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MOTIONS OF A POROUS PARTICLE IN STOKES FLOW PART 2. LINEAR FLOWS NEAR A FLUID INTERFACE

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Abstract—Analytical results are presented for the motion of a porous sphere in the vicinity of a plane fluid-fluid interface. The fluids are assumed to undergo a linear undisturbed flow and the viscosity ratio of the two fluids is assumed to be arbitrary. The analysis consists of the method of reflections, coupled with an application of fundamental singularity solutions for Stokes flow to calculate the hydrodynamic force and torque on the particle. The fundamental relationships for the force and torque are then applied, in combination with the corresponding solutions obtained in earlier publications for the translation and rotation through a quiescent fluid, to determine the motion of a neutrally buoyant particle freely suspended in the flow.

INTRODUCTION

In part 1 of the present pair of papers [1], we have considered the motion of a porous spherical particle in a mean flow through an unbounded single-fluid domain. It is apparent, however, that the motion of a particle in the vicinity of a boundary is often fundamentally different from its motion in an unbounded fluid owing to hydrodynamic wall effect. This type of 'wall' effect plays an important role in a wide range of interesting problems including the Brownian motion of a colloidal particle near a phase boundary, locomotion of micro-organisms and sedimentation phenomena near a fluid-fluid interface. Recently, we considered translation and rotation of a porous particle near a fluid interface when the fluids are at rest at infinity [2]. Although the quiescent problem is of some intrinsic interest, and is a logical starting point for investigation of particle motions near a fluid interface, many problems of practical significance involve particle motions in a mean flow at infinity [3,4]. This is true of 'wall' effects in the rheology of dilute suspensions, studies of structure and breakup of flocs subjected to fluid stresses, and the modeling of polymer molecules to account for the hydrodynamic interactions between polymer segments [5-8]. The solutions obtained in this paper are partly motivated by these and other potential applications, and partly as a contribution to the literature on flow through a porous body of finite size which may be used by comparison to experimental measurements as a basis for testing the applicability of existing continuum models for flow in porous media.

Two distinct methods have been commonly employed to study particle motions in the presence of a flat interface; namely, (1) a standard solution via superposition using the eigensolutions of Laplace's equation in bipolar coordinates, and (2) solution via the reciprocal theorem of Lorentz [9]. The majority of previous analyses of creeping motion near a flat interface were restricted to rigid, *impermeable* spherical particles, and utilized separation of variables in bipolar coordinates.

The reciprocal theorem approach (i.e., singularity method) was pioneered by Lorentz who derived a solution for the fluid motion generated by a point force (i.e., Stokeslet) in the presence of a plane solid wall. This approach, which is essential if the particles are permeable or nonspherical, is to construct solutions using spatial distributions of fundamental singularities. Recently, fundamental singularity solutions were developed by a generalization of Lorentz analysis, and used to solve the creeping motions of a slender, rod-like particle and a *porous* spherical particle through a quiescent fluid in the presence of a flat interface [2,10].

In the present study, we use the singularity method to investigate the hydrodynamic interactions between a porous sphere and a flat fluid-fluid interface in linear flows that are compatible with the presence of a plane interface. The flow of the viscous fluid inside the porous sphere is analyzed via Brinkman's equation which is of the same spatial order as the Stokes' equation, and thus allows the problem of matching the interior and exterior flows to be resolved [11]. The theory yields the components of the hydrodynamic force and torque on the porous sphere at rest in the flow field to the undisturbed flow parameters such as strain rate or shear rate. These solutions are then applied, for illustrative purpose, to calculate the particle trajectories in linear shear and uniaxial axisymmetric straining flows as typical representations.

FORMULATION OF THE PROBLEM

1. Governing equations and boundary conditions

We consider a porous spherical particle immersed in a linear undisturbed flow near a flat fluid-fluid interface of two immiscible fluids 1 and 2. The two continuous fluid phases are assumed to be undergoing the undisturbed linear flow in the form

$$\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{L}^{(i)}\cdot\mathbf{x} \tag{1}$$

in which $\mathbf{U}_i^\infty(\mathbf{x})$ is the undisturbed velocity field in fluid i (=1,2), and $\mathbf{x}$ denotes a position vector measured from the origin that is placed at the interface. Further, the undisturbed interface at $z=0$ is assumed to remain flat, and the particle supposed to be wholly immersed in fluid 2. Figure 1 shows a schematic view of

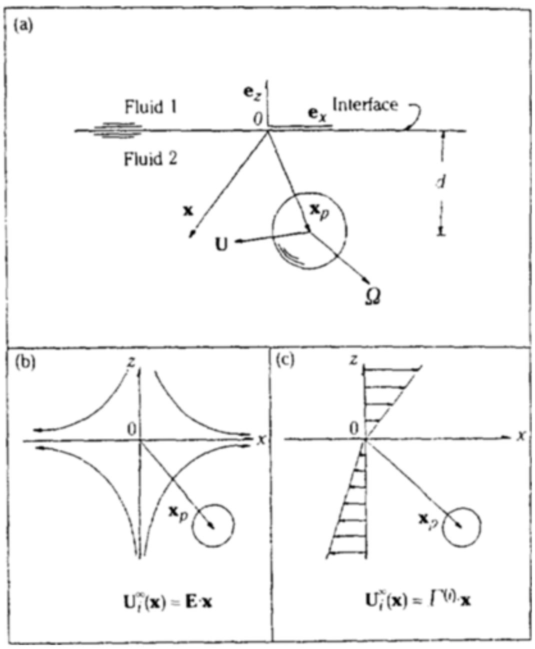

Fig. 1. Schematic diagrams for (a) description of the coordinate system, (b) a uniaxial extensional flow $\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{E}\cdot\mathbf{x}$, and (c) a linear shear flow $\mathbf{U}_i^\infty(\mathbf{x})=\Gamma^{(i)}\cdot\mathbf{x}$.

