HÉNON MAPPINGS IN THE COMPLEX DOMAIN
I: THE GLOBAL TOPOLOGY OF DYNAMICAL SPACE

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CONTENTS

1. Introduction

In 1969, Hénon ([Hé1] and [Hé2]) began the investigation of the mappings

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}, \quad \text{where } a \neq 0,$$

as mappings having roughly the same behavior as a particular Poincaré section of the Lorenz differential equation. Hénon demonstrated numerically that for certain values of the parameters the mappings appeared to have a strange attractor. This has finally been established rigorously by Benedicks and Carleson ([BC], [MV]).

There has since been an enormous amount of work on the dynamics of the Hénon mappings (in particular, see [Ho], [HWh] and [HWi], which give further references). This work is all in the real domain. As far as we know, this paper ([H] was an early version) is the first attempt to understand the Hénon mappings in $\mathbf{C}^2$. Recently others have done work in this area including Friedland and Milnor ([FM] and [M1]), Bedford, Lyubich, and Smillie ([B], [BS1], [BS2], [BS3], [BS4], [BLS], [S]), and Fornæss and Sibony ([FS]).

In the study of iteration of polynomials of one variable, extending to complex values of the variable has been very useful, even when the original polynomials were real. We hope that the same thing will happen here, more or less for the same reason. There is essentially nothing that can be said about real polynomials which is independent of the coefficients, largely because virtually all features independent of conjugation, such as periodic cycles, are likely to disappear under perturbation. In the complex domain, the behavior is far more uniform.

Our work started from a different point of view. In 1982, Calabi suggested that the computer should be used to investigate the basin of attraction of one of the two attractive fixed points of the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}y - \frac{1}{4}(x^3 - 3x) \\ x \end{pmatrix}$$

The reason for examining this was that it provided an example of a *Fatou-Bieberbach domain.* These are open subsets $U \subset \mathbf{C}^n$ which are biholomorphically isomorphic to $\mathbf{C}^n$ and whose complement, $\mathbf{C}^n - U$, has non-empty interior. When an automorphism of $\mathbf{C}^n$ has an attractive fixed point or attractive cycle, the basin is always such a domain. Fatou and Bieberbach ([F], [Bi]) first constructed examples of such domains as basins of attractive fixed points. They have been extensively studied in [BS2] and [FS].

Despite considerable numerical work, we were unable to work out the topology of the closures of the basins and decided to look at simpler automorphisms of $\mathbf{C}^2$, with quadratic polynomials as coordinates. Section 2 shows that the Hénon family encompasses a significant part of this family.

The cast of players. Most of the work on Hénon mappings in the real case has focused on attractors. In the complex, attractors are uninteresting since the only attractors are points. The invariant subsets considered here are inspired by the dynamics of polynomials, as explained below. For any mapping f, let $f^{\circ n}$ denote the n-fold composition of f or f^{-1} depending on whether n is positive or negative.

Our approach has been inspired by the study of complex polynomials of a single variable. Given a polynomial $p(z)$, the natural set to study is

$$K_p = \{ z \mid p^{\circ n}(z) \text{ does not tend to } \infty \text{ as } n \to \infty \}$$

and its boundary $J_p = \partial K_p$, also known as the Julia set of p. Another definition of J_p is

$$J_p = \{ z \mid \text{on no neighborhood of } z \text{ is the sequence } \{ p^{\circ n} \} \text{ normal} \}.$$

The sets studied here are defined in imitation of the one-dimensional case. For a Hénon mapping, the obvious generalization of the Julia set is

$$J_\pm = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \text{ on no neighborhood of } \begin{pmatrix} x \\ y \end{pmatrix} \text{ is the sequence } \{ F^{\circ \pm n} \} \text{ normal} \right\},$$

where a sequence of functions on $U \subset \mathbf{C}^2$ with values in $\mathbf{C}^2$ is defined to be normal if every subsequence has a subsequence which converges uniformly on compact subsets to a function with values in $\mathbf{P}^2$, the complex projective plane.

Define for a Hénon mapping F the following sets:

$$K_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \left\| F^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ does not tend to } \infty \right\} \quad \text{and} \quad U_+ = \mathbf{C}^2 - K_+$$

and

$$K_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \left\| F^{\circ -n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ does not tend to } \infty \right\} \quad \text{and} \quad U_- = \mathbf{C}^2 - K_-.$$

Further, define $J_\pm = \partial K_\pm$, $K = K_+ \cap K_-$, and $J = J_+ \cap J_-$.

It will be seen that K and J are compact and of course invariant under F. These are the spaces which we most wish to understand.

Main results. This paper contains three main results: two concern the structure of $\mathbf{C}^2 - K_\pm$. Topologically, we will show that this set is homeomorphic to a fibration over the reals with fiber a 3-sphere with a solenoid removed (Theorem 6.1). Analytically, $\mathbf{C}^2 - K_\pm$ is isomorphic to a quotient of $(\mathbf{C} - \bar{D}) \times \mathbf{C}$, where $D \subset \mathbf{C}$ is the unit disc, by a group of automorphisms which we determine explicitly (Section 8). The third result gives a compactification $\mathbf{C}^2$ to which the Hénon mappings extend canonically, analogous to compactifying $\mathbf{C}$ by adding a circle at infinity (Theorem 9.1).

The proofs of these results require both some analytic and some topological preliminaries. Most of the topology (Sections 3 and 4) concerns *solenoidal mappings*, one of which plays much the same role with respect to Hénon mappings as multiplying angles by d does for iteration of polynomials. We go into more details than is strictly necessary for our purposes, but we feel that viewing the surrounding countryside makes our particular mappings easier to understand, and the classification of solenoidal mappings (Theorem 3.10) is of independent interest.

For the analytical results, the most important construction is the analog of the Böttcher coordinate ([M2]). When p is a monic polynomial, this is the function φ_p defined in a neighborhood of ∞ such that

$$\varphi_p(p(z)) = (\varphi_p(z))^d$$

and

$$\varphi_p(z) = z + o(1) \text{ near } \infty.$$

The function $\varphi_p(z)$ is constructed by making sense of the following

$$\varphi_p(z) = \lim_{n \to \infty} (p^{\circ n}(z))^{1/d^n}.$$

This is a standard *scattering theory* construction: go toward ∞ via p and return via the unperturbed mapping $z \mapsto z^d$. The fractional power is not *a priori* defined, and has to be dealt with carefully.

The problem of the branches of the roots can be circumvented by defining

$$G_p(z) = \lim_{n \to \infty} \frac{1}{d^n} \log_+ |p^{\circ n}(z)|,$$

where $\log_+(x) = \sup\{\log(x), 0\}$, which is the Green's function of K_p.

This construction generalizes for Hénon mappings as follows. Let a subscript 1 or 2 denote the projection onto the first or second coordinate as in $(F^{\circ n})_1 = pr_1 \circ F^{\circ n}$. Now define the limits

$$G_\pm \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \to \infty} \frac{1}{d^n} \log_+ || F^{\circ \pm n}(x) ||,$$

and

$$\varphi_\pm \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \to \infty} \left[(F^{\circ \pm n})_1 \begin{pmatrix} x \\ y \end{pmatrix} \right]^{1/d^n}.$$

Of course, the matter of where these are defined and the convergence of the limits must be dealt with (and are, in Section 5). Since the first version of this paper was written, much further work on $G_\pm$ has been done, more particularly by considering the closed (1, 1)-currents

$$\mu_\pm = dd^c G_\pm$$

which are analogs of the Brohlin measure ([BS1], [FS]). The measure $\mu = \mu_+ \wedge \mu_-$ has also turned out to be very important.

As far as we know, the complex analytic mappings $\varphi_\pm$ have not received similar attention, but they are even more important to our development.

More particularly, the argument of the Böttcher coordinate has led to the theory of external angles and is fundamental to the combinatorial study of the dynamics of polynomials ([DH], [T]). When the functions $\varphi_\pm$ are combined with the compactification in Section 9, more particularly Corollary 9.4, we find that there is an analogous theory of external angles for Hénon mappings; perhaps we can hope to use the techniques using external rays, etc., to combinatorially describe Hénon mappings. A case in point is the Benedicks-Carleson result in [BC], where the combinatorics is so reminicent of puzzles and tableaux as in [Y], [BH] and [HY].

Continuations of this paper will present results about Hénon mappings as perturbations of polynomials ([HO]). The paper [O] studies the dynamics of complex horseshoes using techniques from these papers.

Acknowledgments. — This paper has taken a long time to write and we have benefited from innumerably many conversations. We particularly want to thank A. Douady, J. Smillie, E. Bedford, B. Branner, J. Milnor, D. Sullivan, C. McMullen, J. C. Yoccoz, M. Shishikura, N. Sibony, L. Carleson, M. Benedicks, C. Bardos, T. Bousch, D. Faught, and L. Ma. Much computer experimentation went into the formulation of the theorems, and we thank H. Smith for the immense amount of programming he performed for us over the years.

No scientific research is possible without financial support: we thank the National Science Foundation for grant DMS 8901729 and earlier grants, the mathematics departments of Cornell University, Yale University, the University of South Florida, SUNY at Stony Brook, the Université de Paris-Sud, the Institute for Advanced Study, the Geometry Project at the University of Minnesota, the Max Planck Institut für Mathematik, the Institut des Hautes Etudes Scientifiques, and the Ecole Normale Supérieure of Paris and Lyon for support and hospitality during the years when this paper was conceived and written.

2. An algebraic characterization of Hénon mappings

The family of mappings on $\mathbf{C}^2$ with quadratic coordinate functions depends *a priori* on 12 parameters. The Hénon mappings

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}, \quad a \neq 0,$$

represent some conjugacy classes of quadratic automorphisms. In this section it is shown that the only other conjugacy classes are represented by the *elementary mappings*,

$$E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 + bx + y^2 \\ k_2 + dy \end{pmatrix}, \quad k_1, k_2 \in \{0, 1\}, \ k_1 k_2 = 0, \ bd \neq 0,$$

where $k_1 = 1$ implies $b = 1$ and $k_2 = 1$ implies either $b = d \neq 1$ or $d = 1$. Note that the elementary mappings consist of several one- and two-parameter families:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bx + y^2 \\ 1 + by \end{pmatrix}, \quad b \neq 0, \ b \neq 1,$$

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bx + y^2 \\ dy \end{pmatrix}, \quad bd \neq 0,$$

$$C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + x + y^2 \\ dy \end{pmatrix}, \quad d \neq 0,$$

$$D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bx + y^2 \\ 1 + y \end{pmatrix}, \quad b \neq 0.$$

Note that every polynomial mapping, G, of degree 2 can be written in the form $G = G_0 + G_1 + G_2$, with each G_k homogeneous of degree k and that every polynomial automorphism has constant Jacobian determinant. The following theorem gives the Jacobian Conjecture in this context, i.e., any polynomial mapping of degree 2 with non-zero constant Jacobian determinant is an automorphism.

Remark. — If G_2 satisfies the non-degeneracy condition $G_2^{-1}(0) = 0$, then the mapping G extends to give an endomorphism of $\mathbf{P}^2$, which will be of degree 4. More generally, if the mapping were given by polynomials of any degree d, and the leading terms G_d were non-degenerate, then the mapping defines an endomorphism of $\mathbf{P}^2$ of degree d^2. Of course, this is incompatible with G being an automorphism. Since G_d is degenerate, $G_d^{-1}(0)$ is a line ℓ_0, and $G_d(\mathbf{C}^2)$ is a line ℓ_1. Either ℓ_0 and ℓ_1 coincide, in which case the mapping is elementary, or they do not coincide, and the mapping is a generalized Hénon mapping.

Theorem **2.1**. — *For every polynomial mapping* $G : \mathbf{C}^2 \to \mathbf{C}^2$ *of degree 2 with constant non-zero Jacobian determinant, the image of* G_2 *and the set on which* G_2 *vanishes are lines through the origin. If these lines are linearly independent, then* G *is conjugate to a Hénon mapping. Otherwise,* G *is conjugate to an elementary mapping.*

Proof. — The general polynomial mapping of degree two is

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 x + c_1 y + d_1 x^2 + e_1 xy + f_1 y^2 \\ a_2 + b_2 x + c_2 y + d_2 x^2 + e_2 xy + f_2 y^2 \end{pmatrix}.$$

The quadratic terms of the Jacobian determinant generally yield the relations

$$\frac{d_1}{d_2} = \frac{e_1}{e_2} = \frac{f_1}{f_2},$$

So the image of the quadratic terms is a line, ℓ_1; assume that $d_2 = e_2 = f_2 = 0$. The linear terms of the Jacobian determinant generally yield the relations

$$e_1^2 b_2 c_2 = 4 d_1 f_1 b_2 c_2.$$

Since G is injective, b_2 and c_2 cannot both be 0. So G_2 vanishes on a line, ℓ_0.

If ℓ_0 and ℓ_1 are linearly independent, then assume $d_1 = 1$ and $e_1 = f_1 = 0$ (sending ℓ_0 to the y-axis). The Jacobian condition shows that $c_2 = 0$ and this is a Hénon mapping.

Otherwise assume that $f_1 = 1$ and $d_1 = e_1 = 0$ by sending ℓ_0 to the x-axis. The Jacobian condition shows that $b_2 = 0$. So G is of the form

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 x + c_1 y + y^2 \\ a_2 \qquad + c_2 y \end{pmatrix}.$$

Note that $b_1 \neq 0$ and $c_2 \neq 0$ are invariant under conjugations which do not introduce new terms. The different cases are listed below:

$$c_2 \neq 1, \quad b_1 = c_2, \quad 2a_2 + c_1(1 - c_2) = 0 \text{ yields B with } b = d = b_1 = c_2,$$
$$c_2 \neq 1, \quad b_1 = c_2, \quad 2a_2 + c_1(1 - c_2) \neq 0 \text{ yields A with } b = b_1 = c_2,$$
$$c_2 \neq 1, \quad b_1 \neq c_2, \quad b_1 \neq 1 \text{ yields B with } b = b_1,\ d = c_2,$$
$$c_2 \neq 1, \quad b_1 = 1, \quad a_1(1 - c_2)^2 + a_2(1 - c_2) + a_2^2 = 0 \text{ yields B with } b = 1,\ d = c_2,$$

$c_2 \neq 1$, $b_1 = 1$, $a_1(1 - c_2)^2 + a_2(1 - c_2) + a_2^2 \neq 0$ yields C with $d = c_2$,

$c_2 = 1$, $a_2 \neq 0$ yields D with $b = b_1$,

$c_2 = 1$, $a_2 = 0$, $b_1 \neq 1$ yields B with $b = b_1$, $d = 1$,

$c_2 = 1$, $a_2 = 0$, $b_1 = 1$, $4a_1 - c_1^2 = 0$ yields B with $b = d = 1$,

$c_2 = 1$, $a_2 = 0$, $b_1 = 1$, $4a_1 - c_1^2 \neq 0$ yields C with $d = 1$. □

Remarks. — The Hénon family of mappings can be written in different forms. For example, Hénon ([Hé1] and [Hé2]) actually studied the family

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 1 - \alpha x^2 \\ \beta x \end{pmatrix}, \quad \alpha\beta \neq 0.$$

Note that $F_{a,c}$ is conjugate to $H_{-c,-a}$. Thus, mappings of the form $F_{a,0}$ were omitted in this other form.

A fixed line is a line which is mapped onto itself (but not necessarily pointwise). Consider the set of lines $y = k$ for all $k \in \mathbf{C}$. Elementary mappings can be understood by how they map these lines: all lines fixed, a unique fixed line, or no fixed line. A fixed line may be fixed pointwise, or there may be a unique fixed point or no fixed point.

3. Solenoidal mappings

This section gives a classification up to conjugacy of *unbraided solenoidal mappings*, $\tau : \mathbf{T} \to \mathbf{T}$ of degree d, satisfying appropriate expansion properties and topological conditions. Solenoidal mappings, which are defined below, are injective mappings of degree $d \geqslant 2$ of the solid torus. The images of such mappings can be braided and quite complicated. We only understand how to classify those which are unbraided.

