AFFINE QUIVERS AND CANONICAL BASES

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Introduction

Let $\mathbf{U}$ be the quantized enveloping algebra attached by Drinfeld and Jimbo to a symmetric generalized Cartan matrix (see [D]); let $\mathbf{U}=\mathbf{U}^+\otimes\mathbf{U}^0\otimes\mathbf{U}^-$ be its triangular decomposition.

In [L], a purely geometric construction of $\mathbf{U}^{\leqslant}=\mathbf{U}^0\otimes\mathbf{U}^-$ (as a Hopf algebra) was given in terms of perverse sheaves on the moduli space of representations of a quiver; the construction gave at the same time a canonical basis of $\mathbf{U}^-$ with very favourable properties. (This amounts, in principle, to a geometric construction of $\mathbf{U}$, since by Drinfeld's quantum double construction [D, § 13], the Hopf algebra $\mathbf{U}$ can be reconstructed in a simple way from the Hopf algebra $\mathbf{U}^{\leqslant}$.)

In the generality of [L], the simple perverse sheaves which enter in the canonical basis of $\mathbf{U}^-$ are only defined in an abstract way, but are not known in a concrete form, except in the simplest case (type A, D, E) when they are exactly the simple perverse sheaves corresponding to orbits.

One of the aims of this paper is to describe in concrete terms the simple perverse sheaves which form the canonical basis in the affine case (that is, the case of a symmetric affine Cartan matrix).

According to an observation of McKay [MK], there is a natural $1-1$ correspondence between symmetric affine Cartan matrices and finite subgroups Γ of $SL(\rho)$, where ρ is a two dimensional $\mathbf{C}$-vector space.

We will show that the construction of $\mathbf{U}^{\leqslant}$ given in [L] can be reformulated in the affine case entirely in terms of the corresponding finite group Γ. Thus, in the affine case, $\mathbf{U}^{\leqslant}$(and hence, as explained above, $\mathbf{U}$) are constructed directly in terms of Γ.

We now describe the content of this paper in some detail. Let Γ be as above; assume that Γ contains the non-trivial element c in the centre of $SL(\rho)$.

Section 1 is concerned with the study of affine roots in terms of the representation theory of Γ; some results in [DR, § 1] are recovered.

In section 2 we give a new treatment of the known theory of representation of affine quivers, emphasizing its connection with the corresponding finite subgroup of $SL(\rho)$; I believe that this is simpler than the earlier treatments.

For type A_1, this theory is due to Kronecker [Kr], who attributes some partial results to Weierstrass. Kronecker's work was concerned with the classification of linear pencils of bilinear forms on a pair of vector spaces. This is equivalent to classifying,

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for any two finite dimensional **C**-vector spaces M^+, M^-, the orbits of the natural action of $GL(M^+) \times GL(M^-)$ on the vector space of all **C**-linear maps

(a) $$M^+ \otimes \rho \to M^-.$$

It is also equivalent to the problem of classifying orbits of pairs of linear maps from M^+ to M^- under the action of the same group, which are just the isomorphism classes of representations of an affine quiver of type A_1.

The classification of (indecomposable) representations of arbitrary affine quivers was given in [GP], [N], [DF], [DR], [R3]. In these references the finite group Γ is not present. The theory becomes much simpler if everything is developed in terms of Γ. In particular, the problem of classifying representations of an affine quiver is the same as classifying the orbits of $GL_\Gamma(M^+) \times GL_\Gamma(M^-)$ on the space of Γ-equivariant linear maps (a), for any pair M^+, M^- of Γ-modules on which c acts as the identity (resp. minus identity). Hence this problem can be regarded as a Γ-equivariant version of the problem studied by Kronecker.

Section 3 collects together some definitions and results of [L] in a form suitable for the purposes of this paper. It also contains a new result (3.6) which is a prerequisite for section 4.

Section 4 is concerned with the study of certain Lagrangian varieties introduced in [L]; we classify the irreducible components of these Lagrangians in the setup of section 2 and show that they index a basis of the (non-quantum) $\mathbf{U}^-$. (This last fact has been conjectured in [L, 12.14].)

The case of cyclic Γ has a special role in the theory; it is studied in section 5, which includes the case where Γ has odd order (excluded in the rest of the paper).

In section 6 we describe explicitly (enumerate) the perverse sheaves which form the canonical basis of $\mathbf{U}^-$ in the affine case, by indicating their support and the corresponding local systems in the framework of section 2; this is the main result of the paper.

Let us now replace Γ by a closed, reductive, infinite subgroup of $SL(\rho)$. (There are three such Γ up to conjugacy: a maximal torus, its normalizer, or the full $SL(\rho)$.) This leads to an infinite graph: of type A_∞ (infinite in both directions), of type D_∞, or of type A_∞ (infinite in one direction, finite in the other). Most results of this paper extend in an obvious way to the infinite case. In fact, some of the difficulties present in the finite case disappear in the infinite case, so that the infinite case is actually simpler.

I wish to thank Ringel for useful discussions and also for explaining to me the results of [DR].

Connections with earlier work. The fact that the algebra $\mathbf{U}^-$ can be constructed in terms of affine quivers has been first shown by Ringel; a brief announcement of his results is in [R2]. The details of his construction are not yet written up, except for affine type A, discussed in his preprint [R4], which I have received one year after submitting this paper. Ringel's construction is quite different from ours; it does not lead to a canonical basis.

This paper has been influenced by ideas in Kronheimer's paper [K]. In [K], Kronheimer has studied the action of $GL_\Gamma(M)$ on the space of Γ-equivariant maps (a) in the case where $M=M^+=M^-$ is the regular representation of Γ, and used this to show that the corresponding two-dimensional Kleinian singularity admits a hyper-Kähler structure.

1. Roots

1.1. Throughout this paper, ρ denotes a fixed two dimensional **C**-vector space with a given non-degenerate symplectic form $\langle\ ,\ \rangle$.

Let Γ be a finite subgroup of the special linear group $SL(\rho)$ which contains the unique non-trivial element c in the centre of $SL(\rho)$.

By a Γ-module we understand a finite dimensional **C**-vector space with a given linear action of Γ. Note that ρ is naturally a Γ-module.

Let I be the set of isomorphism classes of simple Γ-modules. For each $i\in I$ we assume given a simple Γ-module ρ_i in the class i.

Following McKay [MK], we regard I as set of vertices of a graph as follows. For any $i\neq j$ in I, we set $T^i_j=\mathrm{Hom}_\Gamma(\rho_i\otimes\rho,\rho_j)$. We have $\dim T^i_j=\dim T^j_i\in\{0,1,2\}$. If $\dim T^i_j=0$, then i, j are not joined in the graph; if $\dim T^i_j=1$, then i, j are joined by exactly one edge; if $\dim T^i_j=2$ (which only happens when $\Gamma=\{1,c\}$), then i, j are joined by exactly two edges.

This is an *affine Coxeter graph*. Hence the corresponding (affine) roots are defined. The purpose of this section is to reexamine the (known) properties of affine roots from a non-standard point of view, namely from the point of view suggested by McKay's correspondence.

1.2. For two Γ-modules M, M′ we define $(M:M')=\dim\mathrm{Hom}_\Gamma(M,M')$. We also define

$$(M,M')=(M\otimes\mathbf{C}^2:M')-(M\otimes\rho:M')$$

where $\mathbf{C}^2$ is taken with the trivial Γ-action.

Let $\mathcal{G}\Gamma$ be the Grothendieck group of Γ-modules. The elements $\rho_i\,(i\in I)$ form a **Z**-basis of $\mathcal{G}\Gamma$.

Note that $(:)$ and $(,)$ extend uniquely to symmetric bilinear pairings $\mathcal{G}\Gamma\times\mathcal{G}\Gamma\to\mathbf{Z}$. Now (M,M) is an even integer for any $M\in\mathcal{G}\Gamma$ (since $(\rho_i,\rho_i)=2$). We have

(a) $(M,M)\geqslant 0$ for any $M\in\mathcal{G}\Gamma$ and

(b) $(M,M)=0$ if and only if $M=n\mathbf{r}$ for some integer n where $\mathbf{r}=\mathbf{C}[\Gamma]$, regarded as a Γ-module for the left translation.

Indeed, if $M, M'\in\mathcal{G}\Gamma$, we have

(c) $$(M,M')=|\Gamma|^{-1}\sum_{\gamma\in\Gamma}\mathrm{tr}(\gamma,M)\,\mathrm{tr}(\gamma^{-1},M')(2-\mathrm{tr}(\gamma,\rho)).$$

Now $2-\mathrm{tr}(\gamma,\rho)\in\mathbf{R}_{\geqslant 0}$ and $\mathrm{tr}(\gamma, M)\ \mathrm{tr}(\gamma^{-1}, M)\in\mathbf{R}_{\geqslant 0}$ for all γ, and (a) follows. We also see that $(M, M)=0$ if and only if for each $\gamma\neq 1$ we have $\mathrm{tr}(\gamma,M)=0$, and (b) follows.

If M is a Γ-module, we denote by M^* the dual space of M with its natural Γ-module structure; this defines a homomorphism $M\to M^*$ of $\mathcal{G}\Gamma$ onto itself.

1.3. Let R be the set of all vectors $\alpha\in\mathcal{G}\Gamma$ such that $(\alpha,\alpha)=2$. The elements of R are called *roots*.

We denote by $\mathcal{G}\Gamma_+$ the subset of $\mathcal{G}\Gamma$ consisting of elements which can be represented by actual Γ-modules. We have

(a) $$R\subset\mathcal{G}\Gamma_+\cup(-\mathcal{G}\Gamma_+).$$

Otherwise, we can find $\alpha\in R$ and two Γ-modules M, M′ which are non-zero and disjoint such that $\alpha=M-M'$. We have $(M,M')=-(M\otimes\rho:M')\leqslant 0$. Moreover, since M, M′ are disjoint, neither of them can be an integer multiple of $\mathbf{r}$; using 1.2(a), (b), it follows that $(M,M)>0$, $(M',M')>0$ and, these being even integers, we have $(M,M)\geqslant 2$, $(M',M')\geqslant 2$. Thus $2=(\alpha,\alpha)=(M,M)+(M',M')-2(M,M')\geqslant 2+2+0$, a contradiction; this proves (a).

By (a), we have a partition

(b) $$R=R_+\sqcup R_-$$

where $R_+=R\cap\mathcal{G}\Gamma_+$ and $R_-=R\cap(-\mathcal{G}\Gamma_+)$.

The elements of R_+ are said to be *positive roots*. For example, $\rho_i\in R_+$ for all $i\in I$.

1.4. For $\delta=\pm 1$, let I^δ be the set of all $i\in I$ such that the central element $c\in\Gamma$ (of order two) acts on ρ_i as multiplication by δ. We have a partition $I=I^1\sqcup I^{-1}$.

Consider the homomorphism $\nu:\mathcal{G}\Gamma\to\mathbf{Z}$ given by $\nu(M)=\mathrm{tr}(c,M)$.

Let ${}^+R_+$ be the set of all $\alpha\in R_+$ such that $\nu(\alpha)>0$. Similarly, let ${}^-R_+$ be the set of all $\alpha\in R_+$ such that $\nu(\alpha)<0$; let ${}^0R_+$ be the set of all $\alpha\in R_+$ such that $\nu(\alpha)=0$. Thus we have a partition

$$R_+={}^-R_+\sqcup{}^+R_+\sqcup{}^0R_+.$$

1.5. If M is a Γ-module and $\delta = \pm 1$, we denote by M^δ the δ-eigenspace of $c: M \to M$. Then we have a Γ-module decomposition $M = M^1 \oplus M^{-1}$. Note that $\rho = \rho^{-1}$. Now $M \mapsto M^\delta$ extends by linearity to virtual representations; hence it may be regarded as a homomorphism $\mathscr{G}\Gamma \to \mathscr{G}\Gamma$.

We define a homomorphism $\mathbf{c}_\delta : \mathscr{G}\Gamma \to \mathscr{G}\Gamma$ by

(a) $$\mathbf{c}_\delta(M) = M^\delta - M^{-\delta} + M^\delta \otimes \rho.$$

If M_1 satisfies $M_1 = M_1^{-\delta}$, we have $\mathbf{c}_\delta(M_1) = -M_1$. On the other hand, if M_2 satisfies $M_2^\delta \otimes \rho = M_2^{-\delta} \otimes \mathbf{C}^2$, then $\mathbf{c}_\delta(M_2) = M_2$. Now for any M we have $2M = M_1 + M_2$ with M_1, M_2 as above. (We have $M_1 = M^{-\delta} \otimes \mathbf{C}^2 - M^\delta \otimes \rho$ and $M_2 = M^\delta \otimes \mathbf{C}^2 + M^\delta \otimes \rho$.) It follows that

(b) $$\mathbf{c}_\delta^2 = 1.$$

Moreover, using the fact that $(M_1, M_2) = 0$ for any M_1, M_2 as above, we see that

$$(\mathbf{c}_\delta(M), \mathbf{c}_\delta(M')) = (M, M')$$

for any M, M′. In particular, we have

(c) $$\mathbf{c}_\delta(R) \subset R.$$

From the definitions we have

(d) $$\nu(\mathbf{c}_\delta(x)) = -\nu(x)$$

and

(e) $$\dim(\mathbf{c}_\delta(x)) = \dim(x) + 2\delta\nu(x)$$

for all $x \in \mathscr{G}\Gamma$. (Here dim denotes the dimension of a virtual module.)

1.6. Let $M \in R_+$. The following conditions are equivalent.

(a) $\mathbf{c}_\delta M \in R_-$.

(b) $M = \rho_i$ for some $i \in I^{-\delta}$.

(c) $\mathbf{c}_\delta M = -M$.

Indeed, if (a) holds, then $(\mathbf{c}_\delta M)^\delta = M^\delta \in \mathscr{G}\Gamma_-$ hence $M^\delta = 0$. Thus we have $M = \sum_i n_i \rho_i$ where i runs over $I^{-\delta}$ and n_i are integers $\geqslant 0$. For i, i' distinct elements of $I^{-\delta}$ we have $(\rho_i, \rho_{i'}) = 0$; hence $2 = (M, M) = \sum_i 2n_i^2$. It follows that $n_i = 1$ for some index i and $n_j = 0$ for all other indices, so that $M = \rho_i$. Thus, (b) holds.

Conversely, if (b) holds, then from the definition we have $\mathbf{c}_\delta M = -M$ so that (c) holds. The fact that (c) implies (a) is obvious.

1.7. For any $i \in I$ and any $r \in \mathbf{N}$, we set

(a) $$\alpha(i,r) = \mathbf{c}_{(-1)^{r-1}\delta} \ldots \mathbf{c}_{-\delta}\mathbf{c}_{\delta}\rho_i \in \mathscr{G}\Gamma$$

where δ is such that $i \in I^{\delta}$ and the product has r factors.

We have

(b) $$\alpha(i,r) = \rho_i \otimes (S^r\rho \oplus S^{r-1}\rho)$$

where $S^r\rho$ denotes the r-th symmetric power of ρ (as a Γ-module) with the convention that $S^{-1}\rho = 0$. This follows immediately from the definition, by induction on r, using the identity $S^r\rho \otimes \rho = S^{r-1}\rho \oplus S^{r+1}\rho$.

From (b) we see that

(c) $$\nu(\alpha(i,r)) = (-1)^r \nu(\rho_i).$$

From (b), (c) and 1.5 (c) we see that

(d) $$\alpha(i,r) \in {}^+R_+ \text{ if } i \in I^{(-1)^r}, \qquad \text{and} \qquad \alpha(i,r) \in {}^-R_+ \text{ if } i \in I^{-(-1)^r}.$$

Proposition 1.8. — *(a) If $i, i' \in I$ and $r, r' \in \mathbf{N}$, we have $\alpha(i,r) = \alpha(i',r')$ if and only if $i = i'$ and $r = r'$.*

(b) ${}^+R_+$ consists of the elements $\alpha(i,r)$ for various $r \in \mathbf{N}$ and $i \in I^{(-1)^r}$.

(c) ${}^-R_+$ consists of the elements $\alpha(i,r)$ for various $r \in \mathbf{N}$ and $i \in I^{-(-1)^r}$.

We first prove (a). Assume that $\alpha(i,r) = \alpha(i',r')$ and that $i \in I^{\delta}$, $i' \in I^{\delta'}$. We may assume that $r \geqslant r'$. We have $\nu(\alpha(i,r)) = \nu(\alpha(i',r'))$, hence, from 1.7 (c), $(-1)^r\delta \dim \rho_i = (-1)^{r'}\delta' \dim \rho_{i'}$. The two sides of this equality must have the same sign: $(-1)^r\delta = (-1)^{r'}\delta'$. Therefore from our assumption and the definition it follows that $\alpha(i, r-r') = \alpha(i', r'-r')$. Thus we are reduced to the case where $r' = 0$.

If $r > 0$, then both $\alpha(i,r)^1$ and $\alpha(i,r)^{-1}$ are non-zero (they are, up to order, $\rho_i \otimes S^r\rho$, $\rho_i \otimes S^{r-1}\rho$). On the other hand, $\alpha(i',0)^{-\delta'} = 0$. Hence $\alpha(i,r)$ cannot be equal to $\alpha(i',0)$ for $r > 0$. Thus we have $r = r' = 0$. This clearly implies $i = i'$; (a) is proved.

We now prove (b). Let $\alpha \in {}^+R_+$. We define a sequence of roots $\alpha[s]$ $(s = 0, 1, \ldots)$ by $\alpha[0] = \alpha$, $\alpha[s] = \mathbf{c}_{(-1)^s}\alpha[s-1]$ for $s \geqslant 1$. It is enough to show that there exists $s \geqslant 0$ such that $\alpha[s] = \rho_i$ for some $i \in I^{(-1)^s}$.

From 1.5 (d), (e), it follows by induction on s that, for any $s \geqslant 0$, we have $\nu(\alpha[s]) = (-1)^s \nu(\alpha)$ and $\dim \alpha[s] = \dim \alpha - 2s\nu(\alpha)$.

The last quantity is > 0 for $s = 0$ and is < 0 for sufficiently large s (since $\nu(\alpha) > 0$). Hence there exists an $s \geqslant 0$ such that $\dim \alpha[s'] > 0$ for $s' = 0, 1, \ldots, s$ and $\dim \alpha[s+1] \leqslant 0$. We then have $\alpha[s] \in R_+$ and $\alpha[s+1] \in R_-$. We have $\mathbf{c}_{(-1)^{s+1}}\alpha[s] \in R_-$, hence, by 1.6, $\alpha[s] = \rho_i$ for some $i \in I^{(-1)^s}$. Thus, (b) is proved.

We now prove (c). Let $\alpha \in {}^-R_+$. If $\mathbf{c}_1(\alpha) \in R_-$ then, by 1.6, we have $\alpha = \rho_i$ for some $i \in I^{-1}$ so that $\alpha = \alpha(i, 0)$, as required. Assume now that $\mathbf{c}_1(\alpha) \in R_+$. By 1.5 (d), we have $\nu(\mathbf{c}_1(\alpha)) = -\nu(\alpha) > 0$, hence $\mathbf{c}_1(\alpha) \in {}^+\mathrm{R}_+$. By (*b*) we have $\mathbf{c}_1(\alpha) = \alpha(i, r)$ for some $r \in \mathbf{N}$ and some $i \in \mathrm{I}^{(-1)^r}$. We then have $\alpha = \mathbf{c}_1(\alpha(i, r)) = \alpha(i, r+1)$. This proves (*c*).

1.9. In this subsection we assume that Γ is cyclic of order $2n$. Let $\mathrm{M} \in \mathscr{G}\Gamma$; we can write $\mathrm{M} = \sum_i p_i \rho_i$; from the definitions we see that $(\mathrm{M}, \mathrm{M}) = \sum (p_i - p_j)^2$ where the sum is taken over all edges of the graph attached to Γ in 1.1 and i, j denote the two ends of an edge. We see that $\mathrm{M} \in \mathrm{R}$ precisely when there are two edges such that $|p_i - p_j| = 1$ and for all other edges we have $p_i = p_j$. If $n = 1$ it follows that R_+ consists of the elements $s\rho_i + (s+1)\rho_{i'}$ and $(s+1)\rho_i + s\rho_{i'}$ for various $s \in \mathbf{N}$, where $\mathrm{I} = \{i, i'\}$; in particular, ${}^0\mathrm{R}_+$ is empty.

We now assume that $n \geqslant 2$. Then the corresponding graph is a $2n$-gon with set of vertices I.

If $\mathrm{M} = \sum_i p_i \rho_i$ is in R_+, then the previous argument shows that there is an integer $a \geqslant 0$ such that the sets $\mathrm{J} = \{i \in \mathrm{I} \mid p_i = a\}$ and $\mathrm{J}' = \{i \in \mathrm{I} \mid p_i = a+1\}$ have the following properties:

(*a*) J, J′ are non-empty;

(*b*) J, J′ form a partition of I;

(*c*) the full subgraphs $\langle \mathrm{J} \rangle$, $\langle \mathrm{J}' \rangle$ with vertices J or J′ are both connected.

(Conversely, given $a \geqslant 0$ and J, J′ with the properties just described, they define a positive root.) The following fact is easily verified:

(*d*) We have $\mathrm{M} \in {}^0\mathrm{R}_+$ if and only if J and J′ have even cardinals.

We now show the following.

(*e*) If $\mathrm{M} \in {}^0\mathrm{R}_+$, then $\mathrm{M} \neq \mathrm{M}^*$.

Indeed, assume that $\mathrm{M} = \mathrm{M}^*$. Now $\rho_i \to \rho_i^*$ defines an involution of I which takes any edge of our graph to an edge; clearly, this involution has exactly two fixed points, and they are not joined by an edge. Since $\mathrm{M} = \mathrm{M}^*$, this involution must map J into itself and J′ into itself, where J, J′ are as above. Hence it defines graph automorphisms of $\langle \mathrm{J} \rangle$ and $\langle \mathrm{J}' \rangle$ which are both non-trivial (since the two fixed points on I are not joined). Since $\langle \mathrm{J} \rangle$ is a graph of type A_m and m is even (see (d)) our graph automorphism has no fixed points on J. Similarly it has no fixed points on J′. This contradicts the fact that it has fixed points on I.

1.10. For general Γ, we define ${}^0\mathscr{G}\Gamma$ as the subgroup of $\mathscr{G}\Gamma$ consisting of all M such that $\nu(\mathrm{M}) = 0$.

Assume now that Γ is cyclic of order $2n \geqslant 4$. We shall write the trivial one-dimensional Γ-module as ρ_{i_0} with $i_0 \in \mathrm{I}$. Let L′, L″ be the two Γ-stable lines in ρ. Then L′, L″ are naturally one-dimensional Γ-modules and we have $\mathrm{L}' = \rho_{i'}$, $\mathrm{L}'' = \rho_{i''}$

for well-defined elements $i' \neq i''$ of I; these are exactly the elements of I which are joined with i_0 in our graph. Note that

$$\{\rho_i \mid i \in I\} = \{L'^{\otimes j} \mid 0 \leqslant j \leqslant 2n-1\} = \{L''^{\otimes j} \mid 0 \leqslant j \leqslant 2n-1\}.$$

Let ${}^0\mathscr{G}'\Gamma$ (resp. ${}^0\mathscr{G}''\Gamma$) be the subgroup of $\mathscr{G}\Gamma$ generated by the elements $L'^{\otimes(2j)} + L'^{\otimes(2j+1)}$ (resp. $L''^{\otimes(zj)} + L''^{\otimes(2j+1)}$) for $0 \leqslant j \leqslant n-1$.

The following properties are easily verified.

(a) ${}^0\mathscr{G}\Gamma = {}^0\mathscr{G}'\Gamma + {}^0\mathscr{G}''\Gamma$.

(b) ${}^0\mathscr{G}'\Gamma \cap {}^0\mathscr{G}''\Gamma = \{s\mathbf{r} \mid s \in \mathbf{Z}\}$.

(c) $(M', M'') = 0$ for any $M' \in {}^0\mathscr{G}'\Gamma$ and $M'' \in {}^0\mathscr{G}''\Gamma$.

(d) The homomorphism $*: \mathscr{G}\Gamma \to \mathscr{G}\Gamma$ maps ${}^0\mathscr{G}'\Gamma$ onto ${}^0\mathscr{G}''\Gamma$.

We now define ${}^0R_+(L')$ (resp. ${}^0R_+(L'')$ to be the intersection ${}^0R_+ \cap {}^0\mathscr{G}'\Gamma$ (resp. ${}^0R_+ \cap {}^0\mathscr{G}''\Gamma$).

From the results in 1.9 we see that

(e) ${}^0R_+(L')$ consists of the (distinct) elements

$$\sum_{j=2r}^{2r+2m-1} L'^{\otimes j} \in \mathscr{G}\Gamma;$$

${}^0R_+(L'')$ consists of the (distinct) elements

$$\sum_{j=2r}^{2r+2m-1} L''^{\otimes j} \in \mathscr{G}\Gamma;$$

(here r is any integer in $[0, n-1]$ and $m \geqslant 1$ is an integer not divisible by n).