the system. The undisturbed flow field is consistent with the existence of a flat, nondeformable interface at which the normal components of velocities are identically zero (i.e., $\mathbf{U}_i^\infty\cdot\mathbf{e}_z=0$). Thus, the strain rate tensor $\mathbf{L}^{(i)}$ for a uniaxial axisymmetric extensional flow has Cartesian components

$$L_{lm}^{(i)}=E(\delta_{lm}-3\delta_{l3}\delta_{m3}) \tag{2}$$

while that for a simple shear flow parallel to the interface has

$$L_{lm}^{(i)}=\frac{\mu_2}{\mu_i}(\Gamma_{13}\delta_{l1}+\Gamma_{23}\delta_{l2})\delta_{m3}. \tag{3}$$

Here, E and $\Gamma_{l3}(l=1,2)$ are usually denoted as the strain rate and shear rate, respectively, and μ_i is the viscosity of fluid i.

The analysis which we consider is predicated on the neglect of inertia effects in the fluids and in the motion of the porous sphere, thus

$$\mathrm{Re}=\frac{Ea^2\rho_2}{\mu_2}\ \left(\text{or}\ \frac{\Gamma_{13}a^2\rho_2}{\mu_2}\right)\ll 1, \tag{4}$$

where we have chosen $u_c=Ea$(or $\Gamma_{l3}a$) as the characteristic velocity, and the radius of the sphere $l_c=a$, as the characteristic length scale. The appropriate governing equations thus reduce to Stokes' equations in both fluids and to the Brinkman's equation inside the porous sphere and the equation of continuity in each fluid (see Figure 1), i.e., in dimensionless form:

$$\nabla\cdot\mathbf{u}=0,\quad \nabla\cdot\tau=0 \text{ in fluids 1 and 2,} \tag{5}$$

and inside the porous sphere

$$\nabla\cdot\mathbf{u}=0 \quad \nabla\cdot\tau=\frac{a^2}{k}\mathbf{u} \tag{6}$$

with the stress τ and pressure p given by

$$\tau_i=-p_i\mathbf{I}+\frac{\mu_i}{\mu_2}(\nabla\mathbf{u}_i+\nabla\mathbf{u}_i^t)\quad (i=1\text{ and }2) \tag{7}$$

in which k denotes the *permeability* of the porous sphere. The characteristic stress in the nondimensionalization of (5) and (6) is $p_c=\mu_2E$(or $\mu_2\Gamma_{l3}$).

It is convenient for the analysis which follows to decompose the undisturbed flow field $\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{L}^{(i)}\cdot\mathbf{x}$ into a constant vector (i.e., a uniform streaming flow), $\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{L}^{(2)}\cdot\mathbf{x}_p$, and a linear part with vanishing velocity at the body center, $\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{L}^{(i)}\cdot\mathbf{x}-\mathbf{L}^{(2)}\cdot\mathbf{x}_p$. The Stokes' problem for $\mathbf{U}_i^\infty(\mathbf{x})=\mathbf{L}^{(2)}\cdot\mathbf{x}_p$ is precisely equivalent to the problem of particle translation with velocity $\mathbf{U}=-\mathbf{L}^{(2)}\cdot\mathbf{x}_p$ through a fluid *at rest* at infinity. A complete detailed solution is available for this problem for a porous sphere from Yang and Leal [2], who determined the relationship between the hydrodynamic force $\mathbf{F}$ and torque $\mathbf{T}$ on the particle and the translational velocity $\mathbf{U}(=-\mathbf{L}^{(2)}\cdot\mathbf{x}_p)$:

$$\mathbf{F}=\mathbf{K}_T\cdot\mathbf{L}^{(2)}\cdot\mathbf{x}_p \tag{8}$$

$$\mathbf{T}=\mathbf{K}_C\cdot\mathbf{L}^{(2)}\cdot\mathbf{x}_p \tag{9}$$

where the second order tensors $\mathbf{K}_T$ and $\mathbf{K}_C$ denote the translation and coupling tensors, respectively, cf. section 5.

It thus remains only to solve the problem for the linear undisturbed flow $\mathbf{U}_l^\infty(\mathbf{x})=\mathbf{L}^{(l)}\cdot\mathbf{x}-\mathbf{L}^{(2)}\cdot\mathbf{x}_p$. The boundary conditions for this problem are

$$\mathbf{u}_t=\mathbf{L}^{(l)}\cdot\mathbf{x}-\mathbf{L}^{(2)}\cdot\mathbf{x}_p \quad \text{as } |\mathbf{x}|\to\infty \tag{10}$$

and, at the sphere surface defined by a position vector $\mathbf{x}_B\in S$,

$$[|\mathbf{u}_2|]_S=0 \tag{11}$$

$$[|\mathbf{n}\cdot\tau|]_S=0. \tag{12}$$

The symbol $[|\cdot|]_S$ in (11) and (12) represents the jump across the surface S and $\mathbf{n}$ is the outward normal. At the plane interface, the conditions of continuity of velocity and tangential stress plus zero normal velocity must be satisfied.