We will show that up to conjugacy, such mappings, when they are appropriately expanding and contracting, are classified by an integer. Only one of these mappings seems relevant to the study of Hénon mappings. On the other hand, the authors puzzled about these mappings quite a bit while understanding the structure of Hénon mappings, and we feel that it will be clearer if we study them all, if only to contrast the relevant one to the others.

Theorem 3.1 holds for arbitrary mappings of degree d while Propositions 3.3, 3.5, and 3.6 require the mappings to be solenoidal. The construction of solenoids is given before Proposition 3.6. Proposition 3.7 shows that solenoidal mappings of degree 2 are unbraided while Proposition 3.8 requires unbraidedness. Theorem 3.11 is the classification of conjugacy classes.

Theorem 3.11 reduces the determination of a *conjugacy* class to the computation of an *isotopy* class and the verification of a hyperbolicity condition.

Solenoidal mappings. Let D be the disk of radius 2, $\mathbf{T} = S^1 \times D$, and denote by (ζ, z) the coordinates in $\mathbf{T}$.

Definition. — Let C_+ and C_- denote the constant families of cones

$$C_+(\zeta, z) = \{(\xi, u \mid |\xi| \geqslant |u|\} \quad \text{and} \quad C_-(\zeta, z) = \{(\xi, u) \mid |\xi| \leqslant |u|\}$$

in the tangent bundle of $\mathbf{T}$.

Definition. — A *solenoidal mapping* $\tau : \mathbf{T} \to \mathbf{T}$ of degree d is an injective C^1 immersion of degree d, such that, for all $(\zeta, z) \in \mathbf{T}$ and for some constant $K > 1$,

$$d_{(\zeta, z)}\, \tau(C_+(\zeta, z)) \subset C_+(\tau(\zeta, z)),$$

$$(\xi, u) \in C_+(\zeta, z) \text{ and } d_{(\zeta, z)}\, \tau(\xi, u) = (\xi_1, u_1) \text{ imply } |\xi_1| > K\,|\xi|$$

and

$$(\xi, u) \in C_-(\zeta, z) \text{ and } d_{(\zeta, z)}\, \tau(\xi, u) = (\xi_1, u_1) \text{ imply } |u_1| < \frac{1}{K}\,|u|.$$

Remark. — The definition says roughly that the derivatives of a solenoidal mapping preserve the family of cones C_+ and are expanding in the ξ direction and contracting in the u direction in C_+. From the fact that τ is an immersion it follows that the inverses of the derivatives of a solenoidal mapping preserve the family of cones C_-.

Examples. — Let $S^1 = \{\zeta \in \mathbf{C} \mid |\zeta| = 1\}$, $D = \{z \in \mathbf{C} \mid |z| \leqslant 2\}$, and $\mathbf{T} = S^1 \times D$. Define $e_1 = \{1\} \times \partial D$ and $e_2 = S^1 \times \{2\}$, each oriented by the counterclockwise orientation of the circle. We will examine very closely the following mappings, $\tau_{d,k} : \mathbf{T} \to \mathbf{T}$, which are unbraided solenoidal for every integer $d \geqslant 2$ and $k \in \mathbf{Z}$:

$$\tau_{d,k}\begin{pmatrix}\zeta \\ z\end{pmatrix} = \begin{pmatrix}\zeta^d \\ \zeta + \varepsilon z \zeta^{k+1-d}\end{pmatrix}.$$

The reason for the shift in the exponent will become clear later: $\tau_{d,0}$ has much nicer properties than the others.

Theorem **3.1.** — *For every mapping $f : \mathbf{T} \to \mathbf{T}$ of degree $d \geqslant 2$, there exists exactly $d - 1$ continuous functions $\pi : \mathbf{T} \to \mathbf{R}/\mathbf{Z}$ of degree 1 such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{f} & \mathbf{T} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbf{R}/\mathbf{Z} & \xrightarrow{z \mapsto z^d} & \mathbf{R}/\mathbf{Z} \end{array}$$

For any two such mappings π_1 and π_2, there exists ω with $\omega^{d-1} = 1$ and $\pi_1 = \omega\pi_2$.

Proof. — By the Lefschetz Fixed Point Theorem ([D]), f has $1 - d$ fixed points counted with multiplicity. Since $1 - d \neq 0$, there is at least one fixed point t_0.

To avoid difficulties with branches of d-th roots, lift f to $\tilde{f} : \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}$, where

$\tilde{\mathbf{T}} = \mathbf{R} \times \mathrm{D}$ is the universal covering space of $\mathbf{T}$, with base point t_0 and the lift $\tilde{f} : \tilde{\mathbf{T}} \to \mathbf{T}$ satisfying $\tilde{f}(\tilde{t}_0) = \tilde{t}_0$. Let $\gamma : \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}$ be a generator of the fundamental group. Then $\tilde{f}(\gamma(\tilde{t})) = \gamma^{\circ d}(\tilde{f}(\tilde{t}))$ for every $\tilde{t} \in \tilde{\mathbf{T}}$.

Consider the space

$$\Pi = \{ \tilde{\pi} : \tilde{\mathbf{T}} \to \mathbf{R} \mid \tilde{\pi} \text{ is continuous and } \tilde{\pi}(\gamma(t)) = \tilde{\pi}(t) + 1 \}$$

with the uniform metric, δ, which is well-defined because of the periodicity. In this metric Π is a complete metric space. Define a mapping $\varphi : \Pi \to \Pi$ by

$$\varphi(\tilde{\pi})\,(\tilde{t}) = \frac{1}{d}\,\tilde{\pi}(\tilde{f}(\tilde{t})).$$

If $\tilde{\pi}$ satisfies $\tilde{\pi}(\tilde{f}(\tilde{t})) = \tilde{\pi}(\tilde{t}) + 1$, then so does $\varphi(\tilde{\pi})$.

Lemma **3.2.** — *The mapping φ is strongly contracting.*

Proof. — If $\tilde{\pi}_1, \tilde{\pi}_2 \in \Pi$, then

$$|\,\varphi(\tilde{\pi}_1)\,(\tilde{t}) - \varphi(\tilde{\pi}_2)\,(\tilde{t})\,| = \frac{1}{d}\,|\,\tilde{\pi}_1(\tilde{f}(\tilde{t})) - \tilde{\pi}_2(\tilde{f}(\tilde{t}))\,| \leqslant \frac{1}{d}\,\delta(\tilde{\pi}_1, \tilde{\pi}_2).$$

□ (Lemma 3.2)

Let $\tilde{\pi}_0$ be the fixed point of φ and π_0 be the mapping $\mathbf{T} \to \mathbf{R}/\mathbf{Z}$ induced by $\tilde{\pi}_0$. Clearly the mappings $\pi_k = e^{2\pi i k/(d-1)}\,\pi_0$ still semi-conjugate f to $z \mapsto z^d$.

If $\pi' : \mathbf{T} \to \mathbf{R}/\mathbf{Z}$ is any mapping making the diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\ f\ } & \mathbf{T} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathbf{R}/\mathbf{Z} & \xrightarrow{\ z \mapsto z^d\ } & \mathbf{R}/\mathbf{Z} \end{array}$$

commute, then $\pi'(t_0)$ is a fixed point of $z \mapsto z^d$, so it must be one of the $(d-1)$-th roots of 1 and there exists k with $\pi'(t_0) = \pi_k(t_0)$. Now $\pi' = \pi_k$, since the lift of $e^{-2\pi i k/(d-1)}\,\pi'$ in Π is a fixed point of φ, hence is π_0. □ (Theorem 3.1)

Proposition **3.3.** — *If a mapping f of degree d is solenoidal, then the mappings $\pi : \mathbf{T} \to \mathbf{R}/\mathbf{Z}$ in Theorem 3.1 are fibrations with fibers homeomorphic to disks.*

Proof. — Define $\Pi_0 \subset \Pi$, the family of Lipshitz fibrations consisting of those elements of Π whose fibers are disks which are graphs of Lipshitz functions $\alpha : \mathrm{D} \to \mathbf{R}$ (i.e. $|\,\alpha(z_1) - \alpha(z_2)\,| \leqslant |\,z_1 - z_2\,|$). Since $\tilde{f}^{-1}$ preserves the family of cones C_-, the family of Lipshitz fibrations is stable under φ. So the fixed point π_0 is a limit of Lipshitz fibrations.

The space Π_0 is not closed in Π, but the fibers of $\pi \in \overline{\Pi}_0$ are fairly easy to understand.

Lemma **3.4.** — *For any $\pi = \lim \pi_j$, with $\pi_j \in \Pi_0$, and for any $x \in \mathbf{R}$, there exist two Lipshitz functions $\alpha_1(z) \leqslant \alpha_2(z)$ such that $\pi^{-1}(x) = \{(y, z) \mid \alpha_1(z) \leqslant y \leqslant \alpha_2(z)\}$.*

Proof. — We have

$$\pi^{-1}(x_0) = \bigcap_{\varepsilon>0} \overline{\bigcup_{i>0} \bigcap_{j\geqslant i} \pi_j^{-1}([x_0-\varepsilon, x_0+\varepsilon])}.$$

Decreasing intersections of sets of the form $\pi^{-1}(x) = \{(y, z) \mid \alpha_1(z) \leqslant y \leqslant \alpha_2(z)\}$ are still of this form. Hence it is enough to show that

$$\overline{\bigcup_{i>0} \bigcap_{j\geqslant i} \pi_j^{-1}([x_0-\varepsilon, x_0+\varepsilon])}$$

is of this form. Now suppose that

$$\pi_j^{-1}([x_0-\varepsilon, x_0+\varepsilon]) = \{(y, z) \mid \alpha_{j,\varepsilon}(z) \leqslant y \leqslant \beta_{j,\varepsilon}(z)\}.$$

Then

$$\overline{\bigcup_{i>0} \bigcap_{j\geqslant i} \pi_j^{-1}([x_0-\varepsilon, x_0+\varepsilon])} = \{(y, z) \mid \limsup_{j\to\infty} \alpha_{j,\varepsilon}(z) \leqslant y \leqslant \liminf_{j\to\infty} \beta_{j,\varepsilon}(z)\}.$$

We need to show that $\alpha_{j,\varepsilon}(z) \leqslant \beta_{j,\varepsilon}(z)$ for all j and ε sufficiently large. For any fixed z, the function $\tilde{y} \mapsto \tilde{\pi}_0(\tilde{y}, z)$ is surjective, so there exists $\tilde{y}_0$ with $\tilde{\pi}_0(\tilde{y}_0, z) = x_0$. Choose I so large that $|\tilde{\pi}_j - \tilde{\pi}_0| < \varepsilon$ for $j \geqslant i$. Then $\tilde{\pi}_j(\tilde{y}_0, z) \in [x_0-\varepsilon, x_0+\varepsilon]$. So for all $j \geqslant i$, $\alpha_{j,\varepsilon}(z) \leqslant y_0 \leqslant \beta_{j,\varepsilon}(z)$. □ (Lemma 3.4)

Now if a fiber is not a Lipshitz disk, then it has nonempty interior. Also the fibers are compact and their projections onto $\mathbf{R}$ have bounded length. If any two points (x_1, z_1) and (x_2, z_2) satisfy $|z_2 - z_1| < x_2 - x_1$ and $(x'_i, z'_i) = \tilde{f}(x_i, z_i)$, then $x'_2 - x'_1 \geqslant K(x_2 - x_1)$ for some $K > 1$.

If a fiber is not a Lipshitz disk, then let (x_1, z_1) and (x_2, z_2) be two points of the interior satisfying $|z_2 - z_1| < x_2 - x_1$. Let $(x_i^{(n)}, z_i^{(n)}) = \tilde{f}^{\circ n}(x_i, z_i)$. These are still in the same fiber and $x_2^{(n)} - x_1^{(n)} \geqslant K^n(x_2 - x_1)$ for some $K > 1$. This contradicts that the length of the projection onto $\mathbf{R}$ of the fiber is finite. □ (Proposition 3.3)

Proposition **3.5.** — *The components of* $f^{\circ k}(\mathbf{T}) \cap \pi^{-1}(z)$ *have diameters tending to* 0 *as* $k \to \infty$ *for all* $z \in S^1$.

Proof. — Suppose that $x, y \in f^k(\mathbf{T})$ are in the same fiber and realize the maximal vertical diameter, d_k, of $f^k(\mathbf{T})$. Compare this with the maximal vertical diameter, d_{k-1}, of $f^{k-1}(\mathbf{T})$. Let $x_1 = f^{-1}(x)$ and $y_1 = f^{-1}(y)$ and let ℓ be the straight line joining them.

$$d_{k-1} \geqslant \int_\ell |dz| \geqslant K \int_{f(\ell)} |dz| \geqslant K \int_x^y |dz| \geqslant K\, d_k. \quad \square$$

The solenoid Σ_d. Given a space X with a mapping $f\colon X \to X$, consider the *projective limit*

$$\hat{X}_f = \varprojlim(X, f) = \{(\ldots, x_2, x_1, x_0) \mid f(x_{i+1}) = x_i \quad \text{for } i = 0, 1, 2, \ldots\}.$$

When the mapping f is clear, simply write $\hat{X}$ instead of $\hat{X}_f$. This construction is sometimes referred to as the inverse limit construction. A point of this projective limit is a point of $x_0 \in X$ along with a " history of the point " under the iteration of f.

The mapping f induces $\hat{f}: \hat{X} \to \hat{X}$ by

$$\hat{f}(\ldots, x_2, x_1, x_0) = (\ldots, f(x_2), f(x_1), f(x_0)) = (\ldots, x_1, x_0, f(x_0))$$

which is always *bijective* as

$$\hat{f}^{-1}(\ldots, x_2, x_1, x_0) = (\ldots, x_2, x_1).$$

Consider the projective limit of the mapping $\delta: S^1 \to S^1$ given by $\delta(\zeta) = \zeta^d$. Define $\Sigma_d = \varprojlim(S^1, \delta)$ and the bijection $\hat{\delta}: \Sigma_d \to \Sigma_d$ as in the introduction above. This construction was studied carefully by Williams ([W]). The solenoid was first studied by Vietoris ([V]) and van Danzig ([vD]).

Let π_f be one of the $d - 1$ mappings guaranteed by Theorem 3.1.

Proposition **3.6.** — *Let* $\Sigma_f = \bigcap_n f^{\circ n}(\mathbf{T})$. *The mapping*

$$x \mapsto (\ldots, \pi_f(f^{\circ -2}(x)), \pi_f(f^{-1}(x)), \pi_f(x), \pi_f(f(x)))$$

is a homeomorphism $h_\Sigma : \Sigma_f \to \Sigma_d$.

Proof. — Let $\mathbf{z} = (\ldots, z_1, z_0)$ be a point of Σ, and define

$$X_{\mathbf{z},k} = \{ x \in \widetilde{\mathbf{T}} \mid f^{-i}(x) \text{ is defined and } \pi_f(f^{-i}(x)) = z_i \text{ for } 0 \leqslant i \leqslant k \}.$$

Then $f^{\circ k}$ maps $\pi_f^{-1}(z_k)$ bijectively to $X_{\mathbf{z},k}$. In particular, $X_{\mathbf{z},k}$ is a component of $f^{\circ k}(\widetilde{\mathbf{T}}) \cap \pi_f^{-1}(z_0)$. Since these components have diameters tending to 0 as $k \to \infty$, and

$$h_\Sigma(\mathbf{z}) = \bigcap_k X_{\mathbf{z},k},$$

we see that h_Σ is bijective. It is continuous, and the domain is compact, so it is a homeomorphism. □

Definition. — An injective mapping $\tau: \mathbf{T} \to \mathbf{T}$ of degree d is *unbraided* if there exists a fiber homeomorphism $\varphi: \mathbf{T} \to \mathbf{T}$ such that $\varphi \circ \tau$ sends the core circle $S^1 \times \{0\}$ into $S^1 \times S^1$ as a $(d, 1)$-torus (un)knot.