1.11. We still assume that Γ is cyclic of order $2n \geqslant 4$. Let $M \in {}^0\mathscr{G}\Gamma$ be such that $M^* = M$ and $(1:M)$ is even. By 1.10(*a*), we can write $M = \alpha' + \alpha''$ where $\alpha' \in {}^0\mathscr{G}'\Gamma$ and $\alpha'' \in {}^0\mathscr{G}''\Gamma$ (notation of 1.10) From $M = M^*$ we deduce $\alpha' - \alpha''^* = \alpha'^* - \alpha''$; using 1.10(d), we see that both sides of the last expression are contained in ${}^0\mathscr{G}'\Gamma \cap {}^0\mathscr{G}''\Gamma$ hence, by 1.10(b), they are of the form $s\mathbf{r}$ for some integer s. We have the following equalities modulo 2:

$$s = (1 : s\mathbf{r}) = (1 : \alpha') - (1 : \alpha''^*) = (1 : \alpha') + (1 : \alpha'') = (1 : M) = 0.$$

Thus we have $s = 2s'$ for some integer s'. Replacing α', α'' by $\alpha' - s'\mathbf{r}$, $\alpha'' + s'\mathbf{r}$ respectively, we see that we may assume that $s = 0$ so that $\alpha'' = \alpha'^*$.

1.12. In this subsection we assume that Γ is not cyclic, but that it has a normal subgroup Γ_1 which is cyclic of index 2. Then Γ_1 must have order $2n \geqslant 4$. We shall use the notation of 1.10 for Γ_1 instead of Γ; in particular, L′, L″ are the two Γ_1-stable lines in ρ and ${}^0\mathscr{G}'\Gamma_1$, ${}^0\mathscr{G}''\Gamma_1$ are the corresponding subspaces of ${}^0\mathscr{G}\Gamma_1$.

Let $\mathscr{U}$ be the subgroup of $\mathscr{G}\Gamma$ consisting of those M such that $\mathrm{tr}(\gamma, \mathrm{M})=0$ for any $\gamma\in\Gamma-\Gamma_1$ and for $\gamma=c$. Let $\mathscr{U}_1={}^0\mathscr{G}'\Gamma$. Taking induced representations gives a homomorphism

(a) $\mathrm{Ind}: \mathscr{U}_1 \to \mathscr{U}.$

We show that

(b) this is an isomorphism preserving the inner product (,).

Let $\mathrm{M}\in\mathscr{U}$ and let M_1 be its restriction to Γ_1. By assumption we have

$$0=|\Gamma_1|^{-1}\sum_{\gamma\in\Gamma-\Gamma_1}\mathrm{tr}(\gamma,\mathrm{M})=(1:\mathrm{M})-(\sigma:\mathrm{M}),$$

where σ is the non-trivial one-dimensional Γ-module on which Γ_1 acts trivially. It follows that $(1:\mathrm{M})=(\sigma:\mathrm{M})$, hence $(1:\mathrm{M}_1)=(1+\sigma:\mathrm{M})=(1:\mathrm{M})+(\sigma:\mathrm{M})$ is an even integer. If $\gamma\in\Gamma-\Gamma_1$, then $\gamma x\gamma^{-1}=x^{-1}$ for all $x\in\Gamma_1$. It follows that $\mathrm{M}_1=\mathrm{M}_1^*$. We can therefore apply 1.11 to M_1; we see that there exists $\alpha'\in\mathscr{U}_1$ such that $\mathrm{M}_1=\alpha'+\alpha'^*$. This shows that M and $\mathrm{Ind}(\alpha')$ have the same trace at all elements of $\Gamma_1-\{c\}$; from our assumptions they also have the same trace (zero) at c and at elements of $\Gamma-\Gamma_1$, so they are equal. Thus the map (a) is surjective.

Next we assume that α, $\beta\in\mathscr{U}_1$ are mapped by (a) to the same element. It follows that $\alpha+\alpha^*=\beta+\beta^*$ as elements of ${}^0\mathscr{G}\Gamma_1$. Then $\alpha-\beta=-\alpha^*+\beta^*$; using 1.10(d), we see that both sides of the last equality are contained in ${}^0\mathscr{G}'\Gamma_1\cap{}^0\mathscr{G}''\Gamma_1$ hence, by 1.10(b), they are of the form $s\mathbf{r}$ for some integer s which must satisfy $s\mathbf{r}^*=-s\mathbf{r}$. (Hence, $\mathbf{r}$ is relative to Γ_1.) It follows that $s=0$, so that $\alpha=\beta$. Thus the map (a) is injective hence an isomorphism.

Finally, let α, $\beta\in\mathscr{U}_1$. In the following formulas inner products refer either to Γ or Γ_1:

$$(\mathrm{Ind}\,\alpha\otimes\rho:\mathrm{Ind}\,\beta)=((\alpha+\alpha^*)\otimes\rho:(\beta+\beta^*))/2=(\alpha\otimes\rho:\beta+\beta^*).$$
$$(\mathrm{Ind}\,\alpha:\mathrm{Ind}\,\beta)=((\alpha+\alpha^*):(\beta+\beta^*))/2=(\alpha:\beta+\beta^*).$$

Hence

$$(\mathrm{Ind}\,\alpha,\ \mathrm{Ind}\,\beta)=(\alpha,\ \beta+\beta^*)=(\alpha,\ \beta).$$

(The last equality follows from 1.10(d), (c).) Our assertion is verified.

1.13. We now consider a general Γ. Let F be the set of points L in the projective line $\mathrm{P}(\rho)$ whose isotropy group Γ_{L} (a cyclic group) in Γ has order >2. Then F is a finite set. There is a canonical involution $L \mapsto \tilde{L}$ of F defined by the requirement that L, $\tilde{L}$ are distinct and $\Gamma_L = \Gamma_{\tilde{L}}$. This involution commutes with the action of Γ on $P(\rho)$.

Let $\mathscr{X}$ be a set of representatives for the Γ-orbits on F. We may assume that the following condition is satisfied: if $L \in \mathscr{X}$ and $\tilde{L}$ is not in the same Γ-orbit as L, then $\tilde{L} \in \mathscr{X}$.

For $L \in \mathscr{X}$, we shall denote by $\mathscr{U}_L$ the subspace of $\mathscr{G}\Gamma_L$ defined as ${}^0\mathscr{G}'\Gamma$ in 1.10 (but replacing Γ, L' by Γ_L, L); note that Γ_L is a cyclic group of order $\geqslant 4$. Let $\mathrm{Ind}_L : \mathscr{U}_L \to {}^0\mathscr{G}\Gamma$ be the induction homomorphism. Let $\mathscr{V}_L$ be its image.

Proposition 1.14. — (*a*) *For any* $L \in \mathscr{X}$, *the homomorphism* Ind_L *is injective and it preserves the inner product* $(\, ,)$.

(b) *We have* $\sum_{L \in \mathscr{X}} \mathscr{V}_L = {}^0\mathscr{G}\Gamma$.

(c) *If* L, $L' \in \mathscr{X}$ *are distinct then* $\mathscr{V}_L$ *is orthogonal to* $\mathscr{V}_{L'}$ *for* $(\, ,)$.

(d) *For any* $L \in \mathscr{X}$, *we have* $\mathbf{r} \in \mathscr{V}_L$.

Let Γ'_L be the normalizer of Γ_L in Γ. Then either $\Gamma'_L = \Gamma_L$ or Γ_L has index two in Γ'_L. Let $\mathscr{U}'_L$ be the subspace of $\mathscr{G}\Gamma'_L$ generated by the elements M such that $\mathrm{tr}(c, M) = 0$ and $\mathrm{tr}(\gamma, M) = 0$ for all $\gamma \in \Gamma'_L - \Gamma_L$. Let $\mathrm{Ind}'_L \to {}^0\mathscr{G}\Gamma$ be the homomorphism given by inducing from Γ'_L to Γ. Let $\mathscr{V}'_L$ be its image.

Let $\mathscr{X}'$ be a maximal subset of $\mathscr{X}$ with the following property: if L, L' are distinct elements of $\mathscr{X}'$, then Γ_L, $\Gamma_{L'}$ are not conjugate in Γ. Then, if $L \in \mathscr{X}'$ and $\tilde{L} \in \mathscr{X}$, we have $\tilde{L} \notin \mathscr{X}'$.

The following result will be needed in the proof of 1.14.

Lemma 1.15. — (a) *For any* $L \in \mathscr{X}$, *the homomorphism* Ind'_L *is injective, and it preserves the inner product* $(\, ,)$.

(b) *We have* $\sum_{L \in \mathscr{X}'} \mathscr{V}'_L = {}^0\mathscr{G}\Gamma$.

(c) *If* L, $L' \in \mathscr{X}'$ *are distinct, then* $\mathscr{V}'_L$ *is orthogonal to* $\mathscr{V}'_{L'}$ *for* $(\, ,)$.

Let $M \in \mathscr{U}'_L$. The character of $\mathrm{Ind}'_L(M)$ is zero at c and at elements of Γ which are not contained in a conjugate of Γ_L; its value at an element $\gamma \in \Gamma_L - \{1, c\}$ is equal to $\mathrm{tr}(\gamma, M)$. From this we see immediately that Ind'_L preserves $(\, ,)$ and that (c) holds (we use 1.2(c) and the fact that in that formula we may omit the term $\gamma = 1$).

We also see that if $\mathrm{Ind}'_L(M)$ is zero then the character of M is zero at all elements of $\Gamma'_L - \{1\}$ and, clearly, also at 1, so that $M = 0$; thus (a) is proved.

We now prove (b). We consider the following statement:

(d) For any $M \in {}^0\mathscr{G}\Gamma$ and any $L \in F$ there exists $M' \in {}^0\mathscr{G}\Gamma'_L$ such that $\mathrm{tr}(\gamma, M) = \mathrm{tr}(\gamma, \mathrm{Ind}'_L(M'))$ for all $\gamma \in \Gamma_L - \{1\}$ and $\mathrm{tr}(\gamma, M') = 0$ for all $\gamma \in \Gamma'_L - \Gamma_L$.

Granting this, we see that the sum over $L \in \mathscr{X}'$ of the elements $\mathrm{Ind}'_L(M')$ provided by (d) is an element $M'' \in {}^0\mathscr{G}\Gamma$ such that M'', M have the same character values at all elements $\neq 1$, hence $M = M'' + s\mathbf{r}$ for some integer s. Since both M'' and $s\mathbf{r}$ are contained in $\sum_{L} \mathcal{V}'_L$, (b) follows.

It remains to prove (d).

If Γ is cyclic, there is nothing to prove. If Γ is a quaternion group of order 8, the verification of (d) is an easy exercise, left to the reader. Hence we may assume that Γ is non-cyclic, of order $\geqslant 10$ and that (d) is already known when Γ is replaced by one of its proper subgroups.

Consider first the case where Γ'_L is a proper subgroup of Γ so that (d) is known to hold for (Γ'_L, L) instead of (Γ, L). Let M_L be the restriction of M to Γ'_L. Then $M_L \in {}^0\mathcal{G}\Gamma'_L$. By our assumption we can find $M' \in {}^0\mathcal{G}\Gamma'_L$ such that $\mathrm{tr}(\gamma. M') = 0$ for all $\gamma \in \Gamma'_L - \Gamma_L$ and $\mathrm{tr}(\gamma, M) = \mathrm{tr}(\gamma, M')$ for all $\gamma \in \Gamma_L - \{1, c\}$. Then clearly M' is as required in (d) (for Γ, L).

Next we consider the case where $\Gamma'_L = \Gamma$. Then Γ is a binary dyhedral group of order $\geqslant 12$. We may assume that $L \in \mathcal{X}'$. For any $L' \in \mathcal{X}'$ with $L' \neq L$, we have that $\Gamma'_{L'} \neq \Gamma$, hence, by the previous argument, (d) holds for Γ, L'. Hence we can find $\mu_{L'} \in {}^0\mathcal{G}\Gamma$ such that $\tilde{\mu} = \sum_{L' : L' \neq L} \mu_{L'}$ satisfies $\mathrm{tr}(\gamma, M) = \mathrm{tr}(\gamma, \tilde{\mu})$ for all $\gamma \in \Gamma - \Gamma_L$ and $\mathrm{tr}(\gamma, \tilde{\mu}) = 0$ for all $\gamma \in \Gamma_L - \{1\}$.

Let $M' = M - \tilde{\mu}$. Then $\mathrm{tr}(\gamma, M') = 0$ for all $\gamma \in \Gamma - \Gamma_L$ and $\mathrm{tr}(\gamma, M') = \mathrm{tr}(\gamma, M)$ for all $\gamma \in \Gamma_L - \{1\}$. Hence M' is as required; (d) is proved. The lemma follows.

1.16. We now prove 1.14 (a). If $\Gamma'_L = \Gamma_L$, then 1.14 (a) follows from 1.15 (a). If $\Gamma'_L \neq \Gamma_L$, then the homomorphism in 1.14 (a) is the composition of the homomorphism in 1.15 (a) with one as in 1.12 (a), so that 1.14 (a) follows from 1.15 (a) and 1.12 (b).

We now prove 1.14 (b). If $\Gamma'_L = \Gamma_L$, then $\tilde{L} \in \mathcal{X}$ and $\mathcal{V}'_L = \mathcal{V}_L + \mathcal{V}_{\tilde{L}}$ by 1.10 (a). If $\Gamma'_L \neq \Gamma_L$ then $\mathcal{V}'_L = \mathcal{V}_L$ (by 1.12 (b)). Thus $\sum_{L \in \mathcal{X}'} \mathcal{V}'_L = \sum_{L \in \mathcal{X}} \mathcal{V}_L$ so that 1.14 (b) follows from 1.15 (b).

We now prove 1.14 (c). From the proof of 1.14 (b) just given and from 1.15 (c) we see that it is enough to show that $\mathcal{V}_L$, $\mathcal{V}_{\tilde{L}}$ are orthogonal for (,) whenever L, $\tilde{L}$ are both in $\mathcal{X}$. In this case, $\Gamma'_L = \Gamma_L = \Gamma'_{\tilde{L}} = \Gamma_{\tilde{L}}$ is a cyclic group and the desired result follows from 1.15 (c) and 1.10 (c).

Finally, 1.14 (d) follows immediately from definitions. This completes the proof of 1.14.

The following result gives, in conjunction with 1.8, a complete parametrization of the set R_+.

Corollary 1.17. – *For any* $L \in \mathcal{X}$ *we define* ${}^0R_+(L)$ *to be the intersection* ${}^0R_+ \cap \mathcal{V}_L$ *and we define an integer* $n_L \geqslant 2$ *by* $|\Gamma_L| = 2n_L$.

(a) *We have a partition* ${}^0\mathrm{R}_+ = \sqcup_{\mathrm{L}\in\mathscr{X}} {}^0\mathrm{R}_+(\mathrm{L})$.

(b) *For any* $\mathrm{L}\in\mathscr{X}$, *the set* ${}^0\mathrm{R}_+(\mathrm{L})$ *consists of the (distinct) elements*

$$\mu(\mathrm{L}, r, m) = \sum_{j=2r}^{2r+2m-1} \mathrm{Ind}_{\mathrm{L}} \mathrm{L}^{\otimes j} \in \mathscr{G}\Gamma$$

(here r is any integer defined up to a multiple of n_{L} *and* $m \geqslant 1$ *is an integer not divisible by* n_{L}*).*

Let $\alpha\in{}^0\mathrm{R}_+$. By 1.14 (b), we can write $\alpha=\sum_{\mathrm{L}\in\mathscr{X}}\alpha_{\mathrm{L}}$ where $\alpha_{\mathrm{L}}\in\mathscr{V}_{\mathrm{L}}$.

Using 1.14 (c) we have $2=(\alpha, \alpha)=\sum_{\mathrm{L}}(\alpha_{\mathrm{L}}, \alpha_{\mathrm{L}})$. The last expression is a sum of even integers $\geqslant 0$; it follows that there exists $\mathrm{L}\in\mathscr{X}$ such that $(\alpha_{\mathrm{L}}, \alpha_{\mathrm{L}})=2$ and $(\alpha_{\mathrm{L}'}, \alpha_{\mathrm{L}'})=0$ for $\mathrm{L}'\neq\mathrm{L}$. By 1.2 (b) we have $\alpha_{\mathrm{L}'}=s_{\mathrm{L}'}\mathbf{r}$ for some integer $s_{\mathrm{L}'}$ hence $\alpha_{\mathrm{L}'}\in\mathscr{V}_{\mathrm{L}}$ (see 1.14 (d)). It follows that $\alpha\in\mathscr{V}_{\mathrm{L}}$, so that $\alpha\in{}^0\mathrm{R}_+(\mathrm{L})$. Next we assume that $\beta\in{}^0\mathrm{R}_+(\mathrm{L})\cap{}^0\mathrm{R}_+(\mathrm{L}')$ with $\mathrm{L}\neq\mathrm{L}'$ in $\mathscr{X}$. By 1.14 (c) we then have $(\beta, \beta)=0$ contradicting $(\beta, \beta)=2$. This proves (a). Now (b) follows immediately from 1.10 (e), 1.14 (a), 1.14 (c). The corollary is proved.

Corollary 1.18. — *The inclusions define an isomorphism*

$$\oplus_{\mathrm{L}\in\mathscr{X}}(\mathscr{V}_{\mathrm{L}}/\mathbf{Zr})\cong{}^0\mathscr{G}\Gamma/\mathbf{Zr}.$$

This follows immediately from 1.14 (b), (c) and 1.2 (b).

1.19. The involution $*:\mathscr{G}\Gamma\to\mathscr{G}\Gamma$ clearly maps R_+ onto itself. From the definitions we see that this involution leaves stable each of the subsets ${}^+\mathrm{R}_+$, ${}^-\mathrm{R}_+$, ${}^0\mathrm{R}_+$. Moreover, if $\mathrm{L}\in\mathscr{X}$, and $\Gamma_{\mathrm{L}}\neq\Gamma'_{\mathrm{L}}$, we have $\mu(\mathrm{L}, r, m)^*=\mu(\mathrm{L}, r, m)$; if $\mathrm{L}\in\mathscr{X}$, and $\Gamma_{\mathrm{L}}=\Gamma'_{\mathrm{L}}$, we have $\mu(\mathrm{L}, r, m)^*=\mu(\tilde{\mathrm{L}}, r, m)$.

2. Indecomposable representations of affine quivers

2.1. Given $\delta=\pm 1$, we define $\Omega(\delta)$ to be the following orientation of our graph (in 1.1): an edge joining $i\neq j$ is oriented from i to j if $i\in\mathrm{I}^{\delta}$ and $j\in\mathrm{I}^{-\delta}$ (see 1.4); this is well defined since $\mathrm{T}^i_j=0$ (see 1.1) when i, j are both in I^1 or both in I^{-1}. Our graph together with the orientation $\Omega(\delta)$ is an *affine quiver*.

Let $\mathscr{B}^{\delta}$ be the (abelian) category of representations of this affine quiver with the orientation $\Omega(\delta)$.

We recall that an object of $\mathscr{B}^{\delta}$ is an I-graded **C**-vector space $\mathbf{V}=\oplus\mathbf{V}_i$ together with linear maps $x_{i\to j}:\mathbf{V}_i\to\mathbf{V}_j$ for any oriented edge $i\to j$. A morphism from $(\mathbf{V}, x_{i\to j})$ to $(\mathbf{V}', x'_{i\to j})$ is a collection of linear maps $y_i:\mathbf{V}_i\to\mathbf{V}'_i$ $(i\in\mathrm{I})$ such that $y_j x_{i\to j}=x'_{i\to j}y_i$ for all $i\to j$. The objects of $\mathscr{B}^{\delta}$ are also called representations of the affine quiver.

The classification of indecomposable objects of $\mathscr{B}^\delta$ is known. For type A_1 it is due to Kronecker [K]; for the other types, see [GP], [N], [DF], [DR], [R3].

In this section we shall reexamine this classification from a non-standard point of view, namely from the point of view suggested by McKay's correspondence.

This point of view leads to simpler proofs than in the references cited. (However, some of the methods we use are inevitably the same as in those references.)

2.2. We define a category $\mathscr{C}^\delta$ as follows. An object of this category is a pair (M, Δ) where M is a Γ-module and $\Delta: M^\delta \otimes \rho \to M^{-\delta}$ is a Γ-linear map. Note that giving Δ is the same as giving a collection Δ_e ($e \in \rho$) of linear maps $M^\delta \to M^{-\delta}$ depending linearly in e and satisfying $\gamma(\Delta_e(x)) = \Delta_{\gamma(e)}(\gamma(x))$ for all $\gamma \in \Gamma$, $e \in \rho$, $x \in M^\delta$. (Δ_e is related to Δ by $\Delta_e(x) = \Delta(x \otimes e)$.)

A morphism from (M, Δ) to $(\tilde{M}, \tilde{\Delta})$ is a Γ-linear map $\varphi: M \to \tilde{M}$ such that $\varphi(\Delta_e(x)) = \tilde{\Delta}_e(\varphi(x))$ for all $x \in M^\delta$ and all $e \in \rho$.

The categories $\mathscr{C}^\delta$ and $\mathscr{B}^\delta$ are equivalent. An equivalence can be obtained by attaching to an object $(\mathbf{V} = \oplus_i \mathbf{V}_i;\ x_{i \to j})$ of $\mathscr{B}^\delta$ the object (M, Δ) of $\mathscr{C}^\delta$ defined as follows. We take $M = \oplus_i \mathbf{V}_i \otimes \rho_i$ (with Γ acting trivially on $\mathbf{V}_i$). The i, j-component of Δ (for $i \in I^\delta$, $j \in I^{-\delta}$) is the linear map

$$\sum_{i \to j} x_{i \to j} \otimes y_{i \to j} : \mathbf{V}_i \otimes (\rho_i \otimes \rho) \to \mathbf{V}_j \otimes \rho_j$$

where $i \to j$ runs over the oriented edges joining i, j and $y_{i \to j}$ is a fixed basis of T_j^i in 1-1 correspondence with this set of edges. In particular, the category $\mathscr{C}^\delta$ is abelian.

2.3. We now discuss duality. Let $\mathbf{M} = (M, \Delta)$ be an object of $\mathscr{C}^\delta$. We associate to $\mathbf{M}$ the object $\mathbf{M}^* = (M^*, \Delta^*)$ of $\mathscr{C}^{-\delta}$ where M^* is as in 1.2 (so that $(M^*)^{\pm 1}$ is naturally the dual space $(M^{\pm 1})^*$ to $M^{\pm 1}$) and, for any $e \in \rho$, $\Delta_e^*: (M^{-\delta})^* \to (M^\delta)^*$ is the transpose of $\Delta_e: M^\delta \to M^{-\delta}$.

This extends naturally to an equivalence of the category $\mathscr{C}^\delta$ with the category opposed to $\mathscr{C}^{-\delta}$. We have $\mathbf{M}^{**} = \mathbf{M}$.

2.4. We want to define two (full) subcategories $\mathscr{C}^\delta_{su}$, $\mathscr{C}^\delta_{in}$ of $\mathscr{C}^\delta$. Let $\mathbf{M} = (M, \Delta)$ be an object of $\mathscr{C}^\delta$. Let $\Delta': M^\delta \to M^{-\delta} \otimes \rho$ be the Γ-linear map defined by

$$\Delta'(x) = \Delta_{e_1}(x) \otimes e_2 - \Delta_{e_2}(x) \otimes e_1$$

for all $x \in M^\delta$; here e_1, e_2 is any basis of ρ such that $\langle e_1, e_2 \rangle = 1$.

We say that $\mathbf{M}$ is an object of $\mathscr{C}^\delta_{su}$ (resp. of $\mathscr{C}^\delta_{in}$) if Δ is surjective (resp. Δ′ is injective). Note that

$$\mathbf{M} \in \mathscr{C}^\delta_{su} \Leftrightarrow \mathbf{M}^* \in \mathscr{C}^{-\delta}_{in}.$$

2.5. Let $\mathbf{M}=(\mathrm{M}, \Delta)$ be an object of $\mathscr{C}^{\delta}$. Note that $\ker\Delta$ is a Γ-module equal to $(\ker\Delta)^{-\delta}$ and that $\operatorname{coker}\Delta'$ (where Δ' is as in 2.4) is a Γ-module equal to $(\operatorname{coker}\Delta')^{\delta}$.

We associate to $\mathbf{M}$ two objects

$$\mathrm{K}\mathbf{M}=(\mathrm{M}^{\delta}\oplus\ker\Delta, \Pi)$$
$$\mathrm{C}\mathbf{M}=(\mathrm{M}^{-\delta}\oplus\operatorname{coker}\Delta', \Xi)$$

of $\mathscr{C}^{-\delta}$, where the notation is as follows. The map $\Pi:\ker\Delta\otimes\rho\to\mathrm{M}^{\delta}$ is defined by $\Pi((x\otimes e)\otimes e')=\langle e', e\rangle x$ for $x\otimes e\in\ker\Delta\subset\mathrm{M}^{\delta}\otimes\rho$ and $e'\in\rho$; $\Xi:\mathrm{M}^{-\delta}\otimes\rho\to\operatorname{coker}\Delta'$ is the canonical (surjective) map.

It is clear that $\mathrm{C}\mathbf{M}\in\mathscr{C}_{\mathrm{su}}^{-\delta}$ and it is easy to check that $\mathrm{K}\mathbf{M}\in\mathscr{C}_{\mathrm{in}}^{-\delta}$. We may regard K, C naturally as functors

(a) $$\mathrm{K}:\mathscr{C}^{\delta}\to\mathscr{C}_{\mathrm{in}}^{-\delta},\qquad \mathrm{C}:\mathscr{C}^{\delta}\to\mathscr{C}_{\mathrm{su}}^{-\delta}.$$

It is easy to check that

(b) $$(\mathrm{K}\mathbf{M})^{*}=\mathrm{C}(\mathbf{M}^{*})$$

for any $\mathbf{M}$ as above. Clearly,

(c) If $\mathbf{M}\in\mathscr{C}_{\mathrm{in}}^{\delta}$ and $\mathbf{M}'\in\mathscr{C}^{\delta}$ is a subobject of $\mathbf{M}$, then $\mathbf{M}'\in\mathscr{C}_{\mathrm{in}}^{\delta}$.

