NONLINEAR PROBLEMS OF THE VIBRATION OF THIN SHELLS (REVIEW)

V. D. Kubenko and P. S. Koval'chuk

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1. Introduction. More than forty years have passed since the publication of the first studies on the problem of the vibration of three-dimensional shell-type structures with finite deflections (i.e., nonlinear vibrations of such structures). The nonlinear vibration of shells has by now become one of the most important divisions of the mechanics of deformable solids. Solutions have been found to a large number of problems having numerous applications, and unique methods and approaches have been developed to obtain those solutions. The progress that has been made in the given area has been the result of the efforts of leading scientists in theoretical mechanics from many different countries: S. P. Timoshenko, E. I. Grigolyuk, V. V. Bolotin, I. I. Vorovich, I. F. Obraztsov, A. S. Vol'mir, A. N. Guz', Ya. M. Grigorenko, V. I. Gulyaev, E. Reissner, G. Donnell, D. Evanson, G. Schmidt, and others. The scope of the discipline is expanding and new avenues of research are being opened. Efficient numerical-analytical methods adapted for powerful computers are being developed and introduced into practice. At a higher level, modern equipment to excite and record vibrations is being used to perform experimental studies.

Due to the unusually broad and diverse range of topics that comprise the modern nonlinear dynamics of shells, it is difficult to compare most of the scientific investigations that have been made in this area within the scope of a single survey. Below, we examine traditional problems in nonlinear shell dynamics that concern period vibrations of thin shells. Our primary focus will be on free vibrations, as well as shell vibrations excited by periodic external loads — including parametric loads. We will also briefly touch on the related topic of determining (identifying) the nonlinear dynamic characteristics of shells from an analysis of experimental data obtained in vibration tests.

We will not discuss the many other important nonlinear problems of the vibration of shells, including dynamic instability under aperiodic-shock loads, impact loads, and other transient loads, free vibrations (flutter), random vibrations, inverse problems (in standard formulations), and other problems that could be the subject of extensive independent surveys.

2. Nonlinear models of shells. The vibrations of all types of actual thin-walled shell structures are essentially nonlinear, since there are always nonlinear relations in the laws that govern their dynamic deformation. The study of these vibrations as "linear" objects is thus an abstraction of the highest order and is related directly to the process of changing over from actual physical phenomena to their mathematical representations in the form of suitable differential (integrodifferential) equations.

The "nonlinearity" seen in the vibrations of shells can have a "geometric" origin (being due to the nonlinear relationship between the strains and the displacements) or a physical origin (when the strains exceed the range of application of Hooke's law, i.e., when they depend linearly on the forces). In a number of cases, the "nonlinearity" may be caused by the complex character of energy dissipation during vibrations. Certain problems may also involve nonlinear inertia, which is related to the existence of nonlinear inertial terms in the mathematical model of the shells.

As regards thin shells, the "predominant" type of nonlinearity and thus the type most frequently accounted for in mathematical models is geometric nonlinearity due to the flexibility of such shells, i.e., due to their relatively low resistance to bending (which causes the shells to undergo displacements that are comparable to their thickness when deformed under load). Shells are considered to be thin shells (in problems of dynamics) if their thickness h satisfies the following well-known criterion [101]: $\frac{\omega h}{c} \ll 1$. Here, ω is the characteristic frequency of vibration and c is the velocity of the transverse elastic waves.

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 34, No. 8, pp. 3–31, August, 1998. Original article submitted September 18, 1997.

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The fundamental principles used in constructing the corresponding linear theories of thin-walled structures have served as the methodological basis for the development of the nonlinear theory of shells. Many researchers have contributed to the linear theory of deformation of shells. Without discussing the historical aspects of that theory's development, we mention the following well-known monographs devoted to the given problem: [5, 36, 37, 64, 87, 148, 202, 220]. These studies have provided a detailed exposition of the main postulates of the classical theory of shells and other, refined theories. The monographs [5, 9, 10, 26, 65, 73, 80, 81, 132, 151–153, 158, 164, 165, etc.] presented different mathematical methods of solving linear problems of shell vibration along with specific results obtained with the models. Surveys of studies concerning the above class of problems can also be found among these publications.

Various aspects of the vibration of shells with small deflections have also been the subject of special survey articles, reference works, and other informational literature [7, 8, 35, 66, 166, 179].

The principles of the nonlinear theory of shells were laid out in the works of Love and S. P. Timoshenko at the end of the XIX century and the beginning of the XX century, respectively. Kirchhoff's well-known kinematic and mechanical hypotheses were of fundamental importance in the construction of the given theory, having been used earlier to derive the equations of deformation of plates. As is known, the use of these hypotheses essentially reduces the study of the behavior of an element of a shell to the study of the behavior of its middle surface. As a result, a three-dimensional boundary-value problem in the mechanics of deformable bodies is reduced to a two-dimensional problem.

This approach was taken further in the research of V. Z. Vlasov [36], Kh. M. Mushtari and K. Z. Galimov [50, 51, 146], L. Donnell [87], Marguerre [207], V. V. Novozhilov [148], A. L. Gol'denveizer [64], A. I. Lur'e [134], A. S. Vol'mir [37], I. I. Vorovich [44], W. Flugge [202], and others. However, the conditions necessary for the broad practical use of a nonlinear theory of shells based on the classical hypotheses came about thanks to the efforts of L. Donnell, Kh. M. Mushtari, and V. Z. Vlasov, who proposed and substantiated a simple variant of the theory — the theory of so-called shallow shells. These are shells that allow the use of an approximation in which the metric of the middle surface is replaced by the metric of a plane. K. Marguerre later generalized this theory to the case of shells of arbitrary curvature (in the initial variant, only cylindrical shells were considered). The resulting equations, still referred to in the literature as the Donnell–Mushtari–Vlasov equations (or Marguerre–Vlasov equations or sometimes simply the Marguerre equations), are known as the equations of the theory of "thin shallow shells." They were derived from the general equations by ignoring the shear strains in the expressions for the components of the changed curvature of the shell. Inertial shearing forces are also ignored. These equations are often used now to study the characteristics of the nonlinear vibration and dynamic stability of shells.

Other, refined variants of the geometrically nonlinear theory of shells including those based on Timoshenko's kinematic model that are sometimes used to solve various dynamic problems were also discussed in the surveys [1, 4, 74, 183, 212] and the studies [19, 50, 51, 101].

3. Free and forced vibrations. Research into the nonlinear vibration of shells was begun in the middle of the 1950s by E. I. Grigolyuk [70, 71] and E. Reissner [216, 217]. These studies were preceded by just a few publications devoted to vibrations of simpler continuous systems (rods, beams, and plates) with large deflections. The dynamic deformation of shells was studied solely by analyzing their linear mathematical models. The authors of [70, 71, 216, 217] were the first to use nonlinear equations of the following form in calculations of shell vibrations (for cylindrical and spherical shells)

$$\frac{D}{h}\nabla^4 w+\rho\frac{\partial^2 w}{\partial t^2}=L(w,\Phi)+\nabla_k^2\Phi+\frac{N_x}{h}\frac{\partial^2 w}{\partial x^2}+\frac{N_y}{h}\frac{\partial^2 w}{\partial x^2}+\frac{q(x,y,t)}{h};$$

$$\frac{1}{E}\nabla^4\Phi=-\frac{1}{2}L(w,w)-\nabla_k^2 w, \tag{1}$$

where w is the transverse deflection; D is cylindrical stiffness; h is the thickness of the shell; ρ is the density of the material; Φ is the stress function for the stresses in the middle surface; N_x and N_y are the unit forces in the initial stress state; $q(x,y,t)$ is a normal periodic load; $\nabla_k=k_x\frac{\partial^2}{\partial x^2}+k_y\frac{\partial^2}{\partial y^2}$ (k_x and k_y are the curvatures of an element of the shell in the directions of the x and y axes, respectively); ∇^4 and L are known differential operators. Thus, the dynamic deformation of shells was studied mainly in bending (inertial shearing forces were ignored). Researchers chose to represent the solution of Eqs. (1) in the form of monomial ("monomodal") approximations, so that a variational method could be used to reduce the initial problem to the study of a second-order nonlinear differential equation in the amplitude parameter

$$\ddot{f}+\omega^2 f+\delta f^2+\gamma f^3=q_0(t). \tag{2}$$

Here, ω is the natural frequency of the shell; δ and γ are constant parameters. Integration of the given equation by Lindshtedt's method in [70, 71] and Galerkin's method in [216, 217] (in the latter case, the orthogonalization was done only for one-quarter of the period of vibration) yielded the first specific "nonlinear" feature of the deformation of shells with finite deflections — the fact that their vibrations are asymmetric relative to the undeformed middle surface. In contrast to the linear case, the amplitudes of vibration turned out to be different in the directions of outward and inward normals. Meanwhile, the amplitude of the "sag" (along the inward normal) was always greater than the amplitude of the bend (along the outward normal).

The number of publications on the vibration of shells with large deflections grew rapidly after 1955, but all of them were initially theoretical in nature. I. I. Vorovich was the first [43] to take a sufficiently rigorous approach in studying problems related to the existence and uniqueness of the solutions of the nonlinear equations that describe the vibration of shells. The subsequent monograph [44] elaborated on this topic and included an extensive bibliography. (A rigorous proof of the validity of using classical shell models — particularly the Donnell–Mushtari–Vlasov model — in calculations of nonlinear vibrations was also presented by V. I. Sedenko [172, 173] on the basis of more recent developments in the given discipline.)