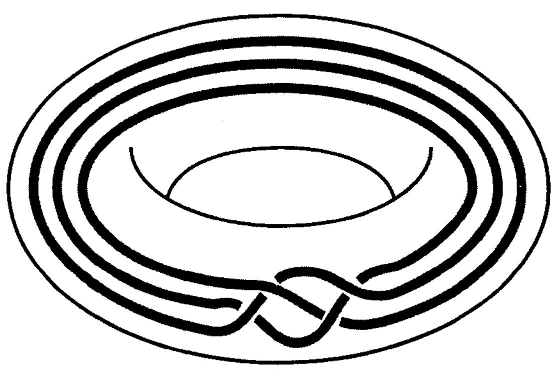


Fig. 3.1

Remark. — No embedding of $\mathbf{T}$ in S^3 is specified. In particular, the $(d, 1)$-torus (un)knot and $(d, d+1)$-torus knot are equivalent from this point of view. Proposition 3.7 shows that if $d = 2$, all solenoidal mappings are unbraided; this is false for $d \geqslant 3$. See Figure 3.1.

Proposition **3.7.** — *For degree $d = 2$, all solenoidal mappings are unbraided.*

Proof. — We will show that there exists a fiber homeomorphism

$$g : (\mathbf{T}, f(S^1 \times \{0\})) \to (\mathbf{T}, \tau_{2,k}(S^1 \times \{0\})).$$

Note that the pair of sets $(\mathbf{T}, \tau_{2,k}(S^1 \times \{0\}))$ is independent of k.

Both $(\mathbf{T}, f(S^1 \times \{0\}))$ and $(\mathbf{T}, \tau_{2,k}(S^1 \times \{0\}))$ are locally trivial fiber bundles over S^1 with fibers homeomorphic to disks with two marked points. Locally trivial fiber bundles over S^1 are classified by the isotopy classes of their monodromy. For the bundles under consideration the monodromy homeomorphisms lie in $\mathcal{H}omeo\ (D, \{a, b\})$.

The mapping $\mathcal{H}omeo\ (D, \{a, b\}) \to \mathcal{P}erm\{a, b\}$ (the latter being the symmetric group on two elements) is surjective with contractible fibers ([Ha]). So the isotopy class of the monodromy of these bundles depends only on how they permute the boundary components. For both bundles, the points are exchanged. So the bundles are fiber homeomorphic. □

Proposition **3.8.** — *For every unbraided solenoidal mapping f there exists a unique integer k and a mapping $h : \mathbf{T} - \mathrm{int}(f(\mathbf{T})) \to \mathbf{T} - \mathrm{int}(\tau_{d,k}(\mathbf{T}))$ such that the following diagram commutes:*

$$\textbf{(3.9)}\qquad \begin{array}{ccc} \partial\mathbf{T} & \xrightarrow{\ h\ } & \partial\mathbf{T} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau_{d,k}} \\ \partial f(\mathbf{T}) & \xrightarrow[h]{} & \partial\tau_{d,k}(\mathbf{T}) \end{array}$$

Proof. — **Step 1.** There exists a fiber homeomorphism

$$g : \mathbf{T} - \mathrm{int}(f(\mathbf{T})) \to \mathbf{T} - \mathrm{int}(\tau_{d,k}(\mathbf{T}))$$

mapping $\partial\mathbf{T}$ to $\partial\mathbf{T}$. Note that the set $\mathbf{T} - \mathrm{int}(\tau_{d,k}(\mathbf{T}))$ is independent of k.

The definition of unbraided says that the bundles of pairs $(\mathbf{T}, f(S^1 \times \{0\}))$ and $(\mathbf{T}, \tau_{d,k}(S^1 \times \{0\}))$ are fiber-homeomorphic. Pick a base-point in S^1, and let D be the fiber above that base-point, $\{a_1, \ldots, a_d\} = D \cap f(S^1 \times \{0\})$, with the points ordered along the circle $f(S^1 \times \{0\})$, $\{b_1, \ldots, b_d\} = D \cap \tau_{d,k}(S^1 \times \{0\})$ ordered similarly. Then the bundles of pairs above are classified by their monodromies m_f and $m_{d,k}$. The definition of unbraided says that there exists a homeomorphism $\varphi : D \to D$ with $\varphi(\{a_1, \ldots, a_d\}) = \{b_1, \ldots, b_d\}$ and conjugating m_f to $m_{d,k}$.

Let $U_1, \ldots, U_d$ be the components of $D \cap f(\mathbf{T})$, labeled so that $a_i \in U_i$, and $V_1, \ldots, V_d$ be the components of $D \cap \tau_{d,k}(\mathbf{T})$, labeled so that $b_i \in V_i$. Now deform φ

so that $\varphi(U_i) = V_i$. This is (unpleasant) 2-dimensional topology. First adjust ε in the definition of $\tau_{d,k}$ so that $\varphi(U_i) \cap V_j = \emptyset$ for $i \neq j$. Therefore the sets $V_i \amalg \varphi(V_i)$ have disjoint neighborhoods D_i homeomorphic to disks.

Lemma **3.10.** — *If* U *and* V *are closed subsets of the open unit disk* D, *with* $0 \in \mathring{U} \cap \mathring{V}$ *and both homeomorphic to closed disks, then there exists a homeomorphism* $\psi : D \to D$ *which is the identity on* ∂D *and with* $\psi(U) = V$.

Proof. — Use conformal mapping to represent both $D - U$ and $D - V$ as standard annuli, giving a system of " polar coordinates " where the radial curves are labeled by the points at which they intersect ∂D and the circular curves by their relative distance to ∂D. Then making points with the same coordinates in $D - U$ and $D - V$ correspond gives a homeomorphism of $\bar{D} - \mathring{U}$ onto $\bar{V} - \mathring{V}$. This can be continued to U and V since any homeomorphism of the boundary of a disk extends to a homeomorphism of the interior, by radial extension, for instance. □ (Lemma 3.10)

Find a homeomorphism $\psi : (D, \{b_1, \ldots, b_d\}) \to V$ such that $\psi \circ \varphi$ is isotopic to φ and $\psi \circ \varphi(U_i) = V_i$. Unfortunately, this mapping does not now conjugate the monodromies, but it does up to isotopy, and that is enough, since bundles are classified by the isotopy classes of their monodromy.

Step 2. Next it will be shown that k can be chosen so that diagram (3.9) commutes on the level of homology. The homology group $H_1(\partial \mathbf{T})$ is isomorphic to $\mathbf{Z}^2$. Choose the basis $\{S^1 \times \{2\}, \{1\} \times 2S^1\}$, the circles oriented counterclockwise in **C**.

Consider the mapping $g \circ f^{-1} \circ g^{-1} \circ \tau_0$. This is a fiber homeomorphism $\partial \mathbf{T} \to \partial \mathbf{T}$, hence induces a mapping given by a matrix $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$ for some integer ℓ on $H_1(\partial \mathbf{T})$. Since the construction of g is unique up to isotopy, ℓ is an invariant of f.

Observe that $\tau_{d,k}$ can be written $\tau_{d,0} \circ w^k$, where w is the *twist* mapping

$$w(\zeta, z) = (\zeta, \zeta z).$$

So $g \circ f^{-1} \circ g^{-1} \circ \tau_{d,k} = g \circ f^{-1} \circ g^{-1} \circ \tau_{d,0} \circ w^k$. Since w^k induces the mapping given by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ on $H_1(\partial \mathbf{T})$, set $k = -\ell$ so that $g \circ f^{-1} \circ g^{-1} \circ \tau_{d,k}$ induces the identity on the homology.

Step 3. Finally, adjust g into h so that the desired diagram commutes.

There exists a homotopy $G_t : \partial \mathbf{T} \times I \to \partial \mathbf{T}$ with $G_0 = \tau_{d,k}^{-1} \circ g \circ f \circ g^{-1}$ and $G_1 = \mathrm{Id}$. Let $U \subset \mathbf{T} - \mathrm{int}(\tau_{d,k}(\mathbf{T}))$ be a narrow thickening of $\partial \mathbf{T}$ homeomorphic to $I \times \partial \mathbf{T}$. Denote points in U by (t, x). Let

$$h(y) = \begin{cases} g(y), & y \notin g^{-1}(U) \\ G_t(x), & y \in g^{-1}(U) \text{ so that } g(y) = (t, x) \end{cases}$$

on $\mathbf{T} - \operatorname{int}(f(\mathbf{T}))$. Now for $y \in \partial\mathbf{T}$, set $x = g(y)$ and compute

$$(f^{-1} \circ h^{-1} \circ \tau_{d,k} \circ h)\,(y) = f^{-1} \circ g^{-1} \circ \tau_{d,k} \circ (\tau_{d,k}^{-1} \circ g \circ f \circ g^{-1})\,(x) = g^{-1}(x) = y. \quad \square \text{ (Proposition 3.8)}$$

Next comes the classification of the conjugacy classes of unbraided solenoidal mappings.

Theorem **3.11.** — *Every unbraided solenoidal mapping is conjugate to one of the $\tau_{d,k}$, and no two of these are conjugate.*

Proof. — The second part was proved above, when it was shown that different values of k lead to different values of ℓ, which are conjugacy invariants.

We wish to extend h from Proposition 3.8 to $\mathbf{T}$. Take $x \in \mathbf{T}$. If $x \in \Sigma_f$, then define $h(x) = \sigma_{\tau_{d,k}}^{-1} \circ \sigma_f \in \Sigma_{\tau_{d,k}}$ and if $x \notin \Sigma_f$, then define $h(x) = \tau_{d,k}^{\circ m} \circ g \circ \tau_{d,k}^{\circ -m}(x)$, where m is such that $f^{\circ -m}(x) \in \mathbf{T} - \operatorname{int}(f(\mathbf{T}))$. If $f^{\circ -m}(x) \in \partial f(\mathbf{T})$ so that $f^{\circ -(m+1)}(x) \in \partial\mathbf{T}$, then both choices, m and $m+1$, give the same value of h by Proposition 3.8. So the mapping is well-defined, bijective, and conjugates f to $\tau_{d,k}$.

It remains to show that h is continuous on Σ_f. Take $x_0 \in \Sigma_f$. The sets

$$\mathrm{U}_{\mathrm{N},\varepsilon}(x_0) = \{ x \in \mathbf{T} \mid f^{\circ -n}(x) \text{ exists for all } n \leqslant \mathrm{N} \text{ and } |\, \pi_f(f^{\circ -n}(x)) - \pi_f(f^{\circ -n}(x_0)) \,| \leqslant \varepsilon \text{ for } n \leqslant \mathrm{N} \}$$

form a basis of closed neighborhoods of x_0 as $\varepsilon \to 0$ and $\mathrm{N} \to \infty$. Clearly h maps this basis of neighborhoods of x_0 to the corresponding basis of neighborhoods of $f(x_0)$. So h is continuous and hence a homeomorphism. $\square$

4. Embeddings of the solenoid in S^3

In this section we will try to describe the inductive limit of $\mathbf{T}$ under $\tau_{d,0}$. Intuitively, this corresponds to taking a solid torus winding around d times in a larger torus which winds around d times in a yet larger torus, etc., and taking the increasing union. This intuitive picture is ambiguous. To make this precise, the embedding mapping each torus into the next must be specified. This is made precise in this section. Smale first studied solenoids as hyperbolic attractors in S^3 ([Sm]).

Recall the mappings $\tau_{d,k}$ from the example near the beginning of section 3.

Proposition **4.1.** — *The mappings $\tau_{d,0}$ extend to orientation-preserving homeomorphisms $h_d : S^3 \to S^3$.*

Remark. — Note that $\tau_{d,k}$ obviously extends to S^3 for some k. After all, one can take a solid torus (think of a bicycle tire tube) and wrap it d times around itself. The outside of the unwound tube and of the wound tube are both unknotted tori, so there exists a homeomorphism between them. This homeomorphism will map the inner rim of the tube to some curve on the wound tube; the object of this proposition is to describe this curve. The skeptical reader might experiment with a tube for $d = 2, 3$.

The proof of Proposition 4.1 depends on the following:

Lemma **4.2.** — *If* T_1 *and* T_2 *are two solid tori, and* $f: \partial T_1 \to \partial T_2$ *is a homeomorphism which sends curves on* ∂T_1 *which bound disks in* T_1 *into curves which bound disks in* T_2, *then* f *extends to a homeomorphism* $T_1 \to T_2$.

Proof of Lemma 4.2. — We may suppose $T_1 = T_2 = \mathbf{T} = S^1 \times D$. The homeomorphisms of a torus are classified up to isotopy by their action on 1-dimensional homology. If a homeomorphism of $\partial\mathbf{T}$ extends to $\mathbf{T}$, then any isotopic homeomorphism extends also. Clearly the linear homeomorphisms mapping curves of the form $\{\zeta\} \times \partial D$ extend. □ (Lemma 4.2)

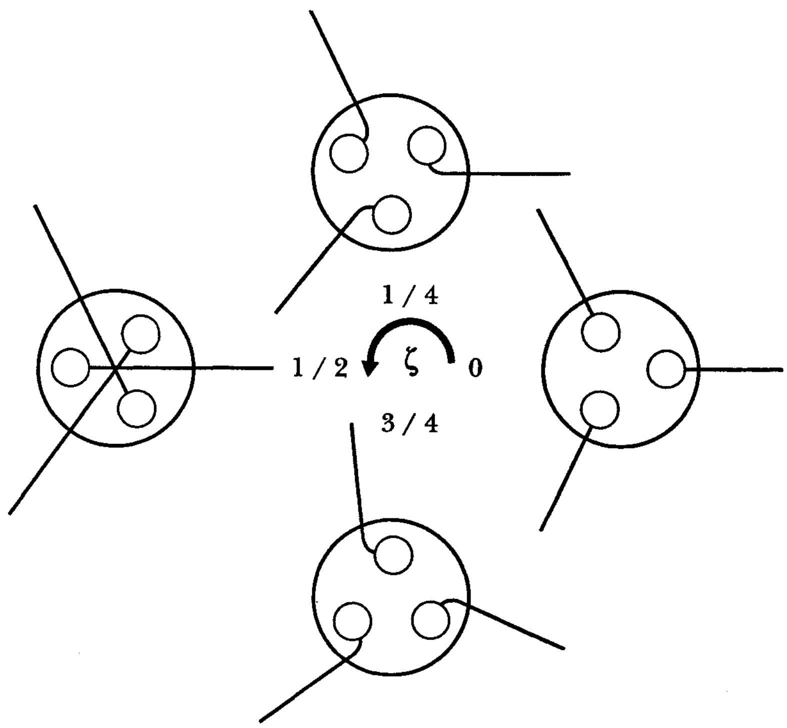


Fig. 4.1

Proof of Proposition 4.1. — The key point is that $\tau_{d,0}$ maps curves on $\partial\mathbf{T}$ which bound embedded disks in $S^3 - \mathbf{T}$ into curves which bound embedded disks in $S^3 - \tau_{d,0}(\mathbf{T})$. This can be seen in Figure 4.1. This is a drawing of $S^1 \times D$, with the disks $\{1\} \times D$, $\{i\} \times D$, $\{-1\} \times D$, $\{-i\} \times D$; the reader is expected to fill in the other slices. Within these disks are d subdisks. The case $d = 3$ is represented, and the triangle formed by these three subdisks rotates by 1/3 of a turn while going around S^1 once. Thus these subdisks represent $\tau_{d,k}(\mathbf{T})$.

The curves drawn on the outside of the disks represent a disk X in $S^3 - \tau_{d,k}(\mathbf{T})$. Verifying that this is indeed a disk is the essence of the proof. We leave to the reader to verify that X is a manifold with boundary $\partial X \subset \partial\tau_{d,k}(\mathbf{T})$. To see that this manifold is simply connected, notice that it is clearly a deformation retract of the subset consisting of $X \cap S^3 - \mathrm{int}(S^3 - \mathbf{T})$, and the star above -1. This is a contractible set: d disks, each with a leash and all leashes connected at one point, as in Figure 4.2.