2. Solution methodology

Let us then consider the solution of the equations (5) and (6), boundary conditions (10)-(12), plus conditions for the presence of a flat interface, for the specific case of a rigid, porous, spherical particle of radius a which is immersed wholly in fluid 2. The fluids are assumed to undergo a linear undisturbed flow defined by $\mathbf{U}_l^\infty(\mathbf{x})=\mathbf{L}^{(l)}\cdot\mathbf{x}-\mathbf{L}^{(2)}\cdot\mathbf{x}_p$ with stagnation point as the sphere center. As indicated in the introduction, we shall approach this problem using the singularity method of Lee, Chadwick and Leal [12] who generalized the reciprocal theorem approach of Lorentz [9], to derive a general lemma for obtaining solutions of Stokes' equations that satisfy continuity of velocity, continuity of tangential stress and zero normal velocity on a flat interface, given only an arbitrary solution of Stokes' equations for an *unbounded* domain with no interface.

In the present paper, we extend the singularity method of Lee et al. [2], to consider the undisturbed linear flow past a stationary porous sphere for the asymptotic limit

$$\varepsilon=\frac{1}{d}\ll 1. \tag{13}$$

When this condition (13) is satisfied, the interface deformation will not only be in quasi-equilibrium, but the magnitude of the deformation will also be asymptotically small. Further, in this case, the singularity method can be simplified to the superposition of fundamental solutions for a point force (i.e., Stokeslet), a potential dipole and higher order singularities (e.g., a stresslet, a rotlet, a potential quadrupole, etc.) at the center of the sphere. Thus, solutions for the problem are constructed in the following manner. First, we put singularities at the center of the sphere which satisfy exactly the boundary conditions at the porous sphere in an unbounded single-fluid domain. The resulting unbounded-domain solution does *not* satisfy the boundary conditions at the flat interface; instead, an error of $O(\varepsilon)$ is generated at the interface. To eliminate this 'error', the simple transformation rule of Lee et al. [12] is used to obtain the corresponding fundamental singularity solutions that satisfy exactly the boundary conditions at the interface. In general, however, these new solutions do not satisfy boundary conditions any longer at the surface of the porous sphere, but induce an error of $O(\varepsilon)$. Additional higher-order singularities must then be included at the sphere center to cancel the induced error of $O(\varepsilon)$ at the sphere surface, and so on. The result of this procedure is an asymptotic approximation, in the form of a series in ε, that is valid in the limit of $\varepsilon\to 0$.

The complete solution for a porous particle immersed in a linear flow $\mathbf{U}_l^\infty=\mathbf{L}^{(l)}\cdot\mathbf{x}$ with stagnation point at the plane interface is obtained by superposition of the corresponding solution for the linear flow $\mathbf{U}_l^\infty=\mathbf{L}^{(l)}\cdot\mathbf{x}-\mathbf{L}^{(2)}\cdot\mathbf{x}_p$ with stagnation point at the body center, and the solution [i.e., (8) and (9)] for the uniform streaming flow $\mathbf{U}_l^\infty=\mathbf{L}^{(2)}\cdot\mathbf{x}_p$.

UNIAXIAL AXISYMMETRIC STRAINING FLOW

Let us begin by considering the creeping motion of a fluid in the vicinity of a stationary spherical porous particle that is located at an arbitrary point $\mathbf{x}_p=(x_p, y_p, -d)$ in fluid 2 when the undisturbed motion is an axisymmetric uniaxial straining flow defined by (2) with stagnation point at the particle center. In an infinite fluid domain with no interface, we determined an exact solution for a porous sphere immersed in the same type of flow [1]. The velocity field outside the porous sphere in this solution can be represented by superposition of the fundamental solutions for a potential quadrupole and a stresslet, both applied at the center of the sphere. For a porous sphere with permeability k, the required singularities are of the form:

$$\text{Stresslet}: \frac{5}{2}A(\sigma)\,\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z,\mathbf{e}_z) \tag{14}$$

$$\text{Potential Quadrupole}: \frac{1}{2}B(\sigma)\,\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z,\mathbf{e}_z). \tag{15}$$

Here, $\mathbf{u}_{SS}(\mathbf{x}, \mathbf{x}_p; \mathbf{e}_z, \mathbf{e}_z)$ and $\mathbf{u}_{PQ}(\mathbf{x}, \mathbf{x}_p; \mathbf{e}_z, \mathbf{e}_z)$ denote the fundamental solutions for a stresslet $(\mathbf{e}_z, \mathbf{e}_z)$ and a potential quadrupole $(\mathbf{e}_z, \mathbf{e}_z)$ located at the center of the sphere in an unbounded single-fluid domain, cf. Chwang and Wu [13] for the specific formulae of $\mathbf{u}_{SS}$ and $\mathbf{u}_{PQ}$. The parameters $A(\sigma)$ and $B(\sigma)$ are defined as

$$A(\sigma) = \frac{\sigma^2 \psi_2(\sigma)}{\psi_0(\sigma) + 10\psi_2(\sigma)}, \tag{16}$$

$$B(\sigma) = -\frac{1}{3} \cdot \frac{2\psi_0(\sigma) - 30\psi_2(\sigma) - 5\sigma^2\psi_2(\sigma)}{\psi_0(\sigma) + 10\psi_2(\sigma)}. \tag{17}$$