Among the studies made in the "early" (to use the classification devised by D. Evensen [219]) stage of investigation, the most typical are those performed by G. Chu [196] and G. Nowinski [210]. Their findings were subsequently the cause of much debate and discussion, which continues to this day. They were the first to pose and examine a question that is fundamental to the mechanics of shells: is the geometric nonlinearity of shells of the "soft" type or the "hard" type. Analyzing the amplitude-frequency characteristics (AFCs) for free and forced vibrations, the authors of those studies came to the conclusion that regardless of the geometric parameters and modes of vibration of shells (as in [70, 216], the authors of the above works examined unimodal approximations of deflections), their skeleton curves always correspond to nonlinearity of the "hard" type (the frequencies of vibration increase with an increase in the amplitudes of vibration). Similar conclusions were reached in [14]. However, the reliability of that result was questioned by D. Evensen [198, 199]. In [198] Evensen proposed a more complex, binomial approximation of deflection for cylindrical shells

$$w=f(t)\sin rx\cos sy-\frac{n^2}{4R}f^2(t)\sin^2 rx. \tag{3}$$

Here, $r=\frac{m\pi}{l}$; $s=\frac{n}{R}$ are the wave generation parameters; R is the radius of the shell; l is its length. In contrast to [196], function (3) satisfies the periodicity condition [38] and vanishes at the ends (in contrast to [210]). In addition to geometric nonlinearity, nonlinear inertia is exhibited in the equation for determining $f(t)$. Their combined effect results in an amplitude-frequency characteristic of the "soft" type. In fact, this type of AFC was observed experimentally soon thereafter [199, 211].

An important new stage in studies of the vibrations of shells was begun during the period 1966–1968 by D. Evensen and R. Fulton [200, 201], who were the first to point out and substantiate the need to qualitatively complicate the approximation of the deflection w. The experimental data which had been obtained up to that time indicated the existence of complex, unconventional stationary waves and modes of deformation in the case of closed circular rings and cylindrical shells subjected to periodic loads. The crux of the problem was accounting for wave (circumferential travelling wave) processes caused by the superposition of "conjugate" bending modes, i.e., geometrically similar modes phase-shifted in the circumferential direction by the angle $\frac{\pi}{2n}$. In the case of a cylindrical shell, the deflection w is represented in the form

$$w=f_1(t)\cos sy\sin rx+f_2(t)\sin sy\sin rx-\frac{n^2}{4R}[f_1^2(t)+f_2^2(t)]\sin^2 rx. \tag{4}$$

(It should be noted that a dual-wave mode of deflection had already been used in calculations of nonlinear vibrations of circular rings [188, 199]). As studies showed, a travelling wave is realized in a shell only within a certain zone of the resonance region in which vibrations in both conjugate modes are stable simultaneously. Outside this zone, unimodal vibrations take place in the mode that was excited directly.

In the study [197] by E. Dowell and C. Ventres, a cylindrical shell was modeled by a system with three degrees of freedom

$$w = f_1(t) \cos sy \sin rx + f_2(t) \sin sy \sin rx + f_3(t) \sin rx. \tag{5}$$

However, the periodicity condition was satisfied only on the average, which would obviously result exclusively in "hard" AFCs for all four modes (n = 2, 4, ...) regardless of the parameters of the shell m, l, h, and R. This approach was developed further by S. Atluri in [194]. The results therein agree qualitatively with [197].

However, directly opposite conclusions were reached by T. Matsuzaki and S. Kobayashi in [208], where a study was made of vibrations of a shell with clamped ends. These conclusions were also supported experimentally by the authors. Nevertheless, in subsequent studies by J. Ginsberg [203], J. Chen and C. Babcock [195], A. I. Telalov [182], and others, it was shown that both types of AFCs — "soft" and "hard" — are possible for cylindrical shells, depending on the modes that are excited.

As follows from the above discussion, complete clarity has yet to be achieved in regard to the state of the given problem as a whole. Still to be definitely determined are the general principles underlying the effects of the physical and geometric parameters of shells, the wave generation parameters, boundary conditions, and other factors on the nonlinear dynamic characteristics of these shells — particularly their natural frequencies. Without a resolution of this issue, it will be impossible to reliably predict resonance situations that might arise when a shell is acted upon by periodic (nearly periodic) external loads.

Nonlinear free and forced vibrations of cylindrical shells, most of them unimodal, were also examined in [16, 17, 96, 155, 156, 170, 204, 213], etc. Some of these investigations considered physical nonlinearity in addition to geometric nonlinearity and analyzed their interaction. The author of [170] attempted to evaluate approximate analytical solutions obtained by the method of harmonic balance (HB). It was shown that the mode of nonlinear vibrations of a shell having a relatively large amplitude differs appreciably from the corresponding harmonic mode and that the HB method can lead to even qualitatively incorrect results in certain cases.

In addition to analytical approaches, numerical methods of solving nonlinear problems of the dynamics of shells were the subject of considerable research in the 1970s. The most important results in this area were obtained by A. S. Vol'mir and his colleagues [38, 39, etc.] and by N. V. Valishvili and V. B. Silkin [34, 33, 176]. The generalized monograph [33] presented different computer algorithms that are based on the straight-line method and were developed to calculate characteristics of free and forced vibrations of spherical (in the case of the "shallow model") shells with large deflections. Dynamic problems ultimately reduce to a Cauchy problem that can be solved by the Runge–Kutta method with automatic selection of the step (depending on the prescribed accuracy). An important conclusion reached on the basis of the results obtained in [33] is that the use of simplified (especially unimodal) models of shells in calculations can lead to serious errors in the determination of the parameters of their vibrations and, thus, the stress-strain state (SSS) as a whole.

Numerical methods of performing dynamic calculations for shell structures on the basis of both linear and nonlinear models were further refined and applied in practice in several subsequent generalizing publications [40, 41, 45, 46, 72, 76, 95, 141, 147, 150, etc.]. The emphasis in most of these studies was on aspects of the nonsteady deformation of such structures, including their interaction with the environment.

As regards investigations of periodic modes of shell deformation with the use of analytical methods, research in this area proceeded in several main directions. The research in one of those directions was begun by A. S. Vol'mir and his followers in [39, 97–100, etc.], where a geometrically nonlinear formulation was used to examine problems on the free and forced vibrations of shallow cylindrical shells with allowance for small initial shape flaws. Fairly complete surveys of early studies in this area were made in the monograph [39] and the article [77]. In these investigations, the initial deflection $w_0 = w_0(x, y)$ is usually treated as a component of the total deflection w ($w = w_0 + w_1$; w_1 is the additional or dynamic deflection). The functions w_1 and w_0 are represented in a form that reflects the character of wave formation in shells during loss of stability [38]

$$w_1 = f_1(t) \sin rx \sin sy + f_2(t) \sin^2 rx;$$

$$w_0 = f_{10} \sin rx \sin sy + f_{20} \sin^2 rx, \quad (6)$$

$$(f_{10}, f_{20} = \mathrm{const}).$$

The second term in w_1 is the "correcting" term and is introduced in order to represent specific features of shell deformation with conical deflections ("preferential inward buckling" [39]). Most problems have been solved by the Bubnov–Galerkin method, with subsequent use of methods from nonlinear mechanics (particularly the harmonic balance, perturbation, Krylov–Bogolyubov, and small parameter methods). It was found that the skeleton curves of the shells usually corresponded to the "soft" type of nonlinearity. For certain wave generation parameters, the frequency characteristics corresponded to the combination ("soft-hard") type. An initial deflection w_0 results in more pronounced asymmetry of the amplitude $f_1(t)$ relative to the undeformed middle surface (with its direction toward the center of curvature).

A similar approach was used later in the studies [56-59, 91, 140, 186, etc.]. V. I. Matsner [140] took actual (determined in special measurements) initial deflections into consideration in the shell equations, which accounted for the good agreement between the theoretical and experimental values of the natural frequencies of the shell (the shell was cylindrical and was reinforced by a set of stringers forming the primary structure). The deformation of shells was described in the studies of M. S. Gershtein and S. S. Khalyuk [57–59] by using equations from the technical theory that were obtained with allowance for transverse shear strain. Here, the Kirchhoff–Love hypothesis is assumed to be valid for each load-bearing layer, while the hypothesis of an assigned distribution of transverse shears is adopted for the shell as a whole. The authors came to the conclusion that the degree of nonlinearity — usually "soft" — which is manifested during shell vibrations is greater, the larger the parameter $\bar{h} = h/R$.

The methods developed for cylindrical shells have also been used to calculate the nonlinear vibrations of shells of other forms, especially spherical shells [11, 78, 88, 125, 171, 191, 215, etc.].

N. I. Zhinzher and V. N. Denisov proposed a special asymptotic method for solving nonlinear problems on the unimodal vibration of shells. The method is based on the notion of dynamic edge effects. It was used to study the effect of geometric nonlinearity on the asymptotic distribution of the natural frequencies of cylindrical panels and shells.