Now going around S^1 once, the angle at which X touches a subdisk rotates by $-(d-1)/d$, so that altogether ∂X is the curve

$$e^{2\pi i t} \mapsto (e^{2d\pi i t}, e^{2\pi i t} + \varepsilon e^{2\pi i t(1-d)}).$$

The mapping $\tau_{d,0}$ maps e_2 (a curve bounding in $S^3 - \mathbf{T}$) to this curve, so $\tau_{d,0}$ extends as required, by Lemma 4.2. □ (Proposition 4.1)

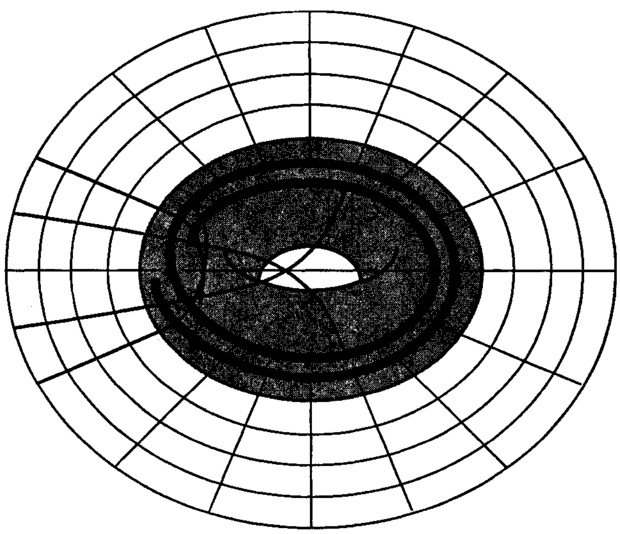


FIG. 4.2

Reflections with respect to a torus. A different way of understanding the extension of $\tau_{d,0}$ to S^3 will be given requiring a definition of reflection with respect to a torus.

The simplest context in which to describe such reflections is to write

$$S^3 = \{(u, v) \mid |u|^2 + |v|^2 = 1\}.$$

Then $S^3 = T' \cup T''$,

where $$T' = \{(u, v) \in S^3 \mid |u| \leqslant 1/\sqrt{2}\} \quad \text{and} \quad T'' = \{(u, v) \in S^3 \mid |v| \leqslant 1/\sqrt{2}\}.$$

These are two unknotted solid tori, and clearly they are exchanged by the mapping

$$\rho : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} v \\ u \end{pmatrix}.$$

To give an intuitive description, we will work in $S^3 = \mathbf{R}^3 \cup \{\infty\}$. So this mapping needs to be translated into a mapping $\mathbf{R}^3 \cup \{\infty\} \to \mathbf{R}^3 \cup \{\infty\}$. Stereographic projection from the point $(0, i)$ maps S^3 to $\mathbf{R}^3 \cup \infty$ according to the formula

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \operatorname{Re} u/(1 - \operatorname{Im} v) \\ \operatorname{Im} u/(1 - \operatorname{Im} v) \\ \operatorname{Re} v/(1 - \operatorname{Im} v) \end{pmatrix}.$$

This mapping takes the torus $|u| = e^{i\theta_1}/\sqrt{2}$, $|v| = e^{i\theta_2}/\sqrt{2}$ to the parametrized torus in $\mathbf{R}^3$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta_1/(\sqrt{2} - \sin\theta_2) \\ \sin\theta_1/(\sqrt{2} - \sin\theta_2) \\ \cos\theta_2/(\sqrt{2} - \sin\theta_2) \end{pmatrix},$$

which just happens to be the torus of revolution obtained by rotating the circle of radius 1 centered at $(\sqrt{2}, 0)$ in the (x, z)-plane around the z-axis. Conjugated by this change of variables, the mapping ρ becomes

$$\rho_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{1}{x^2 + (y-1)^2 + z^2} \begin{pmatrix} 2z \\ x^2 + y^2 + z^2 - 1 \\ 2x \end{pmatrix}.$$

Note that ρ_1 commutes with reflection in the y-axis.

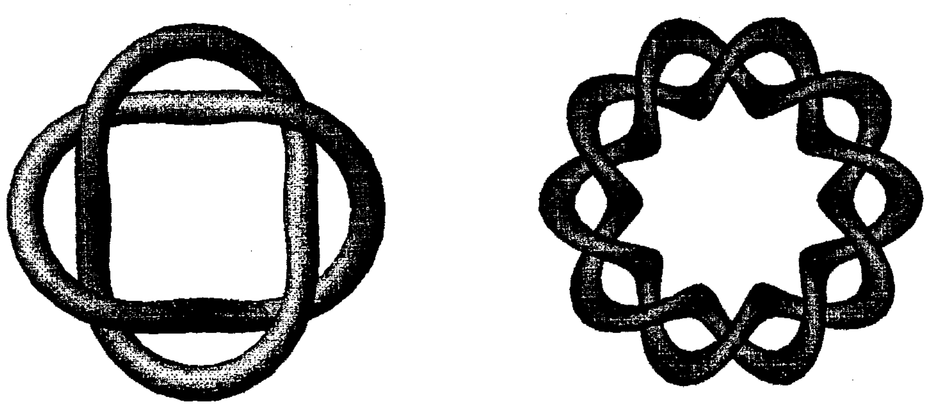


FIG. 4.3

Construction of h_d. Consider two unknotted solid tori $\mathbf{T}_0$ and $\mathbf{T}_1$ embedded in $\mathbf{R}^3$, linked with linking number d, as in Figure 4.3. Then $R_d : S^3 \to S^3$, the rotation by π/d around the z-axis, is a homeomorphism of each onto the other.

For any homeomorphism $\alpha : S^3 \to S^3$ set $\rho_\alpha = \alpha^{-1} \circ \rho_1 \circ \alpha$.

Proposition **4.3.** — *There exists an orientation-preserving homeomorphism*

$$\alpha : \mathbf{R}^3 \cup \{\infty\} \to \mathbf{R}^3 \cup \{\infty\}$$

such that

a) α *maps* $\mathbf{T}_0$ *to* T';
b) α *commutes with reflection in the* y*-axis*;
c) the restriction of $h_d : \rho_\alpha \circ R_d$ *to* $\mathbf{T}_0$ *is a solenoidal mapping conjugate to* $\tau_{d,0}$;
d) h_d *is conjugate to its inverse.*

Proof. — Fiber both $\mathbf{T}_0$ and $\mathbf{T}_1$ over the circle by the radial angles, as measured from the y-axis, and similarly for T'. Choose first the restriction of the homeomorphism α to $\mathbf{T}_0$, so as to map the slice with a given radial angle of $\mathbf{T}_0$ to the corresponding disk of T' and so that α commutes with symmetry with respect to the y-axis.

Next choose a curve γ winding d times around T', symmetric with respect to the y-axis and such that the " radial angle " of T'' is monotone along the curve, and a small tubular neighborhood S around it. Note that this radial angle of T'' will increase by $2d\pi$ along γ. Fiber S by the radial angle, starting at the highest intersection on the y-axis. See Figure 4.4.

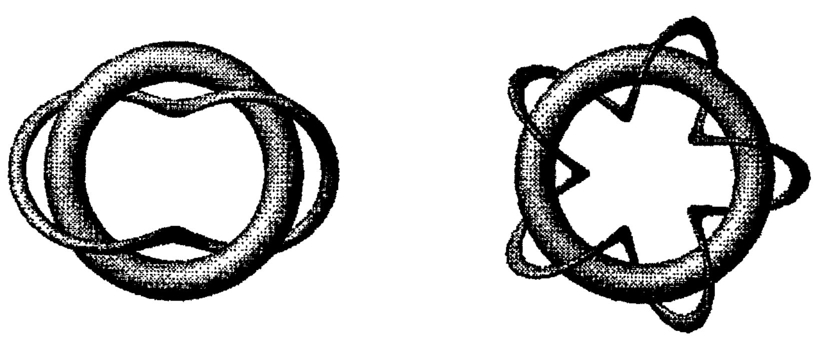


Fig. 4.4

Next, define α on $\mathbf{T}_1$ by sending the slice at a given angle to the slice of S at d times that angle, still preserving the symmetry with respect to the y-axis. Extend the homeomorphism to S^3 so as to preserve the symmetry.

With this choice of α, *a)* and *b)* are clearly true. All the work was designed to satisfy *c)* and *d)*: the restriction of h_d simply multiplies radial angles by d in $\mathbf{T}_0$, hence is expanding in that direction. By choosing the tubular neighborhood S of γ sufficiently thin, h_d can clearly be made contracting in the slices. Since γ is unbraided in T'', the solenoidal mapping $h_d : \mathbf{T}_0 \to \mathbf{T}_0$ is conjugate to $\tau_{d,k}$ for some k, which must be 0 since h_d extends to S^3.

The inverse of h_d is $R_d^{-1} \circ \rho_\alpha$, which is conjugate to $\rho_\alpha \circ R_d^{-1}$. Conjugate the mapping by symmetry around the y-axis. This conjugates R_d to R_d^{-1}, and since the reflection with respect to the y-axis commutes with ρ_α, $R \circ \rho_0$ is conjugated to its inverse. □

This shows that the mapping $h_d : S^3 \to S^3$ is a homeomorphism, which has two invariant solenoids $\Sigma_+ \subset \mathbf{T}_0$ and $\Sigma_- \subset \mathbf{T}_1$, attracting and repelling, respectively; every point is attracted to Σ_+ under forward iteration of h_d and is attracted to Σ_- under iteration of h_d^{-1}.

Inductive limits. Given a space X and a mapping $f: X \to X$, define the inductive limit $\varinjlim(X, f)$ to be

$$\varinjlim(X, f) = X \times \mathbf{N}/\sim,$$

where the equivalence is generated by setting $(x, m) \sim (f(x), m+1)$.

The notion of inductive limit is pathological when f is not injective (the spaces created fail to be Hausdorff). We will use the notion only for injective mappings f, where it really is some sort of increasing union.

Proposition 4.4. — *The inductive limit* $\varinjlim(\mathbf{T}, \tau_{d,0})$ *is homeomorphic to* $S^3 - \Sigma_d$ *and* $\Sigma_d = \bigcap_m \tau_{d,0}^m(\mathbf{T})$.

Proof. — The mapping $(x, m) \mapsto h_d^{-m}(x)$ induces a mapping

$$\varinjlim(\mathbf{T}, \tau_{d,0}) \to \bigcup_m h_d^{-m}(\mathbf{T}_0).$$

The mapping h_d is conjugate to its inverse, and the conjugating mapping is a homeomorphism of $S^3 - h_d^{-m}(\mathbf{T}_0)$ onto $h_d(\mathbf{T}_0)$. □

Corollary 4.5. — *The fundamental group* $\pi_1(S^3 - \Sigma_d)$ *is isomorphic to the additive group* $\mathbf{Z}[1/d]$ *of rational numbers with powers of d in the denominator.*

Proof. — Fundamental groups commute with inductive limits, so, by Proposition 4.4, $\pi_1(S^3 - \Sigma_d)$ is isomorphic to the inductive limit of

$$\mathbf{Z} \xrightarrow{d} \mathbf{Z} \xrightarrow{d} \mathbf{Z} \ldots \quad \square$$

Remark. — It is usually dangerous to speak of fundamental groups without specifying a base point, but in this case the fundamental group is abelian, so there is no ambiguity.

Knots and the mappings $\tau_{d,k}$. We will not need the following results in the sequel, but they may help the reader to understand why the mappings $\tau_{d,k}$ are different. We will only discuss the case $d = 2$, but a similar discussion can be made for arbitrary d, and is a bit simpler in fact when $d > 2$.

Proposition **4.6.** — *The solid tori $\tau^n_{2,k}(\mathbf{T})$ are all unknotted if $k = 0$ and all knotted for $n \geqslant 2$ if $k \neq 0$, except that $\tau^2_{2,-1}(\mathbf{T})$ is unknotted.*

Proof. — The case $k = 0$ is dealt with in Proposition 4.1.

Next show that $\tau^2_{2,k}(\mathbf{T})$ is the $(2, 2k - 1)$ torus knot. This is genuinely knotted unless $k = 0$ or 1. Since $\tau^{n+1}_{2,k}(\mathbf{T})$ is a companion of $\tau^n_{2,k}(\mathbf{T})$, this proves the result for all k except $k = -1$, which requires a separate argument.

Observe that $\tau_{2,k}$ can be written $\tau_{2,0} \circ w^k$, where w is the *twist* $w(\zeta, z) = (\zeta, \zeta z)$. Then $\tau^2_{2,k}(\mathbf{T}) = \tau_{2,0} \circ w^k \circ \tau_{2,0} \circ w^k(\mathbf{T})$. The w^k on the right can be ignored since $\mathbf{T} = w^k(\mathbf{T})$ and since $\tau_{2,0}$ extends to a homeomorphism of S^3, the $\tau_{2,0}$ on the left can be ignored also. The result follows from the computation

$$w^k \circ \tau_{2,0} \begin{pmatrix} \zeta \\ z \end{pmatrix} = \begin{pmatrix} \zeta^2 \\ \zeta^{2k+1} + \varepsilon z \zeta^{2k-1} \end{pmatrix}.$$

The mapping $\zeta \mapsto (\zeta^2, \zeta^{2k+1})$ is a parametrization of the $(2, 2k + 1)$-torus knot, which is indeed knotted unless $2k + 1 = \pm 1$.

To finish the proof, it must be shown that $\tau^3_{2,-1}(\mathbf{T})$ is knotted. As above,

$$\tau^3_{2,-1}(\mathbf{T}) = \tau_{2,0} \circ w^{-1} \circ \tau_{2,0} \circ w^{-1} \circ \tau_{2,0} \circ w^{-1}(\mathbf{T}),$$

and again ignore the w^{-1} on the right and the $\tau_{2,0}$ on the left. The reader may check that the core of the solid torus is then parametrized by $\zeta \mapsto (\zeta^4, \zeta^{-2} + \varepsilon\zeta^{-7})$. We leave it to the reader that this is a parametrization of the $(2, -5)$-torus knot. □

5. The functions $G_\pm$ and $\varphi_\pm$

Recall the generalized Hénon mappings of degree d,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix} = \begin{pmatrix} x^d + q(x) - ay \\ x \end{pmatrix},$$

where $a \neq 0$ and the degree of q is less than d. Recall also the definitions of sets $K_\pm$ and $U_\pm$ from the introduction.

Looking at the formula for the Hénon mappings, note that if x is reasonably large and large with respect to y, then the predominant behavior is that the x-coordinate gets raised to the d-th power. The following definitions are designed to state this rigorously. Set α to be at least as large as the absolute value of the largest root of

$$|x|^d - |q(x)| - (|a| + 2)\, x = 0.$$

If $p(x) = x^2 + c$, then the following value of α works:

$$\alpha = \frac{1}{2}\left(|a| + 2 + \sqrt{(|a| + 2)^2 + 4|c|}\right).$$

Define the regions V_+, V_-, and $W \subset \mathbf{C}^2$ by

$$V_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, |y| \leqslant |x| \text{ and } |x| \geqslant \alpha \right\},$$

$$V_- = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, |x| \leqslant |y| \text{ and } |x| \geqslant \alpha \right\},$$

and

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, |x| \leqslant \alpha \text{ and } |y| \leqslant \alpha \right\}.$$

Lemma **5.1.** — *The sets* V_+ *and* V_- *have the following properties:*

a) $V_\pm \subset U_\pm$ *and* $U_+ = \bigcup_{n\geqslant 0} F^{\circ -n}(V_+)$, $U_- = \bigcup_{n \geqslant 0} F^{\circ n}(V_-)$;

b) $F(V_+) \subset V_+$, $F^{-1}(V_-) \subset V_-$;

c) if $(x,y) \in V_+$, *then* $|(F^{\circ n})_1(x,y)| \geqslant 2^n$ *for* $n = 1, 2, \ldots$ *and if* $(x,y) \in V_-$, *then* $|(F^{\circ -n})_2(x,y)| \geqslant \left(1 + \frac{1}{|a|}\right)^n$ *as* $n = 1, 2, \ldots$;

d) $F(W) \subset W \cup V_+$;

e) if $(x,y) \in V_-$, *then* $|F_2(x,y)| < |y|$ *and if* $(x,y) \in V_+$, *then* $|F_1^{-1}(x,y)| < |x|$.

