THE STOKES PROBLEM FOR A POROUS PARTICLE WITH RADIALLY NONUNIFORM POROSITY

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The flow past a nonuniform porous spherical particle immersed in a uniform steady-state stream is studied in the Stokes approximation. For a power-law radial dependence of the particle permeability coefficient, an analytical solution for the velocity and pressure fields outside and inside the particle is obtained.

The classical problem of slow uniform flow past a rigid sphere (the Stokes problem) and its generalization for a spherical liquid droplet (Hadamard and Rybchinskii) are discussed in many monographs, for example, in [1]. Uniform flow past a porous spherical particle is considered in [2, 3]. In [4], the solution is obtained for a porous particle immersed in an arbitrary shear flow.

In many hydrodynamic chemical engineering processes, the particle porosity is nonuniform. For example, the catalyzer grains produced by calcination consist of layers with different porosities. In particular, a porous grain may have an impermeable core or, on the contrary, a low-permeability outer surface. Purely shear flow past a radially nonuniform particle was considered in [5]. In this paper, we give a solution of the problem of steady-state uniform flow past a porous particle with radially nonuniform porosity.

1. FORMULATION OF THE PROBLEM

Consider a steady-state Stokes flow past a nonuniform spherical porous particle of radius a immersed in a uniform stream of viscous fluid. The origin of the spherical coordinate system r, θ, φ coincides with the particle center. Due to axial symmetry (velocity component $v_\varphi=0$), all the unknown functions are independent of φ.

On the assumption that the Reynolds number is small $\mathrm{Re}=Ua\rho/\mu < 1$, outside the particle the velocity and the pressure are determined by the Stokes equations [1]:

$$\Delta \mathrm{v} = \operatorname{grad} p, \qquad \operatorname{div} \mathrm{v} = 0 \tag{1.1}$$

The fluid flow inside the particle is described by Darcy's law with a permeability coefficient k that depends on the nondimensional radius r:

$$\mathrm{V} = -k(r) \operatorname{grad} P, \qquad \operatorname{div} \mathrm{V} = 0 \tag{1.2}$$

Here, all the variables are nondimensional. The characteristic scales for the length, velocity, and pressure are the particle radius a, the free-stream velocity U, and $p_0=\mu U/a$, where μ is the dynamic viscosity of the fluid. The permeability coefficient is scaled to a^2.

Far away from the particle, the velocity field is uniform:

$$r \to \infty, \qquad v_r = \cos\theta, \qquad v_\theta = -\sin\theta \tag{1.3}$$

On the particle surface, the external normal stress is equal to the internal pressure, the normal velocity component is continuous, and the tangential velocity component has a discontinuity proportional to the derivative of this component with respect to the outward normal [6, 7]:

$$p - 2\frac{\partial v_r}{\partial r} = P, \qquad v_r = V_r, \qquad \lambda\sqrt{k}\,\frac{\partial v_\theta}{\partial v_r} = v_\theta - V_\theta \qquad (r=1) \tag{1.4}$$

The nondimensional constant λ depends on both the physical nature of the porous material and the geometry of its surface. According to the data of [6, 7], $0.25 \le \lambda \le 10$. This dependence is valid only for very small values of the permeability coefficient on the surface.

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2. EXTERNAL AND INTERNAL HYDRODYNAMIC FIELDS

We seek the solution for the external field in the form:

$$v_r(r,\ \theta)=f_1(r)\cos\theta,\quad v_\theta(r,\ \theta)=f_2(r)\sin\theta,\quad p(r,\ \theta)=f_3(r)\cos\theta \tag{2.1}$$

Substituting these formulas in system (1.1) gives:

$$\begin{aligned} r^2f_1'' + 2rf_1' - 4f_1 &= r^2f_3' + 4f_2 \\ r^2f_2'' + 2rf_2' - 2f_2 &= -rf_3 + 2f_1 \end{aligned} \tag{2.2}$$

After applying the operator div to the first of Eqs. (1.1), we obtain the Laplace equation $\Delta p=0$. From this, using (2.1), we obtain:

$$r^2f_3'' + 2rf_3' - 2f_3=0$$

The general solution of this equation is

$$f_3=Ar + Cr^{-2}$$

Since the pressure at infinity is finite, we have $p=Cr^{-2}\cos\theta$.