2.6. Let $\mathbf{M}=(\mathrm{M}, \Delta)$ be an object of $\mathscr{C}^{\delta}$. We have $\mathrm{CK}\,\mathbf{M}=(\mathrm{M}^{\delta}\oplus\operatorname{image}\Delta, \Delta_1)\in\mathscr{C}^{\delta}$ where $\Delta_1:\mathrm{M}^{\delta}\otimes\rho\to\operatorname{image}\Delta$ is induced by Δ. Hence $\mathrm{CK}\mathbf{M}$ is naturally a subobject of M; it admits a complement (S, 0) where S is any Γ-stable subspace of $\mathrm{M}^{-\delta}$ complementary to image Δ. Hence we have a canonical short exact sequence (which is non-canonically split) in $\mathscr{C}^{\delta}$:

(a) $$0\to\mathrm{CK}\,\mathbf{M}\to\mathbf{M}\to(\operatorname{coker}\Delta, 0)\to 0.$$

This shows that

(b) $\mathbf{M}\in\mathscr{C}_{\mathrm{su}}^{\delta}\Leftrightarrow\mathrm{CK}\,\mathbf{M}\to\mathbf{M}$ is an isomorphism.

We associate to $\mathbf{M}$ the element $\mathrm{gr}(\mathbf{M})=\mathrm{M}\in\mathscr{G}\Gamma$. From the definitions we see that

$$\mathbf{M}\in\mathscr{C}_{\mathrm{su}}^{\delta}\Rightarrow\mathrm{gr}(\mathrm{K}\,\mathbf{M})=\mathrm{M}^{\delta}-\mathrm{M}^{-\delta}+\mathrm{M}^{\delta}\otimes\rho$$

or, equivalently:

(c) $$\mathbf{M}\in\mathscr{C}_{\mathrm{su}}^{\delta}\Rightarrow\mathrm{gr}(\mathrm{K}\,\mathbf{M})=\mathbf{c}_{\delta}\,\mathrm{gr}(\mathbf{M}).$$

(see 1.5).

2.7. By 2.5 (c), an object of $\mathscr{C}_{\mathrm{in}}^{\delta}$ is indecomposable in $\mathscr{C}_{\mathrm{in}}^{\delta}$ if and only if it is indecomposable in $\mathscr{C}^{\delta}$. By duality, an analogous statement holds for $\mathscr{C}_{\mathrm{su}}^{\delta}$. Let $\mathscr{P}^{\delta}$

(resp. $\mathscr{P}^{\delta}_{\text{in}}$, $\mathscr{P}^{\delta}_{\text{su}}$) be the set of isomorphism classes of indecomposable objects of $\mathscr{C}^{\delta}$ (resp. $\mathscr{C}^{\delta}_{\text{in}}$, $\mathscr{C}^{\delta}_{\text{su}}$). The following statement follows immediately from definitions.

(a) *For any* $i \in \mathrm{I}$, $\mathrm{P}^{\delta}_i = (\rho_i, 0) \in \mathscr{C}^{\delta}$ *is indecomposable; if* $i \in \mathrm{I}^{\delta}$, *then* P^{δ}_i *is contained in* $\mathscr{C}^{\delta}_{\text{su}}$ *but not in* $\mathscr{C}^{\delta}_{\text{in}}$; *if* $i \in \mathrm{I}^{-\delta}$, *then* P^{δ}_i *is contained in* $\mathscr{C}^{\delta}_{\text{in}}$ *but not in* $\mathscr{C}^{\delta}_{\text{su}}$.

We have

(b) $$\mathscr{P}^{-\delta}_{\text{in}} = \mathscr{P}^{-\delta} - \{\mathrm{P}^{-\delta}_i \mid i \in \mathrm{I}^{-\delta}\},$$

(c) $$\mathscr{P}^{\delta}_{\text{su}} = \mathscr{P}^{\delta} - \{\mathrm{P}^{\delta}_i \mid i \in \mathrm{I}^{-\delta}\}.$$

The left hand side of (c) is contained in the right hand side of (c), by (a). Conversely, let $\mathbf{M}$ be an object in the right hand side of (c). From 2.6 (a) we see, using the indecomposability of $\mathbf{M}$, that either $\mathbf{M} \cong \mathrm{CK}\,\mathbf{M}$ or $\mathbf{M} = (\mathrm{M}, 0)$ with $\mathrm{M} = \mathrm{M}^{-\delta}$. If the first alternative holds, then $\mathbf{M}$, being in the image of C, is in $\mathscr{C}^{\delta}_{\text{su}}$ (see 2.5 (a)), hence it is in the left hand side of (c). If the second alternative holds then by the indecomposability of $\mathbf{M}$, we have $\mathbf{M} \cong \mathrm{P}^{\delta}_i$ for some $i \in \mathrm{I}^{-\delta}$; this is a contradiction and (c) is proved. Now (b) follows from (c) by duality.

(d) *The functor* K *defines a map from the set* (c) *to the set* (b); *the functor* C *defines a map from the set* (b) *to the set* (c); *these two maps are inverse bijections.*

Let $\mathbf{M}$ be in $\mathscr{P}^{\delta}_{\text{su}}$. As we have seen in the proof of (c), we have $\mathbf{M} = \mathrm{CK}\,\mathbf{M}$. In particular, $\mathrm{K}\,\mathbf{M} \neq 0$. By 2.5 (a), we have $\mathrm{K}\mathbf{M} \in \mathscr{C}^{-\delta}_{\text{in}}$. By duality, we also see that for any $\mathbf{M}' \in \mathscr{P}^{-\delta}_{\text{in}}$ we have $\mathbf{M}' = \mathrm{KC}\mathbf{M}'$, hence $\mathrm{C}\mathbf{M}' \neq 0$. Assume now that $\mathrm{K}\mathbf{M}$ is a direct sum of s indecomposable objects $\mathbf{M}'_1, \ldots, \mathbf{M}'_s$ of $\mathscr{C}^{-\delta}_{\text{in}}$, with $s \geqslant 2$. Then $\mathbf{M} = \mathrm{CK}\mathbf{M}$ is isomorphic to the direct sum of $\mathrm{C}\mathbf{M}'_1, \ldots, \mathrm{C}\mathbf{M}'_s$ which are all non-zero, as we have just seen. This contradicts the indecomposability of $\mathbf{M}$. We deduce that $\mathrm{K}\mathbf{M}$ is indecomposable. This establishes the first assertion of (d); the second assertion is obtained from the first, by duality. The third assertion is then obvious from the previous argument.

The following is clear from the definitions.

(e) *If* $i \in \mathrm{I}^{-\delta}$, *then* $\mathrm{CP}^{-\delta}_i = 0$ *and* $\mathrm{KP}^{\delta}_i = 0$.

2.8. For any $\mathbf{M} \in \mathscr{C}^{\delta}$ and any $s \in \mathbf{N}$, we write $\mathrm{K}^s\mathbf{M}$ instead of $\mathrm{K} \ldots \mathrm{K}\mathbf{M}$ (s factors K). We define similarly $\mathrm{C}^s\mathbf{M}$.

(a) *For any* $s \in \mathbf{N}$, *and any* $i \in \mathrm{I}^{(-1)^s\delta}$, $\mathrm{K}^s\mathrm{P}^{(-1)^s\delta}_i$ *is an indecomposable object of* $\mathscr{C}^{\delta}_{\text{su}}$.

For $s = 0$, (a) follows from 2.7 (a). Assume now that $s \geqslant 1$ and that (a) is already known for s replaced by s' with $0 \leqslant s' \leqslant s-1$. We shall write $\Phi_r = \mathrm{K}^r\mathrm{P}^{(-1)^s\delta}_i$. By the induction hypothesis (applied to $-\delta$ instead of δ) we have that Φ_{s-1} is an indecomposable object of $\mathscr{C}^{-\delta}_{\text{su}}$. Using now 2.7 (d), we deduce that $\Phi_s = \mathrm{K}(\Phi_{s-1})$ is indecomposable in $\mathscr{C}^{\delta}$. To show that it is in $\mathscr{C}^{\delta}_{\text{su}}$ it suffices, by 2.7 (c), to show that Φ_s is not of the form P^{δ}_j, with $j \in \mathrm{I}$. Assume that it is of this form; then $\mathrm{gr}(\Phi_s) = \rho_j$.

Applying repeatedly the induction hypothesis and 2.6 (c) we see that $\mathrm{gr}(\Phi_r)=\mathbf{c}_{(-1)^{s-r+1}\delta}\,\mathrm{gr}(\Phi_{r-1})$ for $r=1,\ldots,s$. Hence we have $\mathrm{gr}(\Phi_s)=\alpha(i,s)$ (see 1.7). On the other hand, $\rho_j=\alpha(j,0)$, so that $\alpha(i,s)=\alpha(j,0)$. Using now 1.8 (a), we deduce that $s=0$, a contradiction; this proves (a).

The previous proof yields also the following result:

(b) $$\mathrm{gr}(\mathrm{K}^s\mathrm{P}_i^{(-1)^s\delta})=\alpha(i,s).$$

Using this and 1.8 (a) we deduce:

(c) *Let $s, s'\in\mathbf{N}$ and let $i\in\mathrm{I}^{(-1)^s\delta}$, $i'\in\mathrm{I}^{(-1)^{s'}\delta}$. Then the indecomposable objects $\mathrm{K}^s\mathrm{P}_i^{(-1)^s\delta}$, $\mathrm{K}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta}$ of $\mathscr{C}^\delta$ are isomorphic if and only if $i=i'$ and $s=s'$.*

2.9. The results in this subsection can be deduced from those in the previous section, by duality.

(a) *For any $s\in\mathbf{N}$, and any $i\in\mathrm{I}^{(-1)^{s+1}\delta}$, $\mathrm{C}^s\mathrm{P}_i^{(-1)^s\delta}$ is an indecomposable object of $\mathscr{C}^\delta_{\mathrm{in}}$,*

(b) $$\mathrm{gr}(\mathrm{C}^s\mathrm{P}_i^{(-1)^s\delta})=\alpha(i,s).$$

(c) *Let $s, s'\in\mathbf{N}$ and let $i\in\mathrm{I}^{(-1)^{s+1}\delta}$, $i'\in\mathrm{I}^{(-1)^{s'+1}\delta}$. Then the indecomposable objects $\mathrm{C}^s\mathrm{P}_i^{(-1)^s\delta}$, $\mathrm{C}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta}$ of $\mathscr{C}^\delta$ are isomorphic if and only if $i=i'$ and $s=s'$.*

2.10. We define three subsets ${}^{>}\mathscr{P}^\delta$, ${}^{<}\mathscr{P}^\delta$, ${}^{0}\mathscr{P}^\delta$ of $\mathscr{P}^\delta$ as follows. Let $\mathbf{M}=(\mathrm{M},\Delta)$ be an object of $\mathscr{P}^\delta$. We say that $\mathbf{M}\in{}^{>}\mathscr{P}^\delta$ if $\mathrm{C}^t\mathbf{M}=0$ for some $t\geqslant 1$.

We say that $\mathbf{M}\in{}^{<}\mathscr{P}^\delta$ if $\mathrm{K}^t\mathbf{M}=0$ for some $t\geqslant 1$.

We say that $\mathbf{M}\in{}^{0}\mathscr{P}^\delta$ if $\mathrm{K}^t\mathbf{M}\neq 0$ and $\mathrm{C}^t\mathbf{M}\neq 0$ for all $t\in\mathbf{N}$.

(a) *We have $\mathbf{M}\in{}^{>}\mathscr{P}^\delta$ if and only if $\mathbf{M}$ is isomorphic to an object as in 2.8 (a).*

(b) *We have $\mathbf{M}\in{}^{<}\mathscr{P}^\delta$ if and only if $\mathbf{M}$ is isomorphic to an object as in 2.9 (a).*

(c) *The subsets ${}^{>}\mathscr{P}^\delta$, ${}^{<}\mathscr{P}^\delta$, ${}^{0}\mathscr{P}^\delta$ form a partition of the set $\mathscr{P}^\delta$.*

If $\mathbf{M}$ is as in 2.8 (a), then $\dim\mathrm{M}^\delta-\dim\mathrm{M}^{-\delta}=\dim\rho_i>0$ (see 2.8 (b) and 1.7 (c)). Similarly, if $\mathbf{M}$ is as in 2.9 (a), then $\dim\mathrm{M}^\delta-\dim\mathrm{M}^{-\delta}=-\dim\rho_i<0$ (see 2.9 (b) and 1.7 (c)).

Hence, if (a), (b) are known to hold, then the sets ${}^{>}\mathscr{P}^\delta$, ${}^{<}\mathscr{P}^\delta$ are disjoint and (c) follows.

We now prove (b). If $\mathbf{M}$ is as in 2.8 (b), then from 2.7 (d) we see that $\mathrm{K}^s\mathbf{M}=\mathrm{P}_i^{(-1)^s\delta}$ where $i\in\mathrm{I}^{(-1)^{s+1}\delta}$; using 2.7 (e), it follows that $\mathrm{K}(\mathrm{K}^s\mathbf{M})=0$. Conversely, assume that $\mathrm{K}^{s+1}\mathbf{M}=0$ for some $s\geqslant 0$. We take s to be as small as possible. Then $\mathrm{K}^s\mathbf{M}\neq 0$. By 2.6 (a) (for $\mathrm{K}^s\mathbf{M}$ instead of $\mathbf{M}$) we have an exact sequence in $\mathscr{C}^{(-1)^s\delta}$

$$0\to\mathrm{C}\mathrm{K}^{s+1}\mathbf{M}\to\mathrm{K}^s\mathbf{M}\to(\mathrm{M}_s,0)\to 0$$

where M_s is a Γ-module such that $M_s = M_s^{(-1)^{s-1}\delta}$. In our case this gives an isomorphism $K^s\mathbf{M} \cong (M_s, 0)$. Using 2.7 (d), we deduce that $(M_s, 0)$ is indecomposable and $\mathbf{M} \cong C^s(M_s, 0)$. Since $(M_s, 0)$ is indecomposable, it must be of the form $(\rho_i, 0)$ for some $i \in I^{(-1)^{s-1}\delta}$ and we see that $\mathbf{M}$ is as in 2.9 (a). This proves (b). Now (a) follows from (b) by duality.

2.11. Let $\mathbf{M} = (M, \Delta)$ be an object of $\mathscr{C}^\delta$. We associate to $\mathbf{M}$ an integer

(a) $$a(\mathbf{M}) = \min\{\dim\ker\Delta_e \mid e \in \rho\}.$$

Let U be the set of all lines $L \subset \rho$ such that $\dim\ker\Delta_e = a(\mathbf{M})$ for some (or any) $e \in L - \{0\}$. Then U is an open dense subset in the projective line $P(\rho)$ of ρ. We define the *pseudo-kernel* of Δ to be the subspace $M_0 = \sum_{L \in U} \ker\Delta_{e_L}$ of M^δ; here e_L denotes some non-zero vector in L. We set

(b) $$b(\mathbf{M}) = \dim M_0.$$

It is clear that

(c) $$a(\mathbf{M}) \leqslant b(\mathbf{M})$$

and

(d) $$a(\mathbf{M}) = 0 \Leftrightarrow b(\mathbf{M}) = 0.$$

Now let U′ be an open dense subset of $P(\rho)$ such that $U' \subset U$. We show that, if we replace U by U′ in the definition of M_0, we get again M_0:

(e) $$M_0 = \sum_{L \in U'} \ker\Delta_{e_L}.$$

It suffices to show that for any $L \in U$ we have $\ker\Delta_{e_L} \subset \sum_{L' \in U'} \ker\Delta_{e_{L'}}$. This is a consequence of the following statement whose verification is left to the reader. Let M′, M″ be finite dimensional **C**-vector spaces, and let $A, B : M' \to M''$ be two linear maps such that $\dim\ker(A + tB)$ is independent of t for t in some Zariski open subset T of **C**, containing 0. Then there exists $p \geqslant 1$ such that for any p-element subset T′ of $T - \{0\}$ we have $\ker A \subset \sum_{t \in T'} \ker(A + tB)$.

2.12. We shall prove the inequality

(a) $$b(C\mathbf{M}) \leqslant b(\mathbf{M}) - a(\mathbf{M})$$

for any $\mathbf{M} = (M, \Delta) \in \mathscr{C}^\delta$. We have $C\mathbf{M} = (M^{-\delta} \oplus \operatorname{coker}\Delta', \Xi)$ (see 2.4, 2.5). By definition, an element $x \in M^{-\delta}$ is in the kernel of Ξ_e, $(e \in \rho)$ precisely when $x \otimes e \in$ image

($\Delta' : M^{\delta} \to M^{-\delta} \otimes \rho$). Let us choose a basis e_1, e_2 of ρ such that $\langle e_1, e_2 \rangle = 1$ and such that $\dim \ker \Delta_{e_2} = a(\mathbf{M})$. Let $e = e_1 + le_2$. We see that the condition that $\Xi_e(x) = 0$ can be written in the following four equivalent forms:

$x \otimes e_1 + lx \otimes e_2 = \Delta_{e_1}(y) \otimes e_2 - \Delta_{e_2}(y) \otimes e_1$ for some $y \in M^{\delta}$;

$x = -\Delta_{e_2}(y)$ and $lx = \Delta_{e_1}(y)$ for some $y \in M^{\delta}$;

$x = -\Delta_{e_2}(y)$ and $l\Delta_{e_2}(y) + \Delta_{e_1}(y) = 0$ for some $y \in M^{\delta}$;

$x = -\Delta_{e_2}(y)$ and $\Delta_e(y) = 0$ for some $y \in M^{\delta}$.

We therefore see that the assignment

(b) $$y \mapsto -\Delta_{e_2}(y)$$

defines a surjective linear map $\ker \Delta_e \to \ker \Xi_e$ for any $e = e_1 + le_2$.

Let U′ be the open dense subset of $P(\rho)$ consisting of all lines L such that $e_2 \notin L$, $\dim \ker \Delta_e = a(\mathbf{M})$ and $\dim \ker \Xi_e = a(C\mathbf{M})$ for some (or any) $e \in L - \{0\}$. It then follows that the assignment (b) defines a surjective linear map

(c) $$\sum_{L \in U'} \ker \Delta_{e_L} \to \sum_{L \in U'} \ker \Xi_{e_L}$$

where e_L denotes any non-zero vector in L. By 2.11 (e) this is a linear map from the pseudo-kernel of Δ onto the pseudo-kernel of Ξ. By our choice of e_2, the kernel of Δ_{e_2} is contained in the pseudo-kernel of Δ and is therefore contained in the kernel of the map (c). It follows that the dimension of the pseudo-kernel of Ξ is less than or equal the dimension of the pseudo-kernel of Δ minus $\dim \ker \Delta_{e_2}$. This proves the inequality (a).

2.13. Next, we note the equality

(a) $$a(K\mathbf{M}) = a(\mathbf{M})$$

for any $\mathbf{M} = (M, \Delta) \in \mathscr{C}^{\delta}$. We have $K\,\mathbf{M} = (M^{\delta} \oplus \ker \Delta, \Pi)$, see 2.5. Let e be a non-zero vector in ρ. From the definitions it follows immediately that the assignment $x \mapsto x \otimes e$ is an isomorphism $\ker \Delta_e \cong \ker \Pi_e$. This clearly implies (a).

2.14. Let $\mathbf{M} \in \mathscr{C}^{\delta}$.

(a) We have $b(C^s\mathbf{M}) = 0$ for all $s \geqslant b(\mathbf{M})$.

We argue by induction on $b(\mathbf{M})$. Assume first that $b(\mathbf{M}) = 0$. By 2.12 (a), we have $b(\mathbf{M}) \geqslant b(C\,\mathbf{M}) \geqslant b(C^2\mathbf{M})$... hence $b(C^s\mathbf{M}) = 0$ for all $s \geqslant 0$. Next we assume that $b(\mathbf{M}) > 0$ and that the result is already proved for all $\mathbf{M}'$ with $b(\mathbf{M}') < b(\mathbf{M})$. By 2.11 (d) we have $a(\mathbf{M}) > 0$. Hence, using 2.12 (a), we see that $b(C\mathbf{M}) < b(\mathbf{M})$. Thus, the induction hypothesis is applicable to $C\mathbf{M}$. Now let s be such that $s \geqslant b(\mathbf{M})$. By the previous inequality we then have $s-1 \geqslant b(\mathbf{C}\mathbf{M})$; using the induction hypothesis, it follows that $b(\mathbf{C}^{s-1}\mathbf{C}\mathbf{M})=0$. This completes the inductive proof of (a).

We now prove the following statement.

(b) *If* $\mathbf{M}$ *is indecomposable and* $\mathbf{C}^t\mathbf{M} \neq 0$ *for all* $t \geqslant 0$, *then* $b(\mathbf{M})=0$.

From our assumption and from the results in 2.7 it follows that $\mathbf{K}\mathbf{C}(\mathbf{C}^t\mathbf{M}) \cong \mathbf{C}^t\mathbf{M}$ for all $t \geqslant 0$. Hence $\mathbf{K}^{t+1}\mathbf{C}^{t+1}\mathbf{M} \cong \mathbf{K}^t\mathbf{K}\mathbf{C}\mathbf{C}^t\mathbf{M} \cong \mathbf{K}^t\mathbf{C}^t\mathbf{M}$ for all $t \geqslant 0$, so that $\mathbf{K}^t\mathbf{C}^t\mathbf{M} \cong \mathbf{M}$ for all $t \geqslant 0$.

Let $s \geqslant 0$ be such that $b(\mathbf{C}^s\mathbf{M})=0$ (see (a)). Then $a(\mathbf{C}^s\mathbf{M})=0$ (see 2.11 (d)); using repeatedly 2.13 (a), we have that $a(\mathbf{K}^s\mathbf{C}^s\mathbf{M})=a(\mathbf{C}^s\mathbf{M})$ hence $a(\mathbf{K}^s\mathbf{C}^s\mathbf{M})=0$. As we have seen, we have $\mathbf{K}^s\mathbf{C}^s\mathbf{M} \cong \mathbf{M}$ so that $a(\mathbf{M})=0$. Using again 2.11 (d), we deduce $b(\mathbf{M})=0$.

The following statement can be deduced from (b) by duality.

(c) *If* $\mathbf{M}$ *is indecomposable and* $\mathbf{K}^t\mathbf{M} \neq 0$ *for all* $t \geqslant 0$, *then* $b(\mathbf{M}^*)=0$.

2.15. Assume now that $\mathbf{M}=(\mathbf{M}, \Delta)=\mathbf{K}^s\mathbf{P}_i^{(-1)^s\delta}$ where $i \in \mathbf{I}^{(-1)^{s-1}\delta}$, and $s \geqslant 0$ (see 2.8 (a)). We have

(a) $a(\mathbf{M})=\dim \rho_i$ and $b(\mathbf{M})=\dim \mathbf{M}^\delta=(s+1)\dim \rho_i$.

We argue by induction on s. When $s=0$, we have $\mathbf{M}=\mathbf{P}_i^\delta$ and (a) is obvious. Now assume that $s \geqslant 1$ and that the result is already proved for $s-1$ instead of s. Let $\mathbf{M}'=\mathbf{K}^{s-1}\mathbf{P}_i^{(-1)^s\delta}$. The induction hypothesis is applicable to $\mathbf{M}'$ (with δ replaced by $-\delta$). We have $\mathbf{M}=\mathbf{K}\mathbf{M}'$ hence, by 2.13 (a) and the induction hypothesis, $a(\mathbf{M})=a(\mathbf{M}')=\dim \rho_i$. We also have $\mathbf{M}'=\mathbf{C}\mathbf{M}$, hence by 2.12 (a) and the induction hypothesis

$$s \dim \rho_i = b(\mathbf{M}') \leqslant b(\mathbf{M}) - a(\mathbf{M}) = b(\mathbf{M}) - \dim \rho_i$$

so that $b(\mathbf{M}) \geqslant (s+1) \dim \rho_i$. On the other hand, it is clear that $\dim \mathbf{M}^\delta \geqslant b(\mathbf{M})$. The last two inequalities together with the equality $\dim \mathbf{M}^\delta=(s+1) \dim \rho_i$ (see 2.8 (b) and 1.7 (b)) imply the equalities in (a).

2.16. We will define several (full) subcategories ${}^{>}\mathscr{C}^\delta$, ${}^{<}\mathscr{C}^\delta$, ${}^{0}\mathscr{C}^\delta$ of $\mathscr{C}^\delta$. Let $\mathbf{M}=(\mathbf{M}, \Delta)$ be an object of $\mathscr{C}^\delta$.

We say that $\mathbf{M}$ is an object of ${}^{>}\mathscr{C}^\delta$ if $b(\mathbf{M}^*)=0$ and $b(\mathbf{M})=\dim \mathbf{M}^\delta$.

We say that $\mathbf{M}$ is an object of ${}^{<}\mathscr{C}^\delta$ if $b(\mathbf{M})=0$ and $b(\mathbf{M}^*)=\dim \mathbf{M}^{-\delta}$.

We say that $\mathbf{M}$ is an object of ${}^{0}\mathscr{C}^\delta$ if $b(\mathbf{M})=b(\mathbf{M}^*)=0$.