Proof. — First consider the statements for V_+. To see part *b)* let $(x,y) \in V_+$, and calculate:

$$|p(x) - ay| \geqslant |p(x)| - |a|\,|y| \geqslant |p(x)| - |a|\,|x| \geqslant 2|x|.$$

Thus $F(V_+) \subset V_+$ and

$$\left| (F^{\circ n})_1 \begin{pmatrix} x \\ y \end{pmatrix} \right| \geqslant 2^n |x|$$

for all $n = 1, 2, \ldots$ Hence part *c)* and also part *a)*.

Part *d)* is obvious since $F(W) \subset \{(x,y) \mid |y| < \alpha\}$. Part *e)* is obvious also.

For $(x,y) \in V_-$, the proofs are analogous using

$$\frac{1}{|a|}|p(y) - x| \geqslant \frac{1}{|a|}|(p(y)| - |x|)$$
$$\geqslant \frac{1}{|a|}(|p(y)| - |y|) \geqslant \left(1 + \frac{1}{|a|}\right)|y|. \quad \square$$

Remark. — The proof of part *c)* of Lemma 5.1 shows that the first coordinates of an orbit starting in V_+ grow at least geometrically. This is actually misleading. Since the dominant term of F_1 is of degree d, the growth is like k^{d^n}. Lemma 5.1 shows that every point eventually lands in $V_+ \cup W$.

Proposition **5.2.** — *There exist unique analytic functions* $\varphi_\pm : V_\pm \to \mathbf{C} - \overline{D}$ *such that*

$$\varphi_+\left(F\begin{pmatrix}x\\y\end{pmatrix}\right) = \left(\varphi_+\begin{pmatrix}x\\y\end{pmatrix}\right)^d \quad \textit{and} \quad \varphi_-\left(F^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right) = \left(\varphi_+\begin{pmatrix}x\\y\end{pmatrix}\right)^d$$

and
$$\varphi_+\begin{pmatrix}x\\y\end{pmatrix} \sim x,\ \varphi_-\begin{pmatrix}x\\y\end{pmatrix} \sim y \quad \textit{as} \quad \left\|\begin{pmatrix}x\\y\end{pmatrix}\right\| \to \infty \quad \textit{in } V_+ \textit{ or } V_-, \textit{ respectively.}$$

Proof. — The function φ_+ is constructed below and shown to have the required property. The proof for φ_- is analogous. The proof of uniqueness will be given at the end of section 8.

To simplify the formulas below set the notation $x_n = (F^{\circ(n-1)})_1\,(x,y)$ and $y_n = (F^{\circ(n-1)})_2\,(x,y)$. Note that x_n is a polynomial in x and y of degree d^n whose sole leading term is x^{d^n} and y_n is a polynomial in x and y of degree d^{n-1} whose sole leading term is $x^{d^{n-1}}$.

To define φ_+, meaning must be given to the limit

$$\varphi_+\begin{pmatrix}x\\y\end{pmatrix} = \lim_{n\to\infty} x_n^{1/d^n},$$

or rather the equivalent telescoping infinite product

$$\varphi_+\begin{pmatrix}x\\y\end{pmatrix} = x\,.\,\frac{x_1^{1/d}}{x}\,.\,\ldots\,.\,\frac{x_{n+1}^{1/d^{n+1}}}{x_n^{1/d^n}}\ldots$$

Examine the individual terms of this infinite product:

$$\frac{x_{n+1}^{1/d^{n+1}}}{x_n^{1/d^n}} = \frac{[x_n^d + q(x_n) - ay_n]^{1/d^{n+1}}}{x_n^{1/d^n}}$$
$$= \left[1 + \frac{q(x_n) - ay_n}{x_n^d}\right]^{1/d^{n+1}}.$$

For $(x,y) \in V_+$, $F^{\circ n}(x,y)$ belongs to V_+ and

$$\left|\frac{q(x) - ay}{x^d}\right| \leqslant \frac{|q(x)| + |a|\,|y|}{|x|^d} \leqslant \frac{|q(x)| + |a|\,|x|}{|x|^d} \leqslant \frac{|x|^d - 2\,|x|}{|x|^d}$$
$$= 1 - \frac{2}{|x|^{d-1}} \leqslant 1 - \frac{2}{\alpha}.$$

Now, for the d^n-th root use the principal branch of $(1 + z)^{1/d^n}$. The infinite product converges as the series of the logarithms of the terms in the product converges.

In the product above, consider the factor

$$\left[1 + \frac{q(x_n) - ay_n}{x_n^d}\right]^{1/d^{n+1}}.$$

The terms of highest degree in both polynomials involve only x. Since $|y| \leqslant |x|$, the term $(q(x_n) - ay_n)/x_n^d$ is of order $1/x^{d^n}$. That is, there exists a constant C such that in V_+

$$\frac{q(x_n) - ay_n}{x_n^d} \leqslant \frac{C}{|x|^{d^n}} \tag{5.3}$$

which tends to 0 as $(x, y) \to \infty$ in V_+. Therefore the product is equivalent to its first term x. □

A refinement of this result will be needed, pushing the asymptotic development of $\varphi_\pm$ a bit further. We find it easiest to write

$$\varphi_+(x, y) = u_0(x) + u_1(x) y + \dots$$

as a convergent power series in y, with coefficients Laurent series in x, which is clearly possible by the structure of V_+.

Proposition **5.4.** — *The following asymptotic development holds:*

$$u_0(x) = x + o(|x|) \quad \textit{and} \quad u_1(x) = -\frac{a}{dx^{d-1}} + o\left(\frac{1}{|x|^{d-1}}\right).$$

Proof. — The development of u_0 is already in Proposition 5.2. From (5.3) above, the second and higher factors of the product cannot contribute larger terms than those given, and the first term gives the result. □

Proposition **5.5.** — *The limits*

$$G_\pm\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{d^n} \log_+ \left\| F^{\circ n}\begin{pmatrix} x \\ y \end{pmatrix} \right\|$$

exist, are continuous on $\mathbf{C}^2$, *are pluri-harmonic on* U_+, *and have the properties that*

$$G_+\left(F\begin{pmatrix} x \\ y \end{pmatrix}\right) = dG_+\begin{pmatrix} x \\ y \end{pmatrix} \quad \textit{and} \quad G_-\left(F^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = dG_-\begin{pmatrix} x \\ y \end{pmatrix}.$$

Moreover,

$$U_\pm = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, G_\pm\begin{pmatrix} x \\ y \end{pmatrix} > 0 \right\}.$$

Proof. — Again, the proof will be given for G_+ and the proof for G_- is analogous.

On V_+, define $G_+ = \log|\varphi_+|$. Extend this definition to $(x, y) \in U_+$ as follows. By part *a)* of Lemma 5.1, there exists $n > 0$ such that $F^{\circ n}(x, y) \in V_+$. For such (x, y), define

$$G_+\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{d^n} G_+\left(F^{\circ n}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Further extend G_+ to be zero on K_+.

The definition is consistent: if a higher n had been used, then the result would be the same by Proposition 5.2 and clearly satisfies $G_+(F(x,y)) = dG_+(x,y)$.

The function G_+ is harmonic on U_+ since it is a real part of an analytic function on V_+ and elsewhere the pullback of a pluri-harmonic function by an analytic mapping.

It remains to see that it is continuous. Fix $(x',y') \in J_+$. Then there exists N such that $\| F^{\circ n}(x',y') \| < \alpha$ for all $n \geqslant N$. For any $M > N$, there exists ε so that if $\| (x'',y'') - (x',y') \| < \varepsilon$, then $\| F^{\circ M}(x'',y'') \| < \alpha$. Note that $F(W_+) \subset W_+ \cup V_+$. So the value of G_+ on the first forward image of $F^{\circ N}(x'',y'')$ which is in V_+ is bounded by $C = \sup\{ G_+(F(x,y)) \mid (x,y) \in W \}$. So, $G_+(x'',y'') \leqslant C/d^M$. □

Remark. — The functions $G_\pm$ are obviously subharmonic. This fact has been observed by Bedford and Smillie ([BS2]) and by Fornæss and Sibony ([FS]). They use the fact that $dd^c G_\pm$ are positive (1, 1)-currents supported on $J_\pm$ to derive analogs of the Brolin measure ([Br]) for Hénon mappings. Fornæss and Sibony also prove that $G_\pm$ are Hölder continuous.

6. The global topology of Hénon mappings

The behavior of G_+ is partially described by the following, in which solenoids make their first appearance in this subject.

Theorem **6.1.** — *The mapping* $G_+ : U_+ \to \mathbf{R}_+$ *is a trivial fibration whose fibers are homeomorphic to* $S^3 - \Sigma_{d,0}$, *embedded using the mapping* $\tau_{d,0}$ *as in section 3.*

Proof. — Represent the set $U_+(r) = G_+^{-1}(\log r)$ as the increasing union

$$U_+(r) = V_+(r) \cup F^{-1}(V_+(r^2)) \cup F^{-2}(V_+(r^4)) \cup \dots,$$

where

$$V_+(r) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in V_+ \;\middle|\; G_+\begin{pmatrix} x \\ y \end{pmatrix} = \log r \right\}.$$

Proposition **6.2.** — *a) For large s,* $V_+(s)$ *is homeomorphic to a solid torus, and*

$$\varphi_+ : V_+(s) \to \{ z \mid |z| = s \}$$

is a fibration with fibers homeomorphic to closed disks.

b) The mapping $G_+ : V_+ \to \mathbf{R}_+$ *is a trivial bundle with fibers homeomorphic to solid tori above* (R, ∞) *for* R *sufficiently large.*

Proof of Proposition 6.2. — For any z with $|z| \leqslant 1$ consider the function $\varphi_z(x) = \varphi_+(x, zx)$. The function is defined and analytic for $|x| > \alpha$. By Proposition 5.2, φ_z has a simple pole at ∞. The following lemma, which is an immediate consequence of Montel's Theorem, will be required.

Lemma **6.3.** — *Let* $R > 1$; *then the space of analytic functions* $f: D \to D_R$, *satisfying* $f(0) = 0$ *and* $f'(0) = 1$ *is compact. In particular, there exist numbers* R_1 *and* R_2 *such that all such functions* f *are injective on* D_{R_1}, *and satisfy* $f(D_{R_1}) \supset D_{R_2}$.

Applying Lemma 6.3 to

$$\frac{1}{\varphi_+(1/x, z/x)},$$

which maps the disk of radius $1/\alpha < 1$ to the disk of radius 1, we see that there exist R_1 and R_2 such that if $\zeta > R_2$ and $|z| \leqslant 1$, then there exists a unique x such that $|x| > R_1$ and $\varphi_+(x, zx) = \zeta$. This shows that the mapping $\psi: (x,y) \mapsto (\varphi_+(x,y), y/x)$ is a homeomorphism $V_+(r) \to S^1_r \times D$ for $r \geqslant R_1$. □ (Proposition 6.2)

To compute F in the coordinates (ζ, z), asymptotic developments of x and y as functions of ζ and z must be found.

Proposition **6.4.** — *The following asymptotic development holds*:

$$x = \zeta + o(|\zeta|) + \left(\frac{a}{d\zeta^{d-2}} + o\left(\frac{1}{|\zeta|^{d-2}}\right)\right) z + o(|z|).$$

Proof. — This is a standard inversion of an analytic function from Proposition 5.3. □ (Proposition 6.4)

Now compute F in the coordinates (ζ, z):

$$\begin{pmatrix}\zeta\\ z\end{pmatrix} \mapsto \begin{pmatrix}x(\zeta, z)\\ zx(\zeta, z)\end{pmatrix} \mapsto \begin{pmatrix}p(x(\zeta, z)) - azx(\zeta, z)\\ x(\zeta, z)\end{pmatrix} \mapsto \begin{pmatrix}\zeta^d\\ \dfrac{x(\zeta, z)}{p(x(\zeta, z)) - azx(\zeta, z)}\end{pmatrix}.$$

Only the term ζ^d in the denominator contributes to the leading terms of the development of F, to give

$$F\begin{pmatrix}\zeta\\ z\end{pmatrix} = \begin{pmatrix}\zeta^d\\ \dfrac{1}{\zeta^{d-1}} + o\left(\dfrac{1}{|\zeta^{d-1}|}\right) + \left(\dfrac{a}{d\zeta^{2d-2}} + o\left(\dfrac{1}{|\zeta|^{2d-2}}\right)\right) z + o(|z|)\end{pmatrix}.$$

This mapping is not quite one of the $\tau_{d,k}$'s from section 3, but almost. Change variables once more, to (ζ, η), where $\eta = z\zeta$. In these coordinates, the following expression holds:

$$\text{(6.5)} \qquad F\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \begin{pmatrix}\zeta^d\\ \zeta + o(|\zeta|) + \left(\dfrac{a}{d\zeta^{d-1}} + o\left(\dfrac{1}{|\zeta|^{d-1}}\right)\right)\eta + o(|\eta|)\end{pmatrix}.$$

In particular, it is conjugate to $\tau_{d,0}$.

Consider $\partial V_+(r)$ with r sufficiently large so that Lemma 5.2 applies. Let γ_r be the curve parametrized by $t \mapsto (\zeta = e^{2\pi i t}, \eta = \alpha r^d)$, $0 \leqslant t \leqslant 1$, and let δ_r be the curve parametrized by $t \mapsto (\zeta = r, \eta = \alpha r^d e^{2\pi i t})$, $0 \leqslant t \leqslant 1$.

Proposition **6.6.** — *If $g : V_+(r) \to \mathbf{T}$ is a homeomorphism with $g(\gamma_r)$ a curve on* **T** *bounding a disk in* $S^3 - \mathbf{T}$, *then there exists a homeomorphism $g' : V_+(r^d) \to \mathbf{T}$ such that the diagram*

$$\begin{array}{ccc} V_+(r) & \xrightarrow{g} & \mathbf{T} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{d,0}} \\ V_+(r^d) & \xrightarrow[g']{} & \mathbf{T} \end{array}$$

commutes. Moreover, $g'(\gamma_{r^d})$ is again a curve which bounds in $S^3 - \mathbf{T}$.

Proof of Proposition 6.6. — The existence of g' and its uniqueness up to homotopy is an unpleasant topological generality. The substance of the proposition is in what g' does to γ_{r^d}.

The generality is a consequence of the following lemma.

Lemma **6.7.** — *Let* X *be a 2-sphere with three open disks with disjoint closures removed. Then the space of homeomorphisms of* X *mapping each boundary component to itself is contractible.*

Proof of Lemma 6.7. — See [EE] and [Ha]. □ (Lemma 6.7)

Both $V_+(r^d) - F(V_+(r))$ and $\mathbf{T} - \tau_{d,0}(\mathbf{T})$ are locally trivial fiber bundles over the circle with fibers homeomorphic to the sphere with three holes above. In each case, the functions called ζ are the fibrations. The following shows that these two spaces are fiber-homeomorphic.

Cut the circle at some point, to manufacture two bundles $\widetilde{V_+(r)}$ and $\widetilde{\mathbf{T}}$ of spheres with three holes over the interval I. Both are trivial bundles, and hence homeomorphic to $I \times X$.

