco-elastic nature of a material medium has special significance in wave propagation in a solid medium. The above considerations led the authors to study the Rayleigh waves in a visco-elastic solid semi-space under the recently proposed linear theory of nonlocal elasticity. However, our present investigation involves the definition of internal friction given by Voigt (1887) which may be stated as the visco-elasticity of Voigt-type solids. Here, we consider only first order visco-elasticity. The theoretical results obtained in this paper may be utilised in some relevant practical problems of waves and vibrations in elasto-dynamics which play roles in areas like engineering sciences, earthquake sciences, seismology, geophysics etc. Some numerical calculations, discussions and conclusions have also been presented in their proper places. Authors believe that the problem in its present form was not investigated before.

2. Basic equations

We consider an elastic half-space occupying a region $x_2 \geq O$, with $Ox_1x_2x_3$ as the rectangular Cartesian co-ordinate system where the origin O is situated at any point on the plane boundary and Ox_2 points vertically downwards that is towards the bulk of the material medium. Following Eringen (Ewing *et al* 1957; Das *et al* 1994) the constitutive equations of motion and stress components for the propagation of waves in a first order visco-elastic Voigt-type solid medium under the linear nonlocal theory of elasticity with no body forces may be presented as follows:

$$\tau_{ij,i} = \rho \ddot{u}_j, \tag{1}$$

$$\begin{aligned} \tau_{ij} = {} & \left(\lambda_0 + \lambda_1 \frac{\partial}{\partial t}\right) u_{r,r}\delta_{ij} + \left(\mu_0 + \mu_1 \frac{\partial}{\partial t}\right)\left(u_{i,j} + u_{j,1}\right) \\ & + \int_V \left[\left(\lambda_0{}' + \lambda_1{}' \frac{\partial}{\partial t}\right) u_{r,r'}\delta_{ij} + \left(\mu_0{}' + \mu_1{}' \frac{\partial}{\partial t}\right)\left(u_{i,j} + u_{j,i}\right)\right] \mathrm{d}V\left(\mathbf{x}'\right), \end{aligned} \tag{2}$$

where τ_{ij}, u_i, ρ are stress tensor, displacement vector, mass density respectively.

$\lambda_0, \lambda_1, \mu_0, \mu_1 =$ visco-elastic constants

$\lambda_0{}', \lambda_1{}', \mu_0{}', \mu_1{}' =$ nonlocal visco-elastic moduli, each of which depends on $|\mathbf{x} - \mathbf{x}'|$ for homogeneous solids.

$$\tau_{ij,i} \equiv \frac{\partial \tau_{ij}}{\partial x_i}, u_{i,j} \equiv \frac{\partial u_i(\mathbf{x}, t)}{\partial x_j}, u_{i,j} \equiv \frac{\partial u_i(\mathbf{x}', t)}{\partial x_j'}, u = u_1, \dot{u}_1 \equiv \frac{\partial u_i(\mathbf{x}, t)}{\partial t},$$

$$\begin{aligned} \ddot{u}_i \equiv \partial^2 u_i(\mathbf{x}, t)/\partial t^2, t = \text{time}, \mathbf{x} = (X_1, X_2, X_3), \delta_{ij} &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j. \end{aligned}$$

The basic difference between classical and nonlocal elasticity is in the presence of the volume integral in (2) which indicates that the stress at $(\mathbf{x}, t)$ depends on the strain at all other points $\{\mathbf{x}'\}$ of the body, at time t. This signifies that the distant neighbours of a point $\mathbf{x}$ have an role to play in the propagation of waves.

3. Basic assumptions

To make the problem two-dimensional, one has to consider the domain of X_1 as $-\propto < X_1 < +\propto$ and that of X_2 as $0 < X_2 < +\propto$. Moreover, we assume that everything is uniform in the X_3 direction. We consider here the possibility of a type of wave travelling in the direction of X_1 axis in such a manner that the disturbance is largely confined in the neighbourhood of boundary and at any instant all particles on any line parallel to OX_3 have equal displacement. Due to the first assumption, the wave is a surface one which is an essential condition for Rayleigh waves and the second assumption induces that all partial derivatives with respect to X_3 are zero. In this case, the volume integral in (2) is reduced to a surface integral over $X_1{}'$ and $X_2{}'$ in their ranges.

