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LAW OF FLOW OF A VISCOPLASTIC FLUID THROUGH A POROUS MEDIUM WITH ALLOWANCE FOR INERTIAL LOSSES

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A macroscopic law of flow of a viscoplastic Schwedoff-Bingham fluid through a porous medium is obtained on the basis of percolation theory with allowance for viscous and inertial losses. The asymptotics of the flow law are estimated and expressions for determining the limiting pressure gradient as a function of the microinhomogeneity parameters are given. Satisfactory qualitative agreement between the theoretical and known experimental data is observed.

A nonlinear law of flow of a Newtonian fluid which takes into account inertial hydraulic losses in sudden capillary contractions and expansions was obtained in [1] on the basis of percolation theory. In [2] the value of the hydraulic conductivity of a porous medium was determined with allowance for the viscoplastic properties of the saturating fluid but without allowance for the inertial losses during motion. In the present paper we derive a general nonlinear law of flow of a viscoplastic Schwedoff-Bingham fluid through a porous medium with allowance for viscous and inertial losses. The asymptotics of the flow law and the dependence of the limiting pressure gradient on the microinhomogeneity parameters are obtained. It is shown that for low pressure differences the asymptotics of the flow law are a linear form of the well-known law of flow through a porous medium with a limiting gradient [3, 4]. The theoretical results are compared with experimental data, and satisfactory qualitative agreement between them is observed.

As our model of the pore space, within the framework of the percolation approach [5, 6], we will use a regular spatial network whose nodes and links are, respectively, the pores and the capillary channels between them. The characteristics of this network are the branch length d, the radial pore distribution function (porometric curve) $f(r)$, and the number of branches originating at a single node (lattice coordination number) z.

In order to derive the dependence of the flow parameters on the microcharacteristics of the medium it is usual first to obtain the relation between the external pressure drop in the medium and the fluid flow rate along each conducting path in the system. The fluid is viscoplastic with the limiting shear stress τ_0. In this case the flow along a single capillary can be described by the Buckingham formula [7]

$$q = \pi r^4 \Delta p (8d\mu)^{-1} h(\langle r\rangle), \qquad h(\langle r\rangle) = 1 - 4/3\langle r\rangle + 1/3\langle r\rangle^4 \tag{1}$$

where q is the volume flow rate; Δp is the pressure drop at the capillary ends; μ is the structural dynamic viscosity; $R = \tau_0(2d/\Delta p)$ is the radius of the flow core; and $\langle r\rangle = R/r$ is the relative radius of the core.

Using the exact relation for $h(\langle r\rangle)$ or one of its known approximations [8] leads to quite cumbersome dependences of the pressure gradient on the flow rate which are inconvenient for further investigation. Therefore, we will use the following linear two-point approximation of the function $h(\langle r\rangle)$:

$$h^*(\langle r\rangle) = 1 - 3/4\langle r\rangle: \quad 0 \le \langle r\rangle \le 0.4, \qquad h^*(\langle r\rangle) = 0.778(1 - \langle r\rangle): \quad 0.4 \le \langle r\rangle \le 1 \tag{2}$$

Over the entire interval $0 \le \langle r\rangle \le 1$ the maximum absolute approximation error (2) is not greater than 8.91%. Any accuracy of the approximation of type (2) necessary for practical calculation purposes can be achieved by means of an additional partition of the interval of variation of the quantity $\langle r\rangle$. Substitution of $h^*(\langle r\rangle)$ in place of $h(\langle r\rangle)$ in (1) gives the following expression for the pressure drop on each capillary:

$$\Delta p_k = A\frac{\mu d}{\pi r^4}q + B\frac{\tau_0 d}{r} \tag{3}$$

$$A = 8, \quad B = 8/3: \quad 0 \le \langle r\rangle \le 0.4, \qquad A = 10.729, \quad B = 2: \quad 0.4 \le \langle r\rangle \le 1$$

We will assume that the pressure drop Δp_y at a network node is determined by the density of the fluid ρ, its structural viscosity μ, the limiting shear stress τ_0, the flow rate q, and the radii r_1 and r_2 of the capillaries adjoining the

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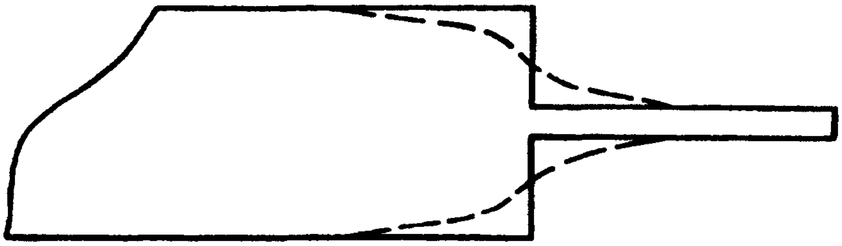

