SEMIANALYTIC AND SUBANALYTIC SETS

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Dedicated to Stanislaw Łojasiewicz for his sixtieth birthday.

CONTENTS

0. Introduction 5
1. The Tarski-Seidenberg theorem and Thom's lemma 7
2. Semianalytic sets 9
3. Subanalytic sets 16
4. Transforming an analytic function to normal crossings by blowings-up 21
5. Uniformization and rectilinearization........ 30
6. Łojasiewicz's inequality; metric properties of subanalytic sets 33
7. Smooth points of a subanalytic set 37

Bibliography 42

0. Introduction

The theory of semianalytic and subanalytic sets originates in the work of Łojasiewicz [19, 20, 21] and (for subanalytic sets) has been elaborated by Gabrielov [11], Hironaka [17, 18] and Hardt [13, 14]. Hironaka, in particular, has used his desingularization and local flattening theorems to prove the following fundamental results: Let M be a real analytic manifold and let X be a subanalytic subset of M.

Theorem **0.1** (Uniformization theorem). — *Suppose that* X *is closed. Then there is a real analytic manifold* N *(of the same dimension as* X*) and a proper real analytic mapping* $\varphi : N \to M$ *such that* $\varphi(N) = X$.

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Theorem **0.2** (Rectilinearization theorem). — *Assume that* M *is of pure dimension m. Let* K *be a compact subset of* M. *Then there are finitely many real analytic mappings* $\varphi_i : \mathbf{R}^m \to M$ *such that*:

(1) *There is a compact subset* L_i *of* $\mathbf{R}^m$, *for each i, such that* $\bigcup_i \varphi_i(L_i)$ *is a neighbourhood of* K *in* M.

(2) *For each i,* $\varphi_i^{-1}(X)$ *is a union of quadrants in* $\mathbf{R}^m$ *(cf. Definition* 5.4*).*

Hironaka has used these theorems to establish the basic properties of subanalytic sets, as well as to give new proofs of Łojasiewicz's theorems on semianalytic sets. Denkowska, Łojasiewicz and Stasica [6, 7, 22], on the other hand, have used Łojasiewicz's "normal partitions" [21] to prove subanalytic analogues of his semianalytic results. Their approach seems motivated partly by an understandable reluctance to use resolution of singularities when it suffices to use techniques whose proofs are completely accessible. But they do not obtain Theorems 0.1 and 0.2 above.

From the point of view of analysis, Theorems 0.1 and 0.2 express the most important aspects of resolution of singularities. However, they are essentially different from resolution of singularities because the morphisms involved are not required to be bimeromorphic. In this article, we give short elementary proofs of Theorems 0.1 and 0.2, using neither desingularization nor local flattening. Our approach (Theorems 4.4 and 5.1) stands in the same relation to local resolution of singularities of real or complex analytic spaces as Zariski's uniformization theorem [28] does to desingularization of algebraic varieties. But our proofs are much simpler than those of [28].

The definition of "subanalytic set" adopted here is "locally, a projection of a relatively compact semianalytic set". (See Section 3.) From this point of departure, Theorem 0.1 is an immediate consequence of the analogous assertion for real analytic sets (Theorem 5.1). (Theorem 5.1 in the complex case would seem already close to resolution of singularities.) For Theorem 0.2 we use, in addition to Theorems 0.1 and 4.4, the fact that, if X is a subanalytic subset of $\mathbf{R}^n$, then the Euclidean distance function $d(x, X)$ has subanalytic graph; this is equivalent to subanalyticity of the complement of a subanalytic set (cf. Theorem 3.10 and Remarks 3.11).

In the various treatments of semianalytic and subanalytic sets, the order of development of the theory is, of course, dictated by the definitions of departure and the techniques employed (normal partitions in the case of Łojasiewicz *et al.*, desingularization in the case of Hironaka, ...). Interest in the theory has recently grown, stimulated partly by applications. But much of the literature is available only as mimeographed notes, and has an aura of technical difficulty which is unjustified.

One of our aims in this article is to describe certain simple techniques from which the fundamental properties of semianalytic and subanalytic sets can be obtained in a systematic way. For this reason, we present an exposition of the basic theory, although we have made some choice of topics to keep the paper of reasonable length. None of the results presented here is original. Neither are the techniques of Sections 1-3: Elementary treatments of semialgebraic sets, based only on the Tarski-Seidenberg theorem and Thom's lemma, have already been given; for example, in the excellent exposition of Coste [4]. We apply the same techniques to semianalytic sets, using the Weierstrass preparation theorem in the simplest possible way. Our proofs of the "fiber-cutting lemma" (Lemma 3.6) and the theorem of the complement for subanalytic sets are essentially those of [6, 7], although we avoid the use of normal partitions.

The reader interested only in the uniformization and rectilinearization theorems can go directly to Sections 4 and 5, referring to Section 3 only for the subanalyticity of the distance function and the fact that any closed subanalytic set is, locally, a proper image of an analytic set of the same dimension (Proposition 3.12).

Sections 6 and 7 illustrate how useful the uniformization theorem is (though it can be avoided in our proof of Łojasiewicz's inequality, which follows an idea attributed to Hörmander by Łojasiewicz). Two simple techniques play important parts in our treatment of subanalytic sets: the use of functions with subanalytic graphs (in particular, the distance function $d(x, X)$), and a fiber-product construction (which greatly simplifies Tamm's proof of subanalyticity of the smooth points of a subanalytic set [26]).

The only prerequisites for this article are the Weierstrass preparation theorem and some related elementary properties of analytic sets. The paper is otherwise self-contained, with the exception of Theorem 1.3, a simple proof of which is given in [4]. The bibliography is not meant to be a complete guide to the literature, but we have tried to include the original sources of all the results presented.

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1. The Tarski-Seidenberg theorem and Thom's lemma

Definition **1.1.** — The class of *semialgebraic subsets* of $\mathbf{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbf{R}^n : P(x) > 0\}$, where $P(x) = P(x_1, \ldots, x_n)$ is a polynomial, which is stable under finite intersection, finite union and complement.

Clearly, $X \subset \mathbf{R}^n$ is semialgebraic if and only if there exist polynomials $f_{ij}(x)$ and $g_{ij}(x)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, such that

$$X = \bigcup_{i=1}^{p} \{ x : f_{ij}(x) = 0, g_{ij}(x) > 0, j = 1, \ldots, q \}.$$

We begin with the approach of Łojasiewicz [21] and P. Cohen [3] to the Tarski-Seidenberg theorem (cf. [4]).

Definition **1.2.** — Let X be a subset of $\mathbf{R}^n$. A function $f: X \to \mathbf{R}$ is *semialgebraic* if, for every semialgebraic subset T of $\mathbf{R}^{p+1}$,

$$\{(t, x) \in \mathbf{R}^{p+n} : x \in X, (t, f(x)) \in T\}$$

is semialgebraic.

This definition implies that X is semialgebraic. Clearly, polynomials are semialgebraic. It is easy to see that differences and products of semialgebraic functions are semialgebraic.

Theorem **1.3.** — *Let* $P(x,y)$, $x = (x_1, \ldots, x_n)$, *be a polynomial. Then there is a semialgebraic partition* $\{A_1, \ldots, A_m\}$ *of* $\mathbf{R}^n$ *such that, for each* $k = 1, \ldots, m$, *either* P *has constant sign* (> 0, < 0, *or* $= 0$) *for all* $x \in A_k$ *and* $y \in \mathbf{R}$, *or there exist finitely many continuous semialgebraic functions* $\xi_1 < \ldots < \xi_{r_k}$ *on* A_k *such that*

(1) $\{\xi_1(x), \ldots, \xi_{r_k}(x)\}$ *is the set of zeros of* $P(x,y)$, *for each* $x \in A_k$;

(2) *the sign of* $P(x,y)$, $x \in A_k$, *depends only on the signs of* $y - \xi_i(x)$, $i = 1, \ldots, r_k$.

Proof. — See [4, Théorème 2.3]. □

Corollary **1.4.** — *Let* $P_1(x,y), \ldots, P_t(x,y)$ *be polynomials, where* $x = (x_1, \ldots, x_n)$. *Then there is a semialgebraic partition* $\{A_1, \ldots, A_m\}$ *of* $\mathbf{R}^n$ *such that, for each* $k = 1, \ldots, m$, *the zeros of* $P_1, \ldots, P_t$ *on* A_k *are given by continuous semialgebraic functions* $\xi_1 < \ldots < \xi_{r_k}$, *and the sign of each* $P_j(x,y)$ *on* A_k *depends only on the signs of* $y - \xi_i(x)$, $i = 1, \ldots, r_k$.

Proof. — Induction on t. Suppose that $P_1, \ldots, P_t$ satisfy the assertion. If P_{t+1} is another polynomial, let $B_1, \ldots, B_p$ denote a partition of $\mathbf{R}^n$ and $\zeta_1, \ldots, \zeta_{s_l}$ the roots of P_{t+1} on B_l, as provided by Theorem 1.3. The assertion for $P_1, \ldots, P_{t+1}$ follows by dividing each $A_k \cap B_l$ into semialgebraic subsets such that all $\xi_i - \zeta_j$ have constant sign on each of them. □

Theorem **1.5.** (Tarski-Seidenberg theorem). — *The image of a semialgebraic set* $X \subset \mathbf{R}^{n+1}$ *by the projection* $\mathbf{R}^{n+1} \to \mathbf{R}^n$ *is semialgebraic.*

Proof. — Say $X = \bigcup_{i=1}^p \bigcap_{j=1}^q \{P_{ij}(x,y)\ \sigma_{ij}\ 0\}$, where each P_{ij} is a polynomial and σ_{ij} denotes either $>$ or $=$. Apply Corollary 1.4 to the P_{ij}; the projection of X is a union of certain A_k. □

Corollary **1.6.** — *A function is semialgebraic if and only if its graph is semialgebraic.*

Proof. — Let $X \subset \mathbf{R}^n$ and let $f: X \to \mathbf{R}$ be a function. Suppose that f is semi-algebraic. Let $T = \{(y, z) \in \mathbf{R}^2 : y = z\}$. Then, according to Definition 1.2, $\operatorname{graph} f = \{(x,y) \in \mathbf{R}^{n+1} : x \in X, (y, f(x)) \in T\}$ is semialgebraic.

Conversely, suppose that graph f is semialgebraic. Let $T \subset \mathbf{R}^{p+1}$ be semialgebraic. Then

$$\{(t, x) \in \mathbf{R}^{p+n} : x \in X, (t, f(x)) \in T\}$$
$$= \pi(\{(t, x, y) : (x,y) \in \operatorname{graph} f, (t, y) \in T\}),$$

where π is the projection $\pi(t, x, y) = (t, x)$, is semialgebraic, by Theorem 1.5. □

Definition **1.7**. — Let $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ be semialgebraic. A mapping $f: X \to Y$ is *semialgebraic* if its component functions are each semialgebraic. (Equivalently, if $\operatorname{graph} f \subset \mathbf{R}^{m+n}$ is semialgebraic, or if $g \circ f$ is semialgebraic for every semialgebraic function $g: Y \to \mathbf{R}$.)

Corollary **1.8**. — *The image of a semialgebraic set by a semialgebraic mapping is semialgebraic.*

It follows from the Tarski-Seidenberg theorem that the closure (and thus the interior) of a semialgebraic set is semialgebraic. The basic properties of semialgebraic sets are all consequences of the Tarski-Seidenberg theorem and the following lemma of Thom. In Section 2 below, we will use the same techniques locally to deduce the basic properties of semianalytic sets, so we say nothing more about semialgebraic sets here.

Lemma **1.9** (Thom's lemma). — *Let* $P_1(x), \ldots, P_m(x)$ *be a finite family of polynomials in one variable, which is stable under differentiation. Let*

$$A = \bigcap_{i=1}^{m} \{ x \in \mathbf{R} : P_i(x)\ \sigma_i\ 0 \},$$

where each σ_i *denotes either* $>$, $<$ *or* $=$. *Then:*

(1) A *is either empty or connected (and therefore a point if* σ_i *is* $=$ *for one nonconstant polynomial* P_i, *or an open interval otherwise).*

(2) *If* $A \neq \emptyset$, *then* $\bar{A} = \bigcap_{i=1}^{m} \{ x : P_i(x)\ \bar{\sigma}_i\ 0 \}$, *where* $\bar{\sigma}_i$ *means* $\geq$, $\leq$ *or* $=$, *according as* σ_i *is* $>$, $<$ *or* $=$.

Proof. — Induction on m. The assertion is trivial when $m = 0$. Suppose it is true for $m - 1$, where $m \geq 1$. Arrange $P_1, \ldots, P_m$ so that P_m has maximal degree in this family. Then $P_1, \ldots, P_{m-1}$ is stable under differentiation. Let $A' = \bigcap_{i=1}^{m-1} \{ x : P_i(x)\ \sigma_i\ 0 \}$, so that $A = A' \cap \{ x : P_m(x)\ \sigma_m\ 0 \}$. Suppose $A' \neq \emptyset$. If A' is a point, the result is clear. If A' is an open interval, then the derivative of P_m has constant sign on A', so that P_m is monotone (or constant) on $\bar{A}'$. The result follows. □

2. Semianalytic sets

Let $\mathcal{A}$ be a ring of real-valued functions defined on a set E. Let $S(\mathcal{A})$ denote the subsets of E which are "described by" $\mathcal{A}$; i.e., the smallest family of subsets of E containing all $\{f(x) > 0\}$, $f \in \mathcal{A}$, which is stable under finite intersection, finite union and complement.

Equivalently, $S(\mathcal{A})$ means the subsets of E of the form $X = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} X_{ij}$, where each X_{ij} is either $\{f_{ij}(x) = 0\}$ or $\{f_{ij}(x) > 0\}$, $f_{ij} \in \mathcal{A}$. (We say that X is "described by" $\{f_{ij}\}$.)

Let M be a real analytic manifold. If U is an open subset of M, let $\mathcal{O}(U)$ denote the ring of real analytic functions on U.

Definition **2.1.** — A subset X of M is *semianalytic* if each $a \in M$ has a neighbourhood U such that $X \cap U \in S(\mathcal{O}(U))$.

Łojasiewicz's version of the Tarski-Seidenberg theorem [21]:

Theorem **2.2.** — *If* $X \in S(\mathcal{A}[t_1, \ldots, t_k])$, *then* $\pi(X) \in S(\mathcal{A})$, *where* $\pi : E \times \mathbf{R}^k \to E$ *is the projection* $\pi(x, t) = x$.

Proof. — Suppose that X is described in $E \times \mathbf{R}^k$ by the functions

$$f_j(x, t) = \sum_{|\alpha| \leqslant N} \lambda_{j,\alpha}(x)\, t^\alpha, \quad j = 1, \ldots, s,$$

where each coefficient $\lambda_{j,\alpha} \in \mathcal{A}$. (We use multiindex notation: $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{N}^k$, where $\mathbf{N}$ denotes the set of all nonnegative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_k$, and $t^\alpha = t_1^{\alpha_1} \ldots t_k^{\alpha_k}$.) Then X is the inverse image of a semialgebraic set X′ by the mapping $\Lambda(x, t) = (\lambda(x), t)$, where $\lambda(x) = (\lambda_{j,\alpha}(x))$. Therefore,

$$\pi(X) = \pi(\Lambda^{-1}(X')) = \{\, x : (\lambda(x), t) \in X', \text{for some } t \,\} = \lambda^{-1}(\pi'(X')),$$

where $\pi'((\lambda_{j,\alpha}), t) = (\lambda_{j,\alpha})$. Since $\pi'(X')$ is semialgebraic, by the Tarski-Seidenberg theorem, then $\pi(X) \in S(\mathcal{A})$. □

In fact, the analysis behind the Tarski-Seidenberg theorem extends to $\mathcal{O}(M)[y]$ and thence (using the Weierstrass preparation theorem) to semianalytic sets:

Definition **2.3.** — Let X be a subset of M. A function $f : X \to \mathbf{R}$ is *semianalytic* if its graph is semianalytic in $M \times \mathbf{R}$.

Proposition **2.4.** — *Let* $f_1(x,y), \ldots, f_t(x,y) \in \mathcal{O}(M)[y]$. *Then there is a semianalytic partition* $\{ A_1, \ldots, A_m \}$ *of* M *such that, for each* $k = 1, \ldots, m$:

(1) *The zeros of* $f_1, \ldots, f_t$ *on* A_k *are given by continuous semianalytic functions* $\xi_1 < \ldots < \xi_{r_k}$.

(2) *The sign of each* $f_j(x,y)$ *on* A_k *depends only on the signs of the* $y - \xi_i(x)$.