4. Boundary conditions

Since the boundary surface $X_2 = 0$ is stress free, we have

$$\tau_{21} = \tau_{22} = 0, \text{ for } X_2 = 0; \tag{3}$$

$$u_1, u_2 \to 0, \text{ as } X_2 \to \propto . \tag{4}$$

5. Fomulation of the problem

In the light of basic assumptions, the dynamical equations of motion for Rayleigh waves may be deduced from (1) as follows.

$$\tau_{11,1} + \tau_{21,2} = \rho \ddot{u}_1, \tag{5}$$

$$\tau_{12,1} + \tau_{22,2} = \rho \ddot{u}_2, \tag{6}$$

where $\tau_{11}, \tau_{21}, \tau_{22}$ are deduced from (2). Hence, the mathematical formulation of the present problem is to solve (5) and (6) under the above boundary conditions.

6. Solution of the problem

To solve the above problem, we apply the Fourier integral transform in the following form

$$u_k(X_1, X_2, t) = \frac{1}{2\pi} \int_{-\propto}^{+\propto} \int_{-\propto}^{+\propto} \bar{u}_k\,(\xi, X_2, \omega)\, e^{-i(\xi x_1 + \omega t)}\, \mathrm{d}\xi\, \mathrm{d}\omega. \tag{7}$$

Substitution of this into (5), (6) and (2) gives

$$-i\xi\,\bar{\tau}_{11} + \bar{\tau}_{21,2} + \rho\omega^2 \bar{u}_1 = 0, \tag{8}$$

$$-i\xi\,\bar{\tau}_{12} + \bar{\tau}_{22,2} + \rho\omega^2 \bar{u}_2 = 0, \tag{9}$$

where,

$$\bar{\tau}_{11} = -i\xi\,\{(\lambda_0 + 2\mu_0) - i\omega\,(\lambda_1 + 2\mu_1)\}\,\bar{u}_1 + (\lambda_0 - i\omega\lambda_1)\bar{u}_{2,2}$$
$$+ \int_0^{\propto} \left[-i\xi\,\left\{\left(\bar{\lambda}_0{}' + 2\bar{\mu}_0'\right) - i\omega\left(\bar{\lambda}_1{}' + 2\bar{\mu}_1'\right)\right\}\bar{u}_1 + \left(\bar{\lambda}_0{}' - i\omega\bar{\lambda}_1'\right)\bar{u}_{2,2'}\right] \mathrm{d}(x_2{}'), \tag{10}$$

$$\bar{\tau}_{22} = -i\xi\,(\lambda_0 - i\omega\lambda_1)\,\bar{u}_1 + \{(\lambda_0 + 2\mu_0) - i\omega\,(\lambda_1 + 2\mu_1)\}\,\bar{u}_{2,2}$$
$$+ \int_0^{\propto} \left[-i\xi\,\left(\bar{\lambda}_0{}' - i\omega\bar{\lambda}_1'\right)\bar{u}_1 + \left\{\left(\bar{\lambda}_0{}' + 2\bar{\mu}_0'\right) - i\omega\left(\bar{\lambda}_1{}' + 2\bar{\mu}_1'\right)\right\}\bar{u}_{2,2'}\right] \mathrm{d}(x_2{}'), \tag{11}$$

$$\bar{\tau}_{12} = (\mu_0 - i\omega\mu_1)\,\bar{u}_{1,2} - i\xi\,(\mu_0 - i\omega\mu_1)\,\bar{u}_2 + \int_0^{\propto} \left[\left(\bar{\mu}_0{}' - i\omega\bar{\mu}_1{}'\right)\bar{u}_{1,2'} - i\xi\left(\bar{\mu}_0{}' - i\omega\bar{\mu}_1{}'\right)\bar{u}_2\right] \mathrm{d}(x_2{}'). \tag{12}$$