A second direction being taken in research is related to the solution of nonlinear problems of shell vibration on the basis of multimodal approximations of the deflections. It became obvious by the end of the 1970s that the previously examined unimodal models of shells were basically unusable for describing many of the phenomena which are observed experimentally and have their origin in the intensive interaction of several modes of vibration that are equivalent in terms of their energy contribution.

In [52, 105, 106, 127, 128, 130, 131, 168], results were presented from theoretical analyses of the bimodal deformation of cylindrical and spherical shells during free and forced vibrations. The "nonlinear" behavior of the shells was studied on the basis of dynamic equations written in mixed form (type (1)). As the interacting modes, conjugate modes were chosen in one case, while arbitrary modes (with different wave parameters) were chosen in another instance. Proceeding on the basis of an analysis of the free vibrations of shells, researchers showed for the first time that, when internal resonances are present (when the natural frequencies are close to one another or are multiples of one another), it is almost impossible to realize individual bending modes of the shells — the excitation of a given bending mode by some means invariably causes the excitation of other modes that are "resonant" with the mode excited first [128, 130]. In this case, the resolvent equations permit first integrals of the form $k_1 a_1^2 + k_2 a_2^2 = C_0$ $(k_1, k_2, C_0 = \mathrm{const})$, where a_1 and a_2 are the amplitudes of vibration corresponding to the resonance modes. This basic fact must be considered when choosing approximations for the deflection w. Taking it into account is particularly important in the case of closed shells of revolution, since they are always characterized by internal resonance (due to the existence of conjugate modes).

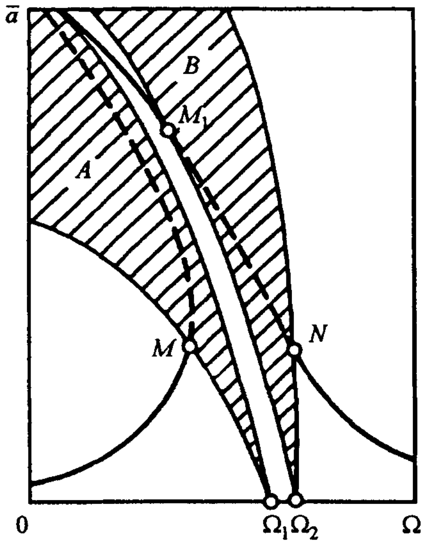


Fig. 1

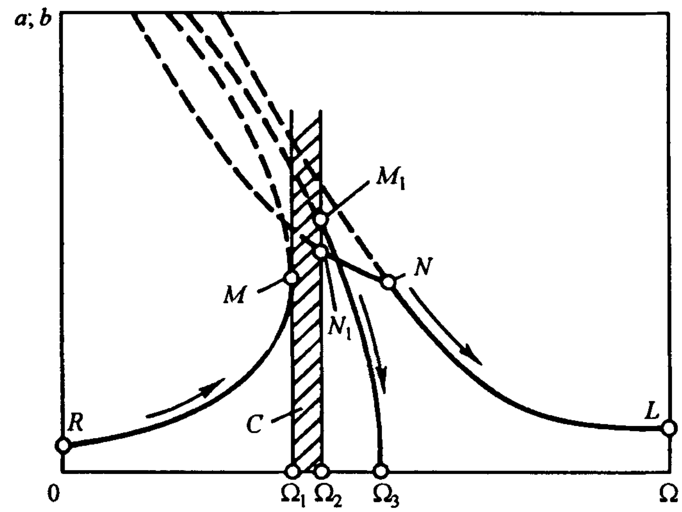


Fig. 2

The studies mentioned above were the first to theoretically explain one other specific effect — the "splitting" of the frequency spectrum of nonideal shells (compared to the case of the corresponding ideal shells, when $w_0 = 0$).

This phenomenon was observed experimentally as early as 1951 by Tobias [221], who showed that there are two preferred nodal configurations for each mode of vibration of a cylindrical shell. In the general case, these configurations have different natural frequencies. The difference between the frequencies can be regarded as a "measure of the initial imperfections in the shell." The presence of two peaks on the experimental AFCs and "splitting" of the frequencies were also described later by Koval [205], A. I. Telalov [182], and the authors of [52, 128]. In [103, 105, 130], a detailed study was made of the effect of initial imperfections on the forced bimodal vibration of cylindrical shells. The case of the interaction of conjugate modes was examined. Dynamic deflection w_1 was represented in the form

$$w_1 = f_1(t)\cos sy \sin rx + f_2(t)\sin sy \sin rx + f_3(t)\sin^2 rx, \tag{7}$$

and corresponded in the linear approximation to the initial deflection. Figure 1 shows a typical AFC corresponding to vibrations in a mode excited directly by the external force. Here, $\bar{a}$ is the amplitude of vibration in this mode; Ω is the frequency of periodic excitation; Ω_1 and Ω_2 are natural frequencies of the shell that correspond to conjugate modes. It can be seen that in addition to the traditional ("classical") region of instability A, there is another region B which is formed as a result of nonlinear coupling of the conjugate modes and the exchange of energy between them. The presence of the second region is responsible for the existence of the section M_1N on the upper branch of the AFC. No unimodal modes of vibration are realized on this section. Studies show that in this zone bimodal vibrations take place (both conjugate modes participate in the deformation of the shell) with the amplitudes $\bar{a}$ and $\bar{b}$, respectively (Fig. 2). A narrow zone C was discovered theoretically at about the same time. Unimodal and bimodal modes are unstable simultaneously within this zone. Certain random modes also exist in the zone, the form of these modes depending on the initial conditions.

Random "jumps" in the amplitudes of forced vibration of shells had already been described experimentally in [195], where it was proposed that the jumps could either be a fundamental phenomenon or could be characteristic only of the specific experiment. However, the authors of [195] did not theoretically examine the stability of the steady-state solutions that they obtained, which casts some doubt on the matter of whether or not they can actually be realized. The region in which the experimentally observed jumps occur probably coincides with the region of instability of those solutions.

Similar "irregularities" in the behavior of full-scale shells subjected to deterministic periodic excitation were also observed in [128, 113]. It is important to note that the manifestation of random modes was preceded by fairly complex periodic modes of the travelling circumferential wave type (the amplitudes and phase velocities of these waves generally evolve over

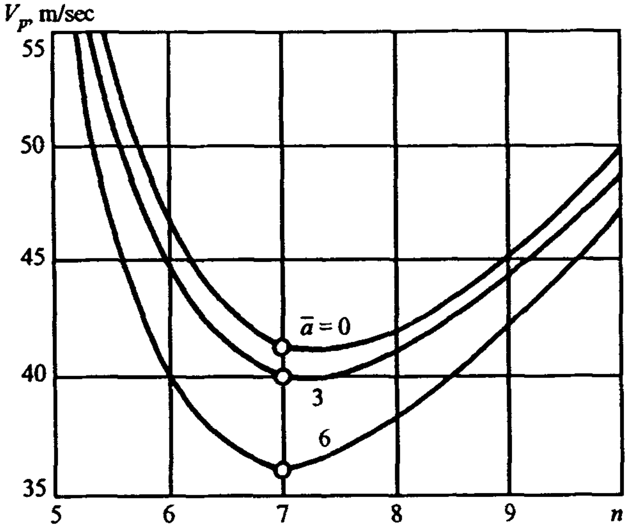


Fig. 3

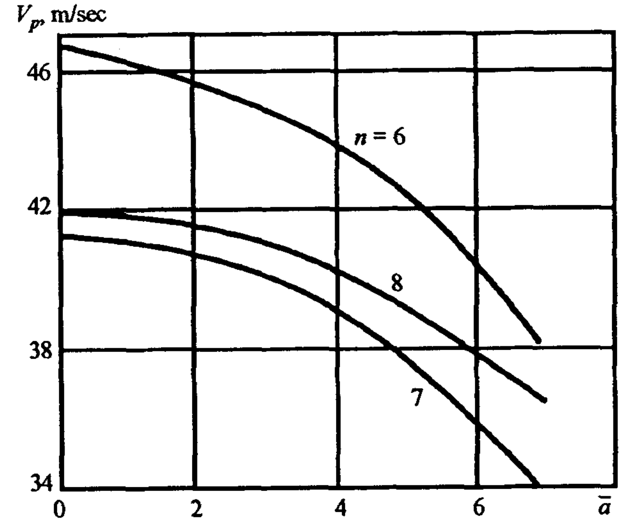


Fig. 4

time, rather than being constant). It was also found in a theoretical analysis that the region of dynamic chaos is always absent in the case of ideal shells (when $w_0 = 0$). Its realization requires that the elastic system possess a certain "asymmetry," and such asymmetry could be caused by initial deflections (which are nonaxisymmetric). It is also necessary that the external load be sufficient to overcome the forces which impede the indirect excitation of a second conjugate mode.

Forced nonlinear vibrations of shells corresponding to the interaction of nonconjugate bending modes (i.e. modes with different wave generation parameters) were examined in [106, 117, 128]. The effects of the interaction in this case are qualitatively similar to the case when conjugate modes are accounted for in the deflection w.