Proof. — Say $f_j(x,y) = \sum_{|\alpha| \leqslant N} \lambda_{j,\alpha}(x)\, y^\alpha$, $j = 1, \ldots, t$. Then each

$$f_j(x,y) = P_j(\lambda(x), y),$$

where $\lambda = (\lambda_{j,\alpha})$ and $P_j(\lambda, y)$ is a polynomial. For the $P_j(\lambda, y)$, consider the semialgebraic partition $\{ A'_k \}$ and, for each k, the continuous semialgebraic functions $\xi'_i(\lambda)$ given by Corollary 1.4. Then $f_j(x, \xi'_i(\lambda(x))) = P_j(\lambda(x), \xi'_i(\lambda(x))) = 0$, when $\lambda(x) \in A'_k$. Take $A_k = \lambda^{-1}(A'_k)$ and $\xi_i = \xi'_i \circ \lambda$. □

There is a several-variable version of Thom's lemma, due to Efroymson [10], which can be extended, locally, to analytic functions ([3]):

Definition **2.5.** — Let U be an open subset of M. A finite family $f_1, \ldots, f_m \in \mathcal{O}(\mathrm{U})$ is *separating* if, for any semianalytic subset A of U of the form

$$\mathrm{A} = \bigcap_{i=1}^{m} \{ x \in \mathrm{U} : f_i(x)\ \sigma_i\ 0 \},$$

where each σ_i is either $>$, $<$ *or* $=$, we have:

(1) A is either empty or connected.

(2) If $\mathrm{A} \neq \emptyset$, then the closure of A in U,

$$\overline{\mathrm{A}} = \bigcap_{i=1}^{m} \{ x \in \mathrm{U} : f_i(x)\ \bar{\sigma}_i\ 0 \},$$

where $\bar{\sigma}_i$ is $\geqslant$, $\leqslant$ *or* $=$, according as σ_i is $>$, $<$ *or* $=$.

It is easy to see that (2) is equivalent to:

(2') If $\mathrm{A} \neq \emptyset$ and B is also given by sign conditions on the f_i, then $\mathrm{B} \subset \overline{\mathrm{A}}$ if and only if every strict sign condition (i.e., $>$ or $<$) on the f_i in B is also satisfied in A.

Theorem **2.6.** — *Any finite family of analytic functions on* M *can be completed, in some neighbourhood of a given point, to a separating family.*

Proof. — Induction on $m = \dim \mathrm{M}$. Let $f_1, \ldots, f_p \in \mathcal{O}(\mathrm{M})$. By the Weierstrass preparation theorem [24, Chapt. II, Th. 2] we can assume that any given point of M admits a coordinate neighbourhood U such that:

(1) $\mathrm{U} = \mathrm{U}' \times \mathrm{I}$, where U' is an open subset of $\mathbf{R}^{m-1}$ and I is an open interval.

(2) Let $(x, y) = (x_1, \ldots, x_{m-1}, y)$ denote the coordinates of $\mathrm{U} = \mathrm{U}' \times \mathrm{I}$. Then each $f_j(x, y) = u_j(x, y)\, g_j(x, y)$, where u_j is an analytic function vanishing nowhere in U, and g_j is a monic polynomial in y whose coefficients are analytic functions on U', such that, for each $x \in \mathrm{U}'$, all real roots of $g_j(x, y)$ belong to I.

Each $g_j \in \mathcal{O}(\mathrm{U}')\,[y] \subset \mathcal{O}(\mathrm{U}' \times \mathbf{R})$. Clearly, it is enough to show that $g_1, \ldots, g_p$ can be completed to a separating family, shrinking U' if necessary. If $m = 1$, this is just Thom's lemma: we get a separating family by adding all nonconstant derivatives of all orders.

In general, we add all nonconstant derivatives of $g_1, \ldots, g_p$ with respect to y of all orders, to get $g_1, \ldots, g_p, g_{p+1}, \ldots, g_{p+q}$, all monic in y (except for constant factors). By Proposition 2.4, there is a semianalytic partition $\{ \mathrm{B}_1, \ldots, \mathrm{B}_s \}$ of U' such that,

([3]) A similar approach is taken by F. Fernandez, T. Recio and J. Ruiz, Generalized Thom's lemma in semianalytic geometry, *Bull. Polish Acad. Sci. Math.* **35** (1987), 297-301.

for each $k = 1, \dots, s$, the zeros of $g_1, \dots, g_{p+q}$ over B_k are given by continuous semianalytic functions $\xi_1 < \dots < \xi_{t_k}$, and the sign of each $g_j(x, y)$ on B_k depends only on the signs of the $y - \xi_i(x)$. After shrinking U' if necessary, each B_k can be described by finitely many analytic functions on U'. By induction (again perhaps shrinking U'), we complete the list of functions which describe the B_k to a separating family, say $g_{p+q+1}(x), \dots, g_{p+q+r}(x)$.

Then $g_1, \dots, g_{p+q+r}$ is a separating family in $U' \times \mathbf{R}$: Suppose that

$$A = \bigcap_{j=1}^{p+q+r} \{(x,y) : g_j(x,y)\ \sigma_j\ 0\},$$

where each σ_j is either $>$, $<$ or $=$. Let

$$B = \bigcap_{j=p+q+1}^{p+q+r} \{(x,y) : g_j(x)\ \sigma_j\ 0\},$$

and let $\xi_1 < \dots < \xi_t$ denote the roots of $g_1, \dots, g_{p+q}$ over B. Let $\pi : U' \times \mathbf{R} \to U'$ be the projection. By Thom's lemma, if $x_0 \in B$, then $A \cap \pi^{-1}(x_0)$ is either empty, or a root $(x_0, \xi_i(x_0))$, or an interval $\{(x_0, y) : \xi_i(x_0) < y < \xi_{i+1}(x_0)\}$ (where, in the latter expression, ξ_i may be $-\infty$ and ξ_{i+1} may be $+\infty$). Then, since the sign of $g_j(x, y)$, $j = 1, \dots, p+q$, on B depends only on the signs of the $y - \xi_i(x)$, A is either empty, or $\{(x, \xi_i(x)) : x \in B\}$, or $\{(x,y) : x \in B, \xi_i(x) < y < \xi_{i+1}(x)\}$. In each case, A is either empty or connected.

Suppose $A \neq \emptyset$. Let

$$A' = \bigcap_{j=1}^{p+q+r} \{(x,y) : g_j(x,y)\ \bar{\sigma}_j\ 0\},$$

where $\bar{\sigma}_j$ is $\geqslant$, $\leqslant$ or $=$, according as σ_j is $>$, $<$ or $=$, and let $\bar{A}$ be the closure of A in $U' \times \mathbf{R}$. Clearly, $\bar{A} \subset A'$. It remains to show that $A' \subset \bar{A}$. By induction,

$$\bar{B} = \bigcap_{j=p+q+1}^{p+q+r} \{x : g_j(x)\ \bar{\sigma}_j\ 0\}.$$

Let $x_0 \in \bar{B}$. Since the $g_j(x,y)$, $j = 1, \dots, p+q$, are monic with respect to y, we can find a neighbourhood V' of x_0 in U' and $K > 0$ such that the roots $\xi_1, \dots, \xi_t$ are bounded in absolute value by K on $B \cap V'$. Thus, for all $x \in B \cap V'$, $\bar{A} \cap (\{x\} \times [-K, K]) \neq \emptyset$, so that $\bar{A} \cap \pi^{-1}(x_0) \neq \emptyset$. By Thom's lemma, there are two possibilities for the fiber of A' over x_0:

(1) A point, which therefore coincides with the fiber of $\bar{A}$ over x_0.

(2) A closed interval with non-empty interior. Suppose that (x_0, y) belongs to its interior. Then surely $g_j(x_0, y)\ \sigma_j\ 0$, $j = 1, \dots, p+q$, where each σ_j is a strict inequality. So $(x_0, y) \in \bar{A}$. Thus the whole closed interval lies in $\bar{A}$.

Therefore, $A' \subset \bar{A}$. □

Corollary **2.7.** — *Let* X *be a semianalytic subset of* M. *Then:*

(1) *Every connected component of* X *is semianalytic.*

(2) *The family of connected components of* X *is locally finite (in particular, finite if* X *is relatively compact).*

(3) X *is locally connected.*

Proof. — It is enough to show that each $a \in M$ has a neighbourhood U such that $X \cap U$ has finitely many connected components, all semianalytic in U. Let U be a neighbourhood of a such that $X \cap U$ can be described using finitely many elements $f_1, \dots, f_p$ of $\mathcal{O}(U)$. By Theorem 2.6, shrinking U if necessary, we can complete $f_1, \dots, f_p$ to a separating family $f_1, \dots, f_p, f_{p+1}, \dots, f_{p+q}$. Then $X \cap U$ is a disjoint union of finitely many connected semianalytic subsets of U, each given by a sign condition on each f_j, $j = 1, \dots, p+q$. □

Corollary **2.8.** — *The closure, and thus the interior, of a semianalytic set is semianalytic.*

Proof. — This is again immediate from Theorem 2.6. □

Corollary **2.9.** — (1) *Let* X *be a semianalytic subset of* M, *and let* $U \subset X$ *be a semianalytic subset of* M *which is open in* X. *Then, locally,* U *is a finite union of semianalytic sets of the form*

$$\{ x \in X : f_1(x) > 0, \dots, f_k(x) > 0 \},$$

where the f_i *are analytic functions.*

(2) *Every closed semianalytic subset of* M *is, locally, a finite union of sets of the form*

$$\{ x : f_1(x) \geqslant 0, \dots, f_k(x) \geqslant 0 \},$$

where the f_i *are analytic functions.*

Proof. — (1) Locally, we can complete a list of analytic functions used to describe X and U to a separating family, say $f_1, \dots, f_k$. Then $U = \bigcup_{i=1}^{p} T_i$, where each $T_i = \bigcap_{j=1}^{k} \{ x : f_j(x) \, \sigma_{ij} \, 0 \}$, $T_i \neq \emptyset$, and σ_{ij} is either $>$, $<$ or $=$. Let V_i be the open semianalytic set given by the intersection of the sets with *strict sign conditions* in the preceding representation of T_i. Then each $T_i \subset V_i$, so that $U \subset X \cap \bigcap_{i=1}^{p} V_i$.

To show $X \cap V_i \subset U$, for each $i : X \cap V_i$ is also a union of semianalytic sets given by sign conditions on each f_j. Let A be one of these sets. By the definition of V_i, every strict sign condition satisfied in T_i is also satisfied in A. Therefore $T_i \subset \overline{A}$ (by condition (2′) following Definition 2.5). Since U is open in X, $U \cap A \neq \emptyset$. Thus, necessarily, $A \subset U$ (U is a disjoint union of sets of the form $\bigcap_{j=1}^{k} \{ f_j(x) \, \sigma_j \, 0 \}$, so that A must either be one of these, or be disjoint from U).

(2) follows from (1). □

Thom's lemma suggests a stronger version of Theorem 2.6:

Proposition **2.10.** — *Let* $f_1, \dots, f_p \in \mathcal{O}(M)$. *Let* $a \in M$. *Then there is a semianalytic open neighbourhood* U *of* a, *and a separating family* $h_1, \dots, h_p, h_{p+1}, \dots, h_{p+s} \in \mathcal{O}(U)$ *such that* $h_j = f_j \mid U$, $j = 1, \dots, p$, *and the collection* $\{A_k\}$ *of subsets of* U *of the form*

$$\bigcap_{j=1}^{p+s} \{ x \in U : h_j(x) \, \sigma_j \, 0 \},$$

where each σ_j *is either* $>$, $<$ *or* $=$, *is a* semianalytic stratification *of* U; *i.e.*:

(1) U *is the disjoint union of the* A_k.

(2) *Each* A_k *is a connected semianalytic subset and analytic submanifold of* M.

(3) *("Condition of the frontier".) If* $A_k \cap \bar{A}_l \neq \emptyset$, *then* $A_k \subset \bar{A}_l$ *and* $\dim A_k < \dim A$.

Proof. — This follows the proof of Theorem 2.6, the notation of which we take up again here. We can assume that U and U′ are semianalytic and, by induction, that the subsets

$$B = \bigcap_{j=p+q+1}^{p+q+r} \{ x \in U' : g_j(x) \, \sigma_j \, 0 \},$$

where σ_j is either $>$, $<$ *or* $=$, form a semianalytic stratification of U′. Shrinking U′ if necessary, we can assume that, for each B, the roots $\xi_1 < \dots < \xi_t$ of $g_1(x,y), \dots, g_{p+q}(x,y)$ over B have graphs which are semianalytic in M. Then U is a disjoint union of semianalytic sets of the form

$$A = \bigcap_{j=1}^{p+q} \{ (x,y) \in U' \times I : x \in B, \, g_j(x,y) \, \sigma_j \, 0 \},$$

where each σ_j is either $>$, $<$ *or* $=$. Since $\{ g_1, \dots, g_{p+q} \}$ is stable under differentiation with respect to y, every root $\xi_i(x)$ of each $g_j(x, y)$ over B is a simple root of one of the g_j. Since B is an analytic manifold, the ξ_i are analytic and A is an analytic manifold. The condition of the frontier is also clear from the proof of Theorem 2.6. □

Corollary **2.11.** — *Let* $\{ X_i \}$ *be a locally finite family of semianalytic subsets of* M. *Then there is a locally finite semianalytic stratification* $\{ A_k \}$ *of* M *which is "compatible" with each* X_i *(i.e. each* X_i *is a union of certain* A_k*).*

Proof. — Each point $a \in M$ admits a neighbourhood in which the X_i are described by finitely many analytic functions $f_1, \dots, f_p$. Then Proposition 2.10 means there is a semianalytic neighbourhood U of a and a finite semianalytic stratification of U which is compatible with $X_i \cap U$. The global assertion follows. □

Remark **2.12.** — Corollary 2.11 allows us to define the *dimension* of a semianalytic set X: If $X = \bigcup_k A_k$ is a stratification, put $\dim X = \max_k \dim A_k$.

This definition is independent of the stratification: $\dim X = d$ if and only if X contains an open set homeomorphic to an open ball in $\mathbf{R}^d$, but not an open set homeomorphic to an open ball in $\mathbf{R}^e$, for $e > d$.

The following theorem of Łojasiewicz [21] (which has a global analogue for semialgebraic sets) distinguishes semianalytic from more general subanalytic sets:

Theorem **2.13.** — *Let* $X \subset M$. *Then* X *is semianalytic of dimension* $\leqslant k$ *if and only if, locally in* M, *there is an analytic set* Z *of dimension* $\leqslant k$ *such that* $X \subset Z$, $\overline{X} - X$ *is semianalytic of dimension* $\leqslant k-1$, *and* $X -$ *(interior of* X *in* Z*) is semianalytic of dimension* $\leqslant k-1$.

Example **2.14** (Osgood). — Let $G(y) = G(y_1, y_2, y_3)$ be a formal power series such that $G(x_1, x_1 x_2, x_1 x_2 e^{x_2}) = 0$. Then $G = 0$. Write $G(y) = \sum_{j=0}^{\infty} G_j(y)$, where $G_j(y)$ is a homogeneous polynomial of order j. Then $0 = G(x_1, x_1 x_2, x_1 x_2 e^{x_2}) = \sum_j x_1^j G_j(1, x_2, x_2 e^{x_2})$. Therefore, for each j, $G_j(1, x_2, x_2 e^{x_2}) = 0$; hence $G_j = 0$.

Let $\varphi : \mathbf{R}^2 \to \mathbf{R}^3$ be the mapping $\varphi(x_1, x_2) = (x_1, x_1 x_2, x_1 x_2 e^{x_2})$. Then

$$X = \varphi(\{x_1^2 + x_2^2 \leqslant 1\})$$

is not semianalytic: Since there are no nontrivial convergent power series relations $G(x_1, x_1 x_2, x_1 x_2 e^{x_2}) = 0$, $\mathbf{R}^3$ itself is the smallest real analytic set containing (the germ at 0 of) X.

Thus, the Tarski-Seidenberg theorem is false for semianalytic sets.

Lemma **2.15.** — *Let* $A \subset B \subset M$, *where* B *is semianalytic. Let* $A_1 = \overline{A} \cap B$ *(the closure of* A *in* B*) and let* $A_2 = A - \overline{B - A}$ *(the interior of* A *in* B*). Then* A *is semianalytic if and only if* $A_1 - A$ *and* $A - A_2$ *are semianalytic.*

Proof. — " Only if " is clear. " If ": $B - A_1$ and A_2 are disjoint open and closed subsets of their union $B - (A_1 - A_2)$, which is semianalytic since

$$A_1 - A_2 = (A_1 - A) \cup (A - A_2).$$

Therefore, A_2 is semianalytic and then $A = A_2 \cup (A - A_2)$ is semianalytic. □

Proof of Theorem 2.13. — " If " is clear. " Only if ": Locally, X is a union of finitely many sets of the form

$$A = \bigcap_{j=1}^{r} \{f_j(x) > 0\} \cap \bigcap_{j=r+1}^{s} \{f_j(x) = 0\},$$

where $f_1, \ldots, f_s$ are analytic. Then $\bigcap_{j=1}^{r} \{f_j(x) > 0\}$ is open and

$$Y = \bigcap_{j=r+1}^{s} \{f_j(x) = 0\}$$

is an analytic set. After perhaps shrinking the local neighbourhood, there is a proper analytic subset Y' of Y such that $Y - Y'$ is an analytic manifold of dimension $= \dim Y$ [24, 27]. Since A is open in Y, if $\dim Y > \dim A$, then $A \subset Y'$. In this case, repeat the argument using Y'... Eventually, A lies in an analytic set Z of dimension $= \dim A$

because any decreasing family of germs of analytic sets stabilizes (as the ring of convergent power series is Noetherian [24]).