Following Eringen (1973), since $\lambda_0{}'$, $\mu_0{}'$ and $\lambda_1{}'$, $\mu_1{}'$ tends to zero rapidly as $|\mathbf{x}' - \mathbf{x}| \to\propto$ we may assume the expressions for $\bar{\lambda}_0{}'$, $\bar{\mu}_0{}'$ and $\bar{\lambda}_1{}'$, $\bar{\mu}_1{}'$ in the following forms

$$\bar{\lambda}_0{}' = \bar{\lambda}_0(\xi)\delta(|x_2{}' - x_2|),\ \bar{\lambda}_1{}' = \bar{\lambda}_1(\xi)\delta(|x_2{}' - x_2|),$$
$$\bar{\mu}_0{}' = \bar{\mu}_0(\xi)\delta(|x_2{}' - x_2|),\ \bar{\mu}_1{}' = \bar{\mu}_1(\xi)\delta(|x_2{}' - x_2|). \tag{13}$$

Using (13) in (10), (11) and (12), one obtains

$$\bar{\tau}_{11} = -i\xi\left[\left\{\left(\lambda_0 + \bar{\lambda}_0\right) + 2\left(\mu_0 + \bar{\mu}_0\right)\right\} - i\omega\left\{\left(\lambda_1 + \bar{\lambda}_1\right) + 2\left(\mu_1 + \bar{\mu}_1\right)\right\}\right]\bar{u}_1 + \left\{\left(\lambda_0 + \bar{\lambda}_0\right) - i\omega\left(\lambda_1 + \bar{\lambda}_1\right)\right\}\bar{u}_{2,2}, \tag{14}$$

$$\bar{\tau}_{22} = -i\xi\left[\left\{\left(\lambda_0 + \bar{\lambda}_0\right) - i\omega\left(\lambda_1 + \bar{\lambda}_1\right)\right\}\right]\bar{u}_1 + \left[\left\{\left(\lambda_0 + \bar{\lambda}_0\right) + 2\left(\mu_0 + \bar{\mu}_0\right)\right\} - i\omega\left\{\left(\lambda_1 + \bar{\lambda}_1\right) + 2\left(\mu_1 + \bar{\mu}_1\right)\right\}\right]\bar{u}_{2,2}, \tag{15}$$

$$\bar{\tau}_{22} = \{(\mu_0 + \bar{\mu}_0) - i\omega\,(\mu_1 + \bar{\mu}_1)\}\left(\bar{u}_{1,2} - i\xi\bar{u}_2\right). \tag{16}$$

Substituting $\bar{u}_k(\xi, x_2, \omega) = \bar{U}_k(\xi, \omega)e^{-\alpha x_2}$, (17)

in (14), (15) and (16) and using these results in (8) and (9), we get the following two equations

$$\left(\alpha^2 - \frac{k^2}{h^2}\xi^2 + k^2\right)\bar{U}_1 + i\alpha\xi\left(\frac{k^2}{h^2} - 1\right)\bar{U}_2 = 0,$$
$$i\alpha\xi\left(\frac{k^2}{h^2} - 1\right)\bar{U}_1 + \left(\alpha^2\frac{k^2}{h^2} - \xi^2 + k^2\right)\bar{U}_2 = 0, \tag{18}$$

where

$$k^2 = \rho\omega^2/\left\{(\mu_0 + \bar{\mu}_0) - i\omega\,(\mu_1 + \bar{\mu}_1)\right\}, \tag{19}$$
$$h^2 = \rho\omega^2/\left[\left\{\left(\lambda_0 + \bar{\lambda}_0\right) + 2\left(\mu_0 + \bar{\mu}_0\right)\right\} - i\omega\left\{\left(\lambda_1 + \bar{\lambda}_1\right) + 2\left(\mu_1 + \bar{\mu}_1\right)\right\}\right].$$

Eliminating $\bar{U}_1$ and $\bar{U}_2$ from (18), we get a quadratic equation in α^2 whose roots are given by

$$\alpha_1^2 = \xi^2 - h^2,$$
$$\alpha_2^2 = \xi^2 - k^2. \tag{20}$$

Since $\bar{u}_1, \bar{u}_2 \to 0$ as $x_2 \to\propto$, the solutions for $\bar{u}_1, \bar{u}_2$ may be taken in the forms

$$\bar{u}_1 = e^{-\alpha_1 x_2}\bar{U}_{11} + e^{-\alpha_2 x_2}\bar{U}_{12},$$
$$\bar{u}_2 = \gamma_1 e^{-\alpha_1 x_2}\bar{U}_{11} + \gamma_2 e^{-\alpha_2 x_2}\bar{U}_{12}, \tag{21}$$

where

$$\gamma_j = \frac{-i\xi\alpha_j(k^2 - h^2)}{\alpha_j^2 - h^2\xi^2 + h^2k^2},\ \ j = 1, 2. \tag{22}$$