An unconventional approach was used in [107, 114, 115, 129] to calculate the nonlinear multimodal deformation of cylindrical shells subjected to periodic excitation. In this approach, the sought deflection w is represented in a mixed (space-time) form

$$w = \sum_{k=0} \sum_{j=1} a_{kj}(t) \cos\left[s_k y - \alpha_{kj}(t)\right] \sin r_j x,$$

$$\left(s_k = \frac{n_k}{R};\ r_j = \frac{m_j \pi}{l}\right) \tag{8}$$

which is typical of wave expansions. In fact, in the general case each term in (8) describes a travelling bending wave whose amplitude a_{kj} and phase α_{kj} are certain functions of time. If a_{kj} = const and $\alpha_{kj} = \omega_{kj} t$, where ω_{kj} is one of the natural frequencies of the shell, then this wave becomes a "classical" travelling wave characterized by a constant amplitude and constant phase velocity. The advantage of the given approach is that it makes it possible to reduce the solution of the initial problem to the analysis of equations that are free of the internal resonances characteristic of conventional approaches. Special methods based on averaging techniques were presented in [112, 125] to determine the parameters of waves — amplitude, phase velocity, and period. The solution for the functions a_{kj} and α_{kj} in (8) is represented in the form:

$$a_{jk} = \sqrt{u_{jk} + v_{jk} \sin 2(\Omega t + \vartheta_{jk})};$$

$$\alpha_{jk} = \varphi_{jk} + \arctan\frac{u_{jk}\tan(\Omega t + \vartheta_{jk}) + v_{jk}}{\sqrt{u_{jk}^2 - v_{jk}^2}}, \tag{9}$$

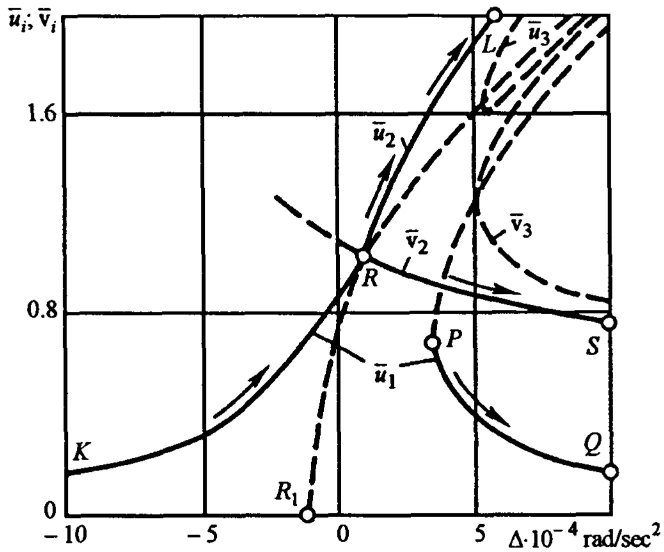


Fig. 5

$$(j = 1, 2, ...; k = 0, 1, 2, ...),$$

where u_{jk}, v_{jk} are the amplitude parameters of the wave and φ_{jk}, ϑ_{jk} are its phase parameters. This solution characterizes a stationary wave in a shell when $u_{jk} = v_{jk}$, while when $u_{jk} \neq v_{jk}$ it characterizes a travelling bending wave with a phase velocity equal to $v_p = \pm \dot{\alpha}_{jk}/n_j$. Figures 3 and 4 show graphs of the dependence of phase velocity on the wave parameter n and the dimensionless amplitudes $\bar{a} = a_{jk}/h$ for a shell with the following parameters:

$$E = 2\cdot 10^{11}\ \text{Pa};\ \frac{l}{R} = 2.5;\ \frac{h}{R} = 3.125\cdot 10^{-3};\ \rho = 7.8\cdot 10^3\ \text{kg/m}^3;\ \mu = 0.3;\ R = 0.16\ \text{m};\ m = 1\ .$$

It can be seen that the minimum velocity of the wave v_p is realized at $n = 7$. It is also interesting to note that the fundamental frequency of this shell corresponds to $n = 6$. The phase velocity of the wave always decreases with an increase in the amplitudes, with the largest decrease being seen for the mode ($m = 1$, $n = 6$).

Figure 5 shows the general form of the AFC

$$\bar{u}_i = \bar{u}_i(\Delta);\ \bar{v}_i = \bar{v}_i(\Delta)\quad \left(\bar{u}_i = \frac{u_{jk}}{h^2};\ \bar{v}_i = \frac{v_{jk}}{h^2};\ \Delta = \omega_{jk}^2 - \Omega^2\right),$$

constructed in the case $m = 1$, $n = 6$, $Q_0 = 60$ Pa [115] (here, $Q_0 = F_0/\rho h$; F_0 is the intensity of the periodic external load, with the period $T = 2\pi/\Omega$). The sections *KR* and *PQ* correspond to unimodal deformation of the shell (in the form of a stationary wave); sections *RL* and *RS* correspond to deformation of the shell in the form of a travelling wave (the frequency curves represented by dashed lines are unstable).

V. I. Gulyaev, V. A. Bazhenov, E. S. Dekhtyaryuk et al. [82, 83, 68, 69, etc.] used a somewhat different ("classical") formulation to examine nonlinear travelling waves in shells of revolution due to moving periodic loads. Along with simplified (traditional) shell models, these researchers used complex models that account for transverse shear and rotational inertia. The emphasis of these investigations was on devising effective algorithms for calculating and then analyzing the stability of periodic wave solutions of the corresponding dynamic equations. The algorithms are based on the combined use of the parameter continuation method, the Newton–Kantorovich method, Floquet's theory, and methods from branching theory. The algorithms made it possible to perform detailed studies of the resonance properties of shells of different geometries, examine complex nonlinear restructuring phenomena due to the resonance interactions of different waves, determine the dependences of the amplitudes of waves on the loading parameters, etc. Certain nonlinear problems on the propagation of free periodic waves in cylindrical shells were also studied.

"Polymodal" approaches to calculating the periodic vibrations of thin shells on the basis of geometrically nonlinear models were further generalized and developed in several investigations [75, 85, 86, 90, 92, 93, 133, 157, etc.] conducted in the Scientific Research Problems Laboratory at the Kiev Institute of Construction Engineering (now Kiev Technical University of Construction and Architecture). Some of these investigations were partially reviewed in [74] by Ya. M. Grigorenko and V. I. Gulyaev. A distinguishing feature of this research was the use of the finite-element method to develop effective computational techniques that allow the construction of finite nonlinear dynamic models of shells. Several (5–10) lower bending modes (obtained from the solution of a linearized problem) were used as the basis functions. Subsequent expansion made it possible to adequately describe the field of membrane displacements that accompanies large deflections. The system of resolvent equations of the shell has the form

$$\ddot{f}_i + 2\varepsilon_i \dot{f}_i + \omega_i^2 f_i + \sum_{j,k,l} \gamma_i^{j,k,l} f_j f_k f_l = q_i(t),$$

$$(i, j, k, l = 1, 2, \ldots N). \tag{10}$$

Here, ε_i are the damping parameters; ω_i are the natural frequencies of the shell; $\gamma_i^{j,k,l}$ are constant coefficients; $q_i(t)$ are periodic functions of time. A "global" (to use the authors' terminology) analysis of this system made it possible to study the entire range of complex modes and regimes of dynamic deformation of the shell which are attributable to mode interactions that take place under harmonic loading. Included in that range are periodic, quasi-periodic, and random modes. It was believed until recently that a shell subjected to deterministic oscillating loading could undergo only harmonic (subharmonic) vibrations. However, deeper investigation of this matter proved otherwise. In addition to regular modes, such shells can also undergo random vibrations whose main property is their unpredictable behavior. The authors of [85, 86, 90] provided a detailed description of mechanisms ("scenarios") for a change from regular vibrations to random vibrations (in the case of cylindrical shells), used qualitative methods to examine the phase trajectories corresponding to random processes, determined the dominant modes whose interaction results in the formation of limiting sets of the "strange attractor" type, and proposed methods of identifying random vibrations.

There has recently been increased interest in more thorough study of problems concerning the dynamic interaction of the bending modes of thin shells under complex resonance conditions, especially for combination resonances. Modern analytical methods are being used in this effort. The process of interaction is being examined within the context of the general theory of nonlinear waves in distributed systems on the basis of such concepts as phase and group synchronism and resonance wave ensembles [167]. The first studies in this area were performed by A. I. Potapov and D. A. Kovrygin for closed circular rings [121, 122, 163]. They examined interaction and energy exchange under conditions of phase synchronism (in this case, the normal frequencies of the ring ω_i and the wave vectors $\{k_i\}$ satisfy the conditions $\omega_1 + \omega_2 = \omega_3$; $k_1 + k_2 = k_3$) between high-frequency (radial) vibrations and a bending wave created by the superposition of conjugate modes. An analysis of the evolutionary equations established that the circumferential mode of vibration becomes unstable (in relation to small perturbations) and results in the resonance excitation of two coupled bending-circumferential waves that travel in opposite directions. Formulas were obtained for the period of energy transfer between the radial vibrations and the travelling waves.