Note that

$$\mathbf{M} \in {}^{<}\mathscr{C}^\delta \Leftrightarrow \mathbf{M}^* \in {}^{>}\mathscr{C}^{-\delta},$$

$$\mathbf{M} \in {}^{0}\mathscr{C}^\delta \Leftrightarrow \mathbf{M}^* \in {}^{0}\mathscr{C}^{-\delta}.$$

By 2.14 (b), (c), any object of ${}^{0}\mathscr{P}^\delta$ is contained in ${}^{0}\mathscr{C}^\delta$.

Now any object $\mathbf{M}=(M, \Delta)$ in ${}^{<}\mathscr{P}^{\delta}$ is as in 2.9 (a) (see 2.10 (b)) hence satisfies the hypothesis of 2.14 (b) so that $b(\mathbf{M})=0$; its dual is as in 2.8 (a) hence we can apply 2.15(a) to it and deduce $b(\mathbf{M}^*)=\dim M^{-\delta}$. Thus, we have $\mathbf{M}\in {}^{<}\mathscr{C}^{\delta}$.

Dually, any object in ${}^{>}\mathscr{P}^{\delta}$ is contained in ${}^{>}\mathscr{C}^{\delta}$.

2.17. Let $\mathbf{M}=(M, \Delta)$ be an object of ${}^{0}\mathscr{C}^{\delta}$. Let Spec $\mathbf{M}$ (*spectrum* of $\mathbf{M}$) be the set of all lines $L\in P(\rho)$ with the following property: for some (or all) $e\in L-\{0\}$, the map $\Delta_e: M^{\delta}\to M^{-\delta}$ is not an isomorphism of **C**-vector spaces. From the definition, it is clear that Spec $\mathbf{M}$ is a finite set. (In particular, we have $\dim M^{\delta}=\dim M^{-\delta}$.)

For any $L\in P(\rho)$ we define subspaces $M_L^{\delta}\subset M^{\delta}$ and $M_L^{-\delta}\subset M^{-\delta}$ as follows. We choose non-zero vectors e, e' of ρ such that $e\in L$ and $\mathbf{C}e'\notin \text{Spec}\,\mathbf{M}$ and we define M_L^{δ} (resp. $M_L^{-\delta}$) to be the set of all $x\in M^{\delta}$ such that $(\Delta_{e'}^{-1}\Delta_e)^N x=0$ (resp. $(\Delta_e\Delta_{e'}^{-1})^N x=0$) for some $N\geqslant 1$. These subspaces are clearly independent of the choice of e. They are also independent of the choise of e'. Indeed, let e'' be another non-zero vector such that $\mathbf{C}e''\notin \text{Spec}\,\mathbf{M}$. If e'' is proportional to e' then it clearly leads to the same subspaces as e'. Assume now that e', e'' are not proportional. Then we can write $e=ae'+be''$ for some $a, b\in\mathbf{C}$ so that $\Delta_e=a\Delta_{e'}+b\Delta_{e''}$. Let $\tau=\Delta_{e'}^{-1}\Delta_{e''}: M^{\delta}\cong M^{\delta}$. We have $(\Delta_{e'}^{-1}\Delta_e)^N=(a1+b\tau)^N$, $(\Delta_{e''}^{-1}\Delta_e)^N=(a\tau^{-1}+b1)^N$ and $(a1+b\tau)^N$, $(a\tau^{-1}+b1)^N$ have the same kernel and M_L^{δ} is well defined. We see similarly that $M_L^{-\delta}$ is well defined.

Clearly, $M_L^{\delta}=0$, $M_L^{-\delta}=0$ if $L\notin \text{Spec}\,\mathbf{M}$. We have a direct sum decomposition $M^{\delta}=\oplus_L M_L^{\delta}$. Indeed, for e', e'', τ as a above, the subspaces M_L^{δ} are precisely the various generalized eigenspaces of τ. Similarly, we have a direct sum decomposition $M^{-\delta}=\oplus_L M_L^{-\delta}$. In particular, Spec $\mathbf{M}$ is non-empty if $\mathbf{M}\neq 0$.

It is clear that, for any $e_1\in\rho$, and any line L in $P(\rho)$, Δ_{e_1} restricts to a linear map of M_L^{δ} into $M_L^{-\delta}$; moreover, this linear map is an isomorphism if $e_1\notin L$.

Next we observe that Γ acts naturally on $P(\rho)$, leaving stable the finite subset Spec $\mathbf{M}$. Note also that $\gamma(M_L^{\delta})\subset M_{\gamma(L)}^{\delta}$ and $\gamma(M_L^{-\delta})\subset M_{\gamma(L)}^{-\delta}$ for all $\gamma\in\Gamma$ and all $L\in P(\rho)$. Let $\mathscr{Z}$ be the set of all subsets of $P(\rho)$ which are orbits of Γ. For each $Z\in\mathscr{Z}$, let $M_Z^{\delta}=\oplus_{L\in Z}M_L^{\delta}$ and $M_Z^{-\delta}=\oplus_{L\in Z}M_L^{-\delta}$. Then M_Z^{δ}, $M_Z^{-\delta}$ are Γ-stable and for any $e_1\in\rho$, Δ_{e_1} restricts to a linear map of M_Z^{δ} into $M_Z^{-\delta}$; moreover, this linear map is an isomorphism if e_1 is not contained in the union of all lines in Z. We may regard $M_Z^{\delta}\oplus M_Z^{-\delta}$ with the restriction of the maps Δ_{e_1} as a subobject $\mathbf{M}_Z$ of $\mathbf{M}$. Then

(a) $$\mathbf{M}=\oplus_{Z\in\mathscr{Z}}\mathbf{M}_Z$$

and for each Z, we have Spec $\mathbf{M}_Z\subset Z$. This decomposition is functorial.

Let ${}^{0}\mathscr{C}_Z^{\delta}$ be the full subcategory of ${}^{0}\mathscr{C}^{\delta}$ consisting of objects with spectrum contained in Z. We see that

(b) ${}^{0}\mathscr{C}^{\delta}$ *is a direct product of the categories* ${}^{0}\mathscr{C}_Z^{\delta}$ $(Z\in\mathscr{Z})$.

2.18. We now fix $Z \in \mathscr{Z}$ and study the category ${}^0\mathscr{C}^\delta_Z$.

The number of elements of the isotropy group in Γ of any element of Z is an even integer $2n$ where $n = n_Z \geqslant 1$. We define a category $\mathscr{C}'_n$ as follows. An object of $\mathscr{C}'_n$ is a $\mathbf{Z}/n\mathbf{Z}$-graded, finite dimensional $\mathbf{C}$-vector space $V = \oplus_{r \in \mathbf{Z}/n\mathbf{Z}} V_r$ together with a nilpotent endomorphism $t: V \to V$ such that $t(V_r) \subset V_{r+1}$ for all r. Morphisms are linear maps respecting the grading and the nilpotent endomorphism.

We will construct an equivalence of categories

(a) ${}^0\mathscr{C}^\delta_Z \cong \mathscr{C}'_n$.

We choose $L \in Z$; let Γ_L be the isotropy group of L in Γ; it is a cyclic group of order $2n$. Let $\hat{\Gamma}_L$ be the group of characters $\Gamma_L \to \mathbf{C}^*$.

We choose a second line L′ in ρ such that $L' \neq L$ and such that L′ is fixed by Γ_L. (If $n > 1$, L′ is uniquely determined by these requirements; if $n = 1$, any line in ρ is fixed by Γ_L.) We choose $e \in L - \{0\}$ and $e' \in L' - \{0\}$. Now Γ_L acts on L through a character $\zeta \in \hat{\Gamma}_L$ and on L′ through ζ^{-1}. Clearly, ζ is a generator of $\hat{\Gamma}_L$.

To an object $\mathbf{M} = (M, \Delta)$ of ${}^0\mathscr{C}^\delta_Z$, we associate an object (V, t) of $\mathscr{C}'_n$ as follows. We set $V = M^\delta_L$. This subspace of M^δ is clearly Γ_L-stable.

For any integer $r \in [0, n-1]$, we denote by V_r the largest Γ_L-stable subspace of V on which Γ_L acts through the character $\zeta^{(1-\delta)/2 + 2r}$.

(Note that the largest Γ_L-stable subspace of V on which Γ_L acts through the character $\zeta^{(1-\delta)/2 + 2r + 1}$ is zero since the value of this character at c is $(-1)^{(1-\delta)/2+1} = -\delta$.)

The V_r form a $\mathbf{Z}/n\mathbf{Z}$-grading of V. We define $t: V \to V$ to be the composition $V = M^\delta_L \to M^{-\delta}_L \to M^\delta_L = V$, where the first map is given by Δ_e and the second map is the inverse of the isomorphism $M^\delta_L \cong M^{-\delta}_L$ given by the restriction of $\Delta_{e'}$ (this last map is an isomorphism since $e' \notin L$).

Now $\Delta_{e'} + \lambda \Delta_e = \Delta_{e' + \lambda e}: M^\delta_L \to M^{-\delta}_L$ is invertible for any $\lambda \in \mathbf{C}$, since we have $e' + \lambda e \notin L$. Hence $1 + \lambda t: V \to V$ is invertible for any $\lambda \in \mathbf{C}$. It follows that $t: V \to V$ is nilpotent.

We now show that $\gamma(t(x)) = \zeta(\gamma)^2 t(\gamma(x))$ for all $x \in V$ and all $\gamma \in \Gamma_L$.

We have $\gamma \Delta_e(x) = \Delta_{\gamma e}(\gamma x) = \Delta_{\zeta(\gamma) e}(\gamma x)$ hence $\gamma(\Delta_e(x)) = \zeta(\gamma) \Delta_e(\gamma x)$.

Similarly, we have $\gamma(\Delta_{e'}(x)) = \zeta(\gamma)^{-1} \Delta_{e'}(\gamma x)$. Replacing here x by $t(x)$ we obtain $\gamma(\Delta_{e'}(t(x))) = \zeta(\gamma)^{-1} \Delta_{e'}(\gamma(t(x)))$. The left hand side is equal to $\gamma(\Delta_e(x))$ hence to $\zeta(\gamma) \Delta_e(\gamma x) = \zeta(\gamma) \Delta_{e'}(t(\gamma(x)))$. It follows that $\zeta(\gamma) \Delta_{e'}(t(\gamma(x))) = \zeta(\gamma)^{-1} \Delta_{e'}(\gamma(t(x)))$. Since $\Delta_{e'}$ is injective on V, it follows that $\zeta(\gamma) t(\gamma(x)) = \zeta(\gamma)^{-1} \gamma(t(x))$, as claimed.

We see that (V, t) is indeed an object of $\mathscr{C}'_n$. Now the assignment $\mathbf{M} \mapsto (V, t)$ extends in an obvious way to a functor ${}^0\mathscr{C}^\delta_Z \to \mathscr{C}'_n$.

We now construct a functor in the opposite direction. We assume given an object (V, t) of $\mathscr{C}'_n$; we shall associate to it an object $\mathbf{M}$ of ${}^0\mathscr{C}^\delta_Z$. We regard V as a Γ_L-module with γ acting on V_r as multiplication by $\zeta(\gamma)^{(1-\delta)/2+2r}$. Let V' be the Γ_L-module with the same underlying space as V but with γ acting on V_r as multiplication by $\zeta(\gamma)^{(1-\delta)/2+2r-1}$.

Let M be the induced Γ-module $\mathbf{C}[\Gamma]\otimes_{\mathbf{C}[\Gamma_L]}(V\oplus V')$. Then $M^\delta=\mathbf{C}[\Gamma]\otimes_{\mathbf{C}[\Gamma_L]}V$ and $M^{-\delta}=\mathbf{C}[\Gamma]\otimes_{\mathbf{C}[\Gamma_L]}V'$. We may regard V, V' naturally as subspaces of M^δ, $M^{-\delta}$. Let $\Delta: M^\delta\otimes\rho\to M^{-\delta}$ be the unique Γ-linear map which extends the Γ_L-linear map $V\otimes\rho\to V'$ given by $x\otimes(ae+be')\mapsto at(x)+bx$. It is clear that $\mathbf{M}=(M, \Delta)$ just defined is an object of ${}^0\mathscr{C}^\delta_Z$. The assignment $(V, t)\to\mathbf{M}$ extends in an obvious way to a functor $\mathscr{C}'_n\to{}^0\mathscr{C}^\delta_Z$. The two functors constructed above provide the desired equivalence of categories.

2.19. Let (V, t) be an indecomposable object of $\mathscr{C}'_n$. On V we have a $\mathbf{Z}/n\mathbf{Z}$ action: the canonical generator of $\mathbf{Z}/n\mathbf{Z}$ acts on V_r as multiplication by $e^{2\pi\sqrt{-1}\,r/n}$. Let m be the smallest integer $\geqslant 1$ such that $t^m=0$. We can find $x\in V$ such that $t^{m-1}(x)\neq 0$; moreover, we can assume that $x\in V_r$ for some integer r (defined up to a multiple of n). Let $\langle x\rangle$ be the subspace of V spanned by $x, tx, \dots, t^{m-1}x$. Clearly, this subspace is t-stable, compatible with the grading and $x, tx, \dots, t^{m-1}x$ is a basis for it. By the theory of (ungraded) nilpotent endomorphisms, there exists a t-stable subspace V_1 of V complementary to $\langle x\rangle$. Let Y be the set of all t-stable subspaces of V which are complementary to $\langle x\rangle$. Thus we have $V_1\in Y$. Let Y' be the vector space of all linear maps $V_1\to\langle x\rangle$ which commute with the action of t. The graph of such a linear map is a subspace of $V_1+\langle x\rangle=V$, which is actually in Y. This gives a bijection $Y'\cong Y$ and shows that Y is an affine space. Now Y is defined purely in terms of $\langle x\rangle$, which is compatible with the grading; hence Y is stable under the natural action of $\mathbf{Z}/n\mathbf{Z}$ on the set of subspaces of V. A finite group acting on an affine space must have a fixed point. Hence, there exists a subspace $V_2\in Y$ which is compatible with the grading. Thus, $\langle x\rangle$ admits a complement which is both t-stable and compatible with the grading. By the indecomposability of V we must then have $V=\langle x\rangle$.

Conversely, given an integer r, defined up to a multiple of n, and an integer $m\geqslant 1$, there is clearly a unique indecomposable object $V_{r,m}=(V, t)$ (up to isomorphism) such that for some $x\in V_r$, $x, tx, \dots, t^{m-1}x$ is a basis of V.

We now see that $V_{r,m}$ form a complete list of indecomposable objects (up to isomorphisms) of $\mathscr{C}'_n$.

2.20. In the setup of 2.18, we denote by $\mathbf{M}_{Z,r,m}$ the indecomposable object of ${}^0\mathscr{C}^\delta_Z$ corresponding to $V_{r,m}$ under 2.18 (a). From the definitions, we have

$$\mathrm{gr}(\mathbf{M}_{Z,r,m})=\mathrm{Ind}^{\Gamma}_{\Gamma_L}(\oplus_{j=2r-1}^{2r+2m-2}L^{\otimes((1-\delta)/2+j)})$$

where $L^{\otimes j}$ is a tensor power of L, $(L \in Z)$ regarded as a 1-dimensional Γ-module.

Thus, if m is divisible by n we have

$$\mathrm{gr}(\mathbf{M}_{Z, r, m}) = (m/n)\,\mathbf{r}.$$

Assume now that m is not divisible by n. Then $L \in F$ and we may assume that $L \in \mathscr{X}$ (see 1.13); we have

$$\mathrm{gr}(\mathbf{M}_{Z, r, m}) = \mu(L, r, m)$$

If $\delta = -1$ and

$$\mathrm{gr}(\mathbf{M}_{Z, r, m}) = \mu(L, -r-m+1, m)^*$$

if $\delta = 1$ (see 1.17).

The previous results can be summarized as follows.

Theorem 2.21. — (*a*) *For any* $P \in \mathscr{P}^\delta$ *we have* $\mathrm{gr}(P) \in R_+ \cup \{\mathbf{r}, 2\mathbf{r}, \ldots\}$.

(*b*) *For any* $\alpha \in R_+$ *there is a unique* $P \in \mathscr{P}^\delta$ *such that* $\mathrm{gr}(P) = \alpha$.

(*c*) *For any integer* $s \geqslant 1$, *the map* $P \mapsto \mathrm{Spec}\, P$ *from* $\{P \in \mathscr{P}^\delta \mid \mathrm{gr}(P) = s\,\mathbf{r}\}$ *to the set of* Γ*-orbits on* $P(\rho)$ *is well defined; its fibre over any* Γ*-orbit* Z *has cardinal equal to half the number of elements in the isotropy group* Γ_L *of any* $L \in Z$.

2.22. *Remark*. — In the language of [DR], the objects of ${}^{>}\mathscr{C}^\delta$, ${}^{0}\mathscr{C}^\delta$, ${}^{<}\mathscr{C}^\delta$ are preinjective, regular, preprojective, respectively.

3. Preparatory results

3.1. In this section we restate some definitions and results of [L] in the special case of affine quivers, in terms of the corresponding group Γ.

Let M be a Γ-module. The corresponding notion in [L] is an I-graded vector space $\mathbf{V} = \oplus\, \mathbf{V}_i$ of finite dimension over $\mathbf{C}$. These are related by $M = \oplus_i \mathbf{V}_i \otimes \rho_i$ (with Γ acting trivially on $\mathbf{V}_i$, as in 2.3).

Let G_M be the (algebraic) group of all linear automorphisms of M commuting with the Γ-action. The corresponding notion in [L] is $G_V = \prod_i \mathrm{Aut}(\mathbf{V}_i)$. (We have $G_M = G_V$.)

Let E_M be the vector space $\mathrm{Hom}_\Gamma(M \otimes \rho, M)$. The corresponding notion in [L] is $\mathbf{E}_V = \oplus\, \mathrm{Hom}(\mathbf{V}_i, \mathbf{V}_j)$ where the sum is taken over all pairs consisting of an edge of our graph and an orientation $i \to j$ of that edge. (These two vector spaces may be identified as in the construction of the equivalence of $\mathscr{C}^\delta$ and $\mathscr{B}^\delta$ in 2.3.)

As in 2.3, giving $\Delta \in \mathrm{Hom}_\Gamma(\mathrm{M} \otimes \rho, \mathrm{M})$ is the same as giving a collection $\Delta_e (e \in \rho)$ of linear maps $\mathrm{M} \to \mathrm{M}$ depending linearly of e and satisfying $\gamma(\Delta_e(x)) = \Delta_{\gamma(e)}(\gamma(x))$ for all $\gamma \in \Gamma$, $e \in \rho$, $x \in \mathrm{M}$. (Δ_e is related to Δ by $\Delta_e(x) = \Delta(x \otimes e)$.)

We have a natural action of $\mathrm{G_M}$ on $\mathrm{E_M}$ given by $(g, \Delta) \mapsto g\Delta$, where $(g\Delta)(x \otimes e) = g(\Delta(g^{-1}x \otimes e))$ for $x \in \mathrm{M}$ and $e \in \rho$. (This corresponds to an action of $\mathrm{G_V}$ on $\mathbf{E}_\mathrm{V}$.)

On $\mathrm{E_M}$ we have a non-degenerate symplectic form $\langle\ ,\ \rangle$ defined by

$$\langle \Delta, \tilde{\Delta} \rangle = \mathrm{tr}(\Delta_{e_1} \tilde{\Delta}_{e_2} - \Delta_{e_2} \tilde{\Delta}_{e_1} : \mathrm{M} \to \mathrm{M}),$$

where e_1, e_2 is any symplectic basis of ρ. (This corresponds to a symplectic form on $\mathbf{E}_\mathrm{V}$ as in [L, 12.1].) This symplectic form is invariant under the $\mathrm{G_M}$-action.

3.2. The *moment map* attached to the $\mathrm{G_M}$-action on the symplectic vector space $\mathrm{E_M}$ is the map $\psi : \mathrm{E_M} \to \mathrm{Hom}_\Gamma(\mathrm{M}, \mathrm{M})$ given by

$$\psi(\Delta) = \Delta_{e_1} \Delta_{e_2} - \Delta_{e_2} \Delta_{e_1} : \mathrm{M} \to \mathrm{M},$$

where e_1, e_2 are as in 3.1. (Compare [L, 12.1] and [K]) Hence we have

$$\psi(\Delta) = 0 \Leftrightarrow \Delta_{e_1} \Delta_{e_2} = \Delta_{e_2} \Delta_{e_1} \text{ for any } e_1, e_2 \in \rho.$$

An element $\Delta \in \mathrm{E_M}$ is said to be *nilpotent* if there exists a number $\mathrm{N} \geqslant 2$ such that for any sequence $e(1), e(2), \ldots, e(\mathrm{N})$ of vectors in ρ, the composition $\Delta_{e(1)} \Delta_{e(2)} \ldots \Delta_{e(\mathrm{N})}$: $\mathrm{M} \to \mathrm{M}$ is zero; an equivalent definition is that there should exist a flag in M which is Δ-stable.

(This corresponds to the notion of nilpotent element of $\mathbf{E}_\mathrm{V}$ given in [L, 1.7, 1.8].)

We define Λ_M to be the set of all nilpotent elements $\Delta \in \mathrm{E_M}$ such that $\psi(\Delta) = 0$. (Compare [L, 12.1].)

Λ_M *is a closed* $\mathrm{G_M}$*-stable subvariety of* $\mathrm{E_M}$ *of pure dimension* $\dim \mathrm{E_M}/2$. (Compare [L, 12.3].)

Let Irr Λ_M be the set of irreducible components of Λ_M.

3.3. For each $i \in \mathrm{I}$ and each $p \in \mathbf{N}$, let $\Lambda_{\mathrm{M}, i, p}$ be the set of all $\Delta \in \Lambda_\mathrm{M}$ such that $(\rho_i : \mathrm{M}) - (\rho_i : \Delta(\mathrm{M} \otimes \rho)) = p$. This is a locally closed subvariety of Λ_M, again of pure dimension $\dim \mathrm{E_M}/2$. (Compare [L, 12.3].) Moreover, for any $p \geqslant 0$, and any $i \in \mathrm{I}$, the union $\bigcup_{p' : p' \leqslant p} \Lambda_{\mathrm{M}, i, p'}$ is open in Λ_M.

3.4. Given $\delta = \pm 1$, we define $\mathrm{E}_\mathrm{M}^\delta = \mathrm{Hom}_\Gamma(\mathrm{M}^\delta \otimes \rho, \mathrm{M}^{-\delta})$.

We can identify in an obvious way $\mathrm{E}_\mathrm{M}^\delta$ with the subspace of $\mathrm{E_M}$ consisting of those $\Delta \in \mathrm{E_M}$ which map $\mathrm{M}^{-\delta} \otimes \rho$ to zero; note that Δ automatically maps $\mathrm{M}^\delta \otimes \rho$ into $\mathrm{M}^{-\delta}$.

We have a direct sum decomposition $\mathrm{E_M} \cong \mathrm{E}_\mathrm{M}^1 \oplus \mathrm{E}_\mathrm{M}^{-1}$.

Thus E_M^1 and E_M^{-1} appear as complementary Lagrangian subspaces of E_M; they are G_M-stable. (Compare [L, 12.8].) In particular, $\langle\ ,\ \rangle$ defines a non-singular pairing $E_M^\delta \otimes E_M^{-\delta} \to \mathbf{C}$ so that E_M is naturally the cotangent bundle of E_M^δ.

In particular, if Y is a submanifold of E_M^δ, then the conormal bundle of Y (a submanifold of the cotangent bundle of E_M^δ) may be naturally regarded as a submanifold of E_M.

(a) If $\Delta \in E_M^\delta$ and $\tilde{\Delta} \in E_M^{-\delta}$, then $\psi(\Delta + \tilde{\Delta}) = 0$ if and only if $\tilde{\Delta}$ is orthogonal with respect to $\langle\ ,\ \rangle$ to the tangent space to the G_M-orbit of Δ (regarded as a vector subspace of E_M^δ). (Compare [L, 12.8 (a)].)

3.5. If N is a Γ-submodule of M and $\Delta \in E_M$, we say that N is Δ-stable if Δ_e maps N intoN for any $e \in \rho$.

Let $M_\dagger = (M_1, M_2, \ldots M_n)$ be a sequence of isotypical Γ-modules such that $M \cong M_1 \oplus \ldots \oplus M_m$ as a Γ-module.

A *flag* in M is by definition a sequence $M = M^{(0)} \supset M^{(1)} \supset \ldots \supset M^{(m)} = 0$ of Γ-submodules such that for any $l = 1, 2, \ldots, m$, the Γ-module $M^{(l-1)}/M^{(l)}$ is isotypical. A *flag of type* $M_\dagger$ is a flag as above such that $M^{(l-1)}/M^{(l)} \cong M_l$ as Γ-modules for $l = 1, 2, \ldots, m$. (Compare [L, 1.4].) A flag as above is said to be Δ-stable (where $\Delta \in E_M$) if each $M^{(l)}$ is Δ-stable.

Given $\Delta \in E_M$ and $M_\dagger$ as above, we denote by $\chi_{M_\dagger}(\Delta)$ the Euler characteristic of the variety of Δ-stable flags of type $M_\dagger$ in M. This variety is empty unless Δ is nilpotent ([L, 1.8]) hence $\chi_{M_\dagger}(\Delta) = 0$ if Δ is not nilpotent. Let $\chi_{M_\dagger} : \Lambda_M \to \mathbf{Z}$ be the function whose value at any $\Delta \in \Lambda_M$ is $\chi_{M_\dagger}(\Delta)$.