Here, ψ_n, a function of the dimensionless permeability defined by $\frac{1}{\sigma^2} = \frac{k}{a^2}$, can be expressed in terms of the modified Bessel function, i.e.,

$$\psi_n(\sigma) = \sqrt{\frac{\pi}{2}}\, \sigma^{-(n+\frac{1}{2})} \mathbf{I}_{(n+1)/2}(\sigma) \tag{18}$$

with the special properties:

$$\psi_0(\sigma) = \frac{\sinh\sigma}{\sigma}, \quad \psi_n(\sigma) = \frac{1}{\sigma}\frac{d}{d\sigma}\psi_{n-1}(\sigma). \tag{19}$$

When $k \to 0$ (or $\sigma \to \infty$), these parameters reduce to the values for an impermeable sphere, i.e., $A(\sigma) \to 1$ and $B(\sigma) \to 1$.

Since we utilize the disturbance-flow formulation defined by (5)-(7) and (10)-(12), and consider only the limit $\varepsilon \equiv \frac{1}{d} \ll 1$, the solution of the full problem, including the interface, is most conveniently obtained via the method of reflections, as explatined in some detail by Lee et al. [12]. The zeroth-order approximation in this procedure, $\mathbf{u}_2^{(0)}$, is the single-fluid unbounded domain solution which satisfies boundary conditions exactly at the sphere surface:

$$\mathbf{u}_2^{(0)}(\mathbf{x}) = \frac{5}{2} A(\sigma)\, \mathbf{u}_{SS}(\mathbf{x}, \mathbf{x}_p; \mathbf{e}_z, \mathbf{e}_z) + \frac{1}{2} B(\sigma)\, \mathbf{u}_{PQ}(\mathbf{x}, \mathbf{x}_p; \mathbf{e}_z, \mathbf{e}_z). \tag{20}$$

Here, in the notation of $\mathbf{u}_2^{(j)}(\mathbf{x})$, the superscript (j) indicates the level of approximation in the context of the method of reflections.

Though the zeroth-order approximation, (20), in the procedure exactly satisfies the boundary conditions at the sphere surface (i.e., continuity of velocity and surface force), it does not satisfy boundary conditions at the flat interface. However, a first correction $\mathbf{u}_2^{(1)}$ which does satisfy these conditions can be obtained by simply utilizing the same form, (20), as in the zeroth-order solution, but with the fundamental solutions $\mathbf{u}_{SS}$ and $\mathbf{u}_{PQ}$ replaced by the corresponding fundamental solutions, $\mathbf{u}_{2,SS}$ and $\mathbf{u}_{2,PQ}$, for a stresslet and a potential quadrupole in the presence of the flat interface, obtained by the generalized reciprocal theorem of Lee et al. [12]. It is convenient to express this solution in the form, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)}$, as a sum of zeroth-order solution plus a 'correction'. Although this solution satisfies the interface boundary conditions, it now does not satisfy the boundary conditions at the sphere surface, and additional singularities are needed at the sphere center in order to cancel the velocity field correction $\mathbf{u}_2^{(1)}$ at the sphere surface: namely, the interface reflection of the potential quadrapole and the stresslet, which is nonzero on the sphere surface. The preceding two steps, leading to the approximation solution, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)}$, can be carried out for arbitrary ε. However, the resulting expression $\mathbf{u}_2^{(1)}$ is highly complicated, and it is not possible for arbitrary ε to determine singularities at the sphere center which precisely satisfy the continuity of velocity and surface force at all points on the sphere surface. Instead, we consider the asymptotic limit $\varepsilon \ll 1$, and then choose singularities to cancel only the first few terms of $\mathbf{u}_2^{(1)}$ at the sphere surface, with $\mathbf{u}_2^{(1)}$ expressed in power of ε. The leading terms of $\mathbf{u}_2^{(1)}$ near the sphere surface, for small ε, are

$$\mathbf{u}_2^{(1)}(\mathbf{x}) = \frac{5}{8} A(\sigma) \left[-\varepsilon^2 \cdot \frac{2+3\lambda}{1+\lambda} \cdot \mathbf{e}_z + \varepsilon^3 \cdot \frac{1+2\lambda}{1+\lambda} \mathbf{E} \cdot (\mathbf{x} - \mathbf{x}_p) \right] + O(\varepsilon^4) \tag{21}$$

in which λ is the viscosity ratio (i.e., $\lambda = \frac{\mu_1}{\mu_2}$) of the two continuous fluid phases 1 and 2. The dimensionless strain rate tensor $\mathbf{E}$ has Cartesian components, $E_{ij} = \delta_{ij} - 3\delta_{i3}\delta_{j3}$ with the origin at the center of the sphere.