Choose trivializations $v : I \times X \to \widetilde{V_+(r)}$ and $u : I \times X \to \widetilde{\mathbf{T}}$. These induce monodromy mappings

$$m_u = (u|_{\{1\}\times X})^{-1} \circ u|_{\{0\}\times X} \quad \text{and} \quad m_v = (v|_{\{1\}\times X})^{-1} \circ v|_{\{0\}\times X}.$$

The mapping $u \circ v^{-1}$ would induce the desired homeomorphism $V_+(r) \to \mathbf{T}$ if $m_u \circ m_v^{-1}$ were the identity. To arrange this, let m_t be a family of homeomorphisms of X such that $m_0 = m_u^{-1} \circ m_v$ and $m_1 = \mathrm{Id}$ and define $m : I \times X \to I \times X$ by $m(t, x) = (t, m_t(x))$. If u is replaced by the trivialization $u_1 = u \circ m$, then the requirement is satisfied.

This manufactures a homeomorphism

$$w = u_1 \circ v^{-1} : V_+(r^d) - F(V_+(r)) \to \mathbf{T} - \tau_{d,0}(\mathbf{T}).$$

It is clear from the construction that its isotopy is unique (among fibered homeomorphisms).

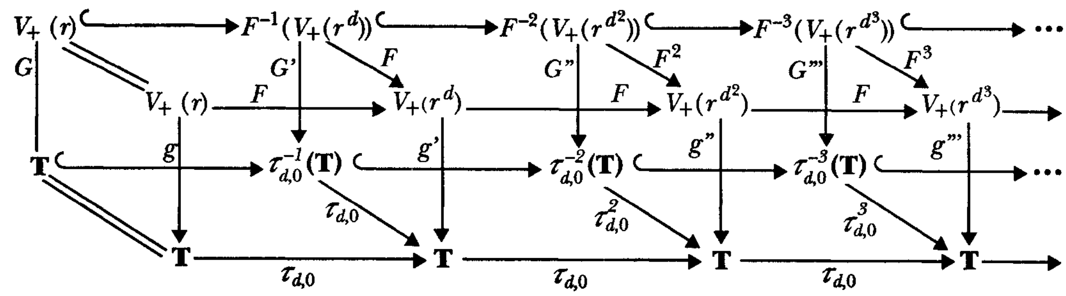
It must be shown that w can be adjusted so as to coincide on $\partial(F(V_+(r)))$ with $\tau_{d,0} \circ g \circ F^{-1}$ and that w maps γ_{r^d} to a curve on $\mathbf{T}$ which bounds a disk in $S^3 - \mathbf{T}$. Both questions are homotopy class questions: the second one obviously and the first because the restriction of w to the boundary can be adjusted to coincide with any homeomorphism in its homotopy class.

Both of these statements follow from the asymptotic expansion (6.5):

$$F\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \begin{pmatrix}\zeta^d\\ \zeta + \dfrac{a\eta}{\zeta^{d-1}} + \varepsilon(\zeta, \eta)\end{pmatrix},$$

where the error term is so small that if a parameter is put in front of it and varied from 1 to 0, then no homotopy classes are changed. Once the parameter is 0, the formula looks exactly like the formula for $\tau_{d,0}$. This is slightly misleading since ζ and η are in the circle of radius r and the disk of radius r, respectively, whereas the arguments of $\tau_{d,0}$ are in the circle of radius 1 and disk of radius 2, respectively. We leave it to the reader to make the appropriate scaling after which the identity is a possible candidate for w. □ (Proposition 6.6)

The proof of Theorem 6.1 is completed by induction. The same construction as above gives a sequence of homeomorphisms $g', g'', \ldots$, where $g^{(k)} : V_+(r^{d^k}) \to \mathbf{T}$. Define $G^{(k)} : F^{-k}(V_+(r^{d^k})) \to \tau_{d,0}^{-k}(\mathbf{T})$ by $G^{(k)} = \tau_{d,0}^{-k} \circ g^{(k)} \circ F^k$. That is, the following diagram commutes:



In the end, $U_+(r) = \bigcup_{k=0}^{\infty} F^{-k}(V_+(r^{d^k}))$ is homeomorphic to

$$\bigcup_{k=0}^{\infty} \tau_{d,0}^{-k}(\mathbf{T}) = S^3 - \Sigma_{d,0}.$$

This proves that the fibers of G_+ are homeomorphic to $S^3 - \Sigma_{d,0}$ for r sufficiently large.

Proposition 6.6 admits parameters: if g_t were a family of homeomorphisms as in the proposition, depending on a parameter t in an interval, then there exists g'_t depending continuously on t and satisfying the conditions of the proposition. In particular, there exist homeomorphisms $g, g', g'', \ldots$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigcup_{r \geq \mathrm{R}} \mathrm{V}_+(r) & \xrightarrow{\mathrm{F}} & \bigcup_{r \geq \mathrm{R}^d} \mathrm{V}_+(r) \ \ldots \\ {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow \\ \mathbf{T} \times [\mathrm{R}, \infty) & \xrightarrow{\tau_{d,0}} & \mathbf{T} \times [\mathrm{R}, \infty) \ \ldots \end{array}$$

Applying this extension of Proposition 6.6 to the inductive proof above shows that the mapping $\mathrm{G}_+ : \mathrm{U}_+ \to \mathbf{R}_+$ is a trivial bundle above (R, ∞), with the same R as in Proposition 6.2. Now $\mathrm{F}^{\circ k}$ is a fiber homeomorphism of $\mathrm{G}_+^{-1}((\mathrm{R}/2^k, \infty))$ to $\mathrm{G}_+^{-1}((\mathrm{R}, \infty))$, covering $x \mapsto 2^k x$. Thus the mapping G_+ is a trivial fibration over any compact subset of $\mathbf{R}_+$, hence locally trivial over $\mathbf{R}_+$, hence trivial since $\mathbf{R}_+$ is contractible. □ (Theorem 6.1)

7. The foliations of U_+

The fibers of G_+ are 3-dimensional manifolds, and not obviously objects of complex analysis. But because G_+ is a pluri-harmonic submersion, $\mathrm{U}_+(s)$ is naturally foliated by Riemann surfaces. We will show that every leaf is isomorphic to $\mathbf{C}$ and dense in $\mathrm{U}_+(s)$. The proof also shows that φ_+ cannot be extended to all of U_+.

Lemma **7.1.** — *Let* W *be open in* $\mathbf{C}^n$ *and let* $h : \mathrm{W} \to \mathbf{R}$ *be a pluri-harmonic submersion. Set* $\mathrm{W}(x) = h^{-1}(x)$. *Then each* $\mathrm{W}(x)$ *is a real* $(2n-1)$*-dimensional manifold, and it is naturally foliated by complex manifolds of dimension* $n-1$, *with tangent space at* $w \in \mathrm{W}(x)$ *given by* $\mathrm{T}_w \mathrm{W}(x) \cap i\mathrm{T}_w \mathrm{W}(x)$.

Proof. — Each $\mathrm{W}(x)$ is a manifold by the Implicit Function Theorem. The uniqueness of the foliation follows from the fact that a real hyperplane T of a complex vector space contains a unique complex hyperplane, namely $\mathrm{T} \cap i\mathrm{T}$.

The existence can be seen by setting locally $h = \mathrm{Re} f$ for some complex analytic function f, which is also a submersion, and observing that $\mathrm{W}(x) = f^{-1}(\{z \mid \mathrm{Re}\, z = x\})$ is naturally foliated by the fibers of f, which are complex manifolds of dimension $n-1$. □

Theorem **7.2.** — *The leaves of the natural foliation of* $\mathrm{U}_+(s)$ *are isomorphic to* $\mathbf{C}$ *and each is dense in* $\mathrm{U}_+(s)$.

Proof. — Choose $\zeta \in \mathbf{C} - \overline{\mathrm{D}}$ with $|\zeta| = s$, with s so large that Proposition 6.2 applies. The leaf through any point of $\varphi_+^{-1}(\zeta)$ can be written

$$\varphi_+^{-1}(\zeta) \cup \mathrm{F}^{-1}(\varphi_+^{-1}(\zeta^d)) \cup \mathrm{F}^{-2}(\varphi_+^{-1}(\zeta^{d^2})) \cup \ldots$$

By Proposition 6.2, this is an increasing union of simply-connected surfaces, hence simply connected.

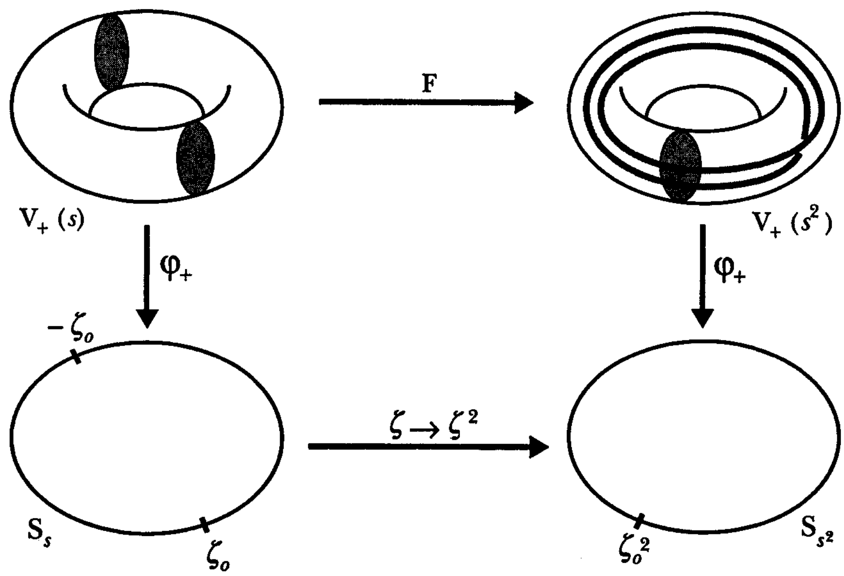


FIG. 7.1

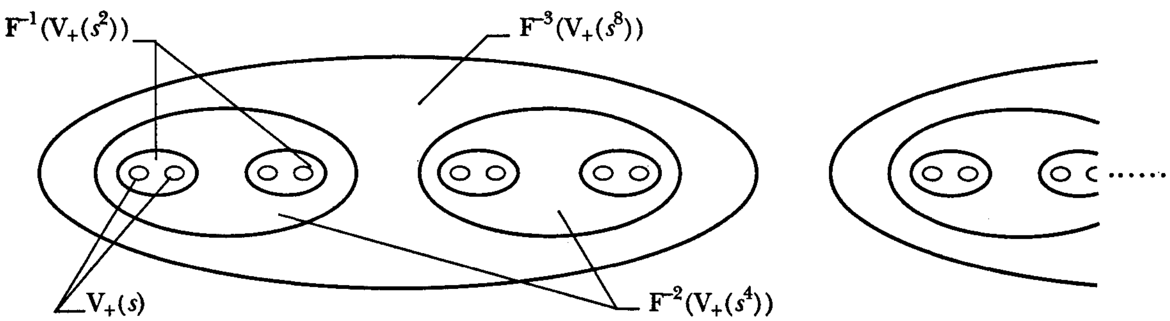


FIG. 7.2

To see that this leaf is dense in $U_+(s)$, note that

$$F^{-1}(\varphi_+^{-1}(\zeta^d)) = \bigcup_{\omega^d=1} \varphi_+^{-1}(\omega\zeta)$$

(see Figure 7) and more generally

$$F^{\circ -k}(\varphi_+^{-1}(\zeta^{d^k})) = \bigcup_{\omega^{d^k}=1} \varphi_+^{-1}(\omega\zeta).$$

Since the d^k-th roots of 1 are dense in the unit circle, each leaf is dense in $V_+(s)$. Applying F repeatedly will make it dense in each term of the increasing union

$$V_+(s) \cup F^{-1}(V_+(s^d)) \cup F^{-2}(V_+(s^{d^2})) \cup \ldots,$$

which occurred in the proof of Theorem 6.1.

It remains to show that the leaves are isomorphic to **C**. This requires the following proposition.

Proposition **7.3.** — *Let* X *be a simply connected Riemann surface, and* $K \subset X$ *a compact connected simply connected subset not reduced to a point. Suppose* $A_0, A_1, \ldots$ *is a sequence of disjoint annuli in* $X - K$ *such that the inclusion of* A_i *into* $X - K$ *is of degree* 1. *If*

$$\sum_{i=0}^{\infty} \operatorname{mod}(A_i) = \infty,$$

then the surface X *is isomorphic to* **C**.

Proof. — The alternative is that X is isomorphic to D and $X - D$ is an annulus with finite modulus M. However, by the subadditivity of moduli of disjoint homotopic annuli ([A], [BH]), $\Sigma_{i=0}^{\infty} \operatorname{mod}(A_i) \leqslant M$. □ (Proposition 7.3)

So find a sequence of annuli in a leaf with a divergent series of moduli. This is actually easy, as the annuli considered grow very rapidly.

Consider the annulus

$$A_\zeta = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in V_+ \;\middle|\; \varphi_+ \begin{pmatrix} x \\ y \end{pmatrix} = \zeta, \frac{1}{2} \leqslant \left| \frac{y}{x} \right| \right\}.$$

For ζ sufficiently large, the function y/x is an isomorphism of A_ζ onto the annulus $1/2 \leqslant |z| \leqslant 1$ of modulus $(\log 2)/(2\pi)$.

The annuli

$$A_\zeta, F^{-1}(A_{\zeta^d}), F^{-2}(A_{\zeta^{d^2}}), \ldots$$

have constant moduli. They are embedded in the leaf which contains $\varphi_+^{-1}(\zeta)$, disjoint by Lemma 5.1, and embedded with degree 1 in the leaf with $\varphi_+^{-1}(\zeta)$ removed. So the leaf is isomorphic to **C** by Proposition 7.3 (see Figure 7.2).

This proves the result for $|\zeta| = s$ sufficiently large. The statement follows in general by observing that F maps bijectively leaves in $U_+(s)$ onto leaves in $U_+(s^d)$. □ (Theorem 7.2)

Proposition **7.4.** — *The mapping*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{\varphi_+ \begin{pmatrix} x \\ y \end{pmatrix}}{\left| \varphi_+ \begin{pmatrix} x \\ y \end{pmatrix} \right|}$$

induces a bijection of the set of leaves onto the (non-Hausdorff) group $\mathbf{R}/\mathbf{Z}[1/d]$.

Proof. — This was already shown in the proof of Theorem 6.2. □

Remark. — There are analogous results for those in Sections 7, 8, and 9 if " F^{-1} " replaces " F ".

8. An analytic description of U_+

In this section the analytic structure of U_+ is analyzed completely. This is done by showing that the " Riemann surface of φ_+ " is a covering space of U_+ isomorphic to $(\mathbf{C} - \bar{D}) \times \mathbf{C}$. Therefore U_+ is a quotient of $(\mathbf{C} - \bar{D}) \times \mathbf{C}$ by some discrete group of automorphisms, isomorphic to $\mathbf{Z}\left[\frac{1}{2}\right]/\mathbf{Z}$. The group of automorphisms of $(\mathbf{C} - \bar{D}) \times \mathbf{C}$ is infinite-dimensional, and since the covering group we are after is only defined up to conjugation, there is a good deal of freedom in the description. The particular choice is algebraically very pleasant, but may not be the best one from a dynamical point of view.

The Riemann surface of φ_+. Let $\tilde{U}_+$ be the smallest quotient of the universal covering space of U_+ on which φ_+ is defined. This covering space should be thought of as the Riemann surface of φ_+, but it cannot be defined as a subset of $U_+ \times \mathbf{C}$ since the fiber above a point of U_+ is a coset of the group of dyadic angles, and hence not discrete in $\mathbf{C}$, so the topology would be wrong.

Being a covering space of an analytic manifold, $\tilde{U}_+$ is a 2-dimensional complex manifold. The set V_+ is naturally embedded as an open subset of $\tilde{U}_+$, using the natural definition of φ_+ on V_+, and of course there is an analytic function $\tilde{\varphi}_+ : \tilde{U}_+ \to \mathbf{C} - \bar{D}$ which extends φ on V_+. This mapping $\tilde{\varphi}_+$ is a submersion, and its fibers are simply connected Riemann surfaces, hence isomorphic to D, $\mathbf{C}$, or the Riemann sphere.

Theorem **8.1.** — *The fibers of* $\tilde{\varphi}_+$ *are isomorphic to* $\mathbf{C}$.