Let $\mathscr{F}_M$ be the **Q**-vector space of functions $\Lambda_M \to \mathbf{Q}$ spanned by the functions $\chi_{M_\dagger} : \Lambda_M \to \mathbf{Z}$ for various $M_\dagger$ as above. This is clearly a finite-dimensional vector space; all functions in $\mathscr{F}_M$ are constructible ([L, 10.18]) and constant on orbits of G_M.

Proposition 3.6. — *Given any* $Y \in \mathrm{Irr}\,\Lambda_M$, *there exists a function* $f \in \mathscr{F}_M$ *such that* (*a*) *for some open dense* G_M-*stable subset* O *of* Y *we have* $f|_O = 1$ *and* (*b*) *for some closed* G_M-*stable subset* $H \subset \Lambda_M$ *of dimension* $< \dim \Lambda_M$ *we have* $f = 0$ *outside* $Y \cup H$.

The result is trivial when $M = 0$. We may therefore assume that $M \neq 0$ and that the result is already proved for Γ-modules of dimension $< \dim M$.

Given $i \in I$, the intersection $Y \cap (\bigcup_{p' : p' \leqslant p} \Lambda_{M, i, p'})$ (see 3.3) is non-empty for some $p \geqslant 0$; for example, for $p = (\rho_i : M)$ it is equal to Y. The smallest integer $p \geqslant 0$ for which this intersection is non-empty is denoted $n(i, Y)$. We have $0 \leqslant n(i, Y) \leqslant (\rho_i : M)$.

According to [L, 12.6], we have $n(i, Y) > 0$ for some $i \in I$.

Hence it is enough to prove (for an $i \in I$ which is fixed from now on) that the proposition holds for any Y such that $n(i, Y) > 0$. This will be proved by descending induction on $n(i, Y)$ (which is bounded from above).

Thus, we may assume that Y is such that $n(i, \mathrm{Y}) = n_0 > 0$ and that the proposition is already proved for all $\tilde{\mathrm{Y}} \in \mathrm{Irr}\,\Lambda_{\mathrm{M}}$ such that $n(i, \tilde{\mathrm{Y}}) > n_0$.

By the definition of n_0, we can find a Γ-module M′ such that M is isomorphic to the direct sum of M′ with n_0 copies of ρ_i.

We define a linear map $\iota: \mathscr{F}_{\mathrm{M}'} \to \mathscr{F}_{\mathrm{M}}$ as follows. Let $f' \in \mathscr{F}_{\mathrm{M}'}$. We must specify the value of $\iota(f')$ at a point $\Delta \in \Lambda_{\mathrm{M}}$. Consider the variety **B** consisting of all Δ-stable Γ-submodules N of M which are isomorphic to M′. If $\mathrm{N} \in \mathbf{B}$ we choose an isomorphism of Γ-modules $\mathrm{N} \cong \mathrm{M}'$. This induces an isomorphism $\Lambda_{\mathrm{N}} \to \Lambda_{\mathrm{M}'}$. Composing $f': \Lambda_{\mathrm{M}'} \to \mathbf{Q}$ with the last isomorphism gives a function $\Lambda_{\mathrm{N}} \to \mathbf{Q}$ whose value at the restriction $\Delta|_{\mathrm{N}}$ is denoted $\tilde{f}'(\mathrm{N}) \in \mathbf{Q}$. (This is independent of the choice of isomorphism $\mathrm{N} \cong \mathrm{M}'$ since f' is constant on $\mathrm{G}_{\mathrm{M}'}$-orbits.) Now $\mathrm{N} \mapsto \tilde{f}'(\mathrm{N})$ is a constructible function on **B**; hence we may associate to it the linear combination of Euler characteristics

$$\sum_{a \in \mathbf{Q}} a \,\mathrm{Euler}\{\mathrm{N} \in \mathbf{B} \mid \tilde{f}'(\mathrm{N}) = a\};$$

this number is by definition $\iota(f')(\Delta)$. (Compare [L, 12.10].) From the definition it follows that

(c) $\iota(f'): \Lambda_{\mathrm{M}} \to \mathbf{Q}$ is a function in $\mathscr{F}_{\mathrm{M}}$ with support contained in $\bigcup_{p \geq n_0} \Lambda_{\mathrm{M}, i, p}$;

(d) if $\Delta \in \Lambda_{\mathrm{M}, i, n_0}$ (so that **B** above is a single point N) then $\iota(f')(\Delta) = f'(\Xi)$ where $\Xi \in \Lambda_{\mathrm{M}', i, 0}$ corresponds to $\Delta|_{\mathrm{N}}$ under some isomorphism of Γ-modules $\mathrm{N} \cong \mathrm{M}'$.

Now let

$$\mathrm{Y}_0 = \mathrm{Y} \cap \Lambda_{\mathrm{M}, i, n_0} = \mathrm{Y} \cap \Big(\bigcup_{p' : p' \leq n_0} \Lambda_{\mathrm{M}, i, p'}\Big).$$

This is an open dense subset of Y and an irreducible component of $\Lambda_{\mathrm{M}, i, n_0}$.

According to [L, 12.5] there exists an irreducible component Y'_0 of $\Lambda_{\mathrm{M}', i, 0}$ such that for any $\Delta \in \mathrm{Y}_0$, we have $\Xi \in \mathrm{Y}'_0$ (where Ξ is related to Δ as in (d)).

Let Y′ be the closure of Y'_0 in $\Lambda_{\mathrm{M}'}$; this is an irreducible component of $\Lambda_{\mathrm{M}'}$.

Since $\dim \mathrm{M}' < \dim \mathrm{M}$, there exists $f' \in \mathscr{F}_{\mathrm{M}'}$ such that for some open dense $\mathrm{G}_{\mathrm{M}'}$-stable subset O′ of Y we have $f'|_{\mathrm{O}'} = 1$ and such that for some closed G_{M}-stable subset $\mathrm{H}' \subset \Lambda_{\mathrm{M}'}$ of dimension $< \dim \Lambda_{\mathrm{M}'}$ we have $f' = 0$ outside $\mathrm{Y}' \cup \mathrm{H}'$.

Let $\mathrm{H}'_0 = \mathrm{H}' \cap \Lambda_{\mathrm{M}', i, 0}$; this is a closed, $\mathrm{G}_{\mathrm{M}'}$-stable subset of $\Lambda_{\mathrm{M}', i, 0}$ of dimension $< \dim \Lambda_{\mathrm{M}'}$.

Replacing if necessary O′ by $\mathrm{O}' \cap \mathrm{Y}'_0$, we may assume that $\mathrm{O}' \subset \mathrm{Y}'_0$.

Let O_0 (resp. H_0) be the set of all $\Delta \in \Lambda_{\mathrm{M}, i, n_0}$ with the following property: any $\Xi \in \Lambda_{\mathrm{M}', i, 0}$ related to Δ as in (d), lies in O′ (resp. in H'_0). Then O_0 is an open, dense, G_{M}-stable subset of Y_0 and H_0 is a closed G_{M}-stable subset of $\Lambda_{\mathrm{M}, i, n_0}$ of dimension $< \dim \Lambda_{\mathrm{M}}$.

Consider $\iota(f') \in \mathscr{F}_{\mathrm{M}}$. By (c), (d) we have

(e) $\iota(f')|_{O_0}=1$; the support of $\iota(f')$ is contained in $(Y_0 \cup H_0) \cup (\bigcup_{p>n_0} \Lambda_{M,i,p})$.

We have

$$\bigcup_{p:p>n_0} \Lambda_{M,i,p} = \bigcup \tilde{Y}$$

where $\tilde{Y}$ runs over the irreducible components of Λ_M such that $n(i, \tilde{Y})>n_0$. By our assumption we can find, for any such $\tilde{Y}$, a function $f[\tilde{Y}] \in \mathscr{F}_M$ satisfying the requirements of the proposition with $O[\tilde{Y}] \subset \tilde{Y}$ and $H[\tilde{Y}]$ of dimension $<\dim \Lambda_M$. The restriction of $\iota(f')$ to $O[\tilde{Y}]$ is a constructible function; since $O[\tilde{Y}]$ is irreducible, there exists an open dense G_M-stable subset $O_1[\tilde{Y}]$ of $O[\tilde{Y}]$ and a number $a_{\tilde{Y}} \in \mathbf{Q}$ such that the restriction of $\iota(f')$ to $O_1[\tilde{Y}]$ is the constant function $a_{\tilde{Y}}$. We may assume that the sets $O_1[\tilde{Y}]$ for various $\tilde{Y}$ are disjoint.

Let

$$f=\iota(f')-\sum_{\tilde{Y}} a_{\tilde{Y}} f[\tilde{Y}].$$

Using (e) and the definitions we see that f has the required properties. The proposition is proved.

3.7. The previous proof gives an f with integral values.

3.8. For any $Y \in \operatorname{Irr} \Lambda_M$, we have a linear function $T_Y: \mathscr{F}_M \to \mathbf{Q}$; it associates to $f \in \mathscr{F}_M$ the (constant) value of f on a suitable open dense subset of Y. We can now define a linear function from $\mathscr{F}_M$ to the $\mathbf{Q}$-vector space of $\mathbf{Q}$-valued functions on $\operatorname{Irr}\Lambda_M$. This linear function is surjective, by 3.6. (This actually holds for any quiver, with the same proof.) In particular, we see that

(a) $|\operatorname{Irr} \Lambda_M| \leqslant \dim \mathscr{F}_M$.

3.9. Let u^- be the -part of the enveloping algebra corresponding to the (affine) Lie algebra associated to our affine Coxeter graph. This is the $\mathbf{Q}$-algebra defined by generators F_i $(i \in I)$ and relations

$$\sum_{p=0}^{N+1} (-1)^p \binom{N+1}{p} F_j^p F_i F_j^{N+1-p}=0$$

for any $i \neq j$ (with $N=\dim T_j^i$, see 1.1).

Consider the $\mathbf{Q}$-vector space $\mathscr{F}=\oplus_M \mathscr{F}_M$ where the sum is over a set of representatives for the isomorphism classes of Γ-modules; the choice of representatives is immaterial since $\mathscr{F}_M$ is canonically isomorphic to $\mathscr{F}_{M'}$ whenever M, M′ are isomorphic.

There is a unique $\mathbf{Q}$-algebra structure on $\mathscr{F}$, together with a surjective algebra homomorphism $u^- \to \mathscr{F}$ such that

(a) $$\mathrm{F}_{i_1}^{s_1}\ldots\mathrm{F}_{i_m}^{s_m}/(s_1!\ldots s_m!) \mapsto \chi_{\mathrm{M}_\dagger}$$

(see 3.5) for any sequence $\mathrm{M}_\dagger = (\mathrm{M}_1, \mathrm{M}_2, \ldots \mathrm{M}_m)$ of isotypical Γ-modules with M_p isomorphic to the direct sum of s_p copies of ρ_{i_p} for all p. (See [L, 12.11].)

For any Γ-module M, we define u_{M}^- to be the subspace of u^- spanned by the left hand sides of (a) such that $\sum_{p : i_p = i} s_p = (\rho_i : \mathrm{M})$ for all i. These give a grading $u^- = \oplus_{\mathrm{M}} u_{\mathrm{M}}^-$ and our homomorphism $u^- \to \mathscr{F}$ clearly respects the gradings. Hence we have

(b) $\dim u_{\mathrm{M}}^- \geqslant \dim \mathscr{F}_{\mathrm{M}}$

for any M.

By the Poincaré-Birkhoff-Witt theorem, we have the equality of formal power series

(c) $$\sum_{\mathrm{M}} \dim u_{\mathrm{M}}^- \mathrm{X}^{\dim \mathrm{M}} = \prod_{\alpha \in \mathbf{R}^+} (1 - \mathrm{X}^{\dim \alpha})^{-1} \prod_s (1 - \mathrm{X}^{s|\Gamma|})^{-|\mathrm{I}|+1}.$$

(The second product is the contribution of the "imaginary" roots.)

4. Irreducible components of a Lagrangian variety

4.1. In this section we study the set of irreducible components of a Lagrangian variety Λ_{M}; we give a combinatorial description of this set, in the setup of section 2.

We first prove a result about vanishing of certain Ext^1-groups.

Proposition 4.2. — *Let* $\mathbf{M}_1, \mathbf{M}_2 \in \mathscr{C}^\delta$ *be as follows*:

(a) $\mathbf{M}_1 = \mathrm{K}^s \mathrm{P}_i^{(-1)^s \delta}$, $\mathbf{M}_2 = \mathrm{K}^{s'} \mathrm{P}_{i'}^{(-1)^{s'} \delta}$ *with* s, $s' \in \mathbf{N}$, $s \geqslant s'$, *and* $i \in \mathrm{I}^{(-1)^s \delta}$, $i' \in \mathrm{I}^{(-1)^{s'} \delta}$, *or*

(b) $\mathbf{M}_1 = \mathrm{C}^s \mathrm{P}_i^{(-1)^s \delta}$, $\mathbf{M}_2 = \mathrm{C}^{s'} \mathrm{P}_{i'}^{(-1)^{s'} \delta}$ *with* s, $s' \in \mathbf{N}$, $s \leqslant s'$, *and* $i \in \mathrm{I}^{(-1)^{s+1} \delta}$, $i' \in \mathrm{I}(-1)^{s'+1\delta}$, *or*

(c) $\mathbf{M}_1 = \mathrm{C}^s \mathrm{P}_i^{(-1)^s \delta}$, $\mathbf{M}_2 = \mathrm{K}^{s'} \mathrm{P}_{i'}^{(-1)^{s'} \delta}$ *with* s, $s' \in \mathbf{N}$, *and* $i \in \mathrm{I}^{(-1)^{s+1} \delta}$, $i' \in \mathrm{I}^{(-1)^{s'} \delta}$, *or*

(d) $\mathbf{M}_1 \in {}^0\mathscr{C}^\delta$, $\mathbf{M}_2 = \mathrm{K}^{s'} \mathrm{P}_{i'}^{(-1)^{s'} \delta}$ *with* $s' \in \mathbf{N}$, *and* $i' \in \mathrm{I}^{(-1)^{s'} \delta}$, *or*

(e) $\mathbf{M}_1 = \mathrm{C}^s \mathrm{P}_i^{(-1)^s \delta}$, $\mathbf{M}_2 \in {}^0\mathscr{C}^\delta$, *with* $s \in \mathbf{N}$, *and* $i \in \mathrm{I}^{(-1)^{s+1} \delta}$, *or*

(f) $\mathbf{M}_1, \mathbf{M}_2 \in {}^0\mathscr{C}^\delta$ *with disjoint spectra.*

Then

$$\mathrm{Ext}^1(\mathbf{M}_1, \mathbf{M}_2) = 0.$$

4.3. In preparation for the proof of 4.2 we note the following. Let

$$0 \to \mathbf{M}' \to \mathbf{M} \to \mathbf{M}'' \to 0$$

be a short exact sequence in $\mathscr{C}^{\delta}_{\mathrm{in}}$. Then the corresponding sequence

$$0 \to \mathrm{C}\mathbf{M}' \to \mathrm{C}\mathbf{M} \to \mathrm{C}\mathbf{M}'' \to 0$$

is exact in $\mathscr{C}^{-\delta}$.

This follows easily from the definitions, using the snake lemma in homological algebra. Similarly, if

$$0 \to \mathbf{M}' \to \mathbf{M} \to \mathbf{M}'' \to 0$$

is a short exact sequence in $\mathscr{C}^{\delta}_{\mathrm{su}}$, then the corresponding sequence

$$0 \to \mathrm{K}\mathbf{M}' \to \mathrm{K}\mathbf{M} \to \mathrm{K}\mathbf{M}'' \to 0$$

is exact in $\mathscr{C}^{-\delta}$.

4.4. We show that any exact sequence

$$0 \to \mathbf{M}' \to \mathbf{M} \to \mathbf{M}'' \to 0$$

in $\mathscr{C}^{\delta}$ such that either

(a) $\mathbf{M}'' = \mathrm{P}^{\delta}_{i}$ with $i \in \mathrm{I}^{-\delta}$

or

(b) $\mathbf{M}' = \mathrm{P}^{\delta}_{i}$ with $i \in \mathrm{I}^{\delta}$,

is split.

Indeed, let $\mathbf{M} = (\mathrm{M}, \Delta)$, $\mathbf{M}' = (\mathrm{M}', \Xi)$ and let W be any Γ-submodule of M complementary to M′. In both cases, W defines a subobject of $\mathbf{M}$, complementary to $\mathbf{M}'$.

4.5. We now prove 4.2. Consider a short exact sequence in $\mathscr{C}^{\delta}$

(a) $$0 \to \mathrm{K}^{s'}\mathrm{P}^{(-1)^{s'}\delta}_{i'} \to \mathbf{M} \to \mathbf{M}_1 \to 0$$

where $s' \in \mathbf{N}$, $i' \in \mathrm{I}^{(-1)^{s'}\delta}$ and $\mathbf{M}_1$ is as in 4.2 (a), (c) or (d). We want to show by induction on s' that this exact sequence is split. For $s' = 0$ this follows from 4.4; hence we may assume that $s' \geqslant 1$ and that our assertion is already proved for $s'-1$, $-\delta$ instead of s', δ.

Now all terms of (a) are contained in $\mathscr{C}^{\delta}_{\mathrm{in}}$. For the first term this follows from 2.5 (a) (since $s' \geqslant 1$). For $\mathbf{M}_1$ this follows from 2.5 (a), if $\mathbf{M}_1$ is as in 4.2 (a) (since $s \geqslant s' \geqslant 1$); from 2.9 (a), if $\mathbf{M}_1$ is as in 4.2 (c); finally, if $\mathbf{M}_1 \in {}^{0}\mathscr{C}^{\delta}$, then it is in $\mathscr{C}^{\delta}_{\mathrm{in}}$

(see 2.7 (b)). Now the middle term of (a) is automatically in $\mathscr{C}^{\delta}_{\mathrm{in}}$ since the other two terms are there.

Using now 4.3, we see that by applying C to (a) we get an exact sequence

$$0 \to \mathrm{CK}^{s'} \mathrm{P}_{i'}^{(-1)^{s'}\delta} \to \mathrm{C}\mathbf{M} \to \mathrm{C}\mathbf{M}_1 \to 0.$$

Since $\mathrm{K}^{s'-1} \mathrm{P}_{i'}^{(-1)^{s'}\delta} \in \mathscr{C}_{\mathrm{su}}^{-\delta}$, by 2.8 (a), we see from 2.6(b) that

$$\mathrm{CK}^{s'} \mathrm{P}_{i'}^{(-1)^{s'}\delta} = \mathrm{K}^{s'-1} \mathrm{P}_{i'}^{(-1)^{s'}\delta}$$

hence the previous exact sequence is

(b) $$0 \to \mathrm{K}^{s'-1} \mathrm{P}_{i'}^{(-1)^{s'}} \to \mathrm{C}\mathbf{M} \to \mathrm{C}\mathbf{M}_1 \to 0.$$

This is an exact sequence of the same type as (a); note that, if $\mathbf{M}_1 = \mathrm{K}^s \mathrm{P}_i^{(-1)^s\delta}$ is as in 4.2 (a), then $\mathrm{C}\mathbf{M}_1 = \mathrm{K}^{s-1} \mathrm{P}_i^{(-)^s\delta}$ (and $s-1 \geqslant s'-1$); if $\mathbf{M}_1$ is in ${}^0\mathscr{C}^{\delta}$, then $\mathrm{C}\mathbf{M}_1$ is in ${}^0\mathscr{C}^{-\delta}$ (see 2.10, 2.16).

By the induction hypothesis, the exact sequence (b) is split.

Now all terms in (a) are fixed by KC (by the result dual to 2.6(b)); hence applying K to the exact sequence (b) gives us back the exact sequence (a). Since (b) is split, it follows that (a) is split and our assertion is proved by induction. Thus 4.2 is proved in the cases (a), (c), (d). Now 4.2 in cases (b), (e) can be obtained from the cases (a), (d) by duality.

Next, assume that

$$0 \to \mathbf{M}_2 \to \mathbf{M} \to \mathbf{M}_1 \to 0$$

is an exact sequence in $\mathscr{C}^{\delta}$ with $\mathbf{M}_1$, $\mathbf{M}_2$ in ${}^0\mathscr{C}^{\delta}$. From the definitions it follows immediately that $\mathbf{M}_1 \in {}^0\mathscr{C}^{\delta}$ and using 2.17 (b) we see that the assertion of 4.2 holds in case (f).

4.6. Now let $\mathbf{M}_1 = (\mathrm{M}_1, \Delta_1)$, $\mathbf{M}_2 = (\mathrm{M}_2, \Delta_2)$ be two objects of $\mathscr{C}^{\delta}$. We define a linear map $\Psi : \mathrm{Hom}_{\Gamma}(\mathrm{M}_1, \mathrm{M}_2) \to \mathrm{Hom}_{\Gamma}(\mathrm{M}_1^{\delta} \otimes \rho, \mathrm{M}_2^{-\delta})$ by associating to any $\varphi \in \mathrm{Hom}_{\Gamma}(\mathrm{M}_1, \mathrm{M}_2)$ the map

$$\Psi(\varphi) : x \otimes e \mapsto \varphi(\Delta_1(x \otimes e)) - \Delta_2(\varphi(x) \otimes e)$$

for all $x \in \mathrm{M}_1^{\delta}$ and $e \in \rho$.

By the definition of morphisms in $\mathscr{C}^{\delta}$ (see 2.3), the kernel of Ψ is exactly $\mathrm{Hom}(\mathbf{M}_1, \mathbf{M}_2)$. Next we define a linear map

$$\Psi' : \mathrm{Hom}_{\Gamma}(\mathrm{M}_1^{\delta} \otimes \rho, \mathrm{M}_2^{-\delta}) \to \mathrm{Ext}^1(\mathbf{M}_1, \mathbf{M}_2)$$

as follows. To a Γ-linear map $\Pi : \mathrm{M}_1^{\delta} \otimes \rho \to \mathrm{M}_2^{-\delta}$ we associate the extension $0 \to \mathbf{M}_2 \to \mathbf{X} \to \mathbf{M}_1$ where $\mathbf{X} = (\mathrm{X}, \Delta) \in \mathscr{C}^{\delta}$ is given by $\mathrm{X} = \mathrm{M}_1 \oplus \mathrm{M}_2$ and

$\Delta : X^{\delta} \otimes \rho \to X^{-\delta}$ is given by a 2×2 matrix with entries Δ_1, Π, Δ_2, 0. The maps $\mathbf{M}_2 \to \mathbf{X}$ and $\mathbf{X} \to \mathbf{M}_1$ are the obvious ones. It is easy to see that Ψ' is a linear map and that its kernel is exactly the image of Ψ.

Proposition 4.7. — *The sequence*

$$0 \to \mathrm{Hom}(\mathbf{M}_1, \mathbf{M}_2) \to \mathrm{Hom}_\Gamma(M_1, M_2) \xrightarrow{\Psi} \mathrm{Hom}_\Gamma(M_1^\delta \otimes \rho, M_2^{-\delta}) \xrightarrow{\Psi'} \mathrm{Ext}^1(\mathbf{M}_1, \mathbf{M}_2) \to 0$$

constructed in 4.6 *is exact.*

This is a special case of a result of Ringel [R1], valid for any quiver.

4.8. Assume that we are given a Γ-module M and an element $\Delta \in E_M^\delta$. Let T be the tangent space to the G_M-orbit of Δ (translated so that it contains zero) and let T′ be the set of vectors in $E_M^{-\delta}$ which are orthogonal to T under $\langle , \rangle$. Consider the exact sequence in 4.7 for $\mathbf{M}_1 = \mathbf{M}_2 = (M, \Delta)$. It is clear that

(a) T is the image of Ψ.

From 4.7 it then follows that

(b) $T' \cong \mathrm{Ext}^1((M, \Delta), (M, \Delta))^*$.

4.9. Assume now that we are given a line L in ρ whose stabilizer Γ_L has order $2n \geqslant 4$, and that $\mathbf{M} = (M, \Delta)$ is an object in ${}^0\mathscr{C}^\delta$ with spectrum contained in the Γ-orbit Z of L. Then $\mathbf{M}$ is isomorphic to a direct sum of indecomposable objects $\mathbf{M}_{Z, r, m}$ (see 2.20) where r runs over $\mathbf{Z}/n\mathbf{Z}$ and $m \geqslant 1$. Let $f(r, m)$ be the number of times $\mathbf{M}_{Z, r, m}$ appears in the decomposition. We say that $\mathbf{M}$ is *aperiodic* if for any $m \geqslant 1$ there exists some $r \in \mathbf{Z}/n\mathbf{Z}$ such that $f(r, m) = 0$. Assume that $\mathbf{M}$ is aperiodic. Let T, T′ be as in 4.8, and let $\Xi \in T'$. We will show that

(a) $\Delta + \Xi \in E_M$ is nilpotent.