Fig. 1. Junction element for capillaries of different radii. The continuous curve corresponds to the idealization of a sudden contraction or expansion. The broken curve corresponds to the real pattern of variation of the radius of a conducting channel in the passage from one capillary to another.

node in the direction of fluid flow. In this case we shall disregard the fact that the capillary junction zone is a non-ideal transition of the sudden expansion or contraction type. (In Fig. 1 we have reproduced a diagram of a junction element for capillaries of different radii in a chain.) Dimensional analysis leads to the expression

$$\Delta p_y = \rho q^2 r_1^{-4} \xi\left(\frac{r_2}{r_1}, \mathrm{Re}, \mathrm{He}\right), \quad \mathrm{Re} = \frac{\rho q}{\mu r_1}, \quad \mathrm{He} = \frac{\rho \tau_0 r_1^2}{\mu^2}$$

Estimates show that for the flow in an individual capillary the Reynolds number Re << Re_{cr} and the Headström number He << 1. This makes it possible to assume the total self-similarity of the function $\xi(r_1/r_2, \mathrm{Re}, \mathrm{He})$ in the parameters Re and He and write

$$\Delta p_y = \rho q^2 r_1^{-4} \xi\left(\frac{r_2}{r_1}\right) \tag{4}$$

The function ξ in dependence (4) can be calculated, for example, on the basis of the Borda formula for a sudden expansion and the Idel'chik formula for a sudden contraction of a capillary [9]

$$\xi(r_1, r_2) = \frac{1}{2\pi^2}\left(1 - \left(\frac{r_2}{r_1}\right)^{-2}\right)^2, \quad r_1 \le r_2; \qquad \xi(r_1, r_2) = \frac{1}{2\pi^2}\left(1 - \left(\frac{r_2}{r_1}\right)^{-2}\right), \quad r_1 > r_2$$

The pressure drop on a conducting path composed of alternating capillaries and nodes (Fig. 2) is

$$\Delta P = \sum_{i=1}^{N} \Delta p_{ki} + \sum_{i=1}^{N} \Delta p_{yi} \tag{5}$$

where i is the number of the capillary and the next node, and N is the total number of capillaries composing the conducting chain in question. The corresponding pressure drops are determined by relations (3) and (4). This pressure drop occurs on the characteristic macroscopic distance $L \approx Nd$; therefore, we can obtain the external pressure gradient $\nabla P = \Delta P/L$ by dividing (5) by L

$$\nabla P = \frac{1}{N}\sum_{i=1}^{N} \frac{\Delta p_{ki} + \Delta p_{yi}}{d} \tag{6}$$

If the minimum radius of a capillary belonging to the given conducting path is equal to r and the capillary radii are distributed independently with a distribution density $f(r)$, then, passing in (6) from summation to integration and taking into account the fact that

$$N = \int_0^\infty f(\omega)\, d\omega$$

we obtain

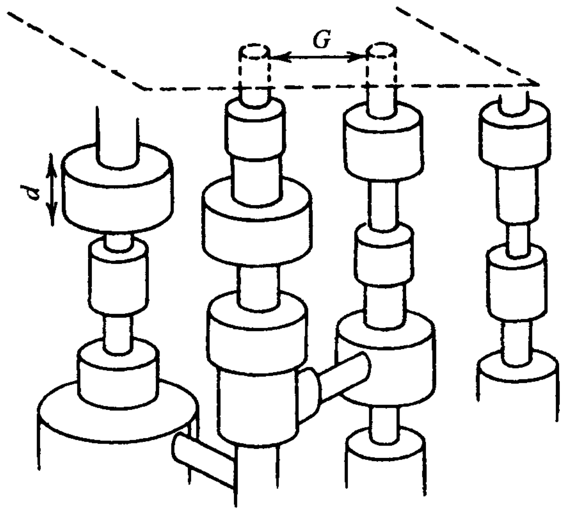

Fig. 2. Formation of fluid-conducting channels from capillaries of different radii.

$$\nabla P = N^{-1}\int_r^\infty \frac{\Delta p_k}{d} f(\omega)\,d\omega + N^{-1}\int_r^\infty \frac{\Delta p_y}{d} f(\omega)\,d\omega$$

or, using (3) and (4)

$$\nabla P = I_0(r) + I_1(r)q + I_2(r)q^2$$

$$I_0(r) = 2B\tau_0 N^{-1}\int_r^\infty \frac{f(\omega)}{\omega}\,d\omega, \qquad I_1(r) = \frac{A\mu}{\pi}N^{-1}\int_r^\infty \frac{f(\omega)}{\omega^4}\,d\omega \tag{7}$$

$$I_2(r) = \frac{1}{d}N^{-2}\int_r^\infty\int_r^\infty \xi(\omega_1, \omega_2) f(\omega_1) f(\omega_2)\,d\omega_1\,d\omega_2$$