Now, $\bar{A} - A$ is semianalytic, and $A -$ (interior of A in Z) is semianalytic, by Lemma 2.15. Since, by Corollary 2.11, we can stratify A and $\bar{A} - A$ simultaneously, the frontier of A has dimension $<$ dim A. Let A_2 be the interior of A in Z. Stratify A and A_2 simultaneously. Then A_2 includes all strata of A of dimension = dim A. (Such a stratum cannot include frontier points of another stratum, by the condition of the frontier.) Thus, $A - A_2$ has dimension $<$ dim A. □

3. Subanalytic sets

Let M denote a real analytic manifold.

Definition **3.1.** — A subset X of M is *subanalytic* if each point of M admits a neighbourhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set (i.e. there is a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $X \cap U = \pi(A)$, where $\pi : M \times N \to M$ is the projection).

From the basic properties of semianalytic sets we obtain: The intersection and union of a finite collection of subanalytic sets are subanalytic. Every connected component of a subanalytic set is subanalytic. The family of connected components is locally finite. A subanalytic set is locally connected. The closure of a subanalytic set is subanalytic.

We will prove that the complement (and thus the interior) of a subanalytic set is subanalytic.

Definition **3.2.** — Let $X \subset M$ and let N be a real analytic manifold. A mapping $f : X \to N$ is *subanalytic* if its graph is subanalytic in $M \times N$.

Clearly, the image of a relatively compact subanalytic set by a subanalytic mapping is subanalytic.

Definition **3.3.** — Let X be a subanalytic subset of M. Let $x \in X$. Then x is a *smooth point* of X (of *dimension k*) if, in some neighbourhood of x in M, X is an analytic submanifold (of dimension k). We say that X is *smooth* if every point of X is a smooth point; i.e., X is an analytic submanifold of M.

In the following four lemmas of [6, 7] (cf. [11]), U and V are finite-dimensional Euclidean spaces, $W = U \oplus V$, and $\pi : W \to U$ denotes the projection.

Lemma **3.4.** — *Let* X *be a relatively compact semianalytic subset of* W. *Then* X *is a finite union of connected smooth semianalytic subsets* A *such that, for each* A:

(1) *The rank* $\mathrm{rk}_x(\pi \mid A)$ *is constant on* A.

(2) *The linear subspaces* $T_x A \cap V$, $x \in A$, *admit a common complement in* V, *and the subspaces* $\pi(T_x A)$, $x \in A$, *admit a common complement in* U. *(Here,* $T_x A$ *denotes the tangent space of* A *at* x.*)*

(3) *There is an analytic function* g *in a neighbourhood of* $\overline{A}$ *such that* $g > 0$ *on* A *and* $g = 0$ *on* $\overline{A} - A$.

Proof. — Let $k = \dim X$. The result is obvious if $k = 0$. If $k > 0$, there is a semianalytic subset Y of X such that $\dim Y < k$ and $X - Y$ consists of smooth points of dimension k. By induction on k, we can assume that the result holds for Y. Therefore, we can assume X is smooth and also connected.

Let $X_0 = \{ x \in X : \mathrm{rk}_x(\pi \mid X) \text{ is maximal} \}$. Then X_0 is semianalytic and $\dim(X - X_0) < k$. Locally, X_0 lies in an analytic set of dimension k; therefore, we can assume there are analytic functions $h_1, \ldots, h_{n-k}$ ($n = \dim W$) defined in a neighbourhood of X_0 such that each h_i vanishes on X_0 and, if $Z = \{ x : \text{the gradients grad } h_i(x)$ are linearly dependent $\}$, then $\dim X_0 \cap Z < k$. By induction, we can assume $\mathrm{rk}_x(\pi \mid X)$ is constant on X and the gradients grad $h_i(x)$ are linearly independent on X.

Let $G_k(W)$ denote the Grassmanian of k-dimensional linear subspaces of W. Given linear subspaces E of U and F of V, let $G_{E,F} = \{ T \in G_k(W) : F$ is complementary to $T \cap V$ in V, and E is complementary to $\pi(T)$ in U $\}$. Clearly, $G_{E,F}$ is an open semialgebraic subset of $G_k(W)$. There exist finitely many such pairs (E, F) such that $G_k(W) = \bigcup G_{E,F}$.

Now $X = \bigcup_{(E,F)} \{ x \in X : T_x X \in G_{E,F} \}$. Each set in this union is open in X; we will have (1) and (2) once we show it is semianalytic. Let $\Sigma = \{(z_1, \ldots, z_{n-k}) \in W^{n-k} : z_1, \ldots, z_{n-k}$ are linear dependent $\}$. If $(z_1, \ldots, z_{n-k}) \in W^{n-k} - \Sigma$, let $S(z_1, \ldots, z_{n-k})$ denote the orthogonal complement of the subspace spanned by $z_1, \ldots, z_{n-k}$. Then $S : W^{n-k} - \Sigma \to G_k(W)$ is a continuous semialgebraic mapping. Put

$$H(x) = (\text{grad } h_1(x), \ldots, \text{grad } h_{n-k}(x)).$$

Then $S^{-1}(G_{E,F})$ is a semialgebraic subset of W^{n-k}, and

$$\{ x \in X : T_x X \in G_{E,F} \} = X \cap H^{-1}(S^{-1}(G_{E,F}))$$

is semianalytic.

To get (3), suppose we have A satisfying (1) and (2). Locally, $\overline{A} - A$ lies in an analytic set Y of dimension $< \dim A$, so, by induction, it suffices to prove (3) for $A - Y$. We can assume $Y = \{ x : g_1(x) = \ldots = g_p(x) = 0 \}$, where the g_j are analytic functions. Take $g = \Sigma g_j^2$. □

Remark **3.5**. — It is clear from Lemma 3.4 (1) how to extend the definition of *dimension* from semianalytic to subanalytic sets. The dimension of a subanalytic set is the highest dimension of its smooth points.

Lemma **3.6** (Fiber-cutting lemma). — *Let* X *be a relatively compact semianalytic subset of* W. *Then there are finitely many smooth semianalytic subsets* B *of* X *such that*:

(1) $\pi(X) = \pi(\bigcup B)$.

(2) *For each* B, $\pi \mid B : B \to U$ *is an immersion.*

(3) *For each* B, *the subspaces* $\pi(T_x B)$, $x \in B$, *have a common complement in* U.

Proof. — Let $k = \dim X$. Write X as a finite union of connected smooth semianalytic subsets A as in Lemma 3.4. For $\bigcup_{\dim A < k} A$, the result holds by induction. On the other hand, each A such that $\dim A = k$ and $\operatorname{rk}(\pi \mid A) = k$ already satisfies (2) and (3). Consider A such that $\dim A = k$ and $\operatorname{rk}(\pi \mid A) < k$. By induction, it is enough to find a semianalytic subset Z of A such that $\dim Z < \dim A$ and $\pi(A) = \pi(Z)$.

It follows from Lemma 3.4 (2) that, for every $x \in A$, the fiber $A_{\pi(x)} = A \cap \pi^{-1}(\pi(x))$ is a submanifold of $\pi^{-1}(\pi(x))$ and, for each connected component C of $A_{\pi(x)}$, $\bar{C} - C \neq \emptyset$. The function g of Lemma 3.4 (3) is positive on C and zero on $\bar{C} - C$. Let $Z = \{ x \in A : d_x g \mid (T_x A \cap V) = 0 \}$, where $d_x g$ denotes the tangent mapping of g at x; i.e. Z is the set of critical points of the restrictions of g to the fibers $A_{\pi(x)}$, $x \in A$. It follows from the first assertion of Lemma 3.4 (2) that Z is semianalytic. For every component C as above, g is not constant on C, so that $\dim Z < \dim A$, and g has a positive maximum on C, so that $Z \cap C \neq \emptyset$ and $\pi(Z) = \pi(A)$. □

Lemma **3.7.** — *Assume that, in* U, *the complement of every subanalytic sets is subanalytic. Let* B *denote a bounded smooth semianalytic subset of* W *such that* $\pi \mid B : B \to U$ *is a local diffeomorphism. For every* $u \in U$, *let* $\mu(u)$ *denote the number of points in the fiber* $B_u = B \cap \pi^{-1}(u)$. *Then* $\mu(u)$ *is bounded on* U.

Proof. — Clearly $\mu(u) < \infty$, for all $u \in U$, and μ is lower semicontinuous. Let $C = \pi(\bar{B} - B)$. Then C is a closed subanalytic subset of U of dimension $< \dim U$; in particular, it is nowhere dense in U. Therefore, it is enough to prove that μ is bounded on $U - C$. By the hypothesis, $U - C$ is subanalytic, hence has finitely many connected components. But $\mu(u)$ is constant on each of them. □

Definition **3.8.** — Let $\varphi : X \to Y$ be a mapping between sets. For any positive integer s, let X^s_φ denote the s-fold fiber product

$$X^s_\varphi = \{ \mathbf{x} = (x^1, \ldots, x^s) \in X^s : \varphi(x^1) = \ldots = \varphi(x^s)\},$$

and let $\boldsymbol{\varphi} : X^s_\varphi \to Y$ denote the induced mapping $\boldsymbol{\varphi}(\mathbf{x}) = \varphi(x^1)$.

Lemma **3.9.** — *Assume that, in* U, *the complement of every subanalytic set is subanalytic. Let* X *be a relatively compact subanalytic subset of* W. *Suppose that the number of points* $\mu(u)$ *in a fiber* $X_u = X \cap \pi^{-1}(u)$ *is bounded on* U. *Then* $W - X$ *is subanalytic.*

Proof. — For each s, let

$$\Delta_s = \{ \mathbf{x} = (x^1, \ldots, x^s) \in W^s_\pi : x^i = x^j \text{ for some } i \neq j \}.$$

Then $X^s \cap (W^s_\pi - \Delta_s)$ is a relatively compact subanalytic subset of W^s. Put $C_s = \{ u \in U : \mu(u) \geq s \}$ and $D_s = \{ u \in U : \mu(u) = s \}$. Then $C_s = \boldsymbol{\pi}(X^s \cap (W^s_\pi - \Delta_s))$ and hence $D_s = C_s - C_{s+1}$ are subanalytic. There exists t such that

$$U = D_0 \cup D_1 \cup \ldots \cup D_t.$$

Now $W - X = \bigcup_{s=0}^{t}(\pi^{-1}(D_s) - X)$. But each

$$\pi^{-1}(D_s) - X = \pi^{-1}(D_s) \cap p((W \times X^s) \cap (W_\pi^{s+1} - \Delta_{s+1})),$$

where $p : W \times W^s \to W$ is the projection. Since $(W \times X^s) \cap (W_\pi^{s+1} - \Delta_{s+1})$ is subanalytic in $W \times W^s$ and " W^s-relatively compact " (i.e. its intersection with $p^{-1}(K)$ is relatively compact, for every compact $K \subset W$), then $\pi^{-1}(D_s) - X$ is subanalytic. Hence $W - X$ is subanalytic. □

Theorem **3.10** (Theorem of the complement). — *Let* M *be a real analytic manifold and let* X *be a subanalytic subset of* M. *Then* M − X *is subanalytic.*

Proof. — We can assume that M is an n-dimensional Euclidean space W and that X is relatively compact. The result is trivial if $n = 0$. We argue by induction on n. There is a finite-dimensional vector space Z and a relatively compact semianalytic subset B of $W \times Z$ such that $X = \pi(B)$, where $\pi : W \times Z \to W$ is the projection. By the fiber-cutting lemma, we can assume that B is smooth, $\pi \mid B$ is an immersion, and the $\pi(T_x B)$, $x \in B$, have a common complement V in W.

Case 1. $\dim B < n$. Let U be a complement of V in W, and let $\pi_0 : W \cong U \oplus V \to U$ be the projection. Since $\dim U < n$, our theorem is true in U, by induction. By Lemma 3.7, the number of points in the fiber $B \cap (\pi_0 \circ \pi)^{-1}(u)$ is bounded on U. Therefore, the number of points in $\pi(B) \cap \pi_0^{-1}(u)$ is bounded. By Lemma 3.9, the complement of $X = \pi(B)$ in W is subanalytic.

Case 2. $\dim B = n$. Then $\pi \mid B$ is a local diffeomorphism. Let $C = \overline{B} - B$. Then $\pi(C)$ is subanalytic and of dimension $< n$, so that $W - \pi(C)$ is subanalytic, from Case 1. Since $W - \pi(\overline{B})$ is open and closed in $W - \pi(C)$, it is also subanalytic. Now $W - \pi(B) = (W - \pi(\overline{B})) \cup (\pi(\overline{B}) - \pi(B)) = (W - \pi(\overline{B})) \cup (\pi(C) - \pi(B) \cap \pi(C))$. Since $\pi(B) \cap \pi(C)$ is subanalytic of dimension $< n$, it follows from Case 1 that $W - \pi(B)$ is subanalytic. □

Remarks **3.11.** — (1) Let X be a subanalytic subset of $\mathbf{R}^n$. Then the distance function $d(x, X) = \min_{z \in \overline{X}} | x - z |$ is subanalytic: We can assume that X is relatively compact. Let $A = \{(x, z, y) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} : z \in \overline{X}, y \geqslant | x - z | \}$. Then A is subanalytic. Let π denote the projection $\pi(x, z, y) = (x, y)$. Then $\{(x, y) \in \mathbf{R}^n \times \mathbf{R} : y \geqslant d(x, X)\} = \pi(A)$ is subanalytic, and the assertion follows from the theorem of the complement.

It is easy to see that, conversely, subanalyticity of the distance function implies the theorem of the complement.

(2) Let M and N be real analytic manifolds and let X and T be subanalytic subsets of M and N, respectively, where T is compact. If $f : X \times T \to \mathbf{R}$ is a continuous subanalytic function, it follows as in (1) that $g(x) = \min_{t \in T} f(x, t)$ is a subanalytic function on X.

Proposition **3.12.** — *Let* M *be a real analytic manifold and let* X *be a closed subanalytic subset of* M. *Then each point of* X *admits a neighbourhood* U *such that* $X \cap U = \pi(A)$, *where*

A is a closed analytic subset of $U \times \mathbf{R}^q$, *for some* q, $\dim A = \dim X \cap U$, *and* $\pi \mid A$ *is proper (where* $\pi : U \times \mathbf{R}^n \to U$ *is the projection).*

Proof. — First assume that X is semianalytic. Let $a \in X$. By Corollary 2.9 (2), a has a neighbourhood U such that $X \cap U$ is a finite union of sets of the form

$$Y = \{ x \in U : f_i(x) \geqslant 0, i = 1, \dots, p \},$$

where each $f_i \in \mathcal{O}(U)$. Let $A \subset U \times \mathbf{R}^p$ be the closed analytic subset

$$A = \{(x,y) = (x, y_1, \dots, y_p) : f_i(x) - y_i^2 = 0, i = 1, \dots, p \}.$$

Then $\dim A = \dim Y$, $Y = \pi(A)$ and $\pi \mid A$ is proper, where $\pi : U \times \mathbf{R}^p \to U$ is the projection.

In fact, we can assume that U is an open neighbourhood of $a = 0$ in $\mathbf{R}^m$. Then there exists $\varepsilon > 0$ such that $D = \{ x = (x_1, \dots, x_m) : \Sigma x_j^2 \leqslant \varepsilon \} \subset U$. Then

$$B = \{(x, y, t) : \Sigma x_j^2 + t^2 = \varepsilon, f_i(x) - y_i^2 = 0, i = 1, \dots, p \}$$

is a compact real analytic subset of $U \times \mathbf{R}^{p+1}$ such that $\dim B = \dim Y \cap D$ and $Y \cap D = \pi'(B)$, where $\pi' : U \times \mathbf{R}^{p+1} \to U$ is the projection.