Proof. — This follows from Theorem 7.2. □

It is unfortunately *not* true that a 2-dimensional complex manifold with a submersion to a subset of $\mathbf{C}$ and with fibers isomorphic to $\mathbf{C}$ is a locally trivial family of copies of $\mathbf{C}$.

Example. — Let $\bar{U} = D \times \bar{\mathbf{C}}$, where $\bar{\mathbf{C}}$ is the Riemann sphere. Choose some non-analytic continuous mapping $\alpha : D \to \bar{\mathbf{C}}$, such as $\alpha(z) = \bar{z}$. Consider the set

$$U = \bar{U} - (\text{graph of } \alpha).$$

The projection $U \to D$ does have all fibers isomorphic to $\mathbf{C}$, but if it were analytically a locally trivial fiber bundle, then the section α would be analytic.

However, with an extra condition, such submersions are locally trivial fiber bundles. Let X, Y be complex manifolds, and $f : Y \to X$ an analytic submersion. Let the vertical tangent bundle $T_{Y/X} = \ker df$. Recall that a vertical 1-form is a section of $\mathcal{H}om(T_{Y/X}, \mathbf{C})$.

Proposition **8.2.** — *If all the fibers of f are isomorphic to* $\mathbf{C}$, *and if* Y *carries an analytic vertical* 1-*form* ω, *such that the integral of* ω *along a path in one fiber vanishes only if the path is closed, then the mapping* $f : Y \to X$ *is a locally trivial fiber bundle.*

Proof. — Choose any $x \in X$, there exists a neighborhood U of x and a section $\sigma : U \to Y$ of f. Now define a mapping $g : f^{-1}(U) \to U \times \mathbf{C}$ by sending y to $\left(f(y), \int_{\gamma(y)} \omega\right)$, where $\gamma(y)$ is a path in $f^{-1}(f(y))$ joining $\sigma(f(y))$ to y. There exists such a path since the fibers are connected, and the integral is independent of the choice since the fibers are simply connected, and all analytic 1-forms on a Riemann surface are closed.

Clearly g makes the diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{g} & U \times \mathbf{C} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

commute, and g is an isomorphism fiber by fiber. Indeed, the hypothesis implies that g is injective on each fiber, and an injective analytic mapping $\mathbf{C} \to \mathbf{C}$ is an isomorphism. □

Theorem **8.3.** — *The projection* $\tilde{\varphi}_+ : \tilde{U}_+ \to \mathbf{C} - \overline{D}$ *is a trivial analytic fiber bundle.*

Proof. — By Cartan's Theorem B ([G]), it is enough to prove that it is locally trivial, since there are no topologically non-trivial affine-line bundles over $\mathbf{C} - \overline{D}$. Moreover, $\mathbf{C} - \overline{D}$ is a Stein domain, so the topological and the analytic classifications of such bundles coincide.

Since $\log \varphi_+$ is well defined up to an additive constant, the 1-form $\omega = d \log \varphi_+$ is well defined on U_+. Moreover, ω has no zeros since any branch of $\log \varphi_+$ is a submersion. Therefore one can locally find a function g on U_+ such that $dg \wedge \omega = dx \wedge dy$. Let ψ be the restriction of $\tilde{\varphi}_+^* \, dg$ to vertical tangent vectors. Since dg is well defined up a multiple of ω, this restriction gives a well defined vertical 1-form.

To avoid conflict of notation with the exterior derivative set $\delta = d$ in the following.

Lemma **8.4.** — *We have* $F^* \psi = (a/\delta) \psi$.

Proof. — Clearly $F^* \omega = \delta\omega$, and $F^* dx \wedge dy = a\, dx \wedge dy$. Thus up to multiples of ω, $F^* dg = (a/\delta)\, dg$. The result follows. □ (Lemma 8.4)

Now to show that the criterion of Proposition 8.2 applies to ψ, project a curve in one fiber of $\tilde{U}_+$ to U_+. This projection will be closed only if the original curve was closed. Further take forward images of the curve until it lies in $V_+(r)$, for sufficiently large r. This will change the integral of ψ by dividing it by a power of a/δ. So it is enough to show that for ζ sufficiently large, the integral

$$\int_\gamma \psi$$

over a curve γ in $\varphi_+(\zeta)$ vanishes only if γ is closed. By Proposition 5.2, $\varphi_+ \sim x$, so that $\omega \sim dx/x$, so that g can be chosen with $dg \sim x\, dy$. Since the path γ is nearly vertical, this term of dg contributes more than all other terms, and hence for such an integral to vanish, the y-coordinates of the endpoints must agree. But this means that the endpoints agree, by Proposition 6.2. □ (Theorem 8.3)

Next the structure of the group $\Gamma \subset \mathrm{Aut}\, \widetilde{\mathrm{U}}_+$ such that $\mathrm{U}_+ = \widetilde{\mathrm{U}}_+/\Gamma$ is examined.

Proposition **8.5.** — *The fibration* $\widetilde{\mathrm{U}}_+ \to \mathrm{U}_+$ *induces an exact sequence of fundamental groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\widetilde{\mathrm{U}}_+) & \longrightarrow & \pi_1(\mathrm{U}_+) & \longrightarrow & \Gamma & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\subset} & \mathbf{Z}\left[\frac{1}{\delta}\right] & \longrightarrow & \mathbf{Z}\left[\frac{1}{\delta}\right]\Big/\mathbf{Z} & \longrightarrow & 0 \end{array}$$

so that Γ *is canonically isomorphic to* $\mathbf{Z}\left[\frac{1}{\delta}\right]\Big/\mathbf{Z}$.

Proof. — By Corollary 4.5, $\pi_1(\mathrm{U}_+)$ is isomorphic to $\mathbf{Z}\left[\frac{1}{\delta}\right]$, where 1 is represented by the canonical generator of $\pi_1(\mathrm{V}_+)$. Since V_+ lifts to $\widetilde{\mathrm{U}}_+$, this verifies that the left square is commutative, and the remainder follows. □

Proposition **8.6.** — *There is a unique lift* $\widetilde{\mathrm{F}}$ *of* F *to* $\widetilde{\mathrm{U}}_+$ *mapping* V_+ *to* V_+, *and it satisfies* $\widetilde{\mathrm{F}}(\gamma(x)) = (\delta\gamma)\, \widetilde{\mathrm{F}}(x)$ *for all* $\gamma \in \Gamma$, *where the composition law of* Γ *is written additively.*

Proof. — Elementary covering space theory shows that the lift exists and is unique. The formula then comes from the fact that $\mathrm{F} : \mathrm{V}_+ \to \mathrm{V}_+$ induces multiplication by δ on the fundamental groups. □

For the remainder of this section, let us restrict ourselves to degree 2, with $p(x) = x^2 + c$. It is possible to find similar formulas in higher degrees, but they require inverting a power series, and the computations are difficult and do not lead to simple expressions.

Theorem **8.7.** — *There exists a unique isomorphism* $\widetilde{\mathrm{U}}_+ \to (\mathbf{C} - \overline{\mathrm{D}}) \times \mathbf{C}$ *such that in that trivialization, the mapping* $\widetilde{\mathrm{F}}$ *is written*

$$\widetilde{\mathrm{F}}(\zeta, z) = \left(\zeta^2, \frac{a}{2}\, z + \zeta^3 - \frac{c}{2}\, \zeta\right).$$

Proof. — Choose a trivialization of the bundle $\widetilde{\mathrm{U}}_+ \to \mathbf{C} - \overline{\mathrm{D}}$ so that the zero section is tangent to a high order to the section $s_0 : \zeta \mapsto (x(\zeta), 0)$ at ∞. There exists such a trivialization: in any trivialization, s_0 is a power series in $\frac{1}{\zeta}$ which converges in some neighborhood of ∞. The sum s_n of the first n terms of this series is an analytic section over all of $\mathbf{C} - \overline{\mathrm{D}}$ and arbitrarily close to s_0 as $n \to \infty$. Now change trivializations so that s_n becomes the zero section.

Next an aymptotic expansion of $\widetilde{\mathrm{F}}$ in this trivialization will be computed.

Lemma **8.8.** — *If n is sufficiently large, then* $\widetilde{F}$ *has an asymptotic expansion*

$$\widetilde{F}(\zeta, z) = \left(\zeta^2, \frac{a}{2} z + \zeta^3 - \frac{c}{2}\zeta + o(1)\right).$$

Proof. — Using a trivialization such that dz corresponds to ψ, then by Lemma 8.4, $\widetilde{F}^* dz = (a/2)\, dz$. This means that $\widetilde{F}$ will act on each fiber as $z \mapsto (a/2)\, z +$ constant, and we are left with computing the constant (which depends on the fiber, i.e., on ζ, of course).

This " constant " can be understood as follows: in the chosen trivialization, take the point $s_n(\zeta)$, and integrate ψ from $s_n(\zeta^2)$ to $\widetilde{F}(s_n(\zeta))$ along a path in the fiber. Return to the definition of ψ above. It was found from a function g on V_+ satisfying $d \log \varphi_+ \wedge dg = dx \wedge dy$; so this integral is just

$$g(\widetilde{F}(s_n(\zeta))) - g(s_n(\zeta^2)).$$

In the formula above, replace s_n by s_0, and only change arbitrarily small terms in the asymptotic expansion. Since φ_+ is to first order x, setting $g(x,y) = xy$ satisfies the equation $d \log \varphi_+ \wedge dg = dx \wedge dy$ to first order. We invite the reader to check that ignoring the other terms of g will not affect the asymptotic expansion above. Setting $(x(\zeta), 0) = s_0(\zeta)$, compute

$$g(x(\zeta)^2 + c, x(\zeta)) - g(x(\zeta^2), 0) = (x(\zeta)^2 + c)\, x(\zeta) + \text{terms to be neglected.}$$

In Proposition 6.4, we started to compute $x(\zeta)$, but dit not quite go to the required precision. In fact, it would have been quite difficult to extract the relevant terms of the $o(|\zeta|)$ in arbitrary degree. The goal is the following formula:

$$x(\zeta) = \zeta - \frac{c}{2\zeta} + o\left(\frac{1}{|\zeta|}\right),$$

which we leave to the reader to verify. Then

$$\begin{aligned} g(x(\zeta)^2 + c, x(\zeta)) - g(x(\zeta^2), 0) &= \left(\left(\zeta - \frac{c}{2\zeta}\right)^2 + c\right)\left(\zeta - \frac{c}{2\zeta}\right) + o(1) \\ &= \zeta^3 - \frac{\zeta}{2c} + o(1). \quad \square \text{ (Lemma 8.8)} \end{aligned}$$

To complete the proof of Theorem 8.7, the $o(1)$ above must be dealt with. This is some function $v(\zeta)$ on $\mathbf{C} - \overline{D}$ which vanishes at infinity. Making a change of trivialization $(\zeta, z) \mapsto (\zeta, z + u(\zeta))$, the expression of $\widetilde{F}$ in the new trivialization is

$$\widetilde{F}\begin{pmatrix}\zeta \\ z\end{pmatrix} = \begin{pmatrix} \zeta^2 \\ \frac{a}{2} z + \zeta^3 - \frac{c}{2}\zeta + v(\zeta) - \frac{a}{2} u(\zeta) + u(\zeta^2)\end{pmatrix}.$$

Next find $u(\zeta)$ so that $v(\zeta) - \frac{a}{2} u(\zeta) + u(\zeta^2) = 0$. This can be done by formal power series, or by setting

$$u(\zeta) = \sum_{m=1}^{\infty} \left(\frac{2}{a}\right)^m v(\zeta^{2^m}).$$

Clearly this series formally solves the problem and it converges since $| v(\zeta^{2^m}) | \leqslant C | \zeta |^{-2^m}$ for some constant C.

This shows the existence of the required trivialization. The uniqueness is clear from the uniqueness to the solution for u above. □ (Theorem 8.7)

Finally the group Γ may be computed.

Theorem **8.9.** — *For each element $j/2^k \in \Gamma$, there exists a unique polynomial $p_{j,k}(\zeta)$ such that in the trivialization above, the element of Γ corresponding to $j/2^k$ is given by*

$$\begin{pmatrix}\zeta \\ z\end{pmatrix} \mapsto \begin{pmatrix} e^{2i\pi j/2^k} z \\ z + p_{j,k}(\zeta)\end{pmatrix}.$$

Proof. — First, compute $\gamma_{1/2}$. By Proposition 8.6,

$$\widetilde{F}\left(\gamma_{1/2}\begin{pmatrix}\zeta \\ z\end{pmatrix}\right) = (2\gamma_{1/2})\, \widetilde{F}\begin{pmatrix}\zeta \\ z\end{pmatrix} = \begin{pmatrix}\zeta^2 \\ \frac{a}{2} z + \zeta^3 - \frac{c}{2}\zeta\end{pmatrix}.$$

This leads immediately to $\gamma_{1/2}(\zeta, z) = \left(-\zeta, z + \frac{4}{a}\left(\zeta^3 - \frac{c}{2}\zeta\right)\right)$.

More generally, suppose that $\gamma_{j/2^{k-1}}$ has been determined. Then Proposition 8.6 gives

$$\widetilde{F}\left(\gamma_{j/2^k}\begin{pmatrix}\zeta \\ z\end{pmatrix}\right) = (\gamma_{j/2^{k-1}})\, \widetilde{F}\begin{pmatrix}\zeta \\ z\end{pmatrix} = \begin{pmatrix}\zeta^2 \\ \frac{a}{2} z + \zeta^3 - \frac{c}{2}\zeta\end{pmatrix}.$$

which can be rewritten

$$\begin{pmatrix} e^{2i\pi j/2^{k-1}} \zeta^2 \\ z + p_{j,k}(\zeta) + (e^{2i\pi j/2^k}\zeta)^3 - \frac{c}{2} e^{2i\pi j/2^k}\zeta \end{pmatrix} = \begin{pmatrix} e^{2i\pi j/2^{k-1}} \zeta^2 \\ z + \zeta^3 - \frac{c}{2}\zeta + p_{j-1,k}(\zeta^2)\end{pmatrix}.$$

This gives us $p_{j,k}(\zeta)$. □

Finally, we fill in a gap in section 5, the uniqueness of $\varphi_\pm$ in Proposition 5.2. This actually requires knowing Theorem 8.7 for Hénon mappings of any degree; the proof goes through with minor changes.

Proof of uniqueness for φ_+. — Suppose $\psi : V \to \mathbf{C} - \bar{D}$ satisfies

$$\psi\left(F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \left(\psi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)^d$$

and $$\psi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \sim x \quad \text{when} \quad \begin{bmatrix} x \\ y \end{bmatrix} \to \infty \text{ in } V_+.$$

First observe that ψ lifts to $\tilde{\psi} : \tilde{U}_+ \to \mathbf{C} - \bar{D}$. This is an application of the lifting criterion for covering spaces. For all n, the space $\tilde{V}_n = \tilde{F}^{-n}(V_+)$ has the homotopy type of a circle, and $\tilde{F} : \tilde{V}_n \to \tilde{V}_{n-1}$ induces multiplication by d on the fundamental groups. Thus ψ can be lifted recursively to all $\tilde{V}_n$ by the formula

$$\psi\left(\widetilde{\begin{bmatrix} x \\ y \end{bmatrix}}\right) = \left(\psi\left(\tilde{F}\left(\widetilde{\begin{bmatrix} x \\ y \end{bmatrix}}\right)\right)\right)^{1/d},$$

and the proper choice of roots will guarantee that all lifts agree with ψ on V_+.