In so far as (21) is concerned, the presence of the interface induces a steady streaming flow at $O(\varepsilon^2)$ normal to the interface. The term of $O(\varepsilon^3)$ in (21) in equivalent to an axisymmetric uniaxial straining flow with stagnation point at the sphere center, and the z-axis as the axis of symmetry. The singularities required to cancel the additional velocity field $\mathbf{u}_2^{(1)}(\mathbf{x})$ of (21) at the sphere surface can be readily evaluated. We have seen previously that an extensional flow of the type represented by the $O(\varepsilon^3)$ term in (21) is generated by superposition of a stresslet and a potential quadrupole. It can be shown that a uniform streaming flow solution is generated in an unbounded single-fluid by a Stokeslet and a potential dipole [1]. To counter the terms of $O(\varepsilon^2)$ in (21), we thus require the superposition of a Stokeslet and a potential dipole at the sphere center. The resulting velocity field is

$$\mathbf{u}^{(2)}_{2,ST} = \frac{15}{32}\cdot\frac{2+3\lambda}{1+\lambda}A(\sigma)\,\varepsilon^2[C(\sigma)\,\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z) - \frac{1}{3}D(\sigma)\,\mathbf{u}_D(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z)] \qquad (22)$$

where

$$C(\sigma) = \frac{\sigma^2\psi_1(\sigma)}{\psi_0(\sigma)+\sigma^2\psi_1(\sigma)+3\psi_1(\sigma)/2} \qquad (23)$$

and

$$D(\sigma) = -\frac{\sigma^2\{2\psi_2(\sigma)-\psi_1(\sigma)\}}{\psi_0(\sigma)+\sigma^2\psi_1(\sigma)+3\psi_1(\sigma)/2}. \qquad (24)$$

Here, $\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\ \mathbf{e}_z)$ and $\mathbf{u}_D(\mathbf{x},\mathbf{x}_p;\ \mathbf{e}_z)$ denote the fundamental solutions for a Stokeslet $\mathbf{e}_z$ and a potential dipole $\mathbf{e}_z$ at the sphere center in an unbounded domain with no interface. It is important to note that the point force (i.e., Stokeslet) velocity of strength $O(\varepsilon^2)$, corresponding to $\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\ \mathbf{e}_z)$, will itself generate a vertical velocity component of $O(\varepsilon^3)$ at the sphere surface when it is 'reflected' from the interface, cf. Yang and Leal [2]. Thus, if we are to consider any correction terms of $O(\varepsilon^3)$ from (21) we must simultaneously include this additional $O(\varepsilon^3)$ correction to the velocity field near the sphere. In order to cancel this $O(\varepsilon^3)$ term at the sphere surface we require an additional point force and potential dipole at the sphere center of the form:

$$\mathbf{u}^{(3)}_{2,ST} = \frac{5}{4}\cdot\{\frac{3}{8}\cdot\frac{2+3\lambda}{1+\lambda}\}^2 A(\sigma)\,C(\sigma)\,\varepsilon^3[C(\sigma)\,\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z) - \frac{1}{3}D(\sigma)\,\mathbf{u}_D(\mathbf{x},\mathbf{x}_P;\mathbf{e}_z)]. \qquad (25)$$

Thus, the complete contribution to the velocity field that is required to cancel the first two terms of $\mathbf{u}_2^{(1)}$ at the sphere surface is a superposition of

$$\text{Stokeslet: } \frac{5}{4}A(\sigma)\,\mathbf{u}_S(\mathbf{x},\mathbf{x}_P;\mathbf{e}_z)\,[\sum_{n=1}^{2}\{\frac{3}{8}C(\sigma)\frac{2+3\lambda}{1+\lambda}\varepsilon\}^n\cdot\varepsilon + O(\varepsilon^4)] \qquad (26)$$

$$\text{Potential Dipole: } -\frac{5}{12}A(\sigma)\frac{D(\sigma)}{C(\sigma)}\mathbf{u}_D(\mathbf{x},\mathbf{x}_P;\mathbf{e}_z)\,[\sum_{n=1}^{2}\{\frac{3}{8}C(\sigma)\frac{2+3\lambda}{1+\lambda}\varepsilon\}^n\varepsilon + O(\varepsilon^4)] \qquad (27)$$

$$\text{Stresslet: } \frac{5}{2}A(\sigma)\,\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_P;\mathbf{e}_z,\mathbf{e}_z)\,[1+\frac{5}{8}A(\sigma)\frac{1+2\lambda}{1+\lambda}\varepsilon^3 + O(\varepsilon^4)] \qquad (28)$$

$$\text{Potential Quadrupole: } \frac{1}{2}B(\sigma)\,\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_P;\mathbf{e}_z,\mathbf{e}_z)\,[1+\frac{5}{8}A(\sigma)\frac{1+2\lambda}{1+\lambda}\varepsilon^3 + O(\varepsilon^4)]. \qquad (29)$$

The complete velocity field, $\mathbf{u}_2^{(0)} + \mathbf{u}_2^{(1)} + \mathbf{u}_2^{(2)} + \mathbf{u}_2^{(3)}$, resulting from the superposition of (26)-(29) now satisfies boundary conditions *exactly* at the interface and boundary conditions to $O(\varepsilon^3)$ at the sphere surface. Higher-order approximations could be obtained by straightforward continuation of the same procedure. However, the solution above is sufficient for present purposes.

The net force exerted on a porous sphere located at the stagnation point in the undisturbed flow field $\mathbf{U}_i^\infty = \mathbf{E}\cdot(\mathbf{x}\text{-}\mathbf{x}_p)$ can be evaluated simply from the Stokeslet distribution:

$$\mathbf{F} = -\varepsilon^2 A(\sigma)\,C(\sigma)\frac{15\pi}{4}\cdot\frac{2+3\lambda}{1+\lambda}\,[1+\frac{3}{8}\frac{2+3\lambda}{1+\lambda}C(\sigma)\,\varepsilon]\mathbf{e}_z + O(\varepsilon^4). \qquad (30)$$

Obviously, the torque is zero owing to the symmetry of the sphere. The force $\mathbf{F}$ is always oriented *away* from the interface, and the magnitude is increased as the viscosity ratio λ becomes larger. Thus a posiive external force $-\mathbf{F}$ would have to be applied to the body to keep it from translating away from the stagnation point $\mathbf{x}_p$ of the flow regardless of the particle position, or the viscosity ratio of the two fluids. It should be understood that, in this flow-field $\mathbf{U}_i^\infty = \mathbf{E}\cdot(\mathbf{x}\text{-}\mathbf{x}_p)$, the interface translates with velocity $-2d\mathbf{e}_z$ toward the stagnation point $\mathbf{x}_p$ at which the body center is held fixed. This 'interface motion' can be viewed as the source of $\mathbf{F}$.