Our assertion for X subanalytic follows using the fiber-cutting lemma. □

There are several equivalent definitions of " subanalytic ":

Proposition **3.13.** — *Let* M *be a real analytic manifold, and let* X *be a subset of* M. *Then the following conditions are equivalent:*

(1) X *is subanalytic.*

(2) *Every point of* M *has a neighbourhood* U *such that*

$$X \cap U = \bigcup_{i=1}^{p} (f_{i1}(A_{i1}) - f_{i2}(A_{i2})),$$

where, for each $i = 1, \dots, p$ *and* $j = 1, 2$, A_{ij} *is a closed analytic subset of a real analytic manifold* N_{ij}, $f_{ij} : N_{ij} \to U$ *is real analytic, and* $f_{ij} \mid A_{ij} : A_{ij} \to U$ *is proper.*

(3) *Every point of* M *has a neighbourhood* U *such that* $X \cap U$ *belongs to the class of subsets of* U *obtained using finite intersection, finite union and complement, from the family of closed subsets of* U *of the form* $f(A)$, *where* A *is a closed analytic subset of a real analytic manifold* N, $f : N \to U$ *is real analytic, and* $f \mid A$ *is proper.*

Proof. — (2) implies (1), by the theorem of the complement. (1) implies (3): Suppose that U is an open subset of M and A is a relatively compact semianalytic subset of $M \times \mathbf{R}^p$ such that $\pi(A) = X \cap U$, where $\pi : M \times \mathbf{R}^p \to M$ is the projection. Let $C = \overline{A} - A$. Then $\pi(A) = \pi(\overline{A}) - (\pi(\overline{A}) - \pi(A)) = \pi(\overline{A}) - (\pi(C) - (\pi(A) \cap \pi(C)))$. Since $\pi(C) - (\pi(A) \cap \pi(C))$ is subanalytic, by the theorem of the complement, and of dimension $< \dim \pi(A)$, the result follows by induction and Proposition 3.12.

(3) implies (2): By (3), every point of M has a neighbourhood U such that $X \cap U$ is a union of sets of the form $X' = \bigcap_{j=1}^{q}(g_{j1}(A_{j1}) - g_{j2}(A_{j2}))$, where each A_{jk} is a closed analytic subset of a real analytic manifold N_{jk}, and $g_{jk} : N_{jk} \to U$ is a real analytic mapping such that $g_{jk} \mid A_{jk}$ is proper. Let $A_1 \subset \prod_{j=1}^{q} N_{j1}$ be the fiber product of the A_{j1} over U, and let $g_1 = g_{11} \circ \pi_1 : \prod_{j=1}^{q} N_{j1} \to U$, where π_1 is the projection to N_{11}. Let A_2 be the disjoint union of the $A_{j1} \times_U A_{j2}$; A_2 is a closed analytic subset of the disjoint union N_2 of the $N_{j1} \times N_{j2}$. Let $g_2 : N_2 \to U$ be the mapping induced by the $g_{j1} \circ \pi_{j1}$, where $\pi_{j1} : N_{j1} \times N_{j2} \to N_{j1}$ is the projection. Then $g_k \mid A_k$ is proper, $k = 1, 2$, and $X' = g_1(A_1) - g_2(A_2)$, as required. □

A bound on the fibers of a subanalytic mapping [8, 13]:

Theorem **3.14.** — *Let* M *and* N *be real analytic manifolds, and let* X *be a relatively compact subset of* M. *Let* $\varphi : X \to N$ *be a subanalytic mapping. Then the number of connected components of a fiber* $\varphi^{-1}(y)$ *is bounded locally on* N.

Proof. — Let $\pi : N \times M \to N$ be the projection. It suffices to prove that if X is a relatively compact subanalytic subset of $N \times M$, then the number of connected components of a fiber $X_y = X \cap \pi^{-1}(y)$ is bounded, $y \in N$. Then we can assume that X is semianalytic and N, M are finite-dimensional vector spaces. We argue by induction on the maximum dimension k of the fibers X_y. Write X as a finite union of connected smooth semianalytic subsets A, as in Lemma 3.4.

First suppose that $k = 0$. For each A, we can write $N = U \oplus V$, where V is a linear complement of $\pi(T_x A)$, for all $x \in A$. Let $\pi_1 : N \to U$ be the projection. Then $(\pi_1 \circ \pi) \mid A$ is a local diffeomorphism, and the result follows from Lemma 3.7.

In general, it suffices to prove the result for each A. Let $k = \dim A - \operatorname{rk}(\pi \mid A)$. Then every component of each fiber A_y, $y \in \pi(A)$, is a submanifold of $\pi^{-1}(y)$ of dimension k. Let $Z = \{ x \in A : (d_x g) \mid (T_x A \cap M) = 0 \}$, where g is the function of Lemma 3.4 (3). We have already shown, in the proof of Lemma 3.6, that, for every $y \in \pi(A)$, Z intersects each component of A_y, and $\dim(Z \cap A_y) < k$. The result follows by induction. □

4. Transforming an analytic function to normal crossings by blowings-up

Let **N** denote the nonnegative integers. Let $\mathbf{K} = \mathbf{R}$ or **C**. For each positive integer m, let $P^{m-1}(\mathbf{K})$ denote the $((m-1)$-dimensional) projective space of lines through the origin in $\mathbf{K}^m$.

Definition and remarks **4.1.** — *Blowing-up*. Let V be an open neighbourhood of 0 in $\mathbf{K}^m$. Put

$$V' = \{(x, \ell) \in V \times P^{m-1}(\mathbf{K}) : x \in \ell \},$$

and let $\pi : V' \to V$ denote the mapping $\pi(x, \ell) = x$. Then π is proper, π restricts to a homeomorphism over $V - \{0\}$, and $\pi^{-1}(0) = P^{m-1}(\mathbf{K})$. The mapping $\pi : V' \to V$ is called the *blowing-up* of V with *center* $\{0\}$.

In a natural way, V' is an algebraic submanifold of $V \times P^{m-1}(\mathbf{K})$: Let $x = (x_1, \ldots, x_m)$ denote the affine coordinates of $\mathbf{K}^m$, and let $\xi = [\xi_1, \ldots, \xi_m]$ denote the homogeneous coordinates of $P^{m-1}(\mathbf{K})$. Then

$$V' = \{(x, \xi) \in V \times P^{m-1}(\mathbf{K}) : x_i \xi_j = x_j \xi_i, i, j = 1, \ldots, m\}.$$

We can cover V' by coordinate charts

$$V'_i = \{(x, \xi) \in V' : \xi_i \neq 0\}, \quad i = 1, \ldots, m,$$

with coordinates $(x_{i1}, \ldots, x_{im})$, for each i, where

$$\begin{aligned} x_{ii} &= x_i, \\ x_{ij} &= \xi_j / \xi_i, \quad j \neq i. \end{aligned}$$

With respect to these local coordinates, π is given by

$$\begin{aligned} x_i &= x_{ii}, \\ x_j &= x_{ii} x_{ij}, \quad j \neq i. \end{aligned}$$

Suppose that $n > m$ and that W is an open subset of $\mathbf{K}^{n-m}$. Then the mapping $\pi \times \mathrm{id} : V' \times W \to V \times W$ is called the *blowing-up* of $V \times W$ with *center* $\{0\} \times W$.

In the same way, if M is an analytic manifold (over $\mathbf{K}$) and Y is a closed analytic submanifold of M, we define the *blowing-up* $\pi : M' \to M$ with *center* Y: M' is an analytic manifold and π is a proper analytic mapping such that:

(1) π restricts to an analytic isomorphism $M' - \pi^{-1}(Y) \to M - Y$.

(2) Let $U \subset M$ be a chart with coordinates given by an analytic isomorphism $\varphi : U \to V \times W$, where V, W are open neighbourhoods of the origins in $\mathbf{K}^m$, $\mathbf{K}^{n-m}$ (respectively), and $\varphi(Y \cap U) = \{0\} \times W$. Let $\pi_0 : V' \to V$ be the blowing-up of V with center $\{0\}$. Then there is an analytic isomorphism $\varphi' : \pi^{-1}(U) \to V' \times W$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi'} & V' \times W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0 \times \mathrm{id}} \\ U & \xrightarrow{\varphi} & V \times W \end{array}$$

Conditions (1) and (2) above define $\pi : M' \to M$ uniquely, up to an isomorphism of M' commuting with π.

Definition **4.2.** — *Local blowing-up.* Let M be an analytic manifold (over $\mathbf{K}$). Let U be an open subset of M, and let Y be a closed analytic submanifold of U. Let $\pi : U' \to M$ denote the composition of the blowing-up $U' \to U$ with center Y, and the inclusion $U \to M$. We call π a *local blowing-up* of M (*over* U, with smooth *center* Y).

We will consider mappings $\pi : W \to M$ obtained as the composition of a finite sequence of local blowings-up; i.e., $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_k$, where, for each $i = 1, \ldots, k$, $\pi_i : U_{i+1} \to U_i$ is a local blowing-up of U_i, and $U_1 = M$, $U_{k+1} = W$.

Definition **4.3.** — Let M be an analytic manifold (over $\mathbf{K}$), and let $\mathcal{O}(M)$ denote the ring of analytic functions on M. Let $f \in \mathcal{O}(M)$. We say that f is *locally normal crossings* if each point of M admits a coordinate neighbourhood U, with coordinates $x = (x_1, \ldots, x_m)$, such that

$$f(x) = x_1^{\alpha_1} \ldots x_m^{\alpha_m} g(x), \quad x \in U,$$

where $g \in \mathcal{O}(U)$, g vanishes nowhere in U, and each $\alpha_i \in \mathbf{N}$.

Theorem **4.4.** — *Let* M *be an analytic manifold (over* $\mathbf{K}$*). Let* $f \in \mathcal{O}(M)$*. (Assume that* f *does not vanish identically on any component of* M.*) Then there is a countable collection of analytic mappings* $\pi_j : W_j \to M$ *such that*:

(1) *Each* π_j *is the composition of a finite sequence of local blowings-up (with smooth centers).*

(2) *There is a locally finite open covering* $\{ U_j \}$ *of* M *such that* $\pi_j(W_j) \subset U_j$, *for all* j.

(3) *If* K *is a compact subset of* M, *then there are compact subsets* $L_j \subset W_j$ *such that* $K = \bigcup_j \pi_j(L_j)$. *(The union is finite, by* (2).*)*

(4) *For each* j, $f \circ \pi_j$ *is locally normal crossings on* W_j.

Remark **4.5.** — We will call a countable collection of mappings $\{ \pi_j : W_j \to M \}$ satisfying (1)-(3) a Σ-*covering* of M. Σ-coverings can be " composed " in the following way: Let $\{ \pi_j : W_j \to M \}$ be a Σ-covering of M, and let $\{ U_j \}$ be as in (2). For each j, suppose that $\{ \pi_{jk} : W_{jk} \to W_j \}$ is a Σ-covering of W_j. If $\{ V_i \}$ is a locally finite covering of M by relatively compact open subsets, then the mappings $\pi_j \mid \pi_j^{-1}(V_i) : \pi_j^{-1}(V_i) \to M$, for all i and j, form a Σ-covering of M; hence we can assume that the U_j are relatively compact. Then, for each j, there is a finite subset $K(j)$ of $\{ k \}$ such that the mappings $\pi_j \circ \pi_{jk} : W_{jk} \to M$, for all j and all $k \in K(j)$, form a Σ-covering of M.

Let $a \in M$. Let $\mathcal{O}_{M,a}$ or $\mathcal{O}_a$ denote the local ring of germs of analytic functions on M at a, and let $\mathfrak{m}_a$ denote the maximal ideal of $\mathcal{O}_a$. Suppose that f is an analytic function on a neighbourhood U of a. Let f_a denote the germ of f at a.

Definition **4.6.** — Assume that f_a is not identically zero. Put

$$\mu_a(f) = \max\{ k \in \mathbf{N} : f_a \in \mathfrak{m}_a^k \},$$

$$\nu_a(f) = \min\{ \mu_a(g) : f_a = g . \prod_{i=1}^{r} \ell_i, \text{ where } g \in \mathcal{O}_a \text{ and } \ell_i \in \mathfrak{m}_a - \mathfrak{m}_a^2, i = 1, \ldots, r \}.$$

(Take $\nu_a(f) = 0$ if $f(a) \neq 0$.)

Clearly, $\nu_a(f) = 0$ if and only if either $f(a) \neq 0$ or f_a is a product of factors $\ell_i \in \mathfrak{m}_a - \mathfrak{m}_a^2$ (" smooth factors "). Both $\mu_a(f)$ and $\nu_a(f)$ are upper semicontinuous as functions of $a \in \mathrm{U}$. (It is easy to see that, in fact, they are upper semicontinuous in the analytic Zariski topology, but we will not use this result.)

Proof of Theorem 4.4. — Induction on $m = \dim \mathrm{M}$. If $m = 1$, then f is already locally normal crossings.

Let $a \in \mathrm{M}$. Suppose $f(a) = 0$. Put $d = \nu_a(f)$. Then, in some neighbourhood U of a, f factors as $f = \ell_1^{n_1} \dots \ell_r^{n_r} g$, where $\mu_a(g) = d$ and the ℓ_i are distinct factors such that $\mu_a(\ell_i) = 1$. Of course, $\mu_a(f) = d + \Sigma n_i$.

If $\mathrm{F}, \mathrm{G} \in \mathcal{O}(\mathrm{U})$ (or $\mathcal{O}_a$), we will say that F is *equivalent to* G (and write $\mathrm{F} \sim \mathrm{G}$) if F equals G times a factor which is invertible in $\mathcal{O}(\mathrm{U})$ (or $\mathcal{O}_a$).

There are local coordinates $x = (x_1, \dots, x_m)$ centered at a such that

$$f_a(0, \dots, 0, x_m) \sim x_m^e,$$

where $e = \mu_a(f)$. It follows that $g_a(0, \dots, 0, x_m) \sim x_m^d$ and each $\ell_{i,a}(0, \dots, 0, x_m) \sim x_m$. By the Weierstrass preparation theorem, we can assume that $\mathrm{U} = \mathrm{V} \times \mathrm{D}$, where V, D are open neighbourhoods of 0 in $\mathbf{K}^{m-1}$, $\mathbf{K}$ (respectively), $a = 0$, and

$$f(x) \sim \ell_1(x)^{n_1} \dots \ell_r(x)^{n_r} g(x),$$

$x = (x_1, \dots, x_m) \in \mathrm{U}$, where:

(1) $$\ell_i(x) = x_m + a_i(x_1, \dots, x_{m-1}), \quad i = 1, \dots, r,$$

$$g(x) = x_m^d + \sum_{j=1}^{d} c_j(x_1, \dots, x_{m-1})\, x_m^{d-j}.$$

(2) The a_i are distinct. For each $i = 1, \dots, r$, $a_i \in \mathcal{O}(\mathrm{V})$ and $a_i(0) = 0$. For each $j = 1, \dots, d$, $c_j \in \mathcal{O}(\mathrm{V})$ and $\mu_0(c_j) \geq j$.

(3) $\{ x \in \mathrm{U} : f(x) = 0 \} = \{ x \in \mathrm{V} \times \mathbf{K} : \ell_1(x)^{n_1} \dots \ell_r(x)^{n_r} g(x) = 0 \}$.

Clearly, we can assume that $\mathrm{M} = \mathrm{U} = \mathrm{V} \times \mathbf{K}$. Put $\widetilde{x} = (x_1, \dots, x_{m-1})$. If $d > 0$, then, after a coordinate transformation

$$x'_k = x_k, \quad k = 1, \dots, m-1,$$

$$x'_m = x_m + \frac{1}{d} c_1(\widetilde{x}),$$

we can further assume that $c_1(\widetilde{x}) \equiv 0$; i.e.,

$$g(x) = x_m^d + \sum_{j=2}^{d} c_j(\widetilde{x})\, x_m^{d-j}.$$

The significance of this representation is that, since $\partial^{d-1} g / \partial x_m^{d-1} = d!\, x_m$, then $\mu_x(g) = d$, $x = (x_1, \dots, x_m)$, only if $x_m = 0$.

If $d = 0$, then after a coordinate transformation

$$x'_k = x_k, \quad k = 1, \dots, m-1,$$

$$x'_m = x_m + a_1(\widetilde{x}),$$

we can assume that $a_1(\widetilde{x}) \equiv 0$.

Let $A_f(\widetilde{x})$ denote the product of all nonzero functions from the following list and all of their nonzero differences:

$$a_i^{d!}, \quad i = 1, \ldots, r,$$
$$c_j^{d!/j}, \quad j = 2, \ldots, d.$$

By induction, there is a Σ-covering $\{ \widetilde{\pi}_k : V_k \to V \}$ such that each $A_f \circ \widetilde{\pi}_k$ is locally normal crossings in V_k. Then $\{ \widetilde{\pi}_k \times \mathrm{id} : V_k \times \mathbf{K} \to V \times \mathbf{K} \}$ is a Σ-covering of U. Therefore, by Remark 4.5, we can assume that $A_f(\widetilde{x})$ is locally normal crossings in V. Shrinking V if necessary, we can assume that $A_f(\widetilde{x})$ is equivalent to a monomial $\widetilde{x}^\theta = x_1^{\theta_1} \ldots x_{m-1}^{\theta_{m-1}}$. Then each nonzero $a_i(\widetilde{x})^{d!} \sim \widetilde{x}^{\alpha^i}$ and each nonzero $c_j(\widetilde{x})^{d!/j} \sim \widetilde{x}^{\gamma^j}$, where $\alpha^i = (\alpha_1^i, \ldots, \alpha_{m-1}^i) \in \mathbf{N}^{m-1}$ and $\gamma^j = (\gamma_1^j, \ldots, \gamma_{m-1}^j) \in \mathbf{N}^{m-1}$. Moreover, by Lemma 4.7 below, *these exponents* α^i, γ^j *are totally ordered with respect to the induced partial ordering from* $\mathbf{N}^{m-1}$ ($\alpha \leqslant \beta$ means $\alpha_k \leqslant \beta_k$, $k = 1, \ldots, m-1$, where $\alpha = (\alpha_1, \ldots, \alpha_{m-1})$ and $\beta = (\beta_1, \ldots, \beta_{m-1})$).