Under the equivalence of categories constructed in 2.18, $\mathbf{M}$ corresponds to an object (V, t) of $\mathscr{C}'_n$ and we clearly have

(b) $\mathrm{Ext}^1(\mathbf{M}, \mathbf{M}) \cong \mathrm{Ext}^1((V, t), (V, t))$

where the last Ext is taken in $\mathscr{C}'_n$. This last Ext-group can be inserted (just like the first one) in an exact sequence as in 4.7 for a cyclic quiver. (The exact sequence in 4.7 makes sense for any quiver). Hence the following analogue of 4.8 (b) holds: $\mathrm{Ext}^1((V, t), (V, t))^*$ is isomorphic to the vector space S consisting of all linear transformations $\sigma : V \to V$ such that $\sigma(V_r) \subset V_{r-1}$ for all r (notation of 2.18) and $t\sigma = \sigma t$. Combining this with (b) and 4.8 (b) we deduce that

(c) $\dim S = \dim T'$.

We now define a map $S \to T'$ by the method in 2.18. Let V', e, e' be as in 2.18. (Recall that $V=V'$ as a vector space.) We may assume that (M, Δ) is constructed from (V, t) as in the end of 2.18. Recall that $M^{\delta}_{L}=V$, $M^{-\delta}_{L}=V'$. Let $\sigma \in S$. There is a unique Γ-linear map $\bar{\sigma}: M^{-\delta} \otimes \rho \to M^{\delta}$ which extends the Γ_L-linear map $V' \otimes \rho \to V$ given by $x' \otimes (ae+be') \mapsto a\,\sigma(t(x'))+b\,\sigma(x')$. One verifies that $\bar{\sigma} \in T'$. It is clear that the map $\sigma \mapsto \bar{\sigma}$ from S to T' is injective; by (c), it is an isomorphism. In particular, we must have $\Xi = =\bar{\sigma}$ for some $\sigma \in S$.

The aperiodicity assumption implies that σ is nilpotent as an endomorphism of V (this is a property of cyclic quivers, see [L, § 15]). Again with the notation of 2.18, we have $M_L = V \oplus V'$. For any $e_1 \in \rho$, $(\Delta+\Xi)_{e_1}: M \to M$ maps M_L into itself; more precisely from the definitions we have $(\Delta+\Xi)_e(x, x')=(\sigma(t(x')), t(x))$ and $(\Delta+\Xi)_{e'}(x, x')=(\sigma(x'), x)$ for all $x, x' \in V=V'$. Since t, σ are commuting nilpotent endomorphisms of V, it follows that a composition of sufficiently many endomorphisms $(\Delta+\Xi)_e$, $(\Delta+\Xi)_{e'}$, will map M_L to zero. Since M is spanned by the Γ-translates of M_L, we see that $\Delta+\Xi \in E_M$ is nilpotent. This proves (a).

4.10. Assume now that we are given a line L in ρ whose stabilizer Γ_L equals $\{1, c\}$ and that $\mathbf{M}=(M, \Delta)$ is an indecomposable object in ${}^0\mathscr{C}^{\delta}$ with spectrum equal to the Γ-orbit of L. Let e, e' be a basis of ρ such that $e \in L$. Let f be the unique element of E^{δ}_{M} such that $f(x \otimes e')=0$, $f(x \otimes e)=\Delta(x \otimes e')$ for all $x \in M^{\delta}_{L}$.

Let T, T' be as in 4.8, and let $\Xi \in T'$. Assume that $\langle f, \Xi \rangle = 0$. We will show that (a) $\Delta+\Xi \in E_M$ is nilpotent.

Under the equivalence of categories constructed in 2.18, $\mathbf{M}$ corresponds to an object (V, t) of $\mathscr{C}'_1$; here t is necessarily a regular nilpotent endomorphism of V.

Just as in 4.9, we have $\dim T' = \dim S$, where S is the vector space of all linear transformations $\sigma: V \to V$ commuting with t.

As in 4.9, we define a map $S \to T'$ by the method in 2.18. Let V' be as in 2.18. (Recall that $V=V'$ as a vector space.) We may assume that (M, Δ) is constructed from (V, t) as in the end of 2.18. Recall that $M^{\delta}_{L}=V$, $M^{-\delta}=V'$. Let $\sigma \in S$. There is a unique Γ-linear map $\bar{\sigma}: M^{-\delta} \otimes \rho \to M^{\delta}$ which extends the Γ_L-linear map $V' \otimes \rho \to V$ given by $x' \otimes (ae+be') \mapsto a\sigma(t(x'))+b\,\sigma(x')$. One verifies that $\bar{\sigma} \in T'$. It is clear that the map $\sigma \mapsto \bar{\sigma}$ from S to T' is injective; by (c), it is an isomorphism.

In particular, we must have $\Xi=\bar{\sigma}$ for some $\sigma \in S$. Since t is regular nilpotent and σ commutes with t, we see that σ is a sum of a nilpotent endomorphism with y times the identity, where $y \in \mathbf{C}$.

We now show that our assumption $\langle f, \Xi \rangle = 0$ implies that $y=0$. For any $x' \in V'$ we have $\Xi_{e'}(x')=\sigma(x')$; hence $(f_e \Xi_{e'} - f_{e'} \Xi_e)(x')=\sigma(x')$. For $x \in V$ we have $(f_e \Xi_{e'} - f_{e'} \Xi_e)(x)=0$. Thus $f_e \Xi_{e'} - f_{e'} \Xi_e$ leaves stable M_L and its trace in there is equal to $\operatorname{tr}\sigma = y \dim V$. Now $f_e \Xi_{e'} - f_{e'} \Xi_e$ is Γ-equivariant hence it must also leave stable M'_L for any L' in the Γ-orbit of L and its trace in there is again $y \dim V$. It follows that $\langle f, \Xi \rangle = y \dim M/2$ and therefore $y=0$. We now see that σ is a nilpotent endomorphism. From this point on the proof of (a) continues exactly as in 4.9.

4.11. Assume now that we are given an object $\mathbf{M}=(M, \Delta)$ of $\mathscr{C}^\delta$. From the results of section 2 we see that there exists a direct sum decomposition $\mathbf{M} = \oplus_{h \in \mathbf{Z}} \mathbf{M}(h)$ where $\mathbf{M}(h) = (M(h), \Delta(h)) \in \mathscr{C}^\delta$ and integers $h_0 \leqslant h_1 \leqslant h_2 \leqslant h_3 \leqslant h_4$ such that the properties (a), (b), (c), (d) below are satisfied.

(a) For any h such that $h_0 < h \leqslant h_1$, we have $\mathbf{M}(h) \cong K^s P_i^{(-)^s \delta}$ for some $s = s(h) \geqslant 0$ and some $i \in I^{(-1)^s \delta}$; moreover, if $h \leqslant h' \leqslant h_1$, then $s(h) \leqslant s(h')$.

(b) For any h such that $h_3 < h \leqslant h_4$ we have $\mathbf{M}(h) \cong C^s P_i^{(-1)^s \delta}$ for some $s = s(h) \geqslant 0$ and some $i \in I^{(-1)^{s+1} \delta}$; moreover, if $h_3 < h \leqslant h'$, then $s(h) \geqslant s(h')$.

(c) For any h with $h_1 < h \leqslant h_3$ we have $\mathbf{M}(h) \in {}^0\mathscr{C}^\delta$; it is non-zero and its spectrum is a single Γ-orbit with isotropy group of order $\geqslant 4$ for $h_1 < h \leqslant h_2$, and of order 2 for $h_2 < h \leqslant h_3$. Moreover, if $h_1 < h < h' \leqslant h_3$ then $\mathbf{M}(h)$, $\mathbf{M}(h')$ have disjoint spectra.

(d) For $h \leqslant h_0$ or $h > h_4$, we have $\mathbf{M}(h) = 0$.

We now make the following assumption on $\mathbf{M}$:

(e) For any h with $h_1 < h \leqslant h_2$, $\mathbf{M}(h)$ is aperiodic. For any h with $h_2 < h \leqslant h_3$, $\mathbf{M}(h)$ is indecomposable.

For each h with $h_2 < h \leqslant h_3$ we choose a line $L_h \subset \rho$ in the spectrum of $\mathbf{M}(h)$. Choose a vector $e' \in \rho$ outisde all these lines L_h, and choose a vector $e \in \rho$, linearly independent from e'. Let $z_h \in \mathbf{C}$ be such that $e + z_h e' \in L_h$. Let f_h be the unique element of $E_{\mathbf{M}(h)}$ such that $f_h(x \otimes e') = 0$ and $f_h(x \otimes e) = \Delta(h)(x \otimes e')$ for all $x \in M(h)^\delta_{L_h}$. Then f_h is like f in 4.10 (for $\mathbf{M}(h)$, L_h, $e + z_h e'$, e' instead of $\mathbf{M}$, L, e, e'); indeed, we have $f_h(x \otimes (e + z_h e')) = 0$ for all $x \in M(h)^\delta_{L_h}$. We may regard f_h as an element of $E^\delta_{\mathbf{M}}$ (equal to zero on $M(h')^\delta \otimes \rho$ for any $h' \neq h$).

In this setup, we have the following result:

Proposition 4.12. — *Associate* T, T′ *to* Δ *as in* 4.8. *Let* $\Xi \in T'$ *be such that* $\langle f_h, \Xi \rangle = 0$ *for each* h *with* $h_2 < h \leqslant h_3$. *Then* $\Delta + \Xi$ *is a nilpotent element of* $E_{\mathbf{M}}$.

The grading $M = \oplus_h M(h)$ of M gives rise to a bigrading $E^{\pm 1}_{\mathbf{M}} = \oplus_{h,h'} E^{\pm 1}_{\mathbf{M}}(h, h')$ where $E^{\pm 1}_{\mathbf{M}}(h, h') = \mathrm{Hom}_\Gamma(M(h)^{\pm 1} \otimes \rho, M(h')^{\mp 1})$.

We consider, for each $h > h'$, the exact sequence 4.7 for $\mathbf{M}_1 = \mathbf{M}(h)$ and $\mathbf{M}_2 = \mathbf{M}(h')$. The Ext^1 term there is zero, by 4.2, hence the map $\Psi : \mathrm{Hom}_\Gamma(M(h), M(h')) \to E^\delta_{\mathbf{M}}(h, h')$ is surjective. The last map Ψ is the restriction of the map with the same name $\Psi : \mathrm{Hom}_\Gamma(M, M) \to E^\delta_{\mathbf{M}}$; it follows that the image of the last map which, by 4.8 (a), is just T, contains $E^\delta_{\mathbf{M}}(h, h')$ for all $h > h'$. In fact, the same holds, even for $h = h'$, except when $h_1 < h \leqslant h_3$ (by the same argument).

Now the annihilator in $E_M^{-\delta}$ (with respect to $\langle\ ,\ \rangle$) of the sum of the $E_M^{\delta}(h, h')$ over $\{(h, h') \mid h \geq h'\} - \{(h, h) \mid h_1 < h \leq h_3\}$ is equal to the sum of the $E_M^{-\delta}(h, h')$ over $\{(h, h') \mid h > h'\} \cup \{(h, h) \mid h_1 < h \leq h_3\}$.

Hence T′ (and so also Ξ) is contained in the last sum. In particular, Ξ maps any $M(h)^{-\delta} \otimes \rho$ into the sum of the $M(h')^{\delta}$ over $\{h' \mid h' \leq h\}$, if $h_1 < h \leq h_3$, and into the sum of the $M(h')^{\delta}$ over $\{h' \mid h' < h\}$, for the other values of h.

Let us now define some Γ-submodules $M[h] = \oplus_{h' \mid h' \leq h} M(h)$ for $h \in \mathbf{Z}$. We have $\ldots \subset M[h-1] \subset M[h] \subset \ldots$ The previous statement can be reformulated as follows:

Ξ maps $M[h]^{-\delta} \otimes \rho$ into $M[h]^{\delta}$, if $h_1 < h \leq h_3$, and into $M[h-1]^{\delta}$, for the other values of h.

For $h_1 < h \leq h_3$, we see that Ξ induces a Γ-linear map $M[h]/M[h-1]^{-\delta} \otimes \rho \to M[h]/M[h-1]^{\delta}$ or, equivalently, $M(h)^{-\delta} \otimes \rho \to M(h)^{\delta}$; this map is denoted $\Xi(h)$.

We clearly have that Δ maps $M(h)^{-\delta} \otimes \rho$ into $M(h)^{-\delta}$ for any h, hence it maps $M[h]^{\delta} \otimes \rho$ into $M[h]^{-\delta}$ for any h. Combining this with the analogous property of Ξ, we deduce that:

$\Delta + \Xi$ maps $M[h] \otimes \rho$ into $M[h]$ for any h.

Hence $\Delta + \Xi$ induces for any h a Γ-linear map $M[h]/M[h-1] \otimes \rho \to M[h]/M[h-1]$ or, equivalently, an element of $E_{M(h)}$. If this element is nilpotent for any h, then, from the definition of nilpotency, it would follow easily that $\Delta + \Xi$ is nilpotent. But from the previous discussion we see that this induced element is $\Delta(h) + \Xi(h)$ if $h_1 < h \leq h_3$, and $\Delta(h)$ for the other values of h.

If $h_1 < h \leq h_2$, then $\Delta(h) + \Xi(h)$ is nilpotent, by the discussion in 4.9. If $h_2 < h \leq h_3$, then $\Delta(h) + \Xi(h)$ is nilpotent, by the discussion in 4.10; indeed, our assumption implies that $\langle f_h, \Xi(h) \rangle = 0$. If $h_0 < h \leq h_1$ or $h_3 < h \leq h_4$, we have $\Delta(h) \in E_{M(h)}^{\delta}$ and all elements of $E_{M(h)}^{\delta}$ are obviously nilpotent in $E_{M(h)}$. This completes the proof.

4.13. Given a Γ-module M, we consider the set $\mathscr{S}(M)^{\delta}$ of all pairs (σ, λ) where $\sigma : \mathscr{P}^{\delta} \to \mathbf{N}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q)$ is a sequence of integers ≥ 1, and the properties (a)-(d) below are satisfied.

(a) σ has finite support.

(b) Let Z be any Γ-orbit in $P(\rho)$ such that the stabilizer of a line in Z has order $2n \geq 4$. For any $m \geq 1$, at least one of the numbers

$$\sigma(\mathbf{M}_{Z, 0, m}), \sigma(\mathbf{M}_{Z, 1, m}), \ldots, \sigma(\mathbf{M}_{Z, n-1, m})$$

is zero. (Notation of 2.20.)

(c) Let Z be any Γ-orbit in $P(\rho)$ such that the stabilizer of a line in Z has order 2. For any $m \geq 1$, we have $\sigma(\mathbf{M}_{Z, 0, m}) = 0$. (Notation of 2.20.)

(d) We have $\sum_{P} \sigma(P) \operatorname{gr}(P) + \sum_{j} \lambda_j \mathbf{r} = M$ in $\mathscr{G}\Gamma$.

Given $(\sigma, \lambda) \in \mathscr{S}(M)^{\delta}$, we consider the subset $X(\sigma, \lambda)$ of E_M^{δ} consisting of all elements Δ such that (M, Δ) is isomorphic in $\mathscr{C}^{\delta}$ to

$$\oplus_P P^{\sigma(P)} \oplus (\mathbf{M}_{Z_1, 0, \lambda_1} \oplus \ldots \oplus \mathbf{M}_{Z_q, 0, \lambda_q})$$

for some distinct Γ-orbits $Z_1, \ldots, Z_q$ as in (c).

In this setup we have the following result:

Proposition 4.14. — (a) $X(\sigma, \lambda)$ *is open, dense, smooth in its closure* $\bar{X}(\sigma, \lambda)$. *It is also irreducible of dimension equal to q plus the dimension of any* G_M*-orbit it contains.*

(b) *Let* $\mathscr{N}(\sigma, \lambda)$ *be the conormal bundle of* $\bar{X}(\sigma, \lambda) \subset E_M^{\delta}$, *regarded as a subvariety of* E_M (*see* 3.4). *Then* $\mathscr{N}(\sigma, \lambda)$ *is an irreducible component of* Λ_M (*see* 3.2).

(c) *If* (σ', λ') *is an element of* $\mathscr{S}(M)^{\delta}$, *distinct from* (σ, λ), *then* $\mathscr{N}(\sigma, \lambda) \neq \mathscr{N}(\sigma', \lambda')$.

The proof of (a) is routine and will be omitted; but the necessary ingredients can be found in the following proof of (b).

Let $\Delta \in X(\sigma, \lambda)$. We can find a decomposition of (M, Δ) as in 4.11. We will use the notation of 4.11 relative to this decomposition.

We will define a smooth submanifold D of E_M^{δ}, whose points are parametrized by vectors $y = (y_h) \in \mathbf{C}^q$; here, the index h is such that $h_2 < h \leqslant h_3$ and q is the number of such indices.

For each y as above we define an element $\Delta^y \in E_M^{\delta}$ by assembling together elements $\Delta^y(h) \in E_{M(h)}^{\delta}$ for all $h \in \mathbf{Z}$. For $h_2 < h \leqslant h_3$, $\Delta^y(h)$ is the unique Γ-linear map $M(h)^{\delta} \otimes \rho \to M(h)^{-\delta}$ such that $\Delta^y(h)_e = \Delta(h)_e - y_h \Delta(h)_{e'}$ and $\Delta^y(h)_{e'} = \Delta(h)_{e'}$ on vectors of $M(h)^{\delta}_{L_h}$. (The subscript L_h refers to Δ.) We have $(M(h), \Delta^y(h)) \in {}^0\mathscr{C}^{\delta}$ and its spectrum is the Γ-orbit of the line $\mathbf{C}(e + (z_h + y_h) e')$.

For all other h, we set $\Delta^y(h) = \Delta(h)$. By definition, $D = \{\Delta^y \mid y \in \mathbf{C}^q\}$. Note that D is an affine space containing Δ; we have $\Delta = \Delta^0$. Clearly, D is contained in $\bar{X}(\sigma, \lambda) \subset E_M^{\delta}$. The tangent space to D at any point of D, translated to zero, is a vector subspace D_0 of E_M^{δ} independent of the chosen point. It is clear that

(d) the vectors f_h of 4.11 form a basis of D_0.

We now associate T, T′ to Δ as in 4.8. Let $\Xi \in E_M^{-\delta}$ be such that Ξ is orthogonal under $\langle\ ,\ \rangle$ to the tangent space to $X(\sigma, \lambda)$ at Δ. Then Ξ is orthogonal to T (the tangent space to the G_M-orbit of Δ at Δ) and to D_0 (the tangent space to D at Δ) hence $\Xi \in T'$ and Ξ is orthogonal to the f_h in (d). Using now 4.12, it follows that $\Delta + \Xi \in E_M$ is nilpotent. Thus, the conormal bundle of $X(\sigma, \lambda)$ is contained in the set of nilpotent elements of E_M. It is also contained in the inverse image of 0 under the moment map (see 3.2) since $X(\sigma, \lambda)$ is a union of G_M-orbits (see 3.4 (a)); hence it is contained in Λ_M. By definition, the conormal bundle of $\bar{X}(\sigma, \lambda)$ is the closure in E_M of the conormal bundle of $X(\sigma, \lambda)$; hence it is also contained in Λ_M (which is closed). It is irreducible, by (a), and being a conormal bundle it has dimension equal to dim $E_M/2$. But $\dim \Lambda_M = \dim E_M/2$ (see 3.2) and (b) follows.

Now (c) follows immediately from (a).

4.15. We now consider the formal power series

$$\Pi = \sum_{M} |\mathscr{S}(M)^{\delta}| X^{\dim M},$$

where M runs over a set of representatives for the isomorphism classes of Γ-modules. From the definitions and from 2.21, we see that

$$\Pi = \prod_{\alpha \in R^{+} - {}^{0}R_{+}} (1 - X^{\dim \alpha})^{-1} \times \prod_{L, m} \Big(\prod_{r} (1 - X^{\dim \mu(L, r, m)})^{-1}$$
$$- \prod_{r} X^{\dim \mu(L, r, m)} \prod_{r} (1 - X^{\dim \mu(L, r, m)})^{-1}\Big)$$
$$\times \prod_{L, s} \big((1 - X^{s|\Gamma|})^{-n_L} - X^{s n_L |\Gamma|} (1 - X^{s|\Gamma|})^{-n_L}\big) \sum_{t \geq 0} p(t) X^{t|\Gamma|},$$

where (with notation of 1.17), L runs over $\mathscr{X}$, m runs over the integers ≥ 1, not divisible by n_L, r runs over $\mathbf{Z}/n_L\mathbf{Z}$, s runs over the integers ≥ 1, and $p(t)$ is the number of partitions of t. Note that $\dim \mu(L, r, m) = m|\Gamma| n_L^{-1}$. We have

$$\Pi = \prod_{\alpha \in R^{+} - {}^{0}R_{+}} (1 - X^{\dim \alpha})^{-1} \prod_{\alpha \in {}^{0}R_{+}} (1 - X^{\dim \alpha})^{-1}$$
$$\times \prod_{L, m} (1 - X^{m|\Gamma|}) \prod_{L, s} (1 - X^{s|\Gamma|})^{-n_L} \prod_{L, s} (1 - X^{s n_L |\Gamma|}) \prod_{s} (1 - X^{s|\Gamma|})^{-1},$$

$$\Pi = \prod_{\alpha \in R^{+}} (1 - X^{\dim \alpha})^{-1} \prod_{s} (1 - X^{s|\Gamma|})^{|\mathscr{X}|}$$
$$\prod_{s} (1 - X^{s|\Gamma|})^{-n'} \prod_{s} (1 - X^{s|\Gamma|})^{-1},$$

where $n' = \sum_{L} n_L$. Thus,

(a) $$\Pi = \prod_{\alpha \in R^{+}} (1 - X^{\dim \alpha})^{-1} \prod_{s} (1 - X^{s|\Gamma|})^{-|I|+1},$$

since

(b) $$\sum_{L}(n_L-1)=|I|-2.$$

(The identity (b) follows from 1.18 by comparing dimensions.) Using 3.9(c), we obtain $\Pi=\sum_{M}\dim u_M^- X^{\dim M}$; using the definition of Π, we deduce

(c) $$\sum_{M:\,\dim M=d}(\dim u_M^- - |\mathscr{S}(M)^\delta|)=0$$

for any $d\geqslant 0$.

Theorem 4.16. — (a) *The algebra homomorphism* $u^-\to\mathscr{F}$ (*see* 3.9) *is an isomorphism.*

(b) *For any* Γ*-module* M, *the map* $(\sigma,\lambda)\mapsto\mathscr{N}(\sigma,\lambda)$ (*see* 4.14) *is a bijection* $\mathscr{S}(M)^\delta\cong \operatorname{Irr}\Lambda_M$.

(c) *For any* $Y\in\operatorname{Irr}\Lambda_M$, *there is a unique function* $f_Y\in\mathscr{F}_M$ *such that for some open dense* G_M*-stable subset* O *of* Y *we have* $f|_O=1$ *and such that for some closed* G_M*-stable subset* $H\subset\Lambda_M$ *of dimension* $<\dim\Lambda_M$ *we have* $f=0$ *outside* $Y\cup H$.

(d) *The functions* f_Y (*for* $Y\in\operatorname{Irr}\Lambda_M$) *form a* **Q**-*basis of* $\mathscr{F}_M$.

For any M, we have

(e) $$|\mathscr{S}(M)^\delta|\leqslant|\operatorname{Irr}\Lambda_M|\leqslant\dim\mathscr{F}_M\leqslant\dim u_M^-;$$

(the first inequality follows from 4.14; the second one is 3.8 (a); the third one is 3.9 (b)). In particular, $\dim u_M^- - |\mathscr{S}(M)^\delta|\geqslant 0$. Introducing this in the equality 4.15 (c), we deduce that $\dim u_M^-=|\mathscr{S}(M)^\delta|$ for all M. This implies that all inequalities in (e) are equalities.

The map $u_M^-\to\mathscr{F}_M$ in 3.9 is surjective; the two spaces involved have the same dimension, hence our map is an isomorphism and (a) follows. (A different proof of (a) is given in [L, 12.13].)

The map in (b) is injective, by 4.14; since the two finite sets involved have the same number of elements, our map is bijective and (b) follows.

Consider the surjective homomorphism from $\mathscr{F}_M$ to the space of all functions $\operatorname{Irr}\Lambda_M\to\mathbf{Q}$ given in 3.8. The two vector spaces involved have the same dimension hence our map is an isomorphism and (c), (d) follow. The theorem is proved.

4.17. The previous theorem provides a natural basis of u^- and of $\mathscr{F}$ indexed by the irreducible components of the various varieties Λ_M and also in purely combinatorial terms, namely in terms of the sets $\mathscr{S}(M)^\delta$.

Corollary 4.18. — *The variety* Λ_M *is Lagrangian.*

Indeed, the map in 4.16(b) being surjective, means (by 4.14) that any irreducible component of Λ_M is a conormal bundle; the corollary follows. (The corollary holds for any quiver, with a different proof.)

5. Cyclic quivers

5.1. In this section we will make a change in our general assumptions: namely Γ will now be a cyclic subgroup of $SL(\rho)$ not necessarily containing c. We assume that $|\Gamma| = N > 1$. All definitions in 1.1-1.3 extend without change; the affine Coxeter graph is now a polygon with a possibly odd number of vertices.

We will show that most results in the earlier sections hold in this case as well.

5.2. We shall fix a Γ-stable line L in ρ. Then we can identify $\mathbf{Z}/N\mathbf{Z}$ with I by $r \mapsto L^{\otimes r}$, regarded as a Γ-module in the obvious way. Consider the set of all pairs (r, m) where r is an integer defined up to a multiple of N and m is an integer $\geqslant 1$ not divisible by N. This set is in 1-1 correspondence with the set of positive roots R_+ by

$$(r, m) \mapsto \alpha_{r, m} = \sum_{j=r}^{r+m-1} L^{\otimes j};$$

this is proved by the argument of 1.9.