Using (21) in the boundary condition (3), we obtain the following equations

$$(\alpha_1 + i\xi\gamma_1)\bar{U}_{11} + (\alpha_2 + i\xi\gamma_2)\bar{U}_{12} = 0, \tag{23}$$
$$\left(i\xi\frac{k^2-2h^2}{h^2k^2} + \frac{\alpha_1\gamma_1}{h^2}\right)\bar{U}_{11} + \left(i\xi\frac{k^2-2h^2}{h^2k^2} + \frac{\alpha_2\gamma_2}{h^2}\right)\bar{U}_{12} = 0.$$

Elimination of $\bar{U}_{11}$ and $\bar{U}_{12}$ from (23) leads to the following frequency equation

$$\left((k^2/\xi^2) - 2\right)^4 = 16\left(1 - (k^2/\xi^2)\right)\left(1 - (h^2/\xi^2)\right). \tag{24}$$

Equation (24) is the dispersion relation for Rayleigh waves propagating in a visco-elastic semispace for nonlocal solids where k^2 and h^2 are given by (19). The form of this frequency equation (24) is identical to the corresponding equation of classical theory which is again identical in form to the frequency equation for Rayleigh waves in a nonlocal elastic solid as per the deduction made by Eringen (1973), though each of k and h represents different expressions for classical elasticity, nonlocal elasticity and nonlocal visco-elasticity.

7. Particular cases

Case (a): For the sake of numerical calculation, we extend the concept of the Poisson material to include $\lambda_0 = \mu_0$, $\lambda_1 = \mu_1$, $\bar{\lambda}_0 = \bar{\mu}_0$, $\bar{\lambda}_1 = \bar{\mu}_1$.

Hence, using the above in (19) and then solving (24), we get

$$k^2/\xi^2 = 0.8453 \text{ (approximately)}. \tag{25}$$

Therefore

$$\omega/\xi = c_{R_1}\left(1 - i\omega\left((\mu_1 + \bar{\mu}_1)/(\mu_0 + \bar{\mu}_0)\right)\right)^{1/2}, \tag{26}$$

where

$$c_{R_1} = 0.9194\left((\mu_0 + \bar{\mu}_0)/\rho\right)^{1/2}.$$

In this case attenuation exponents α_1 and α_2, are given by

$$\alpha_1^2/\xi^2 = 0.7182,$$
$$\alpha_2^2/\xi^2 = 0.1546. \tag{27}$$

For visco-elastic solids, ξ being generally complex, setting $\xi = \eta + i\delta$ it may be noted that for the wave to be physically realistic, we should have $\omega/2\pi$ as frequency, where as, $2\pi/\eta$ is the wavelength, $c = \omega/\eta$ is the phase velocity and δ is a decay coefficient.

Hence, using this value of ξ in (25) and equating real and imaginary parts, we get

$$\eta^2 = \left[\rho\omega^2/0.8453\left[2\mu_0\left(1 + \frac{\bar{\mu}_0}{\mu_0}\right)\left\{1 + \omega^2\left(\frac{\mu_1 + \bar{\mu}_1}{\mu_0 + \bar{\mu}_0}\right)^2\right\}\right]\right] \times \left[1 + \left\{1 + \omega^2\left(\frac{\mu_1 + \bar{\mu}_1}{\mu_0 + \bar{\mu}_0}\right)^2\right\}^{1/2}\right] \tag{28}$$

and

$$\delta^2 = \left[\rho\omega^2/0.8453\left[2\mu_0\left(1+\frac{\bar{\mu}_0}{\mu_0}\right)\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}\right]\right] \times \left[\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}^{1/2}-1\right]. \quad (29)$$