Lemma **4.7**. — *Let* $y = (y_1, \ldots, y_p)$. *Let* $\alpha, \beta, \gamma \in \mathbf{N}^p$ *and let* $a(y)$, $b(y)$, $c(y)$ *be invertible elements of* $\mathbf{K}\{y\}$. *If*

$$a(y)\,y^\alpha - b(y)\,y^\beta = c(y)\,y^\gamma,$$

then either $\alpha \leqslant \beta$ *or* $\beta \leqslant \alpha$.

Proof. — Put $\delta_k = \min(\alpha_k, \beta_k)$, $k = 1, \ldots, p$, where $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_p)$. Let $\delta = (\delta_1, \ldots, \delta_p)$. If $\delta = \alpha$, then $\alpha \leqslant \beta$. Otherwise, choose k such that $\delta_k \neq \alpha_k$. Then, on $\{ y : y_k = 0 \}$, we have $y^{\alpha - \delta} = 0$ and $0 \neq -b(y)\,y^{\beta - \delta} = c(y)\,y^{\gamma - \delta}$. Since b and c are invertible, if follows that $\beta = \gamma$. Then $a(y)\,y^\alpha = (b(y) + c(y))\,y^\beta$, so that $\beta \leqslant \alpha$. □

In view of Remark 4.5, the proof of Theorem 4.4 will be complete once we prove the following two assertions:

Case 1. $d > 0$. There is a (finitely indexed) Σ-covering $\{ \pi_t : W_t \to U \}$ such that, for each t, $\nu_y(f \circ \pi_t) < d$, for all $y \in W_t$.

Case 2. $d = 0$. There is a (finitely indexed) Σ-covering $\{ \pi_t : W_t \to U \}$ such that, for each t, $f \circ \pi_t$ is locally normal crossings on W_t.

Case 1. We will use an inductive argument. To set up the induction, it is convenient to begin with f of the following more general form:

$$f(x) \sim \widetilde{x}^\alpha \ell_1(x)^{n_1} \ldots \ell_r(x)^{n_r} \ldots \ell_s(x)^{n_s} g(x),$$

where $\alpha \in \mathbf{N}^{m-1}$, $r \leqslant s$, the ℓ_i are distinct smooth factors, g and $\ell_1, \ldots, \ell_r$ are as before, and $\ell_{r+1}, \ldots, \ell_s$ vanish nowhere on $\{ x_m = 0 \}$. (At this first stage we really have $\alpha = 0$ and $s = r$.) Of course, $\nu_x(f) = d$ only if $\mu_x(g) = d$. The exponents α^i and γ^j of the nonzero $a_i^{d!}$ $(i = 1, \ldots, r)$ and $c_j^{d!/j}$ are totally ordered.

Let σ denote the smallest among these exponents; say $\sigma = (\sigma_1, \ldots, \sigma_{m-1})$. Then $|\sigma| = \sum_{k=1}^{m-1} \sigma_k \geq d!$. Put

$$Z = \{ x \in U : \mu_x(\ell_1^{n_1} \ldots \ell_r^{n_r} g) = d + \sum_{i=1}^{r} n_i \}.$$

Clearly,

$$\begin{aligned} Z &= \{ x \in U : \mu_x(g) = d \text{ and } \ell_i(x) = 0,\ i = 1, \ldots, r \} \\ &= \{ x \in U : x_m = 0 \text{ and } \sum_{k \in J(x)} \sigma_k \geq d! \}, \end{aligned}$$

where $J(x) = \{ k : x_k = 0, k = 1, \ldots, m-1 \}$. Let S denote the collection of subsets I of $\{ 1, \ldots, m-1 \}$ such that $0 \leq \sum_{k \in I} \sigma_k - d! < \sigma_l$, for all $\ell \in I$; i.e., the minimal subsets I of $\{ 1, \ldots, m-1 \}$ such that $\sum_{k \in I} \sigma_k - d! \geq 0$. For each $I \in S$, put

$$Z_I = \{ x \in U : x_m = 0 \text{ and } x_k = 0,\ k \in I \}.$$

The Z_I, $I \in S$, are the irreducible components of Z.

Let $I \in S$. Let $\pi : U' \to U$ be the blowing-up with center Z_I. Then U' identifies with

$$\{(x, \xi) \in U \times P^{m-1}(\mathbf{K}) : \xi_k = 0,\ k \notin I \cup \{m\}, \text{ and } x_k \xi_l = x_l \xi_k,\ k, l \in I \cup \{m\}\}$$

(cf. Definition 4.1). As in 4.1, U' is covered by coordinate charts

$$U'_k = \{(x, \xi) \in U' : \xi_k \neq 0 \},\quad k \in I \cup \{ m \},$$

where U'_k has coordinates $y = (y_1, \ldots, y_m)$ such that

$$\begin{aligned} x_l &= y_l, \quad \ell \notin I \cup \{ m \}, \\ x_k &= y_k, \\ x_l &= y_k y_l, \quad l \in (I \cup \{ m \}) - \{ k \}. \end{aligned}$$

Since, for every $x \in Z_I$, $a_i(x) = 0$, $i = 1, \ldots, r$, and $\mu_x(c_j) \geq j$, $j = 2, \ldots, d$, it follows that $\nu_y(f \circ \pi) = 0$ at each point y of $U' - \bigcup_{k \in I} U'_k$. Therefore, it suffices to consider $f \circ \pi_k$ for each $k \in I$, where $\pi_k = \pi \mid U'_k$.

Fix $k \in I$. If $y = (y_1, \ldots, y_m) \in U'_k$, put $\tilde{y} = (y_1, \ldots, y_{m-1})$ and $\tilde{\pi}_k(\tilde{y}) = \pi_k(y)^{\sim}$. Clearly, $U'_k = V'_k \times \mathbf{K}$, where $V'_k = \{ y \in U'_k : y_m = 0 \}$, and $\tilde{\pi}_k : V'_k \to V$. Then

$$\begin{aligned} (\ell_i \circ \pi_k)(y) &= y_k(y_m + a'_i(\tilde{y})), \quad i = 1, \ldots, r, \\ (g \circ \pi_k)(y) &= y_k^d g'(y), \end{aligned}$$

where

$$\begin{aligned} a'_i(\tilde{y}) &= \frac{1}{y_k} (a_i \circ \tilde{\pi}_k)(\tilde{y}) \in \mathcal{O}(V'_k), \quad i = 1, \ldots, r, \\ g'(y) &= y_m^d + \sum_{j=2}^{d} c'_j(\tilde{y}) y_m^{d-j}, \\ c'_j(\tilde{y}) &= \frac{1}{y_k^j} (c_j \circ \tilde{\pi}_k)(\tilde{y}) \in \mathcal{O}(V'_k), \quad j = 2, \ldots, d. \end{aligned}$$

It follows that each nonzero $a_i'(\tilde{y})^{d!} \sim \tilde{y}^{\beta^i}$ and each nonzero $c_j'(\tilde{y})^{d!/j} \sim \tilde{y}^{\delta^j}$, where $\beta^i = (\beta_1^i, \dots, \beta_{m-1}^i) \in \mathbf{N}^{m-1}$, $\delta^j = (\delta_1^j, \dots, \delta_{m-1}^j) \in \mathbf{N}^{m-1}$, and

$$\begin{aligned} \beta_\ell^i &= \alpha_\ell^i, \quad \ell \neq k, \\ \beta_k^i &= \sum_{\ell \in I} \alpha_\ell^i - d!, \\ \delta_\ell^j &= \gamma_\ell^j, \quad \ell \neq k, \\ \delta_k^j &= \sum_{\ell \in I} \gamma_\ell^j - d!. \end{aligned}$$

In particular, the exponents β^i and δ^j are totally ordered in the same way as the α^i and γ^j.

If $\nu_y(f \circ \pi_k) < d$, for all $y \in U_k'$, we are done. Suppose $\nu_y(f \circ \pi_k) = d$, for some $y = (y_1, \dots, y_m) \in U_k'$. It follows that $\mu_y(g') = d$, and hence that $y_m = 0$. Therefore, since each $(c_j')^{d!/j} \sim \tilde{y}^{\delta^j}$, we have $\mu_0(g') = d$. Likewise, for each $i = 1, \dots, r$, if $a_i'(\tilde{y}) = 0$, $\tilde{y} \in V_k'$, then $a_i'(0) = 0$.

Let $\tau = (\tau_1, \dots, \tau_{m-1})$ denote the smallest among the nonzero exponents β^i and δ^j; then τ is associated to $f \circ \pi_k$ in the same way as σ is associated to f. Let q denote the number of indices $i = 1, \dots, r$ such that $a_i'(0) = 0$. If $q = r$, then

$$\begin{aligned} \tau_\ell &= \sigma_\ell, \quad \ell \neq k, \\ \tau_k &= \sum_{\ell \in I} \sigma_\ell - d!; \end{aligned}$$

in particular, $|\tau| < |\sigma|$ (while, as before, $|\tau| \geq d!$). In other words, either $q < r$ or $q = r$ and $|\tau| < |\sigma|$. It follows that, after transforming f by a Σ-covering involving finitely many sequences of at most (the integral part of) $|\sigma|/d!$ local blowings-up over successive coordinate charts, as above, either r or d must decrease. Case 1 follows by induction.

Case 2. To set up an appropriate induction, it is again convenient to begin with f of a more general form:

$$f(x) \sim \tilde{x}^\alpha \ell_1(x)^{n_1} \dots \ell_r(x)^{n_r},$$

where $\alpha \in \mathbf{N}^{m-1}$ and the ℓ_i are distinct smooth factors $\ell_i(x) = x_m + a_i(\tilde{x})$, $a_i(0) = 0$, such that $a_1(\tilde{x}) \equiv 0$ and $A_f(\tilde{x}) \sim \tilde{x}^\theta$, $\theta \in \mathbf{N}^{m-1}$, where A_f is the product of all nonzero a_i and their differences. (At this first stage we really have $\alpha = 0$.) In particular, the exponents α^i of the nonzero $a_i(\tilde{x}) \sim \tilde{x}^{\alpha^i}$ are totally ordered.

Let σ denote the smallest among these exponents; say $\sigma = (\sigma_1, \dots, \sigma_{m-1})$. Then $|\sigma| = \sum_{k=1}^{m-1} \sigma_k \geq 1$. Let

$$\begin{aligned} Z &= \{ x \in U : \ell_i(x) = 0, i = 1, \dots, r \} \\ &= \{ x \in U : x_m = 0 \text{ and } \sum_{k \in J(x)} \sigma_k \geq 1 \}, \end{aligned}$$

where $J(x) = \{ k : x_k = 0, k = 1, \dots, m-1 \}$. For each $k = 1, \dots, m-1$, let $Z_k = \{ x \in U : x_k = x_m = 0 \}$. The Z_k where $\sigma_k \geq 1$ are the irreducible components of Z.

Let $\pi : U' \to U$ be the blowing-up with center Z_k, for some $k = 1, \ldots, m-1$ such that $\sigma_k \geqslant 1$. Then

$$U' = \{(x, \xi) \in U \times P^{m-1}(\mathbf{K}) : \xi_\ell = 0, \ell \neq k, m, \text{ and } x_k \xi_m = x_m \xi_k\},$$

and $U' = U'_k \cup U'_m$, where, for $\ell = k, m$, U'_ℓ is the coordinate chart $\{(x, \xi) \in U' : \xi_\ell \neq 0\}$. Let $\pi_\ell = \pi \mid U'_\ell$.

The chart U'_m has coordinates $y = (y_1, \ldots, y_m)$ in which π_m is given by $x_k = y_k y_m$, $x_m = y_m$ and $x_\ell = y_\ell$ when $\ell \neq k, m$. Let

$$X'_m = \bigcup_{i=1}^{r} \{y \in U'_m : 1 + a_i(\pi_m(y)^{\sim})/y_m = 0\}.$$

Then X'_m is a closed analytic subset of U'_m. Clearly, $X'_m \cap (U'_m - U'_k) = \emptyset$ and $f \circ \pi_m$ is locally normal crossings on $U'_m - X'_m$.

The chart U'_k has coordinates $y = (y_1, \ldots, y_m)$ in which π_k is given by $x_k = y_k$, $x_m = y_k y_m$ and $x_\ell = y_\ell, \ell \neq k, m$. Let $\widetilde{y} = (y_1, \ldots, y_{m-1})$ and $\widetilde{\pi}_k(\widetilde{y}) = \pi_k(y)^{\sim}$. Clearly, $U'_k = V'_k \times \mathbf{K}$, where $V'_k = \{y \in U'_k : y_m = 0\}$, and $\widetilde{\pi}_k : V'_k \to V$. Let $f' = f \circ \pi_k$. Then

$$f'(y) \sim \widetilde{y}^{\beta} \ell'_1(y)^{n_1} \ldots \ell'_r(y)^{n_r},$$

where $\beta = (\beta_1, \ldots, \beta_{m-1}) \in \mathbf{N}^{m-1}$, with $\beta_\ell = \alpha_\ell, \ell \neq k$, and $\beta_k = \alpha_k + n_1 + \ldots + n_r$, and where each

$$\ell'_i(y) = y_m + a'_i(y),$$
$$a'_i(y) = \frac{1}{y_k} (a_i \circ \widetilde{\pi}_k)(\widetilde{y}) \in \mathcal{O}(V'_k).$$

Therefore, each nonzero $a'_i(\widetilde{y}) \sim \widetilde{y}^{\beta^i}$, where $\beta^i = (\beta^i_1, \ldots, \beta^i_{m-1}) \in \mathbf{N}^{m-1}$, $\beta^i_k = \alpha^i_k - 1$, and $\beta^i_\ell = \alpha^i_\ell$, $\ell \neq k$.

Suppose that $a'_i(0) = 0, i = 1, \ldots, r$. Then $A_{f'}(\widetilde{y}) \sim \widetilde{y}^{\varphi}$, where $\varphi \in \mathbf{N}^{m-1}$, and the β^i are totally ordered in the same way as the α^i. Let $\tau = \min \beta^i$; say $\tau = (\tau_1, \ldots, \tau_{m-1})$. Then $\tau_k = \sigma_k - 1$ and $\tau_\ell = \sigma_\ell$, $\ell \neq k$, so that $1 \leqslant |\tau| = |\sigma| - 1$. Therefore, after repeating the process of blowing up $|\sigma|$ times, we can assume that $a'_{i_0} \neq 0$, for some $i_0 = 2, \ldots, r$.

Let b^p_m, $p = 1, \ldots, s$, denote the distinct values $-a'_i(0)$, $i = 1, \ldots, r$. Then $2 \leqslant s \leqslant r$, since $a'_1 \equiv 0$. For each p, let $I(p) = \{i : b^p_m = -a'_i(0), i = 1, \ldots, r\}$. Choose $i(p) \in I(p)$. Put $U^p = U'_k$, with coordinates $z = (z_1, \ldots, z_m)$ centered at $b^p = (0, \ldots, 0, b^p_m)$ defined by

$$z_\ell = y_\ell, \quad \ell = 1, \ldots, m-1,$$
$$z_m = y_m + a'_{i(p)}(y_1, \ldots, y_{m-1}).$$

Then, for each $i = 1, \ldots, r$, $\ell'_i(y) = \ell''_i(z)$, where $\ell''_i(z) = z_m + a''_i(\widetilde{z})$, with $a''_i(\widetilde{z}) = a'_i(\widetilde{z}) - a'_{i(p)}(\widetilde{z})$. Put $X^p = \{z \in U^p : \ell''_i(z) = 0$, for some $i \notin I(p)\}$. Since each

$a'_i(\widetilde{z}) - a'_j(\widetilde{z}) \sim \widetilde{z}^{\gamma^{ij}}$, for some $\gamma^{ij} \in \mathbf{N}^{m-1}$, it follows that $X^p \cap \{ z : \ell''_i(z) = 0 \} = \emptyset$, for all $i \in I(p)$. In $U^p - X^p$, f' coincides with an analytic function

$$f^p(z) \sim \widetilde{z}^{\gamma} \prod_{i \in I(p)} \ell''_i(z)^{n_i}, \quad z \in U^p,$$

where $\gamma \in \mathbf{N}^{m-1}$, $a''_{i(p)} \equiv 0$, and $A_{f''}(\widetilde{z}) \sim \widetilde{z}^{\psi}$, for some $\psi \in \mathbf{N}^{m-1}$. But $I(p)$ has fewer than r elements.