Now let us turn to the original problem of calculating the force and torque acting on a stationary sphere that is located at arbitrary point $\mathbf{x}_p$ in fluid 2 which is undergoing the axisymmetric uniaxial extension flow $\mathbf{U}_i^\infty = \mathbf{E}\cdot\mathbf{x}$ with origin at the interface (i.e., Figure 1). As we showed in section 2, the hydrodynamic force and torque exerted in this case can be determined by a superposition of the force and torque for a uniform streaming flow with velocity $\mathbf{U}_i^\infty = \mathbf{E}\cdot\mathbf{x}_p$ and for a uniaxial straining flow $\mathbf{U}_i^\infty = \mathbf{E}\cdot(\mathbf{x}\text{-}\mathbf{x}_p)$ with stagnation point at the sphere center. The resulting force and torque can be expressed in the following form:

$$\mathbf{F} = \mathbf{K}_T\cdot\mathbf{E}\cdot\mathbf{x}_p - \varepsilon^2 A(\sigma)\,C(\sigma)\frac{15\pi}{4}\cdot\frac{2+3\lambda}{1+\lambda}\,[1+\frac{3}{8}\cdot\frac{2+3\lambda}{1+\lambda}\cdot C(\sigma)\cdot\varepsilon]\,\mathbf{e}_z + O(\varepsilon^4) \qquad (31)$$

$$\mathbf{T} = \mathbf{K}_C\cdot\mathbf{E}\cdot\mathbf{x}_p + O(\varepsilon^4). \qquad (32)$$

The components of the translation and coupling tensors $\mathbf{K}_T$ and $\mathbf{K}_C$ were determined up to $O(\varepsilon^3)$ by Yang and Leal [2] for motion of a porous sphere near a plane fluid-fluid interface.

Dukhin and Rulev [14] obtained an exact result for the drag force on a small impermeable spherical particle located at the axis of symmetry in an axisymmetric uniaxial extensional flow $\mathbf{U}_i^\infty = \mathbf{E}\cdot\mathbf{x}$, near a gas-liquid interface (i.e., $\lambda \to 0$), using the eigensolutions of Laplace's equation in bipolar coordinates. It is a simple matter to calculate the drag force $\mathbf{F}$ on the porous *permeable* sphere from the present asymptotic solution (31) with $\mathbf{x}_p = (0, 0, -d)$. The drag ratio, i.e., the drag divided by the Stokes drag, $12\pi\mu_2 adE$, is simply given as

$$\frac{\mathbf{F}}{12\pi\mu_2 adE} = C(\sigma)\,[1 + \sum_{n=1}^{3}\{\frac{3}{8}C(\sigma)\frac{2+3\lambda}{1+\lambda}\cdot\varepsilon\}^n$$

$$+\{\frac{15}{16}C(\sigma)\cdot\frac{2+5\lambda}{1+\lambda} - D(\sigma)\cdot\frac{1}{16}\cdot\frac{1+4\lambda}{1+\lambda} -$$

$$\frac{1}{16}E(\sigma)\cdot\frac{31+79\lambda}{1+\lambda}\}\,\varepsilon^3 - \frac{5}{16d}A(\sigma)$$

$$\frac{2+3\lambda}{1+\lambda}\varepsilon^2\{1+\frac{3}{8}\,\frac{2+3\lambda}{1+\lambda}C(\sigma)\cdot\varepsilon\}\,]\,\mathbf{e}_z + O(\varepsilon^4) \qquad (33)$$

in which $E(\sigma)$ is defined as

$$E(\sigma) = \frac{\sigma^2\{\psi_1(\sigma) - 2\psi_2(\sigma)\}}{\psi_o(\sigma) + \sigma^2\psi_1(\sigma) + 3\psi_1(\sigma)/2}. \qquad (34)$$

It should be noted that, when $\sigma \to \infty$ (or $k \to 0$), the present solution for the drag, (33), reduces to the asymptotic result of Yang and Leal [15] for the case of a rigid *impermeable* sphere. In order to illustrate the effects of hydrodynamic interaction between the particle and the interface the drag ratio of (33) is plotted as a function of the separation distance d for $\frac{k}{a^2} = 0$, 0.1 and 1.0 (i.e., $\sigma^2 = \infty$, 10 and 1.0, respectively). For each value of $\frac{k}{a^2}$, we include two values of the viscosity ratio $\lambda = 0$ (i.e., free surface) and $\lambda = \infty$ (i.e., solid wall). Also shown for comparison are the corresponding exact solution results of Dukhin and Rulev [14] for an *impermeable* sphere near a free surface (i.e., $\lambda = \frac{k}{a^2} \to 0$). It can be seen from Figure 2 that there is very good agreement between the two solutions, except in the region near $d \simeq 1$. As expected, the difference between the two results becomes larger as the sphere approaches the interface owing to the poor convergence of the asymptotic solution (33) in powers of ε. Further, due to the presence of the interface, the magnitude of drag is increased for any values of λ and $\frac{k}{a^2}$ considered here, and this effect is a strong function of the particle position relative to the interface.