5.3. The definition of the orientation $\Omega(\delta)$ given in 2.1 is not applicable here. Instead, we define an orientation Ω_L of our graph as follows: we have $i \to j$ if $i, j \in I$ correspond to $r, r+1 \in \mathbf{Z}/N\mathbf{Z}$ respectively. (This is ambiguous if $N = 2$; in that case we orient the two edges from $i \neq j$ so that one is $i \to j$ and the others is $j \to i$.)

The category $\mathscr{C}'_N$ defined in 2.18 is then a full subcategory of the (abelian) category of representations of this affine quiver (with the orientation Ω_L).

Note that giving a finite dimensional $\mathbf{Z}/n\mathbf{Z}$-graded $\mathbf{C}$-vector space $V = \oplus V_r$ is the same as giving a Γ-module. (We make Γ act on V so that on any V_r it acts via the character given by $L^{\otimes r}$.)

Let $\mathrm{End}_1(V)$ be the space of all linear maps $t: V \to V$ such that $t(V_r) \subset V_{r+1}$ for all r. Let $\mathrm{Aut}_0(V)$ be the group of all automorphisms of V preserving the grading.

If (V, t) is an object of $\mathscr{C}'_N$ we denote by $\mathrm{gr}(V, t)$ the Γ-module V.

Let $\mathscr{P}$ be the set of isomorphism classes of indecomposable objects of $\mathscr{C}'_N$. As in 2.19, $\mathscr{P}$ consists of the objects $V_{r, m}$ where r is an integer defined up to a multiple of N and m is an integer $\geqslant 1$. From the definitions we see that

(a) For any $P \in \mathscr{P}$ we have $\mathrm{gr}(P) \in R_+ \cup \{\mathbf{r}, 2\mathbf{r}, \ldots\}$.

(b) For any $\alpha \in R_+$ there is a unique $P \in \mathscr{P}$ such that $\mathrm{gr}(P) = \alpha$.

(c) For any integer $s \geqslant 1$, the set $\{P \in \mathscr{P} \mid \mathrm{gr}(P) = s\mathbf{r}\}$ has cardinal equal to N.

5.4. Given a Γ-module V we denote by $\mathscr{S}(V)$ the set of all functions $\sigma : \mathscr{P} \to \mathbf{N}$ such that properties (a), (b), (c) below hold:

(a) σ has finite support.

(b) For any $m \geqslant 1$, at least one of the numbers

$$\sigma(V_{0,m}), \sigma(V_{1,m}), \ldots, \sigma(V_{N-1,m})$$

is zero.

(c) We have $\sum_P \sigma(P)\,\mathrm{gr}(P) = V$ in $\mathscr{G}\Gamma$.

Given $\sigma \in \mathscr{S}(V)$ we define $X(\sigma)$ to be the set of all $t \in \mathrm{End}_1(V)$ such that (V, t) is isomorphic in $\mathscr{C}'_N$ to $\oplus_P P^{\sigma(P)}$. This is a single $\mathrm{Aut}_0(V)$-orbit in $\mathrm{End}_1(V)$.

5.5. All definitions and results in section 3 except those in 3.4 remain valid without change. The following is a substitute for 3.4 in the present case: for V as above, we may identify naturally E_V with the cotangent bundle of $\mathrm{End}_1(V)$.

We have the following result. (See [L, 15.5].)

(a) If $\sigma \in \mathscr{S}(V)$, then the closure (in E_V) of the conormal bundle of $X(\sigma)$ is an irreducible component $\mathscr{N}(\sigma)$ of Λ_V.

5.6. We consider the formal power series

$$\Pi = \sum_V |\mathscr{S}(V)| X^{\dim V}$$

where V runs over a set of representatives for the isomorphism classes of Γ-modules. From the definitions and from 5.3, we see that

$$\Pi = \prod_m \Big(\prod_r (1 - X^{\dim \alpha_{r,m}})^{-1} - \prod_r X^{\dim \alpha_{r,m}} \prod_r (1 - X^{\dim \alpha_{r,m}})^{-1}\Big)$$
$$\times \prod_s ((1 - X^{sN})^{-N} - X^{sN^2}(1 - X^{sN})^{-N}),$$

where m runs over the integers $\geqslant 1$, not divisible by N, r runs over $\mathbf{Z}/N\mathbf{Z}$, s runs over the integers $\geqslant 1$.

Note that $\dim \alpha_{r,m} = m$. We have

$$\Pi = \prod_{\alpha \in R_+} (1 - X^{\dim \alpha})^{-1} \prod_m (1 - X^{mN}) \prod_s (1 - X^{sN})^{-N} \prod_s (1 - X^{sN^2}),$$

$$\Pi = \prod_{\alpha \in R_+} (1 - X^{\dim \alpha})^{-1} \prod_s (1 - X^{sN})^{-N+1}.$$

Using now 3.9(c) (which is valid in our case), we obtain

(a) $$\sum_{V\,:\,\dim V=d} (\dim u_V^- - |\mathscr{S}(V)|) = 0$$

for any $d \geqslant 0$. (Compare 4.15(c).) The following result is entirely analogous to 4.16.

Proposition 5.7. — (a) *There is a natural algebra isomorphism* $u^- \to \mathscr{F}$.

(b) *For any* Γ*-module* M, *the map* $\sigma \mapsto \mathscr{N}(\sigma)$ (*see* 5.5(*a*)) *is a bijection* $\mathscr{S}(V) \cong \operatorname{Irr} \Lambda_V$.

(c) $\mathscr{F}$ *has a canonical basis, defined as in* 4.16(*c*), (*d*), *naturally indexed by the elements of* $\operatorname{Irr} \Lambda_V$.

The proof is the same as that in 4.16 except that the first inequality of 4.16(c) is now deduced from 5.5(a) and instead of using 4.15(c) we now use 5.6(a).

5.8. Let V be a Γ-module. For any $\sigma \in \mathscr{S}(V)$ we denote by P_σ the simple perverse sheaf on $\operatorname{End}_1(V)$ whose support is the closure of $X(\sigma)$ and whose restriction to $X(\sigma)$ is $\mathbf{C}$ (up to shift).

Let $\mathbf{P}_V$ be the set of isomorphism classes of simple perverse sheaves on $\operatorname{End}_1(V)$ in the class defined in [L, §2], for a cyclic quiver.

Theorem 5.9. — *For any* Γ*-module* V, *the map* $\sigma \mapsto P_\sigma$ *is a bijection* $\mathscr{S}(V) \cong \mathbf{P}_V$.

According to [L, 13.6], the singular support of any $P \in \mathbf{P}_V$ (a closed subvariety of the cotangent bundle E_V of $\operatorname{End}_1(V)$) is a union of irreducible components of Λ_V. In particular, the conormal bundle of the support of P is an irreducible component of Λ_V. From the description of the components of Λ_V given in 5.7(b) it follows that the support of P is the closure of $X(\sigma)$ for some $\sigma \in \mathscr{S}(V)$. The restriction of P to some open dense subset of $X(\sigma)$ must be an irreducible local system (up to shift). Since P is $\operatorname{Aut}_0(V)$-equivariant, this open set can be assumed to be $\operatorname{Aut}_0(V)$-stable (hence equal to $X(\sigma)$, which is a single orbit) and the local system on it is equivariant (hence equal to $\mathbf{C}$, since the isotropy groups of points in $X(\sigma)$ are connected). Thus, we have $P = P_\sigma$.

We see that there is a well-defined map $\mathbf{P}_V \to \mathscr{S}(V)$ given by $P \mapsto \sigma$, where $P = P_\sigma$; this map is obviously injective.

By [L, 10.17], we have $\dim u_V^- = |\mathbf{P}_V|$ (where u_V^- is as in 3.9) and by 5.7, we have $\dim u_V^- = |\mathscr{S}(V)|$. It follows that $|\mathbf{P}_V| = |\mathscr{S}(V)|$. Hence the injective map $\mathbf{P}_V \to \mathscr{S}(V)$ above must be a bijection. (The surjectivity of our map could also be deduced from results in [R4]. This would avoid reference to [L, 10.17].) The theorem follows.

6. Perverse sheaves

6.1. Let Γ be as in 1.1.

The quantized enveloping algebra corresponding to u^- has a canonical basis, defined in [L] in terms of perverse sheaves. The purpose of this section is to describe explicitly (or enumerate) these perverse sheaves, in the case of affine quivers, by indicating their support and the corresponding local systems. (Such an explicit description is not known in the generality of [L].)

6.2. We begin by a result about vanishing of certain Hom-groups; this is in some sense complementary to 4.2.

(a) If $\mathbf{M}_1, \mathbf{M}_2 \in \mathscr{C}^\delta$ are as in 4.2, and if they are not isomorphic, then $\mathrm{Hom}(\mathbf{M}_2, \mathbf{M}_1)=0$.

The proof is almost the same as that in 4.5. Let $f\colon \mathrm{K}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta} \to \mathbf{M}_1$ be a morphism in $\mathscr{C}^\delta$, where $s' \in \mathbf{N}$, $i' \in \mathrm{I}^{(-1)^{s'}\delta}$ and $\mathbf{M}_1$ is as in 4.2(a), (c) or (d); if $\mathbf{M}_1$ is as in 4.2 (a), we assume in addition that $s>s'$. We want to prove by induction on s' that $f=0$. For $s'=0$, this follows immediately from the fact that $\mathbf{M}_1 \in \mathscr{C}^\delta_{\mathrm{in}}$ (as in 4.5). Hence we may assume that $s' \geqslant 1$ and that our assertion is already proved for $s'-1$, $-\delta$ instead of s', δ. Then f is a morphism in $\mathscr{C}^\delta_{\mathrm{in}}$ (as in 4.5). As in 4.5, applying C to f leads to a morphism $\mathrm{C}f\colon \mathrm{CK}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta} \to \mathrm{C}\mathbf{M}_1$ of the same type as f, and $\mathrm{CK}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta} = \mathrm{K}^{s'-1}\mathrm{P}_{i'}^{(-1)^{s'}\delta}$. Hence, by the induction hypothesis, we have $\mathrm{C}f=0$. Now, as in 4.5, both $\mathrm{K}^{s'}\mathrm{P}_{i'}^{(-1)^{s'}\delta}$ and $\mathbf{M}_1$ are fixed by KC. Hence $f=\mathrm{KC}f=0$.

Thus, (a) holds if $\mathbf{M}_1$, $\mathbf{M}_2$ are as in 4.2 (a), (c), (d). By duality, it also holds if $\mathbf{M}_1$, $\mathbf{M}_2$ are as in 4.2 (b), (e). Finally, it also holds if $\mathbf{M}_1$, $\mathbf{M}_2$ are as in 4.2 (f), using 2.17 (b).

The following result gives a canonical filtration for any object of $\mathscr{C}^\delta$.

Proposition 6.3. – *Let* $\mathbf{M}=(\mathrm{M}, \Delta) \in \mathscr{C}^\delta$. *There are uniquely defined subobjects* $\mathbf{M}' \subset \mathbf{M}''$ *of* $\mathbf{M}$ *such that* $\mathbf{M}' \in {}^{>}\mathscr{C}^\delta$, $\mathbf{M}''/\mathbf{M}' \in {}^{0}\mathscr{C}^\delta$, $\mathbf{M}/\mathbf{M}'' \in {}^{<}\mathscr{C}^\delta$.

The existence of the subobjects $\mathbf{M}'$, $\mathbf{M}''$ as above follows from the results of section 2. From 4.2 it follows that we can find subobjects $\mathbf{N}$, $\mathbf{N}'$ of $\mathbf{M}$ such that $\mathbf{M}''=\mathbf{M}' \oplus \mathbf{N}$ and $\mathbf{M}=\mathbf{M}'' \oplus \mathbf{N}'$.

Now let $\mathbf{M}'_1 \subset \mathbf{M}''_1$ be two subobjects of $\mathbf{M}$ like $\mathbf{M}'$, $\mathbf{M}''$. We must prove that $\mathbf{M}'_1=\mathbf{M}'$, $\mathbf{M}''_1=\mathbf{M}''$.

We can again find subobjects $\mathbf{N}_1$, $\mathbf{N}'_1$ of $\mathbf{M}$ such that $\mathbf{M}''_1=\mathbf{M}'_1 \oplus \mathbf{N}_1$ and $\mathbf{M}_1=\mathbf{M}''_1 \oplus \mathbf{N}'_1$. From the definitions, it is clear that $\mathbf{M}' \cong \mathbf{M}'_1$, $\mathbf{N} \cong \mathbf{N}_1$, $\mathbf{N}' \cong \mathbf{N}'_1$. Hence there exists an automorphism h of the object $\mathbf{M}$ which carries $\mathbf{M}'$, $\mathbf{N}$, $\mathbf{N}'$ respectively onto $\mathbf{M}'_1$, $\mathbf{N}_1$, $\mathbf{N}'_1$.

Now h is given with respect to the direct sum decomposition $\mathbf{M}=\mathbf{M}' \oplus \mathbf{N} \oplus \mathbf{N}'$ by components (morphisms from one summand to another). By 6.2, the components corresponding to $(\mathbf{M}', \mathbf{N})$, $(\mathbf{M}', \mathbf{N}')$, $(\mathbf{N}, \mathbf{N}')$ are zero. Hence h maps $\mathbf{M}'$ and $\mathbf{M}' \oplus \mathbf{N}$ into themselves. It follows that $\mathbf{M}'_1 = \mathbf{M}'$, $\mathbf{M}''_1 = \mathbf{M}''$, as required.

6.4. Let M be a Γ-module and let $M_\dagger = (M_1, M_2, \ldots M_m)$ be a sequence of isotypical Γ-modules such that $M \cong M_1 \oplus \ldots \oplus M_m$ as a Γ-module. Let $\tilde{E}^\delta_M$ be the (smooth) variety of all pairs consisting of an element $\Delta \in E^\delta_M$ and a flag of type $M_\dagger$ in M, stable under Δ (see 3.5). Associating to such a pair the corresponding element Δ gives a (proper) morphism $\pi : \tilde{E}^\delta_M \to E^\delta_M$.

By the decomposition theorem for perverse sheaves, the direct image complex $\pi_!(\mathbf{C})$ on E^δ_M is a direct sum of simple perverse sheaves on E^δ_M with shifts; let $\mathbf{P}_{M, \delta}$ be the set of (isomorphism classes of) simple perverse sheaves which arise in this way (for various $M_\dagger$). (Compare [L, § 2].) This is a finite set. All objects of $\mathbf{P}_{M, \delta}$ are G_M-equivariant.

Let v be an indeterminate. If π is as above, r is an integer and $P \in \mathbf{P}_{M, \delta}$, let $n(P, M_\dagger, r) \in \mathbf{N}$ be the number of times that $P[r]$ appears in a decomposition of $\pi_!(\mathbf{C})$ as a direct sum of simple perverse sheaves with shifts; let $d(M\ ) = \dim \tilde{E}^\delta_M$.

6.5. Following Drinfeld and Jimbo, we consider the quantized enveloping algebra U^- attached to our affine Coxeter graph. It is the algebra over $\mathbf{Q}(v)$ with generators $F_i\ (i \in I)$ and relations

$$\sum_{p=0}^{N+1} (-1)^p [N+1-p, p] F_j^p F_i F_j^{N+1-p} = 0$$

for any $i \neq j$ (with $N = \dim T^i_j$, see 1.1); here we set

$$[a]_! = \prod_{k=1}^{a} \frac{v^k - v^{-k}}{v - v^{-1}}; \qquad [a, a'] = \frac{[a+a']_!}{[a]_! [a']_!}.$$

Let $\mathbf{F}_M$ be the $\mathbf{Q}(v)$ vector space with basis $\mathbf{P}_{M, \delta}$. Let $\mathbf{F} = \oplus_M \mathbf{F}_M$; the sum is over a set of representatives for the isomorphism classes of Γ-modules (the choice of representatives is immaterial since $\mathbf{F}_M$ is canonically isomorphic to $\mathbf{F}_{M'}$ whenever M, M' are isomorphic).

From [L, § 3, § 9] it follows that there is a unique $\mathbf{Q}(v)$-algebra structure on $\mathbf{F}$, together with a surjective algebra homomorphism

(a) $$U^- \to \mathbf{F}$$

such that

(b) $$F_{i_1}^{s_1} \ldots F_{i_m}^{s_m} / ([s_1]_! \ldots [s_m]_!) \mapsto \sum_P \sum_r n(P, M_\dagger, r) v^{r + d(M\ )} P$$

for any sequence $M_\dagger=(M_1, M_2, \ldots, M_m)$ of isotypical Γ-modules such that M_p is isomorphic to the direct sum of s_p copies of ρ_{i_p} for all p; P runs through $\mathbf{P}_{M,\delta}$ where $M=M_1\oplus\ldots\oplus M_m$.

For any Γ-module M, we define $\mathbf{U}_M^-$ to be the subspace of $\mathbf{U}^-$ spanned by the left hand sides of (b) such that $\sum_{p:i_p=i} s_p=(\rho_i: M)$ for all i. These give a grading $\mathbf{U}^-=\oplus_M \mathbf{U}_M^-$ and our homomorphism $\mathbf{U}^-\to\mathscr{F}$ clearly respects the gradings. Hence we have

(c) $\dim \mathbf{U}_M^- \geqslant \dim \mathbf{F}_M$

for any M.

6.6. We shall denote by $\mathscr{Z}'$ the set of Γ-orbits Z in P(ρ) such that the stabilizer of a line in Z has order 2. This is an open dense subset in the variety of all Γ-orbits in P(ρ).

Let M be a Γ-module and let $\sigma: \mathscr{P}^\delta\to\mathbf{N}$ be a function satisfying 4.13 (a), (b), (c) and the property (a) below.

(a) $\sum_{P\in\mathscr{P}^\delta}\sigma(P)\,\mathrm{gr}(P)+p\,\mathbf{r}=M$ in $\mathscr{G}\Gamma$ for some $p\geqslant 0$.

Let X(σ) be the subset of E_M^δ consisting for all elements Δ such that (M, Δ) is isomorphic in $\mathscr{C}^\delta$ to

$$\oplus_P P^{\sigma(P)}\oplus\mathbf{M}_{Z_1,0,1}\oplus\ldots\oplus\mathbf{M}_{Z_p,0,1}$$

for some distinct Γ-orbits $Z_1, \ldots, Z_p$ in $\mathscr{Z}'$.

Then

(b) X(σ) is a locally closed, smooth, irreducible subvariety of E_M^δ of dimension equal to p plus the dimension of any G_M-orbit it contains (a special case of 4.14 (a)).

We call X(σ) the *σ-stratum* of E_M^δ.

We now define a finite covering

$$\tilde{X}(\sigma)\to X(\sigma)$$

as follows. Let $\tilde{X}(\sigma)$ be the variety consisting of all pairs $(\Delta, Z_1, Z_2, \ldots, Z_p)$ where $\Delta\in X(\sigma)$ and $Z_1, Z_2, \ldots, Z_p$ is a sequence of distinct elements of $\mathscr{Z}'$ such that the spectrum of $\mathbf{M}''(\Delta)/\mathbf{M}'(\Delta)\in{}^0\mathscr{C}^\delta$ is the union of $Z_1, Z_2, \ldots, Z_p$ and possibly other Γ-orbits outside $\mathscr{Z}'$. Here, $\mathbf{M}'(\Delta)\subset\mathbf{M}''(\Delta)$ are the subobjects of (M, Δ) provided by 6.3; they remain in fixed G_M-orbits, when Δ varies in X(σ).

Note that $Z_1, Z_2, \ldots, Z_p$ are uniquely determined by Δ up to order; hence the natural map $\tilde{X}(\sigma)\to X(\sigma)$ given by $(\Delta, Z_1, Z_2, \ldots, Z_p)\mapsto\Delta$ is a principal covering with group S_p, the symmetric group in p letters. (By convention, S_p has one element if $p=0$.)

Any irreducible representation χ of S_p gives rise, via this covering, to a local system $\mathscr{L}_\chi$ on $X(\sigma)$.

Let $P_{\sigma,\chi}$ be the simple perverse sheaf on E^δ_M whose support is the closure of $X(\sigma)$ and whose restriction to $X(\sigma)$ is $\mathscr{L}_\chi$ (with a shift).

Proposition 6.7. — *The perverse sheaf* $P_{\sigma,\chi}$ *belongs to* $\mathbf{P}_{M,\delta}$.

The proof will be given in 6.14.

Lemma 6.8. — *For any* Γ*-module* M, *the perverse sheaf* $\mathbf{C}$ (*with a shift*) *belongs to* $\mathbf{P}_{M,\delta}$.

We can find a sequence $M_\dagger=(M_1, M_2, \ldots M_m)$ of isotypical Γ-modules such that for some $m'\leqslant m$ we have $M^\delta\cong M_1\oplus\ldots M_{m'}$ and $M^{-\delta}\cong M_{m'+1}\oplus\ldots M_m$, and such that $(M_h : M_{h'})=0$ for all $h\neq h'$. Then each M_h is isomorphic to a unique Γ-submodule of M and will be identified with it. It is clear that there is exactly one flag of type $M_\dagger$ in M, namely

$$M=M^{(0)}\supset M^{(1)}\supset\ldots\supset M^{(m)}=0,$$

where $M^{(p)}=M_{p+1}\oplus M_{p+2}\oplus\ldots$ If $p\geqslant m'$, then $(M^{(p)})^\delta=0$, while if $p\leqslant m'$, then $(M^{(p)})^{-\delta}=M^{-\delta}$. In both cases, we see that $M^{(p)}$ is Δ-stable. Hence in our case, the map $\pi: \tilde{E}^\delta_M\to E^\delta_M$ (see 6.4) is an isomorphism, so that $\pi_!(\mathbf{C})=\mathbf{C}$. The lemma follows.

6.9. Assume now that we are given a line L in ρ whose stabilizer Γ_L has order $2n\geqslant 4$, and that $\mathbf{M}=(M, \Delta)$ is an object in ${}^0\mathscr{C}^\delta$ with spectrum contained in the Γ-orbit Z of L. Assume also that $\mathbf{M}=(M, \Delta)$ is aperiodic.

Choose L′, *e*, *e*′ as in 2.18. Let $\mathcal{O}\subset E^\delta_M$ be the G_M-orbit of Δ and let P be the simple perverse sheaf on E^δ_M whose support is the closure of $\mathcal{O}$ and whose restriction to $\mathcal{O}$ is $\mathbf{C}$ (up to shift). We will prove the following result.

(a) P *belongs to* $\mathbf{P}_{M,\delta}$.

Let S be the (locally closed) subvariety of E^δ_M consisting of all elements Ξ such that $(M, \Xi)\in {}^0\mathscr{C}^\delta_Z$.

Let $\bar{S}$ be the variety of all pairs (M_L, Φ) consisting of a Γ_L-stable subspace M_L of M and of a vector space isomorphism $\Phi: M^\delta_L\cong M^{-\delta}_L$ such that the properties (b), (c), (d) below are satisfied:

(b) $M=\oplus_\gamma \gamma(M_L)$ where γ runs over a set of representatives for the cosets Γ/Γ_L;

(c) the Γ_L-module M_L belongs to the subspace $\mathscr{U}_L$ of $\mathscr{G}\Gamma_L$ (see 1.13);

(d) $\gamma\Phi(x)=\zeta(\gamma)^{-1}\Phi(\gamma x)$ for all $x\in M^\delta_L$ and $\gamma\in\Gamma_L$.

(Here, $M^{\pm 1}_L=M_L\cap M^{\pm 1}$ and ζ is the character by which Γ acts on L.)

Note that there is a natural action of G_M on $\bar{S}$.

For any $\Xi \in S$ there are associated subspaces M_L^{δ} and $M_L^{-\delta}$ (see 2.17) and $\Xi_{e'}$ defines an isomorphism like Φ above.

This defines a natural G_M-equivariant morphism $S \to \bar{S}$ and shows that $\bar{S}$ is non-empty.

Given two elements of $\bar{S}$, we get two Γ_L-submodules of M; these are isomorphic, by 1.14 (a). It follows easily that the G_M-action on $\bar{S}$ is transitive.

Let $\bar{s}=(M_L, \Phi) \in \bar{S}$ be the image of Δ in $\bar{S}$; let $G_{M,\bar{s}}$ be its stabilizer in G_M and let S' be the fibre of $S \to \bar{S}$ over $\bar{s}$.

By definition, for any $\Xi \in S'$, the subspaces M_L^{δ}, $M_L^{-\delta}$ associated as in 2.17 to Ξ, are independent of the choice of Ξ. We set $V = M_L^{\delta}$.

As in 2.18, the **C**-vector space V is naturally $\mathbf{Z}/n\mathbf{Z}$-graded: $V = \oplus_r V_r$ (using the action of Γ_L). To any $\Xi \in S'$, we associate a nilpotent linear map $t: V \to V$ by the procedure of 2.18 (applied to Ξ instead of Δ). As in 2.18, we see that $\Xi \mapsto t$ establishes an isomorphism of S' onto the variety of nilpotent endomorphisms in $\mathrm{End}_1(V)$.

If $g \in G_{M,\bar{s}}$, then the restriction of g to V is an automorphism of V preserving the grading. This gives an isomorphism of $G_{M,\bar{s}}$ onto $\mathrm{Aut}_0(V)$.