Since the $U^p - X^p$, $p = 1, \ldots, s$, together with $U'_m - X'_m$ cover U', Case 2 follows by induction on r. □

Remark **4.8.** — Our proof of Theorem 4.4 shows that there is a countable collection of analytic mappings $\pi_j : W_j \to M$ satisfying conditions (1)-(4) of the theorem and having the following additional property: Write each π_j as $\pi_{j1} \circ \pi_{j2} \circ \ldots \circ \pi_{j,k(j)}$, where, for each $k = 1, \ldots, k(j)$, $\pi_{jk} : U_{j,k+1} \to U_{jk}$ is a local blowing up of U_{jk} with smooth center Y_{jk}, and $U_{j1} = M$, $U_{j,k(j)+1} = W_j$. Let E_{jk} denote the union of the inverse images in U_{jk} of $Y_{j1}, \ldots, Y_{j,k-1}$, $k = 2, \ldots, k(j) + 1$. Then each E_{jk} is a union of smooth hypersurfaces in U_{jk}; when $k = k(j) + 1$, these hypersurfaces are transverse.

Corollary **4.9.** — *Let* M *be a real analytic manifold. Let* $f \in \mathcal{O}(M)$. *(Assume that* f *is not identically zero on any component of* M.*) Then there is a real analytic manifold* N *and a proper surjective real analytic mapping* $\pi : N \to M$ *such that:*

(1) $f \circ \pi$ *is locally normal crossings on* N.

(2) *There is an open dense subset of* N *on which* π *is locally an isomorphism.*

Proof. — Let $\{ \pi_j : W_j \to M \}$ and $\{ U_j \}$ be as in Theorem 4.4. Suppose $a \in W_j$. Choose a coordinate neighbourhood $V_{j,a}$ of a in W_j, with coordinates $x = (x_1, \ldots, x_m)$ vanishing at a, such that $(f \circ \pi_j)(x) = x_1^{\alpha_1} \ldots x_m^{\alpha_m} g(x)$, $x \in V_{j,a}$, where $g(x)$ is an analytic function vanishing nowhere on $V_{j,a}$, and each $\alpha_i \in \mathbf{N}$. Let S^m_ε denote the sphere $\{(x_1, \ldots, x_m, t) : x_1^2 + \ldots + x_m^2 + t^2 = \varepsilon^2\}$. For sufficiently small ε, there is a mapping $\varphi_{j,a} : S_{j,a} \to V_{j,a}$, where $S_{j,a} = S^m_\varepsilon$, defined by $\varphi_{j,a}(x, t) = x$, $(x, t) \in S_{j,a}$. Clearly, $f \circ \pi_j \circ \varphi_{j,a}$ is locally normal crossings on $S_{j,a}$.

Let $\{ K_i \}$ be a locally finite covering of M by compact subsets. For each i, there is a finite subset $J(i)$ of $\{ j \}$ such that $K_i = \bigcup_{j \in J(i)} \pi_j(L_{ij})$, where each $L_{ij} \subset W_j$ is compact. For each L_{ij}, choose a finite subset $A(i,j)$ of W_j such that $L_{ij} \subset \bigcup_{a \in A(i,j)} \varphi_{j,a}(S_{j,a})$. We can take N to be the disjoint union of the $S_{j,a}$, where $a \in A(i,j)$, $j \in J(i)$, for all i, and take $\pi : N \to M$ to be the mapping given by $\pi_j \circ \varphi_{j,a}$ on each sphere $S_{j,a}$ in this union. □

Remark **4.10.** — We can require that the mapping $\pi : N \to M$ of Corollary 4.9 satisfy the following additional condition:

(3) Every point of M admits an open neighbourhood U such that $\pi \mid \pi^{-1}(U)$ is *relatively algebraic*; i.e., there is a positive integer q and a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\iota} & U \times P^q(\mathbf{R}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\text{projection}} & \\ & U & \end{array}$$

where ι is a closed embedding and the image of ι is defined by homogeneous polynomial equations (in terms of the standard homogeneous coordinates in $P^q(\mathbf{R})$), whose coefficients are real analytic functions on U.

A blowing-up has this property, by Definition 4.1. Corollary 4.9 with the additional assertion (3) can be proved by induction on the lengths of the sequences of local blowings-up involved in the mappings π_j of Theorem 4.4 (cf. [18, Lemmas 7.2.1 and 7.2.2]).

5. Uniformization and rectilinearization

Throughout this section, M denotes a real analytic manifold.

Theorem **5.1** (Uniformization theorem). — *Let* X *be a closed analytic subset of* M. *Then there is a real analytic manifold* N *(of the same dimension as* X*) and a proper real analytic mapping* $\varphi : N \to M$ *such that* $\varphi(N) = X$.

Proof. — Let $a \in M$. Let X_a denote the germ of X at a. Let $f_1, \ldots, f_n$ be real analytic functions defined in a neighbourhood U of a, such that

$$X \cap U = \{ x \in U : f_i(x) = 0,\ i = 1, \ldots, n \}.$$

Let $r = \dim X_a$. We can assume there is a closed analytic subset Z of U such that $\dim Z < r$ and $X \cap U - Z$ is smooth and of pure dimension r. It suffices to find a compact real analytic manifold N such that $\dim N = r$, and a real analytic mapping $\varphi : N \to M$ such that $\varphi(N) \subset X \cap U$ and $\varphi(N)$ includes a neighbourhood of a in $X \cap U - Z$. We will prove this by induction on $\operatorname{codim} X_a = m - r$, where $m = \dim M_a$. If $\operatorname{codim} X_a = 1$, then the result holds, by Theorem 4.4.

Let $f = f_1 \ldots f_n$. By Theorem 4.4, there are finitely real analytic mappings $\pi_j : W_j \to U$ such that:

(1) Each π_j is the composition of a finite sequence of local blowings-up with smooth centers.

(2) There is a compact subset L_j of W_j, for each j, such that $\bigcup_j \pi_j(L_j)$ is a neighbourhood of a in U.

(3) For each j, $f \circ \pi_j$ is locally normal crossings on W_j.

For each j, write π_j as $\pi_{j1} \circ \pi_{j2} \circ \ldots \circ \pi_{j,k(j)}$, where, for each $k = 1, \ldots, k(j)$, $\pi_{jk} : U_{j,k+1} \to U_{jk}$ is a local blowing-up of U_{jk} over an open subset V_{jk} of U_{jk}, with center a closed analytic submanifold Y_{jk} of V_{jk}, and where $U_{j1} = U$, $U_{j,k(j)+1} = W_j$. For each $k = 2, \ldots, k(j)+1$, let E_{jk} denote the union of the inverse images in U_{jk} of $Y_{j1}, \ldots, Y_{j,k-1}$. We can assume, in addition to (1)-(3) above, that each E_{jk} is a union of smooth hypersurfaces in U_{jk} and, when $k = k(j)+1$, these hypersurfaces are transverse (Remark 4.8). Choosing U small enough, we can assume that $V_{j1} = U_{j1} = U$, for each j.

For each j, put $X_{j1} = X \cap U$ and define

$$X_{j,k+1} = \pi_{jk}^{-1}(X_{jk} \cup Y_{jk}), \quad k = 1, \ldots, k(j).$$

We can assume that, for each j and k, there exists $a_{jk} \in U_{jk}$ such that V_{jk} is an open neighbourhood of a_{jk}, small enough so that:

(1) $X_{jk} \cap V_{jk}$ is a finite union of closed analytic subsets $X_{jk\ell}$ of V_{jk}, where the $X_{jk\ell, b}$ are the irreducible components of $X_{jk,b}$, $b = a_{jk}$.

(2) For each ℓ, every connected component of the smooth points of $X_{jk\ell}$ is adherent to a_{jk}.

For each j and k, let $L(j,k)$ denote the set of those ℓ such that $X_{jk\ell,b}$ is not an irreducible component of $E_{jk,b}$, where $b = a_{jk}$. If $\ell \in L(j,k)$, then $\dim X_{jk\ell,b} \leqslant r$. Suppose that $X_{jk\ell,b} \subset Y_{jk,b}$, where $\ell \in L(j,k)$ and $\dim X_{jk\ell,b} = r$. Then $X_{jk\ell} \subset Y_{jk}$. Since $\dim Y_{jk} < m$, the codimension of $X_{jk\ell}$ in Y_{jk} is less than that of $X \cap U = X_{j1}$ in $U = U_{j1}$. By induction, there is a real analytic manifold $N'_{jk\ell}$ of dimension r, and a proper real analytic mapping $\varphi'_{jk\ell} : N'_{jk\ell} \to Y_{jk}$ such that $\varphi'_{jk\ell}(N'_{jk\ell}) \subset X_{jk\ell}$ and $\varphi'_{jk\ell}(N'_{jk\ell})$ includes the smooth points of dimension r of $X_{jk\ell}$. Therefore, there is a compact real analytic manifold $N_{jk\ell}$ of dimension r, and a real analytic mapping $\varphi_{jk\ell} : N_{jk\ell} \to Y_{jk} \subset U_{jk}$ such that $\varphi_{jk\ell}(N_{jk\ell}) \subset X_{jk\ell}$ and $\varphi_{jk\ell}(N_{jk\ell})$ includes the smooth points of dimension r of $X_{jk\ell}$ within some neighbourhood of the image of L_j in U_{jk}.

Now, for each j, $\prod_{i=1}^{n}(f_i \circ \pi_j)(x)$ is locally normal crossings in W_j; therefore, we can find finitely many points a_{jp} of L_j such that:

(1) For each p, there is a coordinate neighbourhood W_{jp} of a_{jp}, with coordinates $x = (x_1, \ldots, x_m)$ centered at a_{jp}, in which $\prod_{i=1}^{n} f_i(\pi_j(x)) = x_1^{\alpha_1} \ldots x_m^{\alpha_m} u(x)$, where each α_i is a nonnegative integer and $u(x)$ is an analytic function vanishing nowhere in W_{jp}.

(2) There is a positive number ε_{jp}, for each p, such that the balls $B_{jp} = \{ x \in W_{jp} : x_1^2 + \ldots + x_m^2 \leqslant \varepsilon_{jp}^2 \}$ cover a neighbourhood of L_j in W_j.

It follows from (1) that each $(f_i \circ \pi_j)(x)$, $x \in W_{jp}$, is a monomial in $(x_1, \ldots, x_m)$ times an invertible factor. For each p, let

$$X_{jp} = \{ x \in W_{jp} : f_i(\pi_j(x)) = 0,\ i = 1, \ldots, n \}.$$

Then X_{jp} is a union of coordinate subspaces of W_{jp}. Write $X_{jp} = X'_{jp} \cup E'_{jp}$, where E'_{jp} is the union of the irreducible components of X_{jp} lying in $E_{j,k(j)+1}$ and X'_{jp} is the union of the remaining irreducible components (each of which must have dimension $\leqslant r$).

Let X_{jpq} denote the irreducible components of X'_{jp} of dimension r; each is a coordinate subspace of W_{jp} of dimension r. For each p and q, let S_{jpq} denote the standard r-dimensional sphere of radius ε_{jp}, and let $\psi_{jpq} : S_{jpq} \to W_j$ denote the standard mapping onto the ball $B_{jp} \cap X_{jpq}$.

We can take N to be the disjoint union of all $N_{jk\ell}$ and S_{jpq}, and take $\varphi : N \to M$ to be the mapping defined by $\pi_{j1} \circ \pi_{j2} \circ \ldots \circ \pi_{j,k-1} \circ \varphi_{jk\ell}$ on each $N_{jk\ell}$ and by $\pi_j \circ \psi_{jpq}$ on each S_{jpq}. □

The uniformization theorem 0.1 for subanalytic sets is an immediate consequence of Theorem 5.1 and Proposition 3.12.

Remark **5.2.** — In Theorem 5.1, we can require that each point of M admit an open neighbourhood U such that $\varphi \mid \varphi^{-1}(U)$ is relatively algebraic. This follows from our proof, because of Remark 4.10. It then follows from our proof of Proposition 3.12 that, if X is a closed semianalytic subset of M, there exists a real analytic manifold N (of the same dimension as X) and a (proper) real analytic mapping $\varphi : N \to M$ such that $\varphi(N) = X$ and each point of M admits an open neighbourhood U such that $\varphi \mid \varphi^{-1}(U)$ is relatively algebraic. Conversely, if $\varphi : N \to M$ is a real analytic mapping satisfying the latter condition, then $\varphi(N)$ is semianalytic, by Theorem 2.2.

Lemma **5.3.** — *Let* $f_1, \ldots, f_q$ *be continuous subanalytic functions on* M. *Then there exist a real analytic manifold* N, *of the same dimension as* M, *and a proper surjective real analytic mapping* $\varphi : N \to M$ *such that each* $f_i \circ \varphi$ *is analytic on* N.

Proof. — Define $f : M \to \mathbf{R}^q$ by $f = (f_1, \ldots, f_q)$. Then f is a subanalytic mapping. By Theorem 0.1, there is a real analytic manifold N such that $\dim N = \dim \operatorname{graph} f = \dim M$, and a proper real analytic mapping $\Phi : N \to M \times \mathbf{R}^q$ such that $\Phi(N) = \operatorname{graph} f$. Write $\Phi = (\varphi, g)$, where $\varphi : N \to M$ and $g : N \to \mathbf{R}^q$; say $g = (g_1, \ldots, g_q)$. Then φ is a proper surjective real analytic mapping and each $f_i \circ \varphi = g$ is analytic. □

Definition **5.4.** — A subset Q of $\mathbf{R}^m$ is a *quadrant* if there is a partition of $\{1, \ldots, m\}$ into disjoint subsets I_0, I_+ and I_-, such that $Q = \{ x = (x_1, \ldots, x_m) \in \mathbf{R}^m : x_i = 0$ if $i \in I_0$, $x_j > 0$ if $j \in I_+$ and $x_k < 0$ if $k \in I_- \}$.

Proof of the rectilinearization theorem 0.2. — We can assume that $M = \mathbf{R}^m$. We can find a neighbourhood U of K, and closed subanalytic subsets X_{ij} of U, $i = 1, \ldots, p$, $j = 1, 2$, such that

$$X \cap U = \bigcup_{i=1}^{p} (X_{i1} - X_{i2}).$$

For each i and j, let d_{ij} denote the distance function $d_{ij}(x) = d(x, X_{ij})$, $x \in U$. Then $X_{ij} = \{ x \in U : d_{ij}(x) = 0 \}$ and d_{ij} is subanalytic, by Remark 3.11 (1). By Lemma 5.3, there is a real analytic manifold V such that $\dim V = m$, and a proper surjective real analytic mapping $\varphi : V \to U$ such that each $d_{ij} \circ \varphi$ is analytic on V. Then, by Corollary 4.9, there is a real analytic manifold N of dimension m, and a proper surjective real analytic mapping $\pi : N \to V$ such that $\prod_{i,j} d_{ij} \circ \varphi \circ \pi$ is locally normal crossings on N. Theorem 0.2 follows easily. □

6. Łojasiewicz's inequality; metric properties of subanalytic sets

To prove Łojasiewicz's inequality, we will use the following result of [21] on subanalytic sets in low dimensions:

Theorem **6.1.** — *Let* M *be a real analytic manifold and let* X *be a subanalytic subset of* M. *Then:*

(1) *If* $\dim X \leqslant 1$, X *is semianalytic.*

(2) *If* $\dim M \leqslant 2$, X *is semianalytic.*

Lemma **6.2.** — *Let* $k \in \mathbf{N}$ *and let* $y(z)$ *be a holomorphic function defined in a neighbourhood of the origin in* **C**. *Let* X *denote the image of* $z \mapsto (z^k, y(z))$. *Then, in some neighbourhood of* $0 \in \mathbf{C}^2$, X *is the zero set of a holomorphic function*

$$f(x,y) = \prod_{\varepsilon^k = 1} (y - y(\varepsilon x^{1/k})),$$

where the product is over the k'th roots of unity.

Proof. — In some neighbourhood of 0 in $\mathbf{C}^2$, f is a well-defined holomorphic function outside $\{x = 0\}$, which extends continuously to $\{x = 0\}$. Therefore, f is holomorphic in a neighbourhood of 0. □

Lemma **6.3.** — *Let* M *be a real analytic manifold.*

(1) *Let* $A \subset M$ *be a semianalytic subset of dimension* 1. *Let* $a \in \bar{A}$. *Assume* $A - \{a\}$ *is locally connected at a. Then there exist* $\varepsilon > 0$ *and a real analytic mapping* $\gamma : (-\varepsilon, \varepsilon) \to M$ *such that* $\gamma(0) = a$ *and* $\gamma((0, \varepsilon))$ *is a neighbourhood of a in* $A - \{a\}$.