The same problems were further examined in [119, 120] for infinitely long cylindrical shells. As in the case of the ring, the interaction of vibrational and wave modes of deformation was studied for free vibrations, which were described using the Flugge–Lur'e–Byrne model. The author demonstrated the connection between that model and the classical Timoshenko and Donnell–Mushtari–Vlasov equations. An analysis was made of mechanisms by which axisymmetric waves in a shell could become unstable, leading to the excitation of bending waves propagating in the longitudinal direction and travelling waves propagating in the circumferential direction.

A. I. Manevich [139] examined a conceptually similar problem concerning the features of nonlinear interaction in the case of fundamental internal resonance ($\omega_1 \approx \omega_2$) of conjugate bending modes of a circular ring undergoing free vibration. Using the method of multiple scales [209] in combination with topological analysis, Manevich was the first to obtain and study the "amplitude-frequency modulation" integral that links the amplitudes of vibrations in conjugate modes with the phase shift between those vibrations. It was shown that one of two outcomes is possible, depending on the initial conditions: the classical travelling wave mode (with constant amplitude and constant phase velocity), with the wave propagating in the circumferential direction; complex amplitude-frequency modulated vibrations (resulting from the superposition of slow modulation waves and a rapid travelling wave). Obviously, the results reported in [139] can easily be generalized to the case of cylindrical and other shells of revolution, since only circumferential modes take part in the formation of the above-described wave motions.

4. Parametrically excited vibrations. Problems concerning parametric vibrations arising from periodic loads applied in the plane of the shell occupy a special place among nonlinear problems on the vibration of thin shells. Since such loads enter into the main differential equations of the shell in the form of parameters, they are called parametric loads. In the scientific literature, problems of this type are usually referred to as problems of dynamic instability, since their resolution mainly involves solving the problem of loss of stability by the shell and subsequent examination of the supercritical (realized after loss of stability) modes of deformation. These two steps in the solution of the overall problem of dynamic instability are related to each other, and the ensuing discussion will be held within this context.

The basic postulates of the theory of nonlinear parametric vibration of different elastic systems, including shells, were laid out in the famous monograph [23] by V. V. Bolotin published more than 40 years ago. The concepts underlying those postulates were later widely used by many investigators. The results that were obtained are reflected in the generalizing monographs by A. S. Vol'mir [38, 39], G. Schmidt [189], S. A. Ambartsumyan [5], A. E. Bogdanovich [19], A. P. Filippov [184], V. O. Kononenko [94], and others.

As was the case for problems of forced vibrations of shells, the first studies in the given area were oriented toward use in calculations of simplified unimodal models obtained within the framework of traditional classical hypotheses. Inertial shearing forces, inertial rotational forces, and transverse shear strains were usually not considered. The problem of determining the stability of the vibrations of shells was reduced to the solution of the Mathieu–Hill problem [135, 136, 190]. This involved the use of one of the most important conclusions reached by V. V. Bolotin, stating that the boundaries of the regions of dynamic instability (RDIs) of shells can be judged on the basis of linearized forms of their equations of deformation. The RDI was constructed by various analytical methods, especially the method of trigonometric series, the small-parameter method, the Krylov–Bogolyubov asymptotic method, and other methods in nonlinear mechanics [22, 23, 135–138, 187, 190, etc.]. Effective numerical-analytical methods of constructing the RDI (the method of transfer matrices and the method of generalized Hill determinants) were also proposed in studies by V. V. Bolotin [24, 25]. These methods are free of the traditional assumptions regarding the smallness of the dissipative terms, the slight modulation of the parameters, and the closeness of the investigated system to the corresponding canonical system.

Among the most notable publications in the early stage of investigation of parametric vibrations of shells is the cycle of studies by V. Ts. Gnuni [60–63, etc.], as well as the works by G. V. Mishenkov [144, 145], S. A. Ambartsumyan, G. E. Bagdasaryan, and V. Ts. Gnuni [6, 13, 15, etc.]. The focus in this research was on anisotropic shells of revolution with symmetrically arranged layers (isotropic shells were examined in [144, 145]). Stability was analyzed and the amplitudes of the parametric vibrations were calculated by analyzing ordinary differential equations with a periodic coefficient

$$\ddot{f} + 2\varepsilon\dot{f} + \omega^2 (1 + 2\mu\cos\nu t) f + F(f, \dot{f}, \ddot{f}, t) = 0\,, \tag{11}$$

obtained by the Bubnov–Galerkin method. Here, ε is the damping parameter; μ is the depth of modulation of the parameter; ν is the frequency of parametric excitation; F is a function which is nonlinear with respect to $f, \dot{f}$, and $\ddot{f}$ and in most cases has the form $F = \delta f^2 + \gamma f^3$ or $F = \delta f^2 + \gamma f^3 + 2\kappa f(\dot{f}^2 + f\ddot{f})$, ($\delta, \gamma, \kappa = \mathrm{const}$).

As calculations showed, vibrations that occur in the region of the principal parametric resonance ($\omega \approx \nu/2$) usually tend toward higher frequencies ν (i.e., the AFC is of the "soft" type). At the same time, long cylindrical shells have AFCs of relatively complex form: vibrations tend toward smaller values of ν for low amplitudes a and in the opposite direction for high amplitudes (Fig. 6).

Parallel with the above-noted investigations, solutions were being obtained to the first problems on the effect of initial imperfections on parametric vibrations of shells [53–55, 102, etc.]. Initial deflection w_0 was assigned in the form

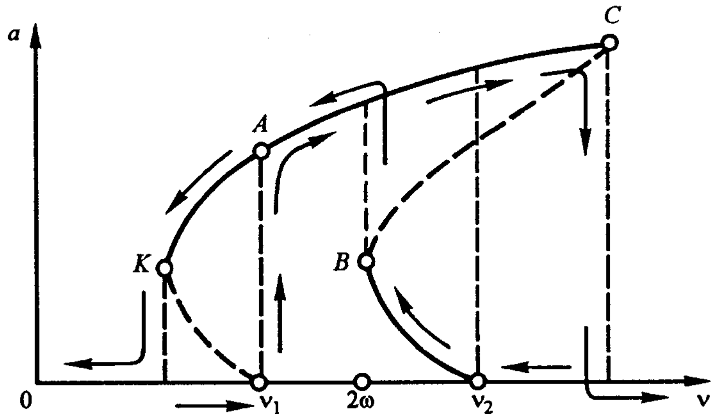


Fig. 6

$w_0 = f_0 \cos rx \cos sy$ in the studies by R. E. Geizenblazen and in the form $w_0 = f_0 \sin rx \sin sy + \psi_0 \sin^2 rx$, ($f_0$, ψ_0 = const) in the investigation by S. Kislyakov.

It was usually assumed that such deflection does not give rise to prestresses in shells and that the deflection can be regarded as being of the same order as the thickness h. The deflection w_1 was approximated in relation to the form of the function $w_0(x, y)$ (corresponding in form to the latter). It was established that the initial deflection w_0 causes a certain expansion of the RDI. The effect of an axisymmetric imperfection $w_0 = w_0(x)$ on this expansion is significantly greater than the corresponding effect of a nonaxisymmetric imperfection $w_0 = w_0(x, y)$.

In the studies by Yao [222, 223], problems on parametric vibrations of cylindrical shells were solved on the basis of more complex approximations of the deflection w. For example, it was assumed in [223] that

$$w = f_1(t) \cos rx \cos sy + f_2(t) \cos 2rx + f_3(t) \cos 2sy + f_4(t). \quad (12)$$

A numerical analysis of equations obtained as usual by the Bubnov–Galerkin method allowed the author to show that only the first term in (12) makes a decisive contribution to the development of parametric vibrations.

Other investigations of parametric vibrations conducted by foreign authors prior to 1970 were surveyed by S. Shu in [219]. Among other studies carried out during this period, we should mention those performed by R. M. Finkel'shtein [185], G. V. Nozhak [149], A. A. Berezovskii and Yu. I. Zharii [18, 89], N. K. Alekseeva [2, 3], G. M. Sal'nikov [169], Zh. M. Sibukaev [174]. Also noteworthy is later research by O. Adams and R. Evan-Iwanovski [192, 193], J. Singer, J. Arboc, and C. Babcock [218], H. Radwan and J. Genin [214], E. F. Sivak [175], and others. In most of these works, solutions to problems on the parametric vibration of shells (mainly cylindrical and spherical shells) were obtained by the Bubnov–Galerkin method and the use of simplified approximations of the deflection. The correctness and usefulness of these solutions depend to a significant extent on a successful choice of deflection approximation.

A number of problems on the parametric vibration of cylindrical shells were solved using the approach proposed by A. S. Vol'mir and A. T. Ponomarev [42]. These authors approximated the deflection function w by means of a trinomial expression

$$w = f(t)\,[\sin rx \sin sy + \psi(t) \sin^2 rx + \varphi(t)]. \quad (13)$$

Here, the function $\varphi(t)$ was determined on the basis of the condition of closure of the shell, while the parameter $\psi(t)$ was found approximately from the solution of the corresponding "quasistatic" problem [38]. Two types of parametric excitation of shells were examined: 1) the action of an axial load $N_x = N_0 + N_1 \cos \nu_1 t$ (in combination with constant radial pressure $q = q_0$); 2) pulsating external pressure $q = q_{10} \cos \nu_2 t$ (in combination with axial static compression $N_x = N_{10}$). The studies conducted in

[42] led to the important conclusion that both "soft" and "hard" excitation of a shell can be achieved by varying the parameters of combination static-dynamic loading (an orthotropic model was studied). An increase in the parameter q_0 (or N_{10}) leads to expansion of the RDI and its simultaneous shifting to a region of lower excitation frequencies ν_1 (ν_2).