Let $t_0: V \to V$ be the nilpotent endomorphism corresponding to Δ and let $\mathcal{O}_0$ be its orbit under $\mathrm{Aut}_0(V)$. Let P_0 be the simple perverse sheaf on $\mathrm{End}_1(V)$ whose support is the closure of $\mathcal{O}_0$ and whose restriction to $\mathcal{O}_0$ is **C** (up to shift).

Our aperiodicity assumption implies that P_0 is a perverse sheaf in the class defined in [L, § 2], for a cyclic quiver. (See 5.9.)

More precisely, let $\mathcal{B}$ be the variety of all sequences $V = V^0 \supset V^1 \supset \ldots \supset V^N = 0$ where, for each p, V^p is a codimension p subspace of V, compatible with the grading. Let $\tilde{\mathcal{B}}$ be the set of all pairs consisting of an element $t \in \mathrm{End}_1(V)$ and a sequence (V^p) as above such that $t(V^p) \subset V^p$ for all p. Let π_0 be the second projection of $\tilde{\mathcal{B}}$ onto $\mathrm{End}_1(V)$ and let $(\pi_0)_!(\mathbf{C})$ be the direct image of **C**. Then

(e) some shift of P_0 is a direct summand of $(\pi_0)_!(\mathbf{C})$.

Now any subspace W of V, compatible with the grading, gives rise to a Γ-submodule $\bar{W} = \oplus_\gamma \gamma(W \oplus \Phi(W))$ of M, where γ runs over a set of representatives for the cosets Γ/Γ_L. Applying this to each member of a flag in V gives an isomorphism of $\mathcal{B}$ onto the variety B' of all sequences of Γ-submodules $M = M^{(0)} \supset M^{(1)} \supset \ldots \supset M^{(N)} = 0$ such that each $M^{(p)}$ is generated by $M^{(p)} \cap M_L$, $\Phi(M^{(p)} \cap M_L^{\delta}) = M^{(p)} \cap M_L^{-\delta}$ and each $M^{(p)}/M^{(p+1)}$ is of the form $\mathrm{gr}(\mathbf{M}_{Z,r,1})$ for some r (see 2.20).

Let $\tilde{B}'$ be the variety of all pairs consisting of a sequence in B' and an element of S', leaving stable each term of that sequence. Let $\pi': \tilde{B}' \to S'$ be the canonical projection. Let $\mathcal{O}'$ be the $G_{M,\bar{s}}$-orbit of Δ and let P' be the simple perverse sheaf on S' whose support is the closure of $\mathcal{O}'$ in S' and whose restriction to $\mathcal{O}'$ is **C** (up to shift).

Then (e) can be reformulated as follows:

(f) some shift of P′ is a direct summand of $(\pi')_!(\mathbf{C})$.

Now let B be the variety of all sequences of Γ-submodules $M=M^{(0)}\supset M^{(1)}\supset\ldots\supset M^{(N)}=0$ such that each $M^{(p)}/M^{(p+1)}$ is of the form $\mathrm{gr}(\mathbf{M}_{Z,r,1})$ for some r (see 2.20).

Let $\tilde{B}''$ be the variety of all pairs consisting of a sequence in B and an element of S, leaving stable each term of that sequence. Let $\pi'': \tilde{B}''\to S$ be the canonical projection.

We note the following fact.

(g) If (M_1,Ξ) is an object of ${}^0\mathscr{C}^\delta$ and M_2 is a Ξ-stable Γ-submodule of M_1 such that $\dim M_2^\delta=\dim M_2^{-\delta}$, then (M_2,Ξ) is again an object of ${}^0\mathscr{C}^\delta$.

Indeed, for some $e_1\in\rho$ we have that $\Xi_{e_1}: M_1^\delta\to M_1^{-\delta}$ is an isomorphism; it restricts to a map $M_2^\delta\to M_2^{-\delta}$ which is necessarily injective, hence an isomorphism, by our assumption on dimensions. If, in addition, (M_1,Ξ) is assumed to have spectrum contained in Z, then the same must hold for (M_2,Ξ).

These remarks can be applied to the members of a sequence in B, assumed to be stable under some element $\Xi\in S'$; it then follows that these members form with Ξ objects of ${}^0\mathscr{C}^\delta$ with spectrum contained in Z. From this we deduce that such a sequence must automatically be contained in B′.

We then see that $\tilde{B}''=G_M\times_{G_{M,\bar{s}}}\tilde{B}'$ in the same way as $S=G_M\times_{G_{M,\bar{s}}}S'$ and $\mathcal{O}=G_M\times_{G_{M,\bar{s}}}\mathcal{O}'$. Therefore from (f) we deduce that

(h) some sift of P″ is a direct summand of $(\pi'')_!(\mathbf{C})$,

where P″ is the simple perverse sheaf on S whose support is the closure of $\mathcal{O}$ in S and whose restriction to $\mathcal{O}$ is **C** (up to shift).

Now let $\tilde{B}$ be the variety of all pairs consisting of a sequence in B and an element of E_M^δ, leaving stable each term of that sequence. Let $\pi:\tilde{B}\to E_M^\delta$ be the canonical projection. From the definition of B it is easy to see that the image $\pi(\tilde{B})$ is the closure of S in E_M^δ. Thus S is open in $\pi(\tilde{B})$ and $(\pi'')_!(\mathbf{C})$ can be regarded as the restriction of $(\pi)_!(\mathbf{C})$ from $\pi(\tilde{B})$ to its open set S. From this and (g) we deduce that some shift of P is a direct summand of $(\pi)_!(\mathbf{C})$.

Now $\tilde{B}$ can be decomposed in connected components; they are obtained by specifying the isomorphism classes of the successive quotients $M^{(p)}/M^{(p+1)}$ as Γ-modules. Then $(\pi)_!(\mathbf{C})$ decomposes accordingly in a direct sum and some shift of P will appear in one of these direct summands. But each of these direct summands is, in the notation of [L, 3.5], an iterated ∗-product of N perverse sheaves (up to shift) of the form **C** on $E_{M'}^\delta$, where M′ are Γ-modules of the form $\mathrm{gr}(\mathbf{M}_{Z,r,1})$ for some r; these are certainly in $\mathbf{P}_{M',\delta}$, by 6.8. Since some shift of P is a direct summand in such an iterated ∗-product, it is contained in $\mathbf{P}_{M,\delta}$, by [L, 3.2, 3.4]. Thus, (a) is proved.

6.10. Assume now that M is a Γ-module isomorphic to the direct sum of p copies of **r**. Let X(0) be the open dense subset of E_M^δ consisting of all Δ such that (M, Δ) is isomorphic to $\mathbf{M}_{Z_1,0,1} \oplus \ldots \oplus \mathbf{M}_{Z_p,0,1}$ for some distinct Γ-orbits $Z_1, \ldots, Z_p$ in $\mathscr{Z}'$. Then X(0) is the special case of X(σ) of 6.6 with σ identically zero. Hence the S_p-covering $\tilde{X}(0) \to X(0)$ and the local system $\mathscr{L}_\chi$ on X(0) corresponding to any irreducible representation χ of S_p are defined as in 6.6.

Let P be the simple perverse sheaf on E_M^δ whose support is E_M^δ and whose restriction to X(0) is $\mathscr{L}_\chi$ (up to a shift). We will show that

(a) P *belongs to* $\mathbf{P}_{M,\delta}$.

Let B be the variety of all sequences of Γ-submodules $M = M^{(0)} \supset M^{(1)} \supset \ldots \supset M^{(p)} = 0$ such that $M^{(k)}/M^{(k+1)} \cong \mathbf{r}$ as a Γ-module, for $k = 0, 1, \ldots, p-1$.

Let $\tilde{B}$ be the variety of all pairs consisting of a sequence in B and an element of E_M^δ, leaving stable each term of that sequence. Let $\pi: \tilde{B} \to E_M^\delta$ be the canonical projection. If $\Delta \in X(0)$ leaves stable a sequence in B, then using 6.9 (g) we see that each member of that sequence forms, together with Δ, an object in ${}^0\mathscr{C}^\delta$. From the definition of X(0) it then follows that there are exactly $p!$ sequences in B left stable by Δ and that the restriction of π defines a covering $\pi^{-1}(X(0)) \to X(0)$ isomorphic to $\tilde{X}(0) \to X(0)$. We see therefore that some shift of P is a direct summand of $\pi_!(\mathbf{C})$.

As in the end of 6.9, $\pi_!(\mathbf{C})$ is an iterated $*$-product of p perverse sheaves (up to shift) of the form **C** on E_r^δ; these are certainly in $P_{M',\delta}$, by 6.8. Since some shift of P is a direct summand in such an iterated $*$-product, it is contained in $\mathbf{P}_{M,\delta}$, by [L, 3.2, 3.4]. Thus, (a) is proved.

6.11. We now return to the setup of 6.6. Let $\Delta(0) \in X(\sigma)$. We can write M as a direct sum $M = \oplus_{h \in \mathbf{Z}} M(h)$ where M(h) are Δ(0)-stable Γ-submodules of M so that, for some $h_1 \leqslant h_2$, the following conditions are satisfied.

(a) For any h such that $h \leqslant h_1$, there exist $s = s(h) \geqslant 0$ and $i = i(h) \in I^{(-1)^s\delta}$ such that (M(h), Δ(0)) is isomorphic to a direct sum of copies of $K^s P_i^{(-1)^s\delta}$; moreover, if $h < h' \leqslant h_1$ then either $s(h) < s(h')$ or $s(h) = s(h')$ and $i(h) \neq i(h')$.

(b) For any h such that $h_2 + 1 < h$, there exist $s = s(h) \geqslant 0$ and $i = i(h) \in I^{(-1)^{s+1}\delta}$ such that (M(h), Δ(0)) is isomorphic to a direct sum of copies of $C^s P_i^{(-1)^s\delta}$; moreover, if $h_2 + 1 < h < h'$, then either $s(h) > s(h')$ or $s(h) = s(h')$ and $i(h) \neq i(h')$.

(c) For any h such that $h_1 < h \leqslant h_2 + 1$, we have $(M(h), \Delta(0)) \in {}^0\mathscr{C}^\delta$; if $h_1 < h \leqslant h_2$, then (M(h), Δ(0)) is non-zero, with spectrum equal to a single orbit outside $\mathscr{Z}'$, and is aperiodic; if $h_1 < h < h' \leqslant h_2$, then (M(h), Δ(0)), (M(h'), Δ(0)) have disjoint spectra. Moreover, $(M(h_2+1), \Delta(0))$ is as in 6.10.

We have necessarily $M(h) = 0$ for large $|h|$.

For any h such that $h \neq h_2 + 1$ and any Γ-module N isomorphic to M(h), we define the σ_h-stratum $X(\sigma_h)$ of E_N^δ to be the set of all $\Xi \in E_N^\delta$ such that there exists an isomorphism of Γ-modules $N \cong M(h)$ which carries (N, Ξ) to $(M(h), \Delta(0))$. For $h = h_2 + 1$ and any Γ-mdoule N isomorphic to $M(h)$, we define the σ_h-stratum $X(\sigma_h)$ of E^δ_N to be the subset $X(0)$ of E^δ_N as in 6.10. Clearly, the σ_h-stratum of N is a single G_N-orbit, if $h \neq h_2 + 1$.

Let B be the variety consisting of all sequences of Γ-submodules of M

(d) $$\ldots M[-1] \subset M[0] \subset M[1] \subset M[2] \subset \ldots$$

such that for any h, $M[h]/M[h-1]$ is isomorphic to $M(h)$ as a Γ-module. Clearly, B is nonempty.

Let $\tilde{B}'$ (resp. $\tilde{B}$) be the variety of all pairs consisting of a sequence (d) in B and an element Δ of E^δ_M, leaving stable each term of that sequence and such that for any h, the restriction of Δ to $(M[h]/M[h-1])$ is in the σ_h-stratum (resp. in the closure of the σ_h-stratum) of $E^\delta_{M[h]/M[h-1]}$.

Clearly, $\tilde{B}'$ is an open subvariety of $\tilde{B}$.

Let $\pi: \tilde{B} \to E^\delta_M$, $\pi': \tilde{B}' \to E^\delta_M$ be the canonical projections. It is clear that π is a proper morphism.

Lemma 6.12. — (*a*) *The restriction of* π' *defines an isomorphism* $\pi'^{-1}(X(\sigma)) \cong X(\sigma)$.

(b) $\tilde{B}$, $\tilde{B}'$ *are irreducible of dimension equal to* $\dim X(\sigma)$.

(c) *The image of* $\pi: \tilde{B} \to E^\delta_M$ *is equal to the closure of* $X(\sigma)$ *in* E^δ_M.

We first prove (a). We will show only that $\pi^{-1}(\Delta)$ is a single point for any $\Delta \in X(\sigma)$. The proof will be along the lines of 6.3.

We may assume that $\Delta = \Delta(0)$ as in 6.11. With notation of 6.11, we set $M((h)) = \oplus_{h' : h' \leqslant h} M(h')$. Then the $M((h))$ together with Δ form an element of $\pi'^{-1}(\Delta)$. We now consider an arbitrary element of $\pi'^{-1}(\Delta)$ formed by a sequence 6.11 (d) together with Δ. By assumption, we have $(M[h]/M[h-1], \Delta) \cong (M(h), \Delta)$ for all $h \neq h_2 + 1$ and $(M[h_2+1]/M[h_2], \Delta) \in {}^0\mathscr{C}^\delta$ has a spectrum disjoint from that of $(M[h]/M[h-1], \Delta)$ for $h_1 < h \leqslant h_2$. Hence we may use the vanishing of Ext-groups in 4.2 to conclude that for each h there exists a Δ-stable Γ-submodule M_h of $M[h]$ such that $M[h] = M[h-1] \oplus M_h$. We have $M = \oplus M_h$ and $(M_h, \Delta) \cong (M(h), \Delta)$ for all $h \neq h_2 + 1$. Then we have automatically $(M_h, \Delta) \cong (M(h), \Delta)$ for $h = h_2 + 1$. Hence we can find an automorphism a of (M, Δ) which maps $M(h)$ onto M_h for all h. Let $a_{h,h'}: M(h) \to M(h')$ be the Γ-linear maps defined by $a(x) = \sum_{h'} a_{h,h'}(x)$ for all $x \in M(h)$.

Since a is compatible with Δ, it follows that $a_{h,h'}$ is compatible with the restrictions of Δ hence it defines a morphism in $\mathscr{C}^\delta$; hence, by 6.2, it must be zero, whenever $h < h'$. It follows that $a(M(h)) \subset \oplus_{h' : h' \leqslant h} M(h') = M((h))$, hence a maps $M((h))$ into itself for any h. Since a is an isomorphism we have $a(M((h)) = M((h))$. On the other hand, from the definition of a we have $a(\mathrm{M}((h))=\mathrm{M}[h]$. Thus $\mathrm{M}[h]=\mathrm{M}((h))$ for all h, as required.

We now prove (b). We consider the second projection $\tilde{\mathrm{B}}' \to \mathrm{B}$ (resp. $\tilde{\mathrm{B}} \to \mathrm{B}$); its fibre at the point 6.11 (d) of B is denoted Φ' (resp. Φ). Since this map is $\mathrm{G_M}$-equivariant and $\mathrm{G_M}$ acts transitively on B, it is enough to prove that Φ and Φ' are irreducible of dimension equal to dim $\mathrm{X}(\sigma)$-dim B.

Clearly,

$$\Phi' \cong \prod_h \mathrm{X}(\sigma_h) \times \prod_{h>h'} \mathrm{Hom}_\Gamma(\mathrm{M}(h)^\delta \otimes \rho, \mathrm{M}(h')^{-\delta})$$

where $\mathrm{X}(\sigma_h)$ are as in 6.11 and Φ is isomorphic to the analogous product in which each $\mathrm{X}(\sigma_h)$ is replaced by its closure. This shows that Φ', Φ are irreducible of the same dimension and

(d) $$\dim \Phi' = \sum_h \dim \mathrm{X}(\sigma_h) + \sum_{h>h'} \dim \mathrm{Hom}_\Gamma(\mathrm{M}(h)^\delta \otimes \rho, \mathrm{M}(h')^{-\delta}).$$

By 6.6 (b) we have

(e) $$\dim \mathrm{X}(\sigma) = p + \dim \mathrm{G_M} - \dim \mathrm{St}_\Delta,$$

where St_Δ is the stabilizer of Δ in $\mathrm{G_M}$. Now St_Δ has the same dimension as its Lie algebra; hence

$$\dim \mathrm{St}_\Delta = \sum_{h,h'} \dim \mathrm{Hom}((\mathrm{M}(h), \Delta), (\mathrm{M}(h'), \Delta)),$$

where the Hom are taken in $\mathscr{C}^\delta$. By 6.2 we have Hom $((\mathrm{M}(h), \Delta), (\mathrm{M}(h'), \Delta))=0$ if $h<h'$, hence

(f) $$\dim \mathrm{St}_\Delta = \sum_{h \geqslant h'} \dim \mathrm{Hom}((\mathrm{M}(h), \Delta), (\mathrm{M}(h'), \Delta)).$$

On the other hand, dim B is equal to dim $\mathrm{G_M}$ minus the dimension of the Lie algebra of the stabilizer of 6.11 (d) in $\mathrm{G_M}$ (by the transitivity of the $\mathrm{G_M}$ action on B). Thus,

(g) $$\dim \mathrm{B} = \dim \mathrm{G_M} - \sum_{h \geqslant h'} \dim \mathrm{Hom}_\Gamma(\mathrm{M}(h), \mathrm{M}(h')).$$

From (d), (e), (f), (g) we see that the difference

$$\dim \Phi' - \dim \mathrm{X}(\sigma) + \dim \mathrm{B}$$

is equal to

$$\sum_h \dim X(\sigma_h) + \sum_{h>h'} \dim \mathrm{Hom}_\Gamma(M(h)^\delta \otimes \rho, M(h')^{-\delta}) - p - \dim G_M$$
$$+ \sum_{h \geqslant h'} \dim \mathrm{Hom}((M(h), \Delta), (M(h'), \Delta)) + \dim G_M$$
$$- \sum_{h \geqslant h'} \dim \mathrm{Hom}_\Gamma(M(h), M(h')).$$

It remains to show that the last expression is zero. By 4.7, the last expression is equal to

$$\sum_h \dim X(\sigma_h) - p + \sum_{h>h'} \dim \mathrm{Ext}^1((M(h), \Delta), (M(h'), \Delta))$$
$$+ \sum_h \dim \mathrm{Hom}((M(h), \Delta), (M(h), \Delta)) - \sum_h \dim \mathrm{Hom}_\Gamma(M(h), M(h)),$$

hence, by the vanishing of Ext-groups 4.2, to

$$\sum_h \dim X(\sigma_h) - p + \sum_h \dim \mathrm{Hom}((M(h), \Delta), (M(h), \Delta)) - \sum_h \dim G_{M(h)}.$$

To prove that this is zero, it is enough to observe that

$$\dim X(\sigma_h) = \sum_h \dim G_{M(h)} - \dim \mathrm{Hom}((M(h), \Delta), (M(h), \Delta))$$

for all $h \neq h_2 + 1$ and

$$\dim X(\sigma_h) = \sum_h \dim G_{M(h)} - \dim \mathrm{Hom}((M(h), \Delta), (M(h), \Delta)) + p$$

for $h = h_2 + 1$. These follow immediately from the definitions and from 6.6 (b). This completes the proof of (b).

We now prove (c). Since π is proper, we see from (b) that the image of π is a closed irreducible subset of E^δ_M of dimension $\leqslant \dim X(\sigma)$. This image contains $X(\sigma)$, by (a), hence it contains the closure of $X(\sigma)$ and therefore it has dimension equal to $\dim X(\sigma)$ and it must coincide with $X(\sigma)$. The lemma is proved.

Lemma 6.13. – *Let* $(M(h), \Delta(0))$ *be as in* 6.11 (a) *or* (b). *The* $G_{M(h)}$-*orbit of* $\Delta(0)$ *in* $E^\delta_{M(h)}$ *is open in* $E^\delta_{M(h)}$.

By 4.8, the codimension of that orbit is equal to

$$\dim \mathrm{Ext}^1((M(h), \Delta(0)), (M(h), \Delta(0)))$$

and this is zero, by 4.2.

6.14. We now prove 6.7. We place ourselves in the setup of 6.11. For any integer h, let $X(\sigma_h) \subset E^{\delta}_{M(h)}$ be defined as in 6.11. Let $\mathscr{L}_h$ be the local system on $X(\sigma_h)$ defined as $\mathbf{C}$ if $h \neq h_2+1$ and as $\mathscr{L}_X$ (see 6.10) if $h=h_2+1$. Let P_h be the simple perverse sheaf on $E^{\delta}_{M(h)}$ whose support is the closure of $X(\sigma_h)$ and whose restriction to $X(\sigma_h)$ is $\mathscr{L}_X$ (up to shift). Then

(a) P_h belongs to $\mathbf{P}_{M(h), \delta}$.

(This follows from 6.9 (a), if $h_1 < h \leqslant h_2$, from 6.10 (a), if $h=h_2+1$ and from 6.13, 6.8 for $h \leqslant h_1$ and for $h > h_2+1$.)

From [L, 3.2, 3.5] it then follows that the complex

(b) $$\ldots * P_2 * P_1 * P_0 * P_{-1} * \ldots \text{ on } E^{\delta}_M$$

(iterated $*$ product) is a direct sum of shifts of simple perverse sheaves in $\mathbf{P}_{M, \delta}$. Hence it is enough to show that some shift of the simple perverse sheaf $P_{\sigma, \chi}$ is a direct summand of (b).

We now review the definition of (b) in our case. Let Φ' be as in the proof of 6.12; from that proof, we have a natural map $\Phi' \rightarrow \prod_h X(\sigma_h)$; we pull back under this map the tensor product of the $\mathscr{L}_h$ and we obtain a local system on Φ'; this extends uniquely to a G_M-equivariant local system $\mathscr{L}'$ on $\tilde{B}'$ (a smooth, irreducible variety, which is open dense in $\tilde{B}$). Let P' be the simple perverse sheaf on $\tilde{B}$ whose support is $\tilde{B}$ and whose restriction to $\tilde{B}'$ is $\mathscr{L}'$, up to shift. Then, by definition, the complex (b) is just $\pi_! P'$ (up to shift).

By 6.12 (b), we have $\dim(\tilde{B}-\tilde{B}') < \dim X(\sigma)$ hence $\dim \pi(\tilde{B}-\tilde{B}') < \dim X(\sigma)$. Thus the set $X' = X(\sigma) - (X(\sigma) \cap \pi(\tilde{B}-\tilde{B}'))$ is an open dense subset of $X(\sigma)$.

By 6.12 (a), the restriction of π is an isomorphism $\pi^{-1}(X') \cong X'$.

Under this isomorphism, the restriction of the local system $\mathscr{L}'$ to the subset $\pi^{-1}(X')$ of $\tilde{B}'$ corresponds to a local system on X' which can be seen to be just the restriction of the local system $\mathscr{L}_0$ defining $P_{\sigma, \chi}$ on $X(\sigma)$.

Thus, the cohomology sheaves of $\pi_! P'$ restricted to X' are equal to $L_0 | X'$ in one degree and zero in all other degrees. Let P'' be the simple perverse sheaf whose support is closure of X' and whose restriction to X' is $\mathscr{L}_0 | X'$, up to shift.

Since $\pi_! P'$ is known to be a direct sum of shifts of simple perverse sheaves and X' is open dense in the support of $\pi_! P'$ (see 6.12 (c)) it follows that some shift of P'' is a direct summand of $\pi_! P'$. We have clearly $P'' = P_{\sigma, \chi}$. Proposition 6.7 is proved.

6.15. Let M be a Γ-module and let (σ, λ) be an element of $\mathscr{S}(M)^{\delta}$ (see 4.3). Recall that $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_q)$. Let $p = \sum_j \lambda_j$. We write $\chi(\lambda)$ for the irreducible representation of S_p corresponding in the usual way to partition λ. (Thus, if $\lambda = (1,, \ldots, 1)$ then $\chi(\lambda)$ is the unit representation.)

Theorem 6.16. — (a) *The algebra homomorphism* $\mathbf{U}^- \to \mathbf{F}$ (*see* 6.5 (a)) *is an isomorphism.*

(b) *For any* Γ-*module* M, *the map* $(\sigma, \lambda) \mapsto \mathrm{P}_{\sigma, \chi(\lambda)}$ (*see* 6.7, 6.15) *is a bijection* $\mathscr{S}(\mathrm{M})^\delta \cong \mathbf{P}_{\mathrm{M}, \delta}$.

For any M, the map in (b) is clearly injective; hence

(c) $$|\mathbf{P}_{\mathrm{M}, \delta}| \geqslant |\mathscr{S}(\mathrm{M})^\delta|.$$

We have

(d) $$\dim \mathbf{U}^-_{\mathrm{M}} \geqslant \dim \mathbf{F}_{\mathrm{M}} = |\mathbf{P}_{\mathrm{M}, \delta}|$$

(see 6.5 (c)),

(e) $$\dim u^-_{\mathrm{M}} \geqslant \dim \mathbf{U}^-_{\mathrm{M}}$$

(since the $\mathbf{Q}$-algebra u^- is a specialization of the $\mathbf{Q}(v)$-algebra $\mathbf{U}^-$), and

(f) $$\dim u^-_{\mathrm{M}} = |\mathscr{S}(\mathrm{M})^\delta|$$

(by 4.16). Combining (c), (d), (e), (f), we see that (c), (d), (e) are equalities. The theorem follows. (Another proof of (a) is given in [L, 10.17].)

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