Expressing q from (7) in terms of ∇P, we obtain the final dependence $q(\nabla P)$

$$q = \frac{2(\nabla P - I_0(r))}{I_1(r) + \sqrt{I_1^2(r) + 4I_2(r)(I_0(r) - \nabla P)}} \tag{8}$$

Relation (8) is a generalization of the formula for viscous Newtonian fluids obtained previously in [1], where on the righthand side of (3) there was no second term, since for a Newtonian fluid the limiting shear stress $\tau_0=0$. Correspondingly, in (7) and (8) there was no term $I_0(r)$ due to the presence of the limiting shear τ_0.

Following [10, 11], we will calculate the number of paths with the minimum radius r_1 per unit cross-sectional area of the flow through the porous medium. The probability of a capillary having a radius $r \geq r_1$ is equal to

$$P_B(r_1) = \int_{r_1}^\infty f(r)\,dr$$

Within the framework of the generalized model [11], the critical radius r_c is related with the threshold of flow through the network by the expression $P_B(r_c)=D/[(D - 1)]$, where D is the space dimension and z is the lattice coordination number. For any $r_1 < r_c$ all the capillaries with radii greater than r_1 form an infinite cluster - a connected pore system through which flow can occur.

Let us set $r_1 < r_c$. The infinite cluster corresponding to this radius has an irregular cellular structure with the characteristic cell dimension [11]

$$G = d\left(\frac{1 - P_B(r_c)}{P_B(r_1) - P_B(r_c)}\right)^{\nu}$$

Here, ν, the index of the correlation radius, is equal to 0.85 and 1.3 for three- and two-dimensional problems, respectively. The number density $n(r_1)$ of the conducting paths collinear with the external effective pressure gradient and composed of pores with radii greater than r_1 (Fig. 2) is

$$n(r_1) = G^{-2} = \frac{1}{d_2}\left[\frac{1}{1 - P_B(r_c)}\int_{r_1}^{r_c} f(r)\,dr\right]^{2\nu}$$

The distribution of the total number of conducting paths (per unit cross-sectional area) over the minimum radius r_1 is

$$F(r_1) = -dn(r_1)/dr_1 \tag{9}$$

Summing the flow rates over all the conducting paths and using (8) and (9), we obtain the nonlinear flow law of the viscoplastic fluid

$$w = \int_0^{r_c} \frac{2F(r)(|\nabla P| - I_0(r))}{I_1(r) + \sqrt{I_1^2(r) + 4I_2(r)(|\nabla P| - I_0(r))}}\,dr \tag{10}$$

Here, w is the seepage velocity of the fluid and $|\nabla P|$ is the absolute value of the external pressure in the medium. Using the collinearity of the vector $\mathbf{w}$ and ∇P, we can readily write relation (10) in vector form. If the total contribution made by the plastic shear factors and the inertial losses to dependence (10) is significantly less than the contribution of the viscous losses so that the inequality

$$I_1^2(r) >> 4I_2(r)(I_0(r) - |\nabla P|) \tag{11}$$

is fulfilled, then from (10) there follows the asymptotic dependence

$$w \cong |\nabla P|\int_0^{r_c}\frac{F(r)}{I_1(r)}\,dr - \int_0^{r_c}\frac{F(r)I_0(r)}{I_1(r)}\,dr \tag{12}$$

The structure of relation (12) corresponds to the well-known form of representation of the law of motion of a viscoplastic fluid through a porous medium [3, 4], the last terms in the expression obtained determining the value of the limiting pressure gradient. If the inequality opposite to (11) is valid, then asymptotics (10) have the form:

$$w \cong \int_0^{r_c}\frac{F(r)\sqrt{|\nabla P| - I_0(r)}}{\sqrt{I_2(r)}}\,dr \tag{13}$$

In the absence of plastic properties ($\tau_0 = I_0(r) = 0$), the asymptotic relations (12) and (13) go over, respectively, into the linear and quadratic laws of flow of a viscous Newtonian fluid [1] which take the inertial losses into account.