The investigation [160] was apparently the first to examine unimodal parametric vibrations of a cylindrical shell with allowance for the imperfect elasticity of its material (the Davidenkov–Pisarenko hysteresis loop [159] was chosen to describe this elasticity). Features of the effect of physical nonlinearity on the characteristics of parametric vibrations were compared with the effect of nonlinear elasticity and inertia.

Parametric vibrations of cylindrical shells of nonlinearly elastic materials in a unimodal regime were also examined by S. B. Sinitsyn [177, 178] and Yu. N. Tamurov [180, 181]. When allowance was made for the energy losses, they were assumed to correspond to frequency-independent (Bock–Sorokin hypothesis) and amplitude-dependent internal friction. It was shown that it is incorrect to use the well-known viscous friction hypothesis (Voigt hypothesis) in calculating parametric vibrations of shells with a hard nonlinear elasticity characteristic, since internal friction decreases with the frequency of vibration. This makes it impossible to establish the maximum amplitude of vibration (the upper branch of the AFC never intersects the lower branch).

The first attempt to investigate multimodal parametric vibrations of shells was made by A. I. Telalov [182], who examined a nonlinear problem concerning the dynamic instability of a cantilever-supported cylindrical glass-plastic shell subjected to kinematic excitation (the top end of the shell was simply supported). The dynamic deflection w was approximated by the expression

$$w = [f_1(t)\cos sy + f_2(t)\sin sy]\,X(x) - \frac{n^2}{4}\,[f_1^2(t) + f_2^2(t)]\,X^2(x), \quad (w_0 = 0), \tag{14}$$

where $X(x)$ are the axial coordinates of the function. The Bubnov–Galerkin method was used to obtain a system of two nonlinear equations with variable parameters

$$\ddot{f}_i + \omega_i^2(1 - 2\mu\cos\nu t)f_i + 2\kappa(f_1\ddot{f}_1 + f_2\ddot{f}_2 + \dot{f}_1^2 + \dot{f}_2^2)f_i = 0, \tag{15}$$

where ω_1 and ω_2 are natural frequencies which are *a priori* assumed to differ somewhat from one another. An analysis of this system established that normal unimodal supercritical modes of deformation can become unstable at certain frequencies of excitation ν.

Substantial progress was made in the cycle of investigations [52, 104, 105, 128, etc.] on multidimensional problems concerning parametric vibrations of cylindrical shells, including shells with initial imperfections. The deflection approximation was constructed by a method typically used for forced vibrations (in the case of hinged support of the ends)

$$w_1 = f_1(t)\cos sy \sin rx + f_2(t)\sin sy \sin rx + f_3(t)\sin^2 rx + f_4(t). \tag{16}$$

The cofunction f_4 describes radial vibrations of points belonging to the end sections (the stress state of the shell is assumed to be momentless prior to loss of stability). The initial deflection w_0 was assigned in a form analogous to Eq. (16) ($f_{40} = 0$).

It was shown that interaction in the region of the main parametric resonance of the modes can give rise to complex buckling modes, particularly "progressive" (with increasing amplitude) travelling waves. Initial deflections in turn result in a certain transformation of the RDI — they shift it in the direction of higher or lower (compared to an ideal shell) excitation frequencies ν. Figures 7 and 8 show typical AFCs $a = a(\nu)$, $b = b(\nu)$ of parametric vibrations of a shell with an nonaxisymmetric initial deflection. These results were obtained for two cases — when the RDIs corresponding to conjugate modes in (16) do not intersect and when such intersection takes place (Fig. 8) (as usual, the stable sections are represented by solid lines and the unstable sections by dashed lines). The occurrence of the "double" vibrational hysteresis seen here has been repeatedly demonstrated experimentally in corresponding vibration tests of smooth metallic and glass-plastic shells [182, 49, 52, 79], shells with apparent additional masses [48, 128, 79], and reinforced shells [79]. The width of the region in

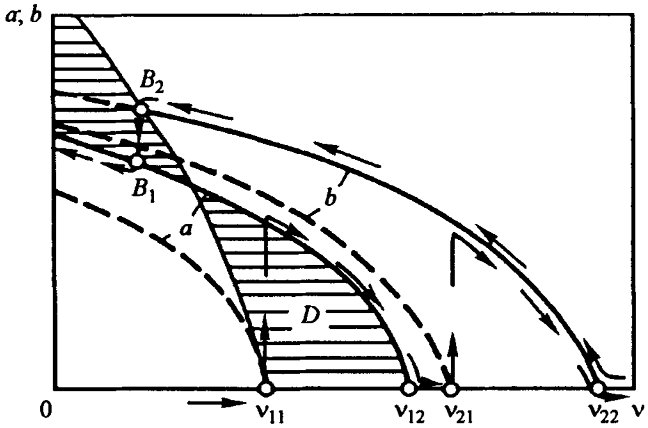


Fig. 7

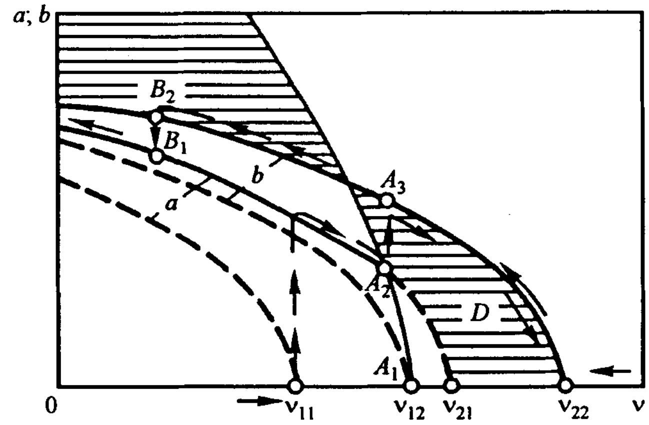
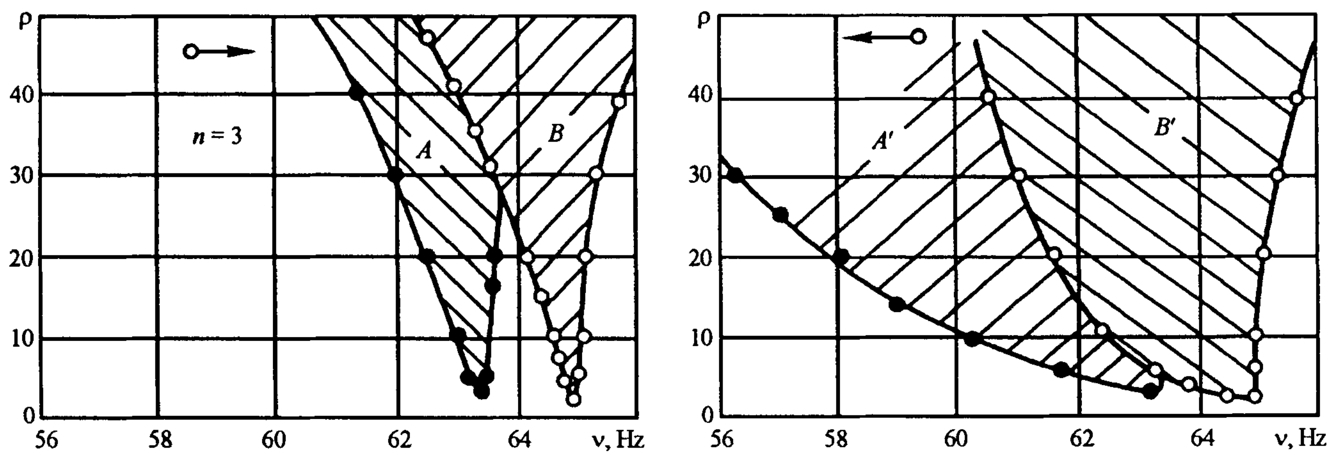


Fig. 8



a

b

Fig. 9

which hysteresis "frequency pulling" takes place is usually many times greater than the width of the region of dynamic instability. This is illustrated by Fig. 9, in which the regions A (A') correspond to excitation of the mode $m = 1$, $n = 3$ of a two-layer glass-plastic shell ($l = 0.96$ m; $R = 0.16$ m; $h = 5\cdot 10^{-3}$ m) that was cantilever-supported in tests. Regions B (B') correspond to the excitation of the corresponding conjugate mode [128] (here, ν is the frequency of vibration of the platform of the vibration stand and ρ is its vibrational acceleration).

The experimental studies also show that the AFCs of all of the shells subjected to parametric excitation have the form typical of nonlinear systems of the "soft" type (except for one or two lower bending modes ($n = 1, 2$)). This result was obtained earlier in a study of unimodal modes of buckling [27, 32, 80].