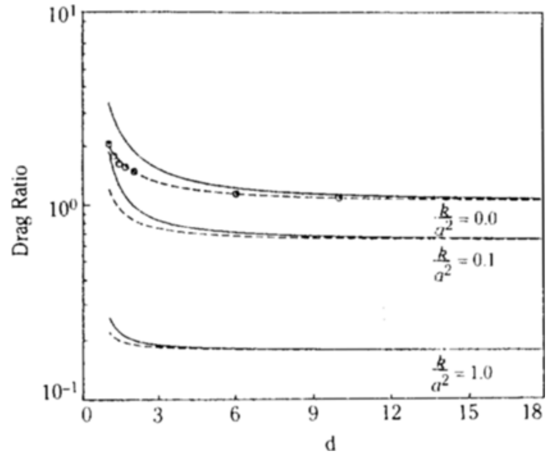

Fig. 2. Drag ratio, $\frac{\mathbf{F}}{12\pi\mu_2 adE}$, as a function of the dimensionless separation distance d for three values of the dimensionless permeability $k/a^2 = 0.0, 0.1$ and 1.0; —, for $\lambda \to \infty$; ----, for $\lambda = 0.0$. Markers are the corresponding exact solution results of Dukhin and Rulev [14] for $\lambda = k/a^2 = 0$.

LINEAR SHEAR FLOW

We now turn to the case of a porous spherical particle located at an arbitrary point $\mathbf{x}_p$ in a simple shear flow $\mathbf{U}_i^\infty = \Gamma^{(i)}\cdot\mathbf{x}$, parallel to the interface as shown in Figure 1. The case in which $\mathbf{U}_i^\infty = \mathbf{C} \neq \mathbf{O}$ at the interface can be treated by superposing a uniform streaming flow past a sphere, $\mathbf{U}_i^\infty = \mathbf{C}$, with the simple shear flow $\mathbf{U}_i^\infty = \Gamma^{(i)}\cdot\mathbf{x}$ [the Cartesian components of shear rate tensor Γ is defined by (3)]. Again the problem can be decomposed into a simple translation of the fluid system including the interface with uniform velocity $\mathbf{U}_i^\infty = \Gamma^{(2)}\cdot\mathbf{x}_p$ past the stationary sphere together with a linear shear flow $\mathbf{U}_i^\infty = \Gamma^{(i)}\cdot\mathbf{x} - \Gamma^{(2)}\cdot\mathbf{x}_p$ with stagnation point at the sphere center. In view of the linearity of the problem and the symmetry of the sphere-interface geometry, we need only solve the case of $\mathbf{U}_i^\infty = \frac{\mu_i}{\mu_2}\Gamma_{13} z\mathbf{e}_x$ corresponding to $\mathbf{L}_{lm}^{(i)} = \Gamma_{lm}^{(i)} = \frac{\mu_2}{\mu_i}\Gamma_{13}\delta_{l1}\delta_{m3}$. In order to analyze the velocity field for a porous sphere in the undisturbed flow $\mathbf{U}_i^\infty = \Gamma_{13}(\frac{\mu_2}{\mu_i}z + d)\,\mathbf{e}_x$ which vanishes at the sphere center, we follow the procedure of the preceding section. As in the preceding analysis, we use the method of reflections, with the solution in an *unbounded* fluid taken from the part 1 of the present series, in which we showed that the solution in an unbounded fluid was simply the superposition of a stresslet, a potential quadrupole, and a rotlet at the center of the sphere, i.e.

$$\text{Stresslet}: -\frac{5}{6}\Gamma_{13}A(\sigma)\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z) \quad (35)$$

$$\text{Potential Quadrupole}: -\frac{1}{6}\Gamma_{13}G(\sigma)\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z) \quad (36)$$

$$\text{Rotlet}: -\frac{1}{2}\Gamma_{13}H(\sigma)\mathbf{u}_R(\mathbf{x},\mathbf{x}_p;\mathbf{e}_y) \quad (37)$$

in which $\mathbf{u}_R(\mathbf{x},\mathbf{x}_p;\mathbf{e}_y)$ denotes the fundamental solution for a rotlet $\mathbf{e}_y$ located at $\mathbf{x}_p$, the center of the sphere, and the parameters $G(\sigma)$ and $H(\sigma)$ are difined as:

$$G(\sigma)=\frac{\sigma^2\{\psi_2(\sigma)-2\psi_3(\sigma)\}}{\psi_0(\sigma)+10\psi_2(\sigma)} \quad (38)$$

and

$$H(\sigma)=\frac{\sigma^2\psi_2(\sigma)}{\psi_0(\sigma)} \quad (39)$$

The unbounded-domain solution represented by (35)-(37) satisfies exactly boundary conditions at the surface of the sphere, but generates an error of $O(\varepsilon)$ at the flat interface.