(2) *Conversely, let* $\gamma : I \to M$ *be a real analytic mapping, where* I *is an interval containing* 0 *in* **R**. *If* $\gamma \not\equiv 0$, *then there exists* $\varepsilon > 0$ *such that* $\gamma((0, \varepsilon))$ *is a (smooth) semianalytic subset of* M.

Proof. — (1) is immediate from Theorem 0.1. (Alternatively, it can be proved by induction on dim M, using Puiseux's theorem.)

(2) We can assume that $M = \mathbf{R}^n$ and $\gamma(0) = 0$. Write $\gamma(s) = (\gamma_1(s), \ldots, \gamma_n(s))$. If $\gamma \not\equiv 0$, we can assume $\gamma_1(s) = s^k$, for some positive integer k. If $n = 2$, the result follows from Lemma 6.2. When $n > 2$, therefore, there exists $\varepsilon > 0$ such that, for each $i = 2, \ldots, n$, the image of $(0, \varepsilon)$ by the mapping $x_1 = s^k$, $x_i = \gamma_i(s)$, $x_j = 0$, $j \neq 1, i$, is semianalytic. Thus $\gamma((0,\varepsilon)) = \bigcap_{i=2}^n \{x = (x_1, \ldots, x_n) : x_1 = s^k, x_i = \gamma_i(s), s \in (0, \varepsilon)\}$ is semianalytic. □

Proof of Theorem 6.1. — (1) Let N be a real analytic manifold and let $\pi : M \times N \to M$ be the projection. By Lemma 3.6, it is enough to prove that if X is a relatively compact semianalytic subset of $M \times N$ and $\dim X = 1$, then $\pi(X)$ is semianalytic. By Lemma 6.3 (1), X is locally a union of finitely many sets of the form $A = \gamma((0, \varepsilon))$, where $\gamma : (-\varepsilon, \varepsilon) \to M \times N$ is a nonconstant analytic mapping, perhaps together with a point. Each $\pi(A) = (\pi \circ \gamma)((0, \varepsilon))$ is semianalytic, by Lemma 6.3 (2).

(2) $\overline{X} - \operatorname{int} X$ and $\overline{X} - X$ are each subanalytic of dimension $\leqslant 1$, hence semianalytic, by (1). But $\overline{X}$ is the union of $\overline{X} - \operatorname{int} X$ and certain components of its complement, hence semianalytic. Therefore, $X = \overline{X} - (\overline{X} - X)$ is semianalytic. □

Theorem **6.4** (Łojasiewicz's inequality). — *Let* M *be a real analytic manifold and let* K *be a subset of* M. *Let* $f, g : K \to \mathbf{R}$ *be subanalytic functions with compact graphs. If* $f^{-1}(0) \subset g^{-1}(0)$, *then there exist* $c, r > 0$ *such that, for all* $x \in K$,

$$|f(x)| \geqslant c\,|g(x)|^r.$$

Remark **6.5.** — In particular, if $M = \mathbf{R}^n$, $X = f^{-1}(0)$ and $g(x) = d(x, X)$, $x \in K$, we get

$$|f(x)| \geqslant c\, d(x, X)^r, \quad x \in K.$$

Proof of Theorem 6.4. — Let $L = \{(u, v) \in \mathbf{R}^2 : u = |g(x)|,\ v = |f(x)|, \text{ for some } x \in K\}$. Then L is a compact semianalytic subset of $\mathbf{R}^2$, by Theorem 6.1. Let $\pi(u, v) = u$ be the projection. We can assume that $0 \in \pi(L)$ and 0 is not an isolated point of $\pi(L)$. By Lemma 6.3 (1), there exist $\varepsilon > 0$ and a parametrized analytic curve $\gamma(s) = (u(s), v(s))$, $s \in (-2\varepsilon, 2\varepsilon)$, such that $u(0) = 0$, $u(s) > 0$ if $s > 0$, and $L \cap ([0, u(\varepsilon)) \times \mathbf{R})$ is bounded below by $\gamma([0, \varepsilon))$. By a change of the parameter s, we can assume that $u(s) = s^k$, for some positive integer k. Then $v(s)$ is strictly positive on $(0, \varepsilon)$ since, for all $u \in (0, \varepsilon^k)$, $\{x \in K : |g(x)| = u\}$ is a compact set on which $|f(x)|$ does not vanish, so has a nonzero minimum. Let $\delta = \varepsilon^k$. Then $|f(x)| \geqslant v(|g(x)|^{1/k}) > 0$, whenever $0 < |g(x)| < \delta$. Therefore, there exist $c, r > 0$ such that $|f(x)| \geqslant c\,|g(x)|^r$, whenever $|g(x)| \leqslant \delta/2$. But $\{x \in K : |g(x)| \geqslant \delta/2\}$ is a compact set on which $|f(x)|$ does not vanish, so the inequality is satisfied on all of K, after perhaps reducing c. □

Definition **6.6.** — Let U be an open subset of $\mathbf{R}^n$ and let X, Y be closed subsets of U. We say that X and Y are *regularly situated* if, for all $x_0 \in X \cap Y$, there exist a neighbourhood V of x_0 and $c, r > 0$ such that, for all $x \in V$,

$$d(x, X) + d(x, Y) \geqslant c\, d(x, X \cap Y)^r.$$

Corollary **6.7.** — *Let* U *be an open subset of* $\mathbf{R}^n$. *Then any two closed subanalytic subsets of* U *are regularly situated.*

Proof. — We can assume the two closed subsets X and Y are compact. The functions $f(x) = d(x, X) + d(x, Y)$ and $g(x) = d(x, X \cap Y)$, restricted to a compact neighbourhood of X ∪ Y, have compact subanalytic graphs. Clearly, $f^{-1}(0) \subset g^{-1}(0)$. The result follows from Theorem 6.4. □

Proposition **6.8** [21]. — *Let g be a real analytic function on a neighbourhood of the origin in* $\mathbf{R}^n$, *such that* $g(0) = 0$. *Then there are constants c, r such that* $0 < r < 1$ *and*

$$|\operatorname{grad} g(x)| \geqslant c\,|g(x)|^r$$

in some neighbourhood of 0. *(Here* $|\operatorname{grad} g(x)| = (\Sigma_{i=1}^n (\partial g/\partial x_i)^2)^{1/2}$, $x = (x_1, \ldots, x_n)$.*)*

Proof (cf. [5, (3.40)]). — Let K be a ball centered at 0 in which $\operatorname{grad} g(x) = 0$ only if $g(x) = 0$. By Theorem 6.4 (with $f(x) = |\operatorname{grad} g(x)|$), there exist $c, r > 0$ such that

$$|\operatorname{grad} g(x)| \geqslant c\,|g(x)|^r, \quad x \in K.$$

Following the proof of Theorem 6.4, we can assume that $r = \mu/k$, where μ is the order $\mu_0(v)$ of v at 0 (cf. Definition 4.6). By Lemmas 3.6 and 6.3, there is an analytic curve $x = \sigma(t)$ such that $\sigma(0) = 0$, $g(\sigma(t)) \neq 0$ if $t \neq 0$, and $|\operatorname{grad} g(\sigma(t))| = v(|g(\sigma(t))|^{1/k})$. Then, for t in a neighbourhood of 0,

$$\left|\frac{d}{dt} g(\sigma(t))\right| \leqslant c'\,|\operatorname{grad} g(\sigma(t))| = c'\,v(|g(\sigma(t))|^{1/k}),$$

where c' is a constant. From the Taylor expansions of $g(\sigma(t))$ and $v(s)$ at 0, it is clear that $r < 1$. □

" Whitney regularity " of a subanalytic set [1, 15]:

Definition **6.9.** — Let X be a compact subset of $\mathbf{R}^n$ and let p be a positive integer. We say that X is *p-regular* if there exists $C > 0$ such that any two points $x, y \in X$ can be joined by a rectifiable curve γ in X of length

$$|\gamma| \leqslant C\,|x - y|^{1/p}.$$

Theorem **6.10.** — *Let* X *be a compact connected subanalytic subset of* $\mathbf{R}^n$. *Then there is a positive integer p such that* X *is p-regular (where the curves can be chosen semianalytic).*

Lemma **6.11.** — *Let* U *be an open subset of* $\mathbf{R}^m$, *and let* $\varphi : U \to \mathbf{R}^n$ *be a real analytic mapping,* $\varphi = (\varphi_1, \ldots, \varphi_n)$. *If* γ *is a rectifiable curve in* U, *then*

$$|\varphi(\gamma)| \leqslant \sqrt{mn}\,|\gamma| \,.\, \sup_{\substack{x \in \gamma \\ i, j}} \left|\frac{\partial \varphi_i}{\partial x_j}(x)\right|.$$

Proof. — Let $a, b \in U$ and let $[a, b]$ denote the line segment from a to b. By the mean value theorem, if $[a, b] \subset U$, then, for each $i = 1, \ldots, n$,

$$|\varphi_i(a) - \varphi_i(b)| \leqslant \sqrt{m}\,|a - b| \,.\, \sup_{\substack{x \in [a, b] \\ 1 \leqslant j \leqslant m}} \left|\frac{\partial \varphi_i}{\partial x_j}(x)\right|.$$

Therefore, if γ is a piecewise linear curve joining a and b,

$$|\varphi_i(a) - \varphi_i(b)| \leqslant \sqrt{m}\,|\gamma|\,.\sup_{\substack{x\in\gamma\\ 1\leqslant j\leqslant m}}\left|\frac{\partial\varphi_i}{\partial x_j}(x)\right|.$$

This formula holds for any rectifiable curve γ, by passage to the limit. The assertion follows. □

Lemma **6.12.** — *Let* X *and* Y *be compact subsets of* $\mathbf{R}^n$ *such that* $X \cap Y = \emptyset$. *Suppose that* X *and* Y *are regularly situated, so that (by Definition* 6.6*)* $d(x, Y) \geqslant c\, d(x, X \cap Y)^r$, *for all* $x \in X$, *where* $c > 0$ *and* r *is a positive integer. If* X *and* Y *are each p-regular, then* $X \cup Y$ *is pr-regular.*

Proof. — Choose C as in Definition 6.9, common for X and Y. Suppose that $x \in X$, $y \in Y$. Choose $z \in X \cap Y$ such that $d(x, X \cap Y) = |x - z|$. Let γ_1 and γ_2 be curves in X and Y (respectively) joining x to z and y to z (respectively), such that $|\gamma_1| \leqslant C\,|x - z|^{1/p}$ and $|\gamma_2| \leqslant C\,|y - z|^{1/p}$. Then $|x - y| \geqslant d(x, Y) \geqslant c\, d(x, X \cap Y)^r = c\,|x - z|^r$. Therefore, $|x - z| \leqslant (|x - y|/c)^{1/r}$ and $|y - z| \leqslant |x - y| + (|x - y|/c)^{1/r}$. An estimate on the length of $\gamma_1 \cup \gamma_2$, as required, follows. □

Proof of Theorem 6.10. — By Theorem 0.1, there exist $m \in \mathbf{N}$, a real analytic mapping $\varphi : \mathbf{R}^m \to \mathbf{R}^n$, and a disjoint union K of finitely many spheres in $\mathbf{R}^m$, such that $\varphi(K) = X$. By Lemma 6.12, we can assume that K is a single sphere. Clearly, there is $c_1 > 0$ such that any two points $x, x' \in K$ can be joined by a semianalytic curve of length $\leqslant c_1\,|x - x'|$.

Consider the following subanalytic functions on $K \times K \subset \mathbf{R}^m \times \mathbf{R}^m$:

$$f(a, x) = |\varphi(a) - \varphi(x)|$$
$$g(a, x) = \min_{(a',x')\in f^{-1}(0)} (|a - a'| + |x - x'|),$$

where $(a, x) \in K \times K$. Then $f^{-1}(0) \subset g^{-1}(0)$. By Łojasiewicz's inequality, there exist $c_2 > 0$ and a positive integer p such that, for all $(a, x) \in K \times K$,

$$|f(a, x)| \geqslant c_2\,|g(a, x)|^p.$$

Let $b, y \in X = \varphi(K)$; say $b = \varphi(a)$, $y = \varphi(x)$, where $a, x \in K$. Choose $a', x' \in K$ such that $\varphi(a') = \varphi(x')$ and $g(a, x) = |a - a'| + |x - x'|$. Let γ_1, γ_2 be semianalytic curves in K joining a and a', x and x', respectively, such that $|\gamma_1| \leqslant c_1\,|a - a'|$ and $|\gamma_2| \leqslant c_2\,|x - x'|$. Then $\gamma = \varphi(\gamma_1) \cup \varphi(\gamma_2)$ is a semianalytic curve joining b, y in X, and

$$|\gamma| \leqslant |\varphi(\gamma_1)| + |\varphi(\gamma_2)|$$
$$\leqslant c_3(|\gamma_1| + |\gamma_2|),$$

where $c_3 = \sqrt{mn}\,\sup_{z\in K, i, j}\left|\frac{\partial\varphi_i}{\partial x_j}(z)\right|$, by Lemma 6.11, so that

$$\begin{aligned}|\gamma| &\leqslant c_1 c_3(|a-a'|+|x-x'|)\\ &\leqslant \frac{c_1 c_3}{c_2^{1/p}}|\varphi(a)-\varphi(x)|^{1/p}\\ &= \mathrm{C}|b-y|^{1/p},\end{aligned}$$

where $\mathrm{C} = c_1 c_3/c_2^{1/p}$. □

7. Smooth points of a subanalytic set

In this final section, we prove Tamm's theorem that the set of smooth points of a subanalytic set is subanalytic [26]. As Tamm does, we use Malgrange's idea of "graphic points", but in a more direct way.

Let N denote a real analytic manifold and let X denote a subanalytic subset of N.

Definition **7.1.** — The *singular set* of X, Sing X, is the complement in X of the smooth points of the highest dimension (cf. Definition 3.3).

Theorem **7.2.** — *For each $k \in \mathbf{N}$, the set of smooth points of* X *of dimension k is subanalytic. In particular,* Sing X *is a closed subanalytic subset of* X.

Remark **7.3.** — For each $k \in \mathbf{N}$, the set of smooth points of dimension k of a semialgebraic (respectively, semianalytic) set is semialgebraic (respectively, semianalytic): The semialgebraic result can be proved as in this section. For semianalytic sets, Proposition 7.4 below is not useful because the distance function is not necessarily semianalytic; nevertheless, the analogue of Theorem 7.2 can be proved using the graphic point techniques of this section together with Remark 5.2 and Proposition 2.10. However, the singular set of a real algebraic set is not necessarily algebraic! For example, if

$$\mathrm{X} = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : y_3^4 - y_1 y_2 y_3^2 - y_2^3 = 0\},$$

then Sing X is the non-positive y_1-axis. (X is the image of the mapping $y_1 = x_1$, $y_2 = x_2(x_2^3 + x_1 x_2)$, $y_3 = x_2^3 + x_1 x_2$.)

Proof of Theorem 7.2. — The smooth points of X (of dimension k) are the smooth points of $\overline{\mathrm{X}}$ (of dimension k) which do not lie in the closure of $\overline{\mathrm{X}} - \mathrm{X}$. Therefore, we can assume that X is closed. The set of smooth points of X of a given dimension k is open and closed in the set of all smooth points. We can assume that $\mathrm{X} \subset \mathbf{R}^n$. Of course, X is the zero set of the distance function $d(x, \mathrm{X})$, which is continuous and subanalytic. Then, by Proposition 7.4 below, our assertion is a consequence of Theorem 7.5 following (with $g(x) = d(x, \mathrm{X})^2$). □

Proposition **7.4** (Poly-Raby [25]). — *Let* X *be a closed subset of* $\mathbf{R}^n$ *and let* $\delta(x)$ *be the distance function* $d(x, \mathrm{X})$. *Let* $a \in \mathrm{X}$. *Then* δ^2 *is analytic in some neighbourhood of* a *if and only if* X *is an analytic submanifold in some neighbourhood of* a.

Theorem **7.5.** — *Let* N *be a real analytic manifold, and let* $g : N \to \mathbf{R}$ *be a continuous subanalytic function. Then* $\{ x \in N : g$ *is analytic at* $x \}$ *is a subanalytic subset of* N.

Proof of Proposition 7.4. — First suppose that X is an analytic manifold near a. We can assume that $a = 0$ and that, near 0, X is the graph of an analytic mapping $\varphi : U \to \mathbf{R}^{n-p}$, where U is an open neighbourhood of 0 in $\mathbf{R}^p$, such that $\varphi(0) = 0$ and $D\varphi(0) = 0$. (Here, $D\varphi(0)$ denotes the derivative or tangent mapping of φ at 0.) Given x, choose $y \in X$ such that $\delta(x) = | x - y |$; if x is sufficiently close to 0, then $y \in$ graph φ and $x - y$ is normal to X at y (since the tangent mapping of $h(z) = | x - z |^2$, $z \in X$, vanishes at y).