Having solved Eq. (2.2) using (1.3), we obtain the following formulas for the fluid velocity components outside the particle:

$$v_r=(1 + Cr^{-1} + 2Br^{-3}/3)\cos\theta,\qquad v_\theta=(-1 - Cr^{-1}/2 + Br^{-3}/3)\sin\theta$$

We will now find the hydrodynamic field inside the particle. After taking the divergence of both sides of the first of Eqs. (1.2), we obtain:

$$\operatorname{grad} k\cdot\operatorname{grad} P + k(r)\,\Delta P=0$$

In the spherical coordinates, this equation takes the form:

$$\frac{\partial k}{\partial r}\frac{\partial P}{\partial r} + k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\right)=0$$

If $P(r,\ \theta)=g(r)\cos\theta$, we have

$$kr^2g'' + (2kr + k'r^2)g' - 2kg=0$$

We will rewrite this equation in the form containing the differential self-conjugate Sturm-Liouville operator [8]:

$$-(kr^2g') + 2kg=0,\qquad |g(0)| < \infty,\qquad k(r)\geq 0 \tag{2.3}$$

From the physical meaning of the permeability coefficient we have $k(r)\geq 0$. Hence, for $r\in[0,\ 1]$ we have $kr^2\geq 0$ and, at the point $r=0$, the coefficient of the higher derivative in Eq. (2.3) has a zero point not less than second order. Using the properties of the Sturm-Liuoville operator, we choose from the fundamental system of solutions of the equation that function which is finite at the zero point: $g(r)=C_1h(r)$. From Eq. (1.2), we obtain the fluid pressure and velocity inside the particle in the form:

$$P=C_1h(r)\cos\theta,\qquad V_r=-C_1k(r)h'(r)\cos\theta,\qquad V_\theta=-C_1\frac{k(r)h(r)}{r}\sin\theta$$

The coefficients C_1, C, and B in the expressions for the velocity and the pressure are obtained from the boundary conditions (1.4) for $r=1$. The second of the conditions (1.4) gives: $C_1=(3C + 4B)/h(1)$. Using this expression, from the rest of the conditions (1.4) we obtain the following system of linear equations for the coefficients C and B:

$$3(1 + 3k_1m)C + 2(1 + 6k_1m)B=-3,\quad 3(1 + \lambda\sqrt{k_1} + 6k_1)C - 2(1 + 3\lambda\sqrt{k_1} - 12k_1)B=-6$$

Here, $k_1=k(1)$ and $m=h'(1)/h(1)$.

The solution of this system is $C=-3K_1$ and $B=3K_2/2$; accordingly, for C_1 we have: $C_1=-3K_3/h(1)$. Here,

$$K_1(k_1, m)=\frac{1 + \lambda k_1^{1/2} + 4(m - 1)k_1}{2 + 4\lambda k_1^{1/2} + 3(3m - 2)k_1 + 15\lambda m k_1^{3/2}}$$
$$K_2(k_1, m)=\frac{1 - \lambda k_1^{1/2} + 6(m - 1)k_1}{2 + 4\lambda k_1^{1/2} + 3(3m - 2)k_1 + 15\lambda m k_1^{3/2}} \tag{2.4}$$

$$K_3(k_1, m)=2K_2 - 3K_1=\frac{1 + 5\lambda k_1^{1/2}}{2 + 4\lambda k_1^{1/2} + 3(3m - 2)k_1 + 15\lambda m k_1^{3/2}} \tag{2.5}$$

Thus, the general solution for the external and internal hydrodynamic fields is as follows:

$$p=-3K_1 r^{-2}\cos\theta, \quad P=-3K_3\frac{h(r)}{h(1)}\cos\theta$$
$$v_r=(1 - K_1 r^{-1} + K_2 r^{-3})\cos\theta, \quad V_r=3k(r)K_3\frac{h'(r)}{h(1)}\cos\theta \tag{2.6}$$
$$v_\theta=-\left(1 - \frac{3}{2}K_1 r^{-1} - \frac{1}{2}K_2 r^{-3}\right)\sin\theta, \quad V_\theta=-3K_3\frac{k(r)h(r)}{rh(1)}\sin\theta$$

Here, the coefficients K_1, K_2, and K_3 are given by the formulas (2.4) and (2.5).