An attempt was made in [109] to theoretically explain the poor correlation between RDIs obtained by experiment and calculation. Bimodal parametric vibrations of a cylindrical shell with small holes were examined in [116]. The problem was solved by replacing part of the multiply connected structure by a "continuous" model with nonuniform bending stiffness and nonuniform density [164]. Combination parametric resonances occurring in the vibration of cylindrical shells (including shells with apparent additional masses) were studied in [189, 123, 111]. The authors described a specific "destabilization" phenomenon (expansion of the RDI) that was attributable to the combined effect of damping forces and initial deflections.

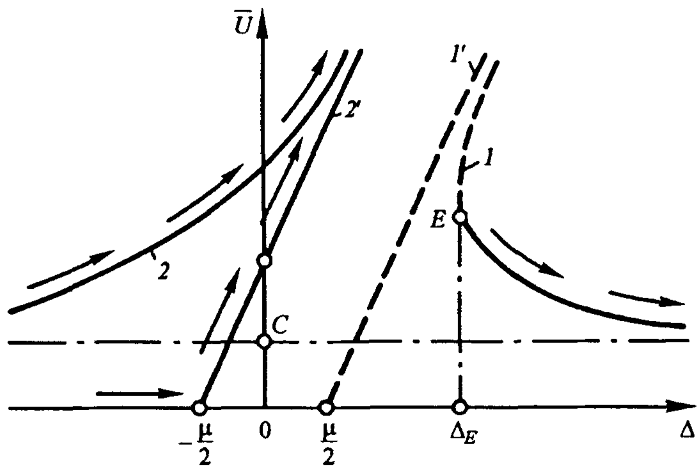


Fig. 10

A wave approach to the calculation of bimodal parametric vibrations of shells (with the deflection w being represented in the form of a generalized travelling wave) was used in [112, 162]. The resonance solution for the amplitude and phase of the wave process has the form (9), with allowance for the substitution $\Omega = \nu/2$. The AFC in this case (curves *1* and *2* in Fig. 10) differs in form from the traditional frequency characteristics for parametric vibrations (straight lines *1'* and *2'* exist only when $\bar{u} > C = \frac{a^2(0)\,\dot{\alpha}(0)}{\omega}$) (the notation used in the figure: $\bar{u}$ is the amplitude parameter of the wave; $\Delta = \omega - \nu/2$; μ is the level of parametric excitation). It can be seen that the excitation of bending modes in the shell requires that an initial energy "push" be provided by assigning certain nontrivial values to amplitude and phase velocity.

A similar formulation was used in [108] to study features of parametric vibrations of cylindrical shells additionally loaded by periodic radial pressure that was nonuniformly distributed over the lateral surface. The authors examined the case when the frequencies of both loads (longitudinal and transverse) satisfy the resonance relation and thus establish the prerequisites for the excitation of parametric and forced vibrations with a large amplitude. The AFC constructed for this case consists of sections characteristic both of purely forced vibrations (sections *CPQ* and *LDM* in Fig. 11) and parametric vibrations (*VK*). The branch of the AFC corresponding to parametric vibration *VK* is realized (is stable) only at frequencies ν such that $\nu < \nu_D = \nu_2$ (point *D* is the point of intersection of curves *2* and *3*). The phase of the parametric vibrations is unique for each steady-state value of amplitude. The complex travelling-wave type mode of deformation typical of forced vibrations is realized in the regions of instability of the unimodal modes (i.e., in the region $\nu_2 < \nu < \nu_3$).

Another approach to calculation of the nonlinear multimodal parametric vibrations of cylindrical shells was proposed by A. E. Bogdanovich and E. G. Feldman [19–21, etc.]. Dynamic problems are solved in two stages. First the spectrum of the RDIs of the shells is found by analyzing the linear equations that describe their deformation. Then the necessary approximation of the "nonlinear" deflection is chosen on the basis of the following principle: if the assigned characteristics of the external load correspond to a point in the plane of the parameters μ, ν (where ν is the frequency of excitation and μ is modulation depth) that is located only in one RDI (such as inside region A in Fig. 12), then it will suffice to approximate the deflection w by means of one term of a Fourier series: $w = f_1(t)\sin r_1 x \cos s_1 y$ $(r_1 = m_1\pi/l, s_1 = n_1/R)$. If two RDI intersect (region C), then the deflection should be chosen accordingly: $w = f_1(t)\sin r_1 x \cos s_1 y + f_2(t)\sin r_2 x \cos s_2 y$, $(r_2 = m_2\pi/l, s_2 = n_2/R)$. A similar procedure is used in the case of the intersection of three or more RDIs. A question that is yet to be answered is why the authors of the above-cited studies approximated the total dynamic deflection w by an incomplete Fourier series — only expansions in the functions $\cos(ny/R)$ were included in that series. This resulted in loss of the complete RDI spectrum corresponding to the functions $\sin(ny/R)$. Both of these expansions should be taken into account in nonlinear problems on parametric vibration, in light of the

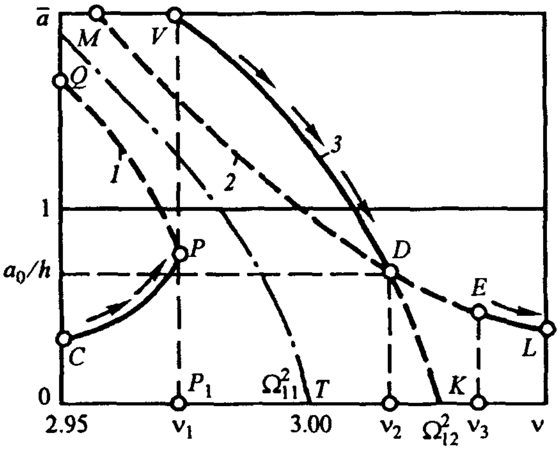


Fig. 11

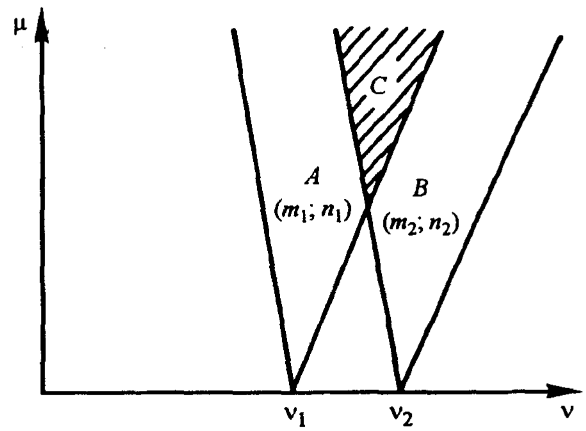


Fig. 12

already-stated fact that excitation of the bending mode $\sin\frac{m\pi x}{l}\cos\frac{ny}{R}$ by an external source inevitably "involves" the mode $\sin\frac{m\pi x}{l}\sin\frac{ny}{R}$ in the process as well (due to internal resonance).

The authors of [19–21] also deem it inexpedient to allow for "correcting" terms, particularly dual harmonics, in the deflection function. The theoretical AFCs that they obtained for composite shells thus always corresponded to "hard" characteristics, which is inconsistent with the experimental data (discussed above).

The experiments also show that the vibrations of points of the shell in orthogonal sections which occur after loss of stability occur relative to a certain line inside the shell (the line is sometimes referred to as the displaced position of dynamic equilibrium) [32, 79, 128]. This line becomes farther removed from the initial (undeformed) middle surface of the shell as the modes become more complex, which is an aspect of nonlinear deformation that should also be considered in the deflection approximation.

Unfortunately, in many current theoretical studies of the nonlinear parametric vibration of shells (this applies to problems involving free and forced vibrations) the deflection functions continue to often be chosen on the basis of representations of the linear theory characteristic of the early stage of research. For example, researchers are still using unimodal models of shells, which is incorrect in the case of finite strains; no allowance is made for "conjugate" modes in the deflection; when the deflection is approximated by several modes, those modes are chosen arbitrarily without allowance for the character of the relations ("strong," "weak") between them in the vibration process, etc. The final results thus are often of no practical value and contradict numerous experimental findings from vibration tests of shells.

5. Certain inverse problems. The nonlinear problems of shell dynamics that were discussed above belong to the class of so-called "direct" problems: the external loads on a shell and its characteristics are assigned and its stress-strain state is unknown. However, in a number of cases, certain "internal" characteristics of a shell (such as the damping parameters or the nonlinearities formed during vibration) are unknown beforehand. Thus, in essence the mathematical model that describes the dynamic behavior of the shell is also unknown. The problem of constructing the model on the basis of certain types of "output" data (which is assumed to be given) belongs to the class of "inverse" problems of dynamics. Information obtained from special dynamic tests of a full-scale object can be used as the "output" data.

As is known, the term "inverse problem" encompasses a range of problems, the main features of which were described in [41, 124, 142, 143, 161, etc.]. The inverse problems that will be discussed below are linked with the term "identification" which was used in [124] and involves the use of modal approaches ("modal identification"). The ultimate goal of these approaches is to construct differential equations that describe the vibrations of an elastic object (in the present case, a shell) in each of its normal modes (in the nonlinear case, quasi-normal modes). It is assumed *a priori* that there are no internal resonances in the elastic region and that all of the nonlinearities are small compared to the linear terms of the indicated equations. This makes it possible to use a conventional method (generally the "resonance" method) [94, 142, 143] to excite one mode of vibration of the object. It is therefore always possible to write an equation of the form

$$\ddot{f}_k + \omega_k^2 f_k = \varepsilon F_k (f_k, \dot{f}_k, \ddot{f}_k) + \varepsilon Q_0^{(k)} \cos \Omega t, \tag{17}$$

where F_k (...) is a function that describes energy dissipation and the nonlinear elastic and inertial characteristics of the object (for the kth mode); $0 < \varepsilon << 1$, $Q_0^k = \text{const}$.