Therefore

$$c^2/\beta^2 = \left[0.8453\left\{2\left(1+(\bar{\mu}_0/\mu_0)\right)\left(1+k^2\right)\right\}\right]/\left\{1+\left(1+k^2\right)^{1/2}\right\}, \quad (30)$$

where

$$\beta^2 = \frac{\mu_0}{\rho} \text{ and } k = \omega\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right).$$

For nonlocal elastic solids in the absence of viscous nature of the material medium (28) gives

$$\eta^2 = \rho\omega^2/0.8453\mu_0\left(1+\bar{\mu}_0/\mu_0\right), \quad (31)$$

so that Rayleigh wave velocity under the theory of nonlocal elasticity without viscous effect is given by

$$c_{R_1} = 0.9194(\mu_0/\rho)^{1/2}\left(1+(\bar{\mu}_0/\mu_0)\right)^{1/2}. \quad (32)$$

Replacing μ_0 by μ, the Lame elastic constant, it is observed that (32) is in agreement with that obtained by Eringen (1973). This equation expresses the fact that Rayleigh waves are definitely dispersive, which is a deviation from our idea of classical elasticity.
Case (b): For another particular case, we extend the concept of the assumptions made by Caloi (1948) and include

$$\lambda_i + (2/3)\mu_i = 0,$$
$$\bar{\lambda}_i + (2/3)\bar{\mu}_i = 0,\ i = 0, 1. \quad (33)$$

Using (33) in (19) and then by solving (24) one obtains

$$k^2/\xi^2 = 0.4746 \text{ (approximately)}, \quad (34)$$

Therefore

$$\omega/\xi = c_{r_1}\left(1 - i\omega\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^{1/2}, \quad (35)$$

where

$$c_{r_1} = 0.6889((\mu_0+\bar{\mu}_0)/\rho)^{1/2}.$$

And the attenuation exponents, α_1 and α_2 in this case are given by

$$\alpha_1^2/\xi^2 = 0.6441, \quad (36)$$
$$\alpha_2^2/\xi^2 = 0.5254.$$

Now, setting $\xi = \eta + i\delta$ in (34) and equating real and imaginary parts, we get

$$\eta^2 = \left[\rho\omega^2/0.4746\left[2\mu_0\left(1+\frac{\bar{\mu}_0}{\mu_0}\right)\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}\right]\right] \times \left[1+\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}^{1/2}\right] \quad (37)$$

and

$$\delta^2 = \left[\rho\omega^2/0.4746\left[2\mu_0\left(1+\frac{\bar{\mu}_0}{\mu_0}\right)\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}\right]\right] \times \left[\left\{1+\omega^2\left(\frac{\mu_1+\bar{\mu}_1}{\mu_0+\bar{\mu}_0}\right)^2\right\}^{1/2} - 1\right]. \quad (38)$$

Therefore

$$c^2/\beta^2 = \left[0.4746\left\{2\left(1+(\bar{\mu}_0/\mu_0)\right)\left(1+k^2\right)\right\}\right] / \left\{1+\left(1+k^2\right)^{1/2}\right\}, \quad (39)$$

where

$$\beta^2 = \mu_0/\rho \text{ and } k = \omega\left((\mu_1+\bar{\mu}_1)/(\mu_0+\bar{\mu}_0)\right).$$

If we do not consider the viscous effect, (37) gives

$$\eta^2 = \rho\omega^2/0.4746\mu_0\left(1+(\bar{\mu}_0/\mu_0)\right) \quad (40)$$

which is valid for nonlocal elastic solids.

Hence the Rayleigh wave velocity for nonlocal elastic solids in the above particular case is

$$c_{r_1} = 0.4889(\mu_0/\rho)^{1/2}(1+(\bar{\mu}_0/\mu_0))^{1/2}. \quad (41)$$

Equations (30) and (39) indicate that further dispersion of Rayleigh waves, over the dispersion due to the nonlocal character of the elastic medium, occurs when the medium is a visco-elastic one.

8. Conclusions

The formulae obtained in the bulk of the paper, which are valid for nonlocal visco-elastic solids, may be used for the estimation of dispersion of general wave forms.

It is observed that dispersion of the Rayleigh wave velocity is further modulated in the case of visco-elastic solids.

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