Both in the neighborhood of the particle center ($r \to 0$) and near the surface ($r \to 1$) the behavior of P, V_r, and V_θ depends on the particular form of the dependence $k(r)$. Since the function $h(r)$ is bounded, if the particle center is impermeable ($k(0)=0$) the velocity components V_r and V_θ tend to zero as $r \to 0$.

Thanks to the axial symmetry, we can introduce the stream function. As can be easily shown, inside and outside the particle the nondimensional stream function has the form:

$$\psi(r, \theta)=-0.5(r^2 - 3K_1 r + K_2 r^{-1})\sin^2\theta,$$

$$\Psi(r, \theta)=-(3/2)K_3 k(r) r^2[h'(r)/h(1)]\sin^2\theta$$

We can now find the tangential stress $\tau_{r\theta}$ on the sphere surface and the drag force F exerted on the porous sphere by the free stream:

$$\tau_{r\theta}=\left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right)\Bigg|_{r=1}=-3K_2\sin\theta$$

$$F=\int_0^\pi(-\tau_{r\theta}\sin\theta - p\cos\theta)2\pi\sin\theta\, d\theta=4\pi(2K_2 + K_1)=6\pi K_4$$

$$K_4=\frac{2}{3}(2K_2 + K_1)=\frac{2}{3}\frac{3 - \lambda k_1^{1/2} + 16(m - 1)k_1}{2 + 4\lambda k_1^{1/2} + 3(3m - 2)k_1 + 15\lambda m k_1^{3/2}}$$

When $k_1=0$ and the coefficient $K_4=1$, this gives the well-known Stokes formula for the drag on an impermeable sphere.

We will now calculate the mass flux of fluid entering the particle through its surface per unit time:

$$Q=-\int_S(\mathrm{v}, \mathrm{n})\, dS=-\int_S v_r|_{r=1}\, dS=-2\pi(1 - 3K_1 + K_2)\int_{\pi/2}^{\pi}\cos\theta\sin\theta\, d\theta=\pi k_1 m K_3$$

Here, the integration is carried out over that part of the particle surface (S) on which the normal velocity component is negative. The mass flux is proportional to the boundary value of the permeability k_l and the radial derivative of the pressure on the surface $m=h'(1)/h(1)$ which, according to (2.3), also depends on the function $k(r)$.