In accordance with [124], the structure of the nonlinear forces in (17) is assumed to be known. It can be determined by so-called qualitative identification methods, which are based on the use of certain geometric constructs or "standard portraits" such as AFCs, APFCs (amplitude-phase-frequency characteristics), and diagrams of decaying free vibrations constructed on the basis of suitable vibration tests of the objects being studied. These portraits are compared to standard portraits corresponding to specific systems with typical nonlinear characteristics for the elastic, inertial, and dissipative forces. If the structure of the nonlinear forces is unknown, specific values of those forces can be found by special methods of analytical identification [124, 161].

The "weakest" link in the construction of theoretical models of shells is the determination of their damping characteristics, since the associated energy losses are affected by many factors — energy dissipation in the material of the shell itself, structural dissipation, and external resistance forces (such as aerodynamic or hydrodynamic damping).

Some results of experimental studies of integral vibration-damping characteristics of shells were reported in [28, 31, 47, 79, 142] for cylindrical shells and in [67] for cylindrical and spherical shells. Tests were conducted with cantilever support of the shells; both isotropic and composite (glass-plastic) shells were studied. Damping capacity was determined on the basis of logarithmic vibration decrements. On the whole, the energy dissipation characteristics depend in a fairly complex manner on the amplitudes of flexural vibrations and the wave parameters. The vibration decrements can either increase or decrease with an increase in amplitude [47, 67]. A similar pattern is seen in the variation of the parameters that characterize wave formation. It was established that the vibration decrements decrease significantly as the bending mode becomes more complex (i.e., as the parameter n increases). The decrease in the integral vibration decrements with an increase in the amplitude of deformation can be attributed in general to an increase in the role of structural energy dissipation in the overall balance of energy losses [67]. No general laws governing "nonlinear" energy dissipation in shells have yet been established. It is clear only that the resolution of this problem will require simultaneous consideration of effect of several factors — the amplitudes of vibration, the "geometry" of the mode being excited, boundary conditions, the effect of interaction of the modes, etc.

The studies [30, 118] were apparently the first to solve the problem of qualitatively identifying the deforming forces in forced nonlinear vibrations of cylindrical shells in the unimodal mode. It was assumed that these forces could satisfy one of the following relations

$$R(f, \dot{f}) = \pm \frac{\eta}{k} [a \pm f)^{n_0} - 2^{n_0 - 1} a^{n_0}] \tag{18}$$

(the Pisarenko–Davidenkov hypothesis [159]);

$$R(f, \dot{f}) = \pm \alpha_0 a^{n_0} \sqrt{1 - f^2/a^2} \tag{19}$$

(the Panovko hypothesis [154]);

$$R(f, \dot{f}) = (\varepsilon_0 + 4 \varepsilon_1 f^2) \Omega \dot{f} \tag{20}$$

(nonlinear viscous friction [23]), where η, k, n_0, ε_0, ε_1, $\Omega = \text{const}$; a is the amplitude of the vibrations.

The model represented by Eq. (17) simultaneously considered the nonlinear elastic force $Q_{el} = \gamma f^3$ and the inertial force $Q_i = 2\kappa f (f \ddot{f} + \dot{f}^2)$ (γ, $\kappa = \text{const}$). A comparison of the APFCs of velocity for Eq. (17) and for a full-size two-layer glass-plastic shell established that the physical essence of the damping forces in the shell is most accurately reflected by hypothesis (19). It was also determined that nonlinear inertia has a dominant effect on the natural frequencies (compared to nonlinear elasticity).

This method was generalized in [12] to the case when parametric excitation is used as the external load on the object in tests. Other methods of determining the damping forces in the vibration of elastic bodies (including shells) were presented in [29, 110], these methods being based on an analysis of decaying free vibrations of the object. The first integral formulas for determining the unknown nonlinear functions of the resistance forces for a specified (established experimentally) "decay angle" $\theta = \theta(a)$ were obtained in [29]. Relations were derived in [110] to determine the unknown damping factors of a shell from experimental vibration decrements and the initial imperfections of the shell.

Certain problems on identifying the dynamic characteristics of aeroelastic thin-walled structures were discussed in the monograph [41] by A. S. Vol'mir. Studies by V. O. Kononenko and N. P. Plakhtienko [124] and other researchers (references can be found in [41, 124]) established the general theoretical principles of analytical identification methods that can be used to construct nonlinear mathematical models of thin-walled shell systems.

On the whole, it should be noted that research into the problems involved in constructing reliable theoretical dynamic models of shells deformed with finite deflections is still in its initial stage and needs to be taken farther.

6. Conclusion. It follows from an analysis of the investigations mentioned above that many of the problems connected with the nonlinear vibration of thin shells have already been exhaustively studied. Many of the results from these investigations have proven useful in practical applications and have served as a scientific foundation for improving the designs and weight characteristics of many different types of objects. At the same time, it is also possible to identify several unresolved problems in the nonlinear dynamics of shells that are important and could be the subject of future investigations.

1. First of all, it is necessary to devise **general methodological approaches** to the construction of adequate nonlinear theoretical dynamic models of shells. These approaches should be based on the principle of "reasonable compromise" — the model should be as simple as possible while still accurately depicting the actual processes that take place in a specific elastic object. To solve this problem, it will be necessary to use modern methods of theoretical analysis and modern computer technology to conduct basic research into the energy coupling of bending modes of free vibration accompanied by internal resonances created by the nonlinearities accounted for in the model. This will make it possible to construct the deflection function (and, thus, the theoretical model itself) with allowance for the relative contribution of each mode to the over vibration process (for actual shells, these modes do not necessarily follow one another with an increase in the wave parameters). The damping forces should be determined on the basis of experimental analysis of the vibrations.

2. In the area of **forced** vibrations, it is necessary to improve research into the multimodal deformation of shells by using refined variants of nonlinear dynamic equations and examining more complex cases — especially polyharmonic external loading. Considerable attention should be given to the analysis of nonlinear vibrations of spherical and conical shells modeled by systems with many degrees of freedom. Also important is the problem of calculating vibrations (free and forced) of shells with large deflections with allowance for different types of nonlinearities (geometric, physical, inertial).

3. In the area of **parametric** vibrations, it is necessary to develop special analytical methods of study oriented toward the case of substantial (arbitrary) parameter modulation depths. Nearly all nonlinear problems on the dynamic stability of shells have been solved with the assumption that the parametric loads are small (of the same order as the nonlinear terms in the dynamic equations), which in many cases is not justified.

Among the other problems of practical interest are those concerning multimodal and wave modes of loss of stability by shells subjected to periodic combination loading — a combination of radial excitation (nonuniformly distributed over the lateral surface) and harmonic loading in the plane of the shell.

4. Substantial progress must be made in problems on the nonlinear vibration of shells that have **initial shape imperfections**. It is not yet clear which practical methods of accounting for initial deflections in the equations of shells (as part of the total deflection or through a change in curvature) will provide the most reliable results. There has not been enough study of the mechanisms by which initial imperfections of shells affect their frequency spectrum, the regions of parametric instability, the character of the loss of stability, the amplitudes of vibration, the realization of wave modes of deformation, and other issues. In calculating the nonlinear vibrations of nonideal shells, broad use should be made of a new approach in which the deflection functions account for the actual modes of such shells ("mixed mode"). These modes are calculated with allowance for initial deflections.

5. Methods of solving **inverse** problems of shell dynamics must be further refined. Such methods have generally not been developed for multimodal idealizations of shells, when the deformation process takes place with allowance for the nonlinear interaction of many different modes.

6. Research also continues to be promising in regard to problems on **random vibrations** of shells undergoing deterministic vibrations. The study of this phenomenon, surprising in its nature, is important for reasons besides obtaining new basic insights into the nonlinear mechanics of shells. For practical purposes, it is extremely important to know how this phenomenon affects the stress-strain state of shell structures, their dynamic strength, and their reliability in service. Meanwhile, there is a danger that the many of the different theoretical studies of chaos now being performed will not be of any practical value for specific shells because the theoretical model does not adequately conform to the full-scale object. Allowing for actual damping in the models (the damping being essentially nonlinear in the case of vibrations of shells with large deflections) may in fact make it physically impossible to realize chaos. Thus, appropriate experimental studies must play an important role in the study of random vibrations of shells. It will be necessary to develop special methods that can "recognize" random modes among other complex modes — quasi-periodic, subharmonic, etc. As regards theoretical approaches, it will be necessary to more thoroughly study the effects of linear and nonlinear damping of shells on the mechanics of chaos, the regions in which chaos exists, scenarios by which a transition from regular to irregular modes can take place, and spectral properties.

In conclusion, we note that the problems discussed above are important for calculating the nonlinear vibrations of all shells, whether they be smooth or reinforced, uniform or anisotropic, elastic, viscoelastic, or elastoplastic, or have holes and apparent additional masses, etc.

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