On each fiber of $\tilde{\varphi}_+$, the function $\tilde{\psi}$ must be constant, since it is a mapping $\mathbf{C} \to \mathbf{C} - \bar{D}$. Thus we can write $\tilde{\psi} = \alpha \circ \tilde{\varphi}_+$, where $\alpha : \mathbf{C} - \bar{D} \to \mathbf{C} - \bar{D}$ is an analytic function; by restriction to V_+, $\psi = \alpha \circ \varphi_+$. But a look at the functional equation shows that α must be of the form $\zeta \mapsto a\zeta^k$, with k a positive integer and $a^{d-1} = 1$. Now the asymptotic expansion shows that $a = 1$ and $k = 1$. □

9. The canonical compactification of K_+

Let $\bar{\bar{\mathbf{C}}}$ be the compactification of $\mathbf{C}$ adding a circle at infinity. Then any polynomial extends continuously to $\bar{\bar{\mathbf{C}}}$ and its restriction to the circle at infinity multiplies angles by the degree of the polynomial.

This section contains a description of an analogous compactification $\bar{\bar{\mathbf{C}}}^2$ of $\mathbf{C}^2$, to which Hénon mappings extend continuously. A 3-sphere is added at infinity and the mapping extends as the solenoidal mapping σ on S^3. This further emphasizes the similar role which solenoidal mappings play for Hénon mappings and angle doubling plays for quadratic polynomials.

In particular, the closures of K_- and K_+ in $\bar{\bar{\mathbf{C}}}^2$ are the solenoids Σ_+ and Σ_- respectively (note the reversal). This sometimes allows us to measure " angles of external rays " in K_- in the solenoid Σ_+. This will turn out to be important in the description of the topology of these sets.

Theorem **9.1.** — *There exists a compact Hausdorff space* X *homeomorphic to a closed four-ball with underlying set* $\mathbf{C}^2 \sqcup S^3$ *such that*

a) the induced topologies on $\mathbf{C}^2$ *and* S^3 *are the standard topologies;*

b) $\mathbf{C}^2$ *is dense in* X;

c) the Hénon mapping F *extends to a homeomorphism* $\bar{\bar{F}} : X \to X$; *and*

d) the restriction of $\bar{\bar{F}}$ *to* S^3 *is the solenoidal mapping* τ_0.

Remark. — This is a surprisingly difficult result to prove, considering that the analogous results have already been worked out at finite distance. The difficulty is that if a 3-sphere is added at infinity in the obvious way, with points corresponding to oriented directions in $\mathbf{C}^2$, then all non-vertical directions are mapped to horizontal directions. In particular, the Hénon mapping does not extend continuously, and even where it is defined, it fails to be injective. To make the extension of the Hénon mapping injective, the horizontal (and vertical) directions will be examined with a microscope.

More precisely, a delicate blowup of the circle at infinity in the x-axis will be made, replacing a point p by a way of approaching p, the method of approach which we focus on being the images of straight lines.

Proof. — **Step 1. Blowing up a circle in** S^3. Consider a compact differentiable curve $\Gamma \subset S^3$, and define the oriented blowup $\widetilde{S}^3_\Gamma$ of S^3 along Γ as follows. First choose a tubular neighborhood U of Γ such that there exists a unique geodesic of S^3 joining any point of U to Γ in U. For $x \in U$, let ξ_x be the tangent vector at x to the geodesic joining x to Γ. Now define the blowup $\widetilde{U}_\Gamma$ as a subset of the unit tangent bundle $T_1(U)$ to $U \subset S^3$ to be

$$\widetilde{U}_\Gamma = \left\{ (x, \xi) \in T_1(S^3) \,\middle|\, \left\{ \begin{array}{ll} \xi = k\xi_x \text{ for some } k > 0 & \text{if } x \in U - \Gamma \\ \xi \text{ is perpendicular to } \Gamma & \text{if } x \in \Gamma \end{array} \right\} \right\}.$$

The obvious mapping $\pi : \widetilde{U}_\Gamma \to U$ is an isomorphism on $\widetilde{U}_\Gamma - \pi^{-1}(\Gamma)$, so glue $S^3 - \Gamma$ onto $\widetilde{U}_\Gamma$ to make $\widetilde{S}^3_\Gamma$. Above Γ, there is a torus, mapping to Γ as a bundle of circles.

Step 2. A first microscope. Consider the solid torus T_1 parametrized by

$$\{(\lambda, \mu) \in \mathbf{C}^2 \mid |\lambda| = 1, |\mu| \leqslant 2\}.$$

The point (λ, μ) of this solid torus will be " at the end of " the ray

$$t \mapsto \begin{bmatrix} \lambda t^2 \\ \mu t \end{bmatrix}.$$

Let Γ_1 be the curve of equation $y = 0$ in S^3. Glue $\partial T_1 = \{(\lambda, \mu) \mid |\lambda| = 1, |\mu| = 2\}$ to $\widetilde{S}^3_{\Gamma_1}$ as follows: choose the circle in $\widetilde{S}^3_{\Gamma_1}$ above the point of Γ_1 corresponding to the point in the circle at infinity on the x-axis in the direction λ. Radial projection of the curve

$$t \mapsto \begin{bmatrix} \lambda t^2 \\ \mu t \end{bmatrix}$$

onto the sphere at infinity gives a curve which approaches the circle orthogonally in a definite direction. Identify (λ, μ) with this direction.

Let $\bar{S}^3$ be the 3-sphere blown up along Γ_1 with the torus T_1 attached as above.

Step 3. A second microscope. Unfortunately, distinguishing these " eventually horizontal curves " do not resolve the images of straight lines. Blow up the circle Γ_2 of equation $\mu^2 = \lambda$ in this torus further to see the constant term in a ray of the form

$$t \mapsto \begin{bmatrix} \mu^2 t^2 \\ \mu t + \nu \end{bmatrix}.$$

More formally, consider the solid torus $T_2 = \{(\mu, \nu) \mid |\mu| = 1, |\nu| \leqslant |a|\}$. Glue T_2 to $\widetilde{S}^3_{\Gamma_2}$ by identifying the point $(\mu, \nu) \in \partial T_2$ (i.e. $|\nu| = |a|$) to the unit vector in the direction of ν at (μ^2, μ).

Let $\overline{\overline{S}}^3$ be $\overline{S}^3$ blown up along Γ_2 with the torus T_2 attached as above.

Step 4. A topology on $\mathbf{C}^2 \sqcup \overline{\overline{S}}^3$. A basis of neighborhoods of each point of $\overline{\overline{S}}^3$ is needed. There is no difficulty at those points which correspond to points of S^3 not on the x-axis: take the cone over a neighborhood of such a point, and cut it off at some radius.

It is not much harder to define a basis of neighborhoods of a point (λ_0, μ_0) in T_1 which is not on the circle Γ_2 or on the boundary of T_1. Take a neighborhood

$$V = \{(\lambda, \mu) \mid |\lambda - \lambda_0| < \varepsilon, |\mu - \mu_0| < \varepsilon\}$$

of $(\lambda, \mu) \in T_1$ and let the neighborhood consist of V and the points which can be written $(\lambda t^2, \mu t)$ for $t > T$ and $(\lambda, \mu) \in V$.

An analogous description is possible for the points inside T_2. Given μ_0 with $|\mu_0| = 1$ and ν_0 with $|\nu_0| < |a|$, choose a neighborhood W of (μ_0, ν_0) in the solid torus T_2 defined by $|mu - \mu_0| < \varepsilon_1, |\nu - \nu_0| < \varepsilon_2$ and a number $T > 0$. Then a neighborhood of (μ_0, ν_0) will consist of W and the points of $\mathbf{C}^2$ which can be written $(\mu^2 t^2, \mu t + \nu)$ with $(\mu, \nu) \in W$ and $t > T$.

It is a good bit harder to define a neighborhood basis for a point on the boundary ∂T_1 or ∂T_2. Let P be the solid paraboloid of points in $\mathbf{C}^2$ which can be written $(\lambda t^2, \mu t)$ with $|\mu| < 2$ (i.e. the set defined by the inequality $|y|^2 \leqslant 4|x|$). Choose $(\lambda_0, \mu_0) \in \partial T_1$, i.e. $|\lambda_0| = 1$ and $|\mu_0| = 2$, and a neighborhood W_1 of (λ_0, μ_0) in T_1. The intersection $W_1 \cap \partial T_1$ corresponds to a set of unit vectors normal to Γ_1. Set W_2 to be the set of points in S^3 which are obtained by traveling a distance less than ε from Γ_1 on the geodesic tangent to such a normal vector. Now a neighborhood of (λ_0, μ_0) consists of $W_1 \cup W_2$, and the points which can be written $(\lambda t^2, \mu t)$ with $(\lambda, \mu) \in W_1$ and $t > T$, and the points on rays through W_2 of norm greater than T and outside the paraboloid P.

We will leave to the reader the analogous construction for the boundary ∂T_2, as well as the proof of the following lemma.

Lemma 9.2. — *The space* $\mathbf{C}^2 \sqcup \overline{\overline{S}}^3$ *with the topology above is compact Hausdorff.*

Step 5. The space $\overline{\overline{\mathbf{C}}}^2$. The compactification of $\mathbf{C}^2$ constructed so far is adequate to do the Hénon mapping once in appropriate regions. It needs to be adapted to the

Hénon mapping as a dynamical system. One way of doing this would consist of making an infinite sequence of " blow ups " as above, so as to resolve the images of parabolic rays, etc., and taking the projective limit. A different method will be used, inspired by the fact that a model for the locus at infinity already exists, as a dynamical system.

Let S^3_1 be a " new " copy of the 3-sphere, and $\sigma : S^3_1 \to S^3_1$ a solenoidal mapping. Let $T'_0 \subset S^3_1$ be a solid torus, such that $T'_2 = \sigma(T'_0)$ is contained in the interior of T'_0. The reason for this funny notation will become clear below.

Back in $\overline{\overline{S}}^3$, consider the solid torus T_0 of equation $|y| \leqslant |x|$. A ray in the cone over the boundary of T_0 can be written $(t\alpha, t\beta)$ with $|\alpha| = |\beta| = 1$. The image of such a ray is the parametrized curve $t \mapsto (\alpha^2 t^2 + c - a\beta t, \alpha t)$, which is asymptotic to the curve

$$s \mapsto \left(\alpha^2 s^2, \alpha s + \frac{1}{2}\frac{a\beta}{\alpha}\right). \tag{9.3}$$

This last curve converges as $s \to \infty$ to a point in ∂T_2, and it is easy to see that the first is sufficiently close to the second so it converges to the same point.

Choose a homeomorphism $h : T'_0 - \mathring{T}'_2 \to T_0 - \mathring{T}_2$, conjugating the mapping $\sigma : \partial T'_0 \to \partial T'_2$ to the mapping $T_0 \to T_2$ induced by the Hénon mapping as above. This is possible, by the classification of solenoidal mappings and the formula (9.3). Now put a topology on $\mathbf{C}^2 \sqcup S^3_1$ as follows. Attach $T'_0 - T'_2$ to $\mathbf{C}^2$ by h. For any point p of $S^3_1 - (\Sigma_+ \cup \Sigma_-)$, choose n such that $\sigma^n(p) \in T'_0 - T'_2$, choose a neighborhood U of $\sigma^n(p)$ and define a neighborhood of p to be $F^{\circ -n}(U \cap \mathbf{C}^2) \cup \sigma^{\circ -n}(U \cap S^3_1)$.

This defines a neighborhood basis for all points in $\mathbf{C}^2 \sqcup S^3_1$, except for those of the solenoids Σ_+ and Σ_-. Recall that both solenoids are canonically homeomorphic to $\varprojlim(S^1, 2)$, and that there exists a unique mapping $\psi_+ : T_0 \to S^1$ which semi-conjugates σ to angle-doubling (Theorem 3.1). A neighborhood of $p = (\ldots, p_2, p_1, p_0) \in \Sigma_+$ is the union of the set of $(x, y) \in \mathbf{C}^2$ such that

$$F^{\circ k}\begin{pmatrix} x \\ y \end{pmatrix} \in V_+ \quad \text{for all } k \leqslant N$$

and

$$\left| \frac{\varphi_+(F^{\circ -k}(x,y))}{|\varphi_+(F^{\circ -k}(x,y))|} - p_k \right| < \varepsilon \text{ for all } k \leqslant N$$

and the set of points p in

$$\bigcap_{k \leqslant N} \sigma^{\circ k} T_0$$

such that $|\psi_+(\sigma^{\circ -k}(p) - p_k| < \varepsilon$.

The union $\mathbf{C}^2 \sqcup S^3_1$, with this topology, is the space $\overline{\overline{\mathbf{C}}}^2$.

Step 6. Compactness of $\overline{\overline{\mathbf{C}}}^2$. It remains to verify that $\overline{\overline{\mathbf{C}}}^2$ is compact, Hausdorff, and that the mapping $\overline{\overline{F}} : \overline{\overline{\mathbf{C}}}^2 \to \overline{\overline{\mathbf{C}}}^2$ which is F on $\mathbf{C}^2$ and σ on S^3_1 is continuous.

To show compactness, take a sequence in $\overline{\overline{\mathbf{C}}}^2$. Suppose that the sequence is in $\mathbf{C}^2$ and that the norms of the elements (x_n, y_n) tend to infinity, as otherwise the sequence obviously has a convergent subsequence. Further, assume that the sequence lies in V_+.

Either there exists k and a subsequence (x_{n_i}, y_{n_i}) such that

$$F^{\circ -k}(x_{n_i}, y_{n_i}) \notin V_+,$$

or there isn't. In the first case, recall the set P from step 4, and choose k and a subsequence, which will still be called (x_{n_i}, y_{n_i}) such that

$$F^{\circ -k}(x_{n_i}, y_{n_i}) \in V_+ - P.$$

If a subsequence can be chosen so that the rays through these points converge to a non-horizontal ray, then this subsequence converges to the point of S^3_1 corresponding to this ray. If the rays through these points tend to the horizontal, then choose a subsequence so that the directions of the rays tend to Γ_1 on a curve orthogonal to Γ_1. This direction is then a point of S^3_1, which is the limit of the subsequence. This shows that the $F^{\circ -k}(x_{n_i}, y_{n_i})$ have a subsequence which converges, and hence so does (x_{n_i}, y_{n_i}), by the second part of sept 5.

Now suppose that the number of times F^{-1} can be iterated on points of the sequence and stay in V_+ tends to infinity. Then by the compactness of the circle and a diagonal argument, a subsequence of the (x_n, y_n) can be chosen so that the sequences

$$\frac{\varphi_+(F^{\circ -k}(x_n, y_n))}{|\varphi_+(F^{\circ -k}(x_n, y_n))|}$$

converge for all k, say to p_k. Clearly $p_k^2 = p_{k-1}$. Let $\boldsymbol{p} = (\dots, p_2, p_1, p_0)$. The subsequence clearly converges to $\boldsymbol{p} \in \Sigma_+$. This shows compactness.

Step 7. The space $\overline{\overline{\mathbf{C}}}^2$ is Hausdorff. It remains to show that distinct points of $\overline{\overline{\mathbf{C}}}^2$ have disjoint neighborhoods. Clearly only points in S^3_1 need to be considered, and Lemma 9.2 shows that only points in the solenoids need to be considered. Even here there is no problem. If $p_0 \neq p_0'$, then let $\varepsilon = \frac{1}{2} | p_0 - p_0' |$. Then the ε-neighborhoods of $\boldsymbol{p}$ and $\boldsymbol{p}'$ are disjoint. If $p_i = p_i'$ for $i < k$ and $p_k \neq p_k'$, then $| p_k - p_k' | = 2$ and so the $\frac{1}{k}$-neighborhoods are disjoint.

Since clearly the mapping $\overline{\overline{F}}$, given by F on $\mathbf{C}^2$ and σ on S^3, is continuous (the bases of neighborhoods are invariant under F), this ends the proof of Theorem 9.1. □

Corollary **9.4.** — *The closure of* K_+ *in* $\overline{\overline{\mathbf{C}}}^2$ *is* $K_+ \cup \Sigma_-$, *and the closure of* K_- *is* $K_- \cup \Sigma_+$.

Proof. — Clearly points of K_- and large norm are points where σ^{-1} can be iterated many times staying in V_+. Such points are in smaller and smaller neighborhoods of points of Σ_+. □

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Manuscrit reçu le 5 janvier 1993