In Fig. 3 we have plotted graphs of the dependence $w(|\nabla P|)$ for media corresponding to two different porometric curves (curves *1–4*). The calculations were carried out on the basis of formulas (10) (curves *1* and *3*) and (12) (curves *2* and *4*) for a model power distribution function of the form:

$$f(r) = 0:\ r \le r_{\min},\quad r \ge r_{\max};\qquad f(r) = (n+1)r^n/(r_{\max}^{n+1} - r_{\min}^{n+1}):\qquad r_{\min} \le r \le r_{\max} \tag{14}$$

Curves *1* and *2* correspond to the exponent $n = -1.15$ and curves *3* and *4* to the exponent $n = -1.25$. The capillary radii were varied from $1.5 \cdot 10^{-7}$ to 10^{-5} m, the limiting shear stress $\tau_0 = 0.007$ Pa and the structural viscosity $\mu = 7.75$ mPa·s, and the other parameters corresponded to the characteristics of the Arlan oil field [12]. The nature of the dependences obtained coincides qualitatively with the experimental results obtained in [12]. The values of the calculated seepage velocity and pressure gradients vary within the same limits as in [12]; however, direct comparison with the experimental data is difficult since in the experiments the limiting shear stress τ_0 and the porometric curve $f(r)$ were not measured. Obviously, a decrease in the proportion of thin capillaries (increase in the exponent n) leads to a sharper

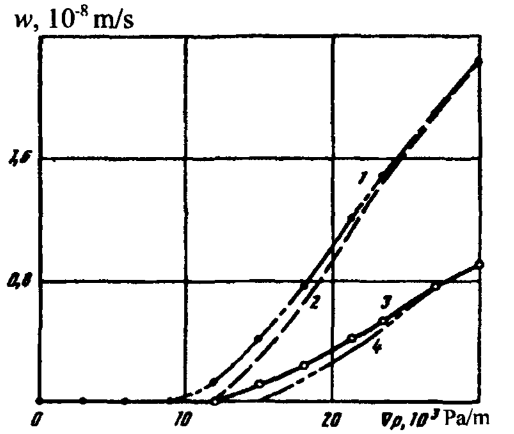

Fig. 3

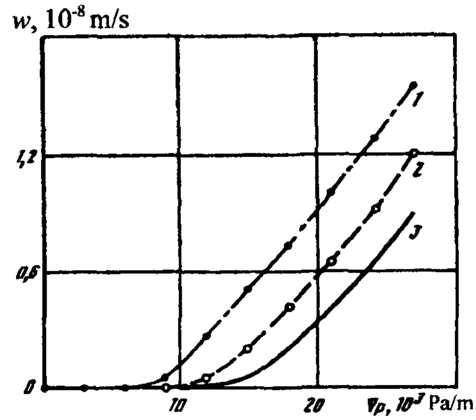

Fig. 4

Fig. 3. Dependence of the seepage velocity on the pressure gradient in media with different exponential porometric curves: curves *1* and *2* correspond to $n = -1.25$ and curves *1* and *4* to $n = -1.15$; $\tau_0 = 0.007$ Pa (curves *2* and *4* correspond to the linear asymptotics of (12)).

Fig. 4. Dependence of the seepage velocity on the pressure gradient for various values of the limiting shear stress ($n = -1.2$) $\tau_0 = 0.005$, 0.007, and 0.009 (curves *1–3*, respectively).

increase in the seepage velocity with the external pressure gradient. In Fig. 4 the $w(|\nabla P|)$ dependences calculated from formula (10) for $f(r)$ in the form (14) with $n = -1.2$ at τ_0 equal to 0.005, 0.007, and 0.009 Pa are shown by curves *1–3*, respectively.

These calculations show that an increase in the limiting shear stress leads to a decrease in the seepage velocity. This is physically reasonable.

An adequate comparison between the theoretical results obtained and the experimental data is possible only when the latter include both the limiting shear τ_0 for the percolating fluid and the porometric curve $f(r)$ of the porous medium considered.

Summary. A percolation model of the transport of a viscoplastic fluid through a porous medium is constructed with allowance for the inertial losses in the process of interaction between the fluid and inhomogeneities of the surface of pore space. An analytic expression for the law of flow of a Bingham plastic through a porous medium is obtained. This general expression includes the linear and quadratic asymptotics, respectively, for low and high pressure gradients, the flow law for a Newtonian fluid when $\tau_0 = 0$, and the explicit form of the relation for calculating the macroscopic limiting pressure gradient as a function of the microinhomogeneity parameters.

In principle, the outlined method of analytic derivation of the macroscopic flow laws can be extended to the case of motion of a fluid with any other rheological properties. In this case the basic principle is the possibility of describing the viscous and inertial losses of pressure in the flow along a single capillary using flow rate approximations of not higher than second order. In this case it is not necessary for the dependence determining the inertial losses to satisfy the hypothesis of total self-similarity with respect to all the similarity parameters other than the capillary radius ratio r_1/r_2. In this case the possibility of describing this dependence within the framework of the above-mentioned approximation remains a necessary condition.

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