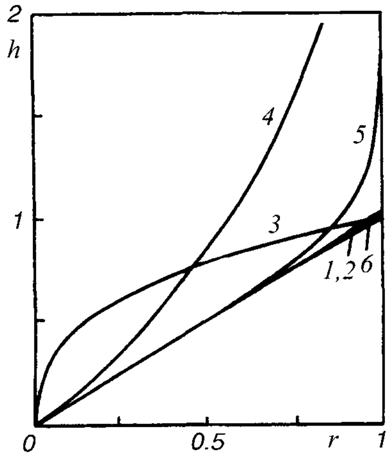

Fig. 1

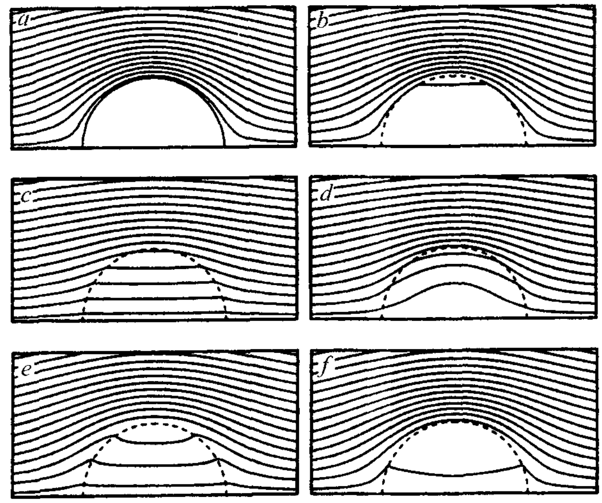

Fig. 2

Fig. 1. Radial dependence of the pressure inside the porous particle for (k_0, k_1)=(0.00125, 0.001): (0.125, 0.1); (0, 0.1); (0.1, 0.001); ($k_0=k_1$=const) – curves *1–6*; (b=4: curves *1–4*; b=0.25: curve *5*).

Fig. 2. Streamline patterns for a power-law particle porosity distribution (*a*: impermeable sphere, *b–e*: cases *1–5* in Fig. 1).

3. POWER LAW FOR THE POROSITY

A typical law for the porosity which, on the one hand, is often encountered in practice and, on the other hand, admits an analytical solution of our problem is the power law:

$$k(r)=(k_1 - k_0)r^b + k_0 \qquad (k_0 \geq 0,\ k_1 > 0,\ b > 0) \tag{3.1}$$

Here, k_0 and k_1 are the permeability coefficient values in the particle center (r=0) and on the particle surface (r=1). If $k_0=k_1$, the particle permeability is constant $k(r)=k_1$. If $k_0 > k_1$, the porosity decreases from the center to the periphery and, if $k_0 < k_1$, the porosity increases with distance from the center. When $k_0=0$, $k(r)=k_1r^b$ and the particle center is impermeable.

Given the power law (3.1) for $k(r)$, Eq. (2.3) takes the form:

$$r^2((k_1-k_0)r^b+k_0)g''+((b+2)(k_1-k_0)r^{b+1}+2k_0r)g'-2((k_1-k_0)r^b+k_0)g=0 \tag{3.2}$$

When $k_0\neq0$, we introduce the new variables: $g(r)=r^c\eta(\xi)$, $\xi=dr^b$. Here, $d=(k_0 - k_1)/k_0$ and c is one of the roots of the equation $c^2 + c - 2=0$, i.e. $c_{1,2}=\{-2, 1\}$ (see [9]). Then, the latter equation reduces to the well-known hypergeometric equation for the function $\eta(\xi)$:

$$\xi(\xi - 1)\eta'' + ((\alpha + \beta + 1)\xi - \gamma)\eta' + \alpha\beta\eta=0$$

$$\gamma=1 + \frac{c_2 - c_1}{b}=1 + \frac{3}{b}, \quad \alpha=\frac{c_2 + B_1}{b}=\frac{1 + B_1}{b}, \quad \beta=\frac{c_2 + B_2}{b}=\frac{1 + B_2}{b}$$

$$B_{1,2}=0.5(b + 1 \mp \sqrt{(b + 1)^2 + 8})$$

The solution of Eq. (3.2) bounded at r=0 can be expressed in terms of the hypergeometric function $F(\alpha, \beta, \gamma; \xi)$ [10] as follows:

$$g(r)=C_1 h(r)=C_1 r F\left(\frac{1+B_1}{b},\ \frac{1+B_2}{b},\ 1+\frac{3}{b};\ dr^b\right)$$

Using this expression for $h(r)$, we obtain:

$$h'(r)=F\left(\frac{1+B_1}{b},\ \frac{1+B_2}{b},\ 1+\frac{3}{b};\ dr^b\right)+\frac{db}{b+3}r^b F\left(\frac{1+B_1}{b}+1,\ \frac{1+B_2}{b}+1,\ 2+\frac{3}{b};\ dr^b\right)$$

$$m=1+\frac{db}{b+3}F\left(\frac{1+B_1}{b}+1,\ \frac{1+B_2}{b}+1,\ 2+\frac{3}{b};\ d\right)\Big/F\left(\frac{1+B_1}{b},\ \frac{1+B_2}{b},\ 1+\frac{3}{b};\ d\right) \tag{3.3}$$

These formulas for $h(r)$, $h'(r)$, and m together with (2.6) give a closed solution for the velocity and pressure fields in the problem of steady-state flow past a porous particle with a power law for the permeability coefficient (3.1).