Let $u \in U$. Then the normal space to X at $(u, \varphi(u))$ is $\{(- D\varphi(u)^* w, w) : w \in \mathbf{R}^{n-p} \}$, where $D\varphi(u)^*$ denotes the adjoint of the linear mapping $D\varphi(u)$. Define $\Phi : U \times \mathbf{R}^{n-p} \to \mathbf{R}^p \times \mathbf{R}^{n-p}$ by $\Phi(u, w) = (u, \varphi(u)) + (- D\varphi(u)^* w, w)$. Since $D\Phi(0)$ is the identity, then Φ is an analytic isomorphism near 0. Thus, for x in a sufficiently small neighbourhood of 0, there is a unique y such that $\delta^2(x) = | x - y |^2$: if $x = \Phi(u, w)$, then $y = (u, \varphi(u))$; say $y = \pi(x)$. So $\delta^2(x) = | x - \pi(x)|^2$ is analytic.

Conversely, suppose that $\delta^2(x)$ is analytic near $a \in X$. All first partial derivatives of δ^2 vanish on X (since δ^2 is nonnegative, and zero on X). Let M be an analytic manifold of minimal dimension containing a neighbourhood of a in X. If $\delta^2 \equiv 0$ in a neighbourhood of a in M, then X coincides with M near a, and we are done. Otherwise, there is a sequence $\{ x_m \} \subset M$ such that $\lim x_m = a$ and $\delta^2(x_m) \neq 0$, for each m. Choose $y_m \in X$ such that $\delta^2(x_m) = | x_m - y_m |^2$. Then $\delta^2(x'_m) = | x'_m - y_m |^2$ for all x'_m on the line segment between x_m and y_m. Therefore, the second derivative of δ^2 at y_m, in the direction $x_m - y_m$, is 2. Passing to a subsequence if necessary, $x_m - y_m$ tends to a limiting direction in the tangent space $T_a M$, and the second derivative of δ^2 in this limiting direction is 2, by continuity. Therefore, the first derivative of δ^2 in the limiting direction defines a smooth analytic hypersurface H near a; $H \supset X$ since all first partial derivatives of δ^2 vanish on X. But H is transverse to M near a, so $H \cap M$ is a manifold of smaller dimension than M containing X near a; contradiction. □

We will prove Theorem 7.5 using Malgrange's idea of " graphic points ": Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\Phi = (\varphi, f) : M \to N \times \mathbf{K}$ be an analytic mapping, where M, N are analytic manifolds (over $\mathbf{K}$). Assume that N is connected, $\dim N = n$, and that φ has generic rank n (i.e. maximal rank n on each component of M).

Definition **7.6.** — A point $a \in M$ is *graphic (with respect to* Φ*)* if there exists a germ of an analytic function g at $\varphi(a)$ such that $f_a = g \circ \varphi_a$. (Here f_a and φ_a denote the germs at a of f and φ, respectively.)

Notation. — Let $a \in M$. Let $\mathcal{O}_{M,a}$ or $\mathcal{O}_a$ denote the ring of germs of analytic functions on M at a. Let $\varphi_a^* : \mathcal{O}_{\varphi(a)} \to \mathcal{O}_a$ denote the homomorphism $\varphi_a^*(g) = g \circ \varphi_a$, where $g \in \mathcal{O}_{\varphi(a)}$. Let $\widehat{\mathcal{O}}_a$ denote the formal completion of $\mathcal{O}_a$, and $\widehat{\varphi}_a^* : \widehat{\mathcal{O}}_{\varphi(a)} \to \widehat{\mathcal{O}}_a$ the induced homomorphism: Consider any local coordinate systems $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ centered at a in M and at $\varphi(a)$ in N, respectively. Then $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) identifies with the ring of convergent (respectively, formal) power series $\mathbf{K}\{x\} = \mathbf{K}\{x_1, \dots, x_m\}$ (respectively, $\mathbf{K}[[x]] = \mathbf{K}[[x_1, \dots, x_m]]$). If $G(y) \in \hat{\mathcal{O}}_{\varphi(a)} = \mathbf{K}[[y]]$, $y = (y_1, \dots, y_n)$, then $\hat{\varphi}_a^*(G)$ is given by formally substituting for y the Taylor series without constant term $y = \Sigma_\alpha D^\alpha \varphi(a)\, x^\alpha/\alpha! - \varphi(a)$. Here $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$, $\alpha! = \alpha_1! \dots \alpha_m!$, $x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}$, and $D^\alpha \varphi(a) = (\partial^{|\alpha|} \varphi/\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m})(a)$, where $|\alpha| = \alpha_1 + \dots + \alpha_m$. We use $f \mapsto \hat{f}$ to denote the inclusion $\mathcal{O}_a \to \hat{\mathcal{O}}_a$.

Theorem **7.7.** — *Let $a \in M$. Then a is graphic if and only if a is* formally graphic; *i.e., there exists $G \in \hat{\mathcal{O}}_{\varphi(a)}$ such that $\hat{f}_a = \hat{\varphi}_a^*(G)$.*

Theorem 7.7 follows from:

Lemma **7.8** [12, 23]. — *Let $\psi = (\psi_1, \dots, \psi_n)$, where $\psi_j \in \mathbf{C}\{x\} = \mathbf{C}\{x_1, \dots, x_m\}$ and $\psi_j(0) = 0, j = 1, \dots, n$. Suppose that ψ has generic rank n. If $G \in \mathbf{C}[[y]] = \mathbf{C}[[y_1, \dots, y_n]]$ and $\hat{\psi}^*(G) = G \circ \psi$ converges, then G converges.*

Remark **7.9.** — If ψ has generic rank n (i.e., a representative in a neighbourhood of 0 has generic rank n), then $\psi^* : \mathbf{C}\{y\} \to \mathbf{C}\{x\}$ is injective since, otherwise, Ker ψ^* defines a germ of a proper analytic subset of $\mathbf{C}^n$ at 0. It then follows from Lemma 7.8 that $\hat{\psi}^*$ is injective.

Proof of Lemma 7.8 (cf. [2, Prop. 1.6]). — Let

$$\delta(x) = \det\left(\frac{\partial \psi_j}{\partial x_i}\right)_{i, j = 1, \dots, n}.$$

By reordering the x_i if necessary, we can assume that $\delta(x) \not\equiv 0$. Suppose $f(x) = G(\psi(x))$, where $f(x) \in \mathbf{C}\{x\}$ and $G(y) \in \mathbf{C}[[y]]$. By the chain rule,

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} \left(\frac{\partial G}{\partial y_j} \circ \psi\right) \frac{\partial \psi_j}{\partial x_i}, \quad i = 1, \dots, m.$$

$(\partial G/\partial y_j$ denotes the formal derivative.) Let $f^{(j)}(x) = ((\partial G/\partial y_j) \circ \psi)(x)$, $j = 1, \dots, n$. By Cramer's rule and the faithful flatness of $\mathbf{C}[[x]]$ over $\mathbf{C}\{x\}$ (cf. [29, Chapt. 8, § 4]), each $f^{(j)}(x) \in \mathbf{C}\{x\}$. Proceed inductively: For each $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$, there exists $f^\beta(x) \in \mathbf{C}\{x\}$ such that $f^0 = f$ and

$$\frac{\partial f^\beta}{\partial x_i} = \sum_{j=1}^{n} f^{\beta + (j)} \frac{\partial \psi_j}{\partial x_i}, \quad i = 1, \dots, m$$

(where (j) denotes the multiindex with 1 in the j-th place and zeros elsewhere). (In fact, $f^\beta(x) = ((\partial^{|\beta|} G/\partial y^\beta) \circ \psi)(x)$.)

Let U be a neighbourhood of 0 in $\mathbf{C}^m$ such that f and each ψ_j converge in U, and every irreducible component of the hypersurface $X = \{x \in U : \delta(x) = 0\}$ passes through 0. Then each f^β converges in U (since its poles form a codimension 1 analytic subset of U contained in $\{\delta(x) = 0\}$).

Let $G_{(x)}(y) = \Sigma_\beta f^\beta(x) y^\beta/\beta!$, where $x \in U$. Clearly, $G_{(0)}(y) = G(y)$ and, if $\delta(x) \neq 0$, then $G_{(x)}(y) \in \mathbf{C}\{y\}$. In fact, $G_{(x)}(y)$ defines a holomorphic function $H(x, y)$ in a neighbourhood of $\{(x,y) \in U \times \mathbf{C}^n : y = 0, \delta(x) \neq 0\}$: Near any point a such that $\delta(a) \neq 0$, ψ admits a local holomorphic section σ; i.e., a holomorphic mapping σ in a neighbourhood of $\psi(a)$ such that $\psi \circ \sigma$ is the identity. For x near a, $G_{(x)} \circ \hat{\psi}_x \in \mathcal{O}_x$ is the Taylor expansion of f at x, so that $G_{(x)} \circ \hat{\psi}_x \circ \hat{\sigma}_{\psi(x)} \in \mathcal{O}_{\psi(x)}$ is the Taylor series of $g = f \circ \sigma$. Clearly, $G_{(x)}(y) = g(\psi(x) + y)$.

Then $G_{(x)}(y) \in \mathbf{C}\{y\}$, for all $x \in U$ (in particular, $G \in \mathbf{C}\{y\}$, as required) as follows: There is an analytic subset Σ of X, of complex codimension at least 2 in U, such that $X - \Sigma$ is a complex submanifold of U of codimension 1. Let $a \in X - \Sigma$. Choose coordinates $(x_1, \ldots, x_m)$ centered at a in U such that $\{x_1 = 0\}$ defines $X - \Sigma$. Define a holomorphic function $\tilde{H}(x, y)$ in a neighbourhood of $a = 0$ in $U \times \mathbf{C}^n$, by

$$\tilde{H}(x,y) = \frac{1}{2\pi i} \int_\gamma \frac{H(\zeta, x_2, \ldots, x_m; y)}{\zeta - x_1}\, d\zeta,$$

where γ is a positively oriented circle around 0 in the x_1-plane. Then, for all $\beta \in \mathbf{N}^n$,

$$\left(\frac{\partial^{|\beta|} \tilde{H}}{\partial y^\beta}\right)(x, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f^\beta(\zeta, x_2, \ldots, x_m)}{\zeta - x_1}\, d\zeta = f^\beta(x),$$

so that $\tilde{H}(x, y)$ is an extension of $H(x, y)$. We can proceed by induction (or use Hartog's theorem) to extend $H(x, y)$ to be holomorphic in a neighbourhood of $U \times \{0\}$. □

Theorem **7.10** (Malgrange). — *Consider* $\Phi = (\varphi, f) : M \to N \times \mathbf{K}$ *as before (*N *is connected,* $\dim N = n$, *and* φ *has generic rank* n*). Then the set* E *of non-graphic points is a closed analytic subset of* M, *contained in the critical set of* φ.

Proof. — We can assume that $\mathbf{K} = \mathbf{C}$, M and N are open subsets of $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, and $\delta(x) = \det(\partial\varphi_j/\partial x_i)_{i,j=1,\ldots,n} \not\equiv 0$, where $\varphi = (\varphi_1, \ldots, \varphi_n)$. If $x \in M - E$, then there exists $g^x \in \mathcal{O}_{\varphi(x)}$ such that $f_x = g^x \circ \varphi_x$. Thus, for each $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, we have $f^\beta \in \mathcal{O}(M - E)$ defined by

$$f^\beta(x) = ((D^\beta g^x) \circ \varphi)(x).$$

As in the proof of Lemma 7.8, each f^β extends to a meromorphic function on M (the quotient of a global holomorphic function by a power of $\delta(x)$).

For each $k = 1, 2, \ldots$, let P_k denote the subset of M where some f^β, $|\beta| \leq k$, has a pole. Then each P_k is a complex analytic subset of M; in fact, locally, each P_k is the union of the zero sets of certain factors of δ, so the sequence $P_1 \subset P_2 \subset \ldots$ is locally stationary. Therefore, $P = \bigcup_{k=1}^\infty P_k$ is a closed analytic subset of M. Obviously, P lies in the critical set of φ.

Clearly, $P \subset E$. On the other hand, if $x \notin P$, then all f^β are holomorphic and hence continuous at x, so $x \notin E$ by the following lemma. □

Lemma **7.11.** — *$x \notin E$ if and only if there exists a sequence $\{x^\ell\} \subset M - \{x : \delta(x) = 0\}$ such that $x = \lim x^\ell$ and, for all $\beta \in \mathbf{N}^n$, $\lim f^\beta(x^\ell)$ exists.*

Proof. — "Only if" has already been seen. "If": By Theorem 7.7, it suffices to find $G \in \hat{\mathcal{O}}_{\varphi(x)}$ such that $\hat{f}_x = \hat{\varphi}_x^*(G)$. For each ℓ, since x^ℓ is graphic, there exists $g^{x^\ell} \in \mathcal{O}_{\varphi(x^\ell)}$ such that $f_{x^\ell} = g^{x^\ell} \circ \varphi_{x^\ell}$. By differentiating, we get:

$$f(x^\ell) = g^{x^\ell}(\varphi(x^\ell)) = f^0(x^\ell),$$

$$\frac{\partial f}{\partial x_i}(x^\ell) = \sum_{j=1}^{n} \frac{\partial g^{x^\ell}}{\partial y_j}(\varphi(x^\ell)) \frac{\partial \varphi_j}{\partial x_i}(x^\ell) = \sum_{j=1}^{n} f^{(j)}(x^\ell) \frac{\partial \varphi_j}{\partial x_i}(x^\ell),$$

$$\dots$$

Let ℓ tend to ∞. The resulting equations mean $\hat{f}_x = \hat{\varphi}_x^*(G)$, where $G(y)$ is the formal power series whose coefficients are the $\lim f^\beta(x^\ell)/\beta!$. □

Remark **7.12.** — Let γ be a curve in $M - E$, with endpoints a and a', say. Then $f_a = g^a \circ \varphi_a$, $f_{a'} = g^{a'} \circ \varphi_{a'}$, where $g_a \in \mathcal{O}_{\varphi(a)}$, $g_{a'} \in \mathcal{O}_{\varphi(a')}$. Clearly, $g^{a'}$ is obtained by analytic continuation of g^a along $\varphi(\gamma)$. In particular g^a is constant on connected components of the fibers of φ (which clearly must lie entirely in $M - E$ or E).

Now, let s be a positive integer. Let M_φ^s be the s-fold fiber product of M over N, and let $\boldsymbol{\varphi} : M_\varphi^s \to N$ be the induced mapping (Definition 3.8). (M_φ^s is a closed analytic subset of M^s.) We say that $\mathbf{a} \in M_\varphi^s$ is an *s-fold graphic point* if there exists $g^{\mathbf{a}} \in \mathcal{O}_{\boldsymbol{\varphi}(\mathbf{a})}$ such that $f_{a^i} = g^{\mathbf{a}} \circ \varphi_{a^i}$, $i = 1, \dots, s$, where $\mathbf{a} = (a^1, \dots, a^s)$. Let $\mathbf{E} \subset M_\varphi^s$ denote the set of non-s-fold graphic points.

Corollary **7.13.** — *With the hypotheses of Theorem* 7.10, $\mathbf{E}$ *is a closed analytic subset of* M_φ^s.

Proof. — We can assume that $\mathbf{K} = \mathbf{C}$. Let X be an irreducible component of (the germ at some point of) M_φ^s, and let $Y = \{\mathbf{a} = (a^1, \dots, a^s) \in X : a^i \in E$, for some $i = 1, \dots, s\}$. Then $Y \subset \mathbf{E}$ and Y is a closed analytic subset of X, by Theorem 7.10. Since X is irreducible, then $X - Y$ is connected [24]. Consider $\mathbf{a} \in X - Y$, $\mathbf{a} = (a^1, \dots, a^s)$. Then each a^i is graphic. If $\mathbf{a}$ is an s-fold graphic point, then $\mathbf{a}'$ is s-fold graphic, for all $\mathbf{a}' \in X - Y$, by Remark 7.12. □

Proof of Theorem 7.5. — Let U be a relatively compact connected open subanalytic subset of N. By Theorem 0.1, there is a compact real analytic manifold M, and a real analytic mapping $\Phi = (\varphi, f) : M \to N \times \mathbf{R}$ such that $\Phi(M) = \operatorname{graph} g \mid \overline{U}$. We can assume that φ has generic rank $n = \dim U$ (on each component of M). By Theorem 3.14, there is a bound s on the number of connected components of the fibers of φ. Let $\boldsymbol{\varphi} : M_\varphi^s \to N$ and $\mathbf{E}$ be as above. Suppose that $y \in U$. Clearly, g is analytic at y if and only if $y \notin \boldsymbol{\varphi}(\mathbf{E})$ (cf. Remark 7.12). The theorem follows immediately. □

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