When k=0, we have $k(r)=k_1 r^b$ and Eq. (2.3) takes the form:

$$r^2 g'' + (2+b) r g' - 2g = 0$$

Its solution can be represented in the form of ordinary power functions:

$$g(r)=C_1 r^{\delta} + C_2 r^{\chi}, \qquad \{\delta, \chi\}=B_{1,2}=0.5(-(b+1) \pm \sqrt{(b+1)^2+8})$$

Taking the boundedness of the function $g(r)$ at r=0 into account, we obtain:

$$h(r)=r^{\delta}, \qquad h'(r)=\delta r^{\delta-1}, \qquad m=h'(1)/h(1)=\delta$$

Using the formulas obtained, we calculated several variants for porous particles with a power law for the porosity. The graphs of the function $h(r)$ are presented in Fig. 1, and the corresponding streamline patterns are shown in Fig. 2. Since the solution is axisymmetric, in Fig. 2 we show only the top part of the flow region ($z > 0$).

Figure 2*a* shows the streamlines of the flow past an impermeable rigid particle (k_1=0). When the values of k_0 and k_1 are similar ($|d| < 0.2$), no matter what the value of b the flow pattern differs only slightly from that for the flow past a particle with constant porosity. If $k < 0.001$, the flow pattern is similar to that for the flow past a rigid particle (Fig. 2*b*), but a slight filtration of fluid through the particle still takes place. For $k_1 > 0.1$, the flow more closely resembles the undisturbed stream (Fig. 2*c*). Figure 2*d* corresponds to a power-law porosity distribution with k=0, k_1=0.1, and b=4. Since the porosity decreases with decrease in the distance from the center, a certain displacement of the streamlines occurs and the qualitative flow pattern is similar to that for the flow past a particle with a radius smaller than a. Figure 2*e* shows almost the limiting case ($d \to 1$): k_0=0.1, k_1=0.001 (d=0.99), and b=4. In this case, the particle has a low-permeability "crust" and much greater internal porosity. The fluid filtering through the dense surface enters the more permeable middle zone of the sphere, which explains the streamline deformation inside the sphere. In the neighborhood of r=1, the pressure varies very sharply (curve *4* in Fig. 1, $m \approx 39.3$).

When the quantities k_0 and k_1 are substantially different, the parameter b also has a noticeable effect on the internal flow pattern. The exponent b affects the mean permeability of the medium:

$$\langle k\rangle=\int_0^1 k(r)\,dr=k_0(b+1-d)/(b+1)$$

For $b \gg 1$, the value of $\langle k\rangle$ is closer to k_0 and, for $b \ll 1$, the mean permeability is closer to k_1. In Fig. 2*f* the streamlines are plotted for the same values of k_0 and k_1 as in Fig. 2*e* but for b=0.25. In this case, the particle has a thicker low-permeability "crust", which results in a slower increase in the pressure as $r \to 1$ (curve *5* in Fig. 1, $m \approx 6.0$), and the flow pattern more closely resembles that for an impermeable sphere.

The parameter λ has only a slight impact on the flow pattern. Thus, a decrease in λ from 10 to 0.25 results in only a certain change in the streamline refraction on the particle surface. In all the cases considered above, λ was taken equal to 1.0.

When b=0 or k_1=k_0, both cases of the power law for the porosity considered above give the well-known solution for the particle with constant permeability $k(r)$=k_1 obtained in [2]. This solution is described by formulas (2.6), (2.4), and (2.5) after the substitutions $h(r)$=r, $h'(r)$=1, and m=1 (curve 6 in Fig. 1).

Summary. In the Stokes approximation, the external and internal hydrodynamic fields for an arbitrary axisymmetric flow (a combination of uniform and shear streams) past a spherical particle with radially nonuniform porosity are superpositions of the solutions obtained in the present study and in [5] for uniform and pure-deformation flows, respectively.

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