As in the preceding example, the first correction for the presence of the interface in the reflections expansion can now be obtained easily from the unbounded-domain solution, (35)-(37), by simply replacing the fundamental solutions $\mathbf{u}_{SS}$, $\mathbf{u}_{PQ}$ and $\mathbf{u}_R$ (which pertain to an unbounded fluid) with the corresponding fundamental solutions $\mathbf{u}_{2,SS}$, $\mathbf{u}_{2,PQ}$ and $\mathbf{u}_{2,R}$ that satisfy boundary conditions on the flat interface (and are generated using the lemma of Lee et al. [12]). The result is the first two terms in the reflections expansion, i.e., $\mathbf{u}_2^{(0)}+\mathbf{u}_2^{(1)}$. The first correction, $\mathbf{u}_2^{(0)}+\mathbf{u}_2^{(1)}$, for the presence of the interface does not satisfy the boundary conditions (11) and (12) at the sphere surface, because the interface reflection $\mathbf{u}_2^{(1)}(\mathbf{x})$ is nonzero at the sphere surface. Following section 3, we examine the leading terms of the reflected velocity field at the sphere surface as a power series in ε.

$$\mathbf{u}_2^{(1)}(\mathbf{x})=\frac{\varepsilon^2}{16}\cdot\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}\Gamma_{13}\mathbf{e}_x+\varepsilon^3\cdot\Gamma^*\cdot(\mathbf{x}-\mathbf{x}_p) \quad (40)$$

where the nonzero components of the second-order shear rate tensor are given by

$$\Gamma_{13}^*=\frac{5\lambda A(\sigma)-(2-\lambda)H(\sigma)}{16(1+\lambda)}\Gamma_{13} \quad (41)$$

$$\Gamma_{31}^*=\frac{5(1+2\lambda)A(\sigma)-(1+4\lambda)H(\sigma)}{16(1+\lambda)}\Gamma_{13}. \quad (42)$$

It can be seen from (40)-(42) that the presence of the interface in this case is equivalent in its effect on the porous particle to a steady streaming flow at $O(\varepsilon^2)$ parallel to the interface, and a linear shear flow at $O(\varepsilon^3)$ either normal or parallel to the interface.

In order to satisfy the conditions of continuity of velocity and surface force at the sphere surface, i.e., boundary conditions of (11) and (12), we need additional singularities at the sphere center that produce a velocity field at the sphere surface of opposite sign. For the term of $O(\varepsilon^2)$, a point force and a potential dipole are required, which have the intensity and orientation:

$$\mathbf{u}_{2,ST}^{(2)}=-\frac{3}{64}\cdot\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}\Gamma_{13}\varepsilon^2\cdot[C(\sigma)\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)-\frac{1}{3}D(\sigma)\mathbf{u}_D(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)]. \quad (43)$$

By induction, we also know that the interface 'reflection' of the point force and potential dipole solutions corresponding to (43) will yield a nonzero contribution of $O(\varepsilon^3)$ to the x-component of velocity at the sphere surface. In order to satisfy the boundary conditions on the sphere surface to $O(\varepsilon^3)$, we thus require an additional point force and potential dipole at the sphere center with magnitude and orientation:

$$\mathbf{u}_{2,ST}^{(3)}=\frac{3}{16}\frac{3}{64}\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}\frac{2-3\lambda}{1+\lambda}\Gamma_{13}C(\sigma)\varepsilon^3\cdot[C(\sigma)\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)-\frac{1}{3}D(\sigma)\mathbf{u}_D(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)]. \quad (44)$$

Further, the singularities required to counter the $O(\varepsilon^3)$ contribution in (40) can be evaluated by determining the corresponding solution for the linear flow in an unbounded fluid domain. It can be shown that a stresslet, a potential quadrupole, and a rotlet are necessary to produce such flow in an unbounded single-fluid domain. Thus,

$$\mathbf{u}_{2,SH}^{(3)}=-\Gamma_{13}^*\varepsilon^3[\frac{5}{6}A(\sigma)\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z)+\frac{1}{2}H(\sigma)\mathbf{u}_R(\mathbf{x},\mathbf{x}_p;\mathbf{e}_y)+\frac{1}{6}G(\sigma)\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z)]$$
$$-\Gamma_{31}^*\varepsilon^3[\frac{5}{6}A(\sigma)\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z,\mathbf{e}_x)-\frac{1}{2}H(\sigma)\mathbf{u}_R(\mathbf{x},\mathbf{x}_p;\mathbf{e}_y)+\frac{1}{6}G(\sigma)\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z,\mathbf{e}_x)] \quad (45)$$

in which the reflected shear components Γ_{13}^* and Γ_{31}^*

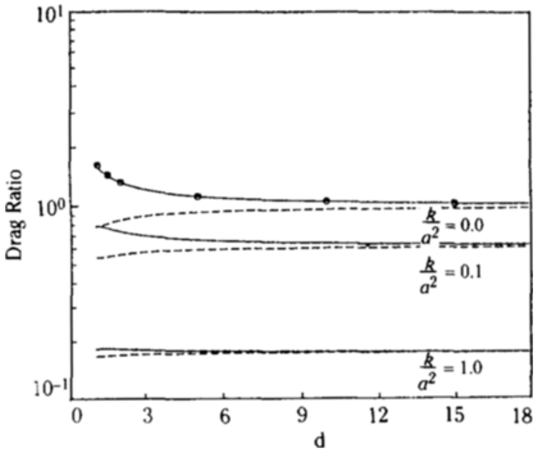

Fig. 3. Drag ratio, $-\frac{F}{6\pi\mu_2 ad\Gamma_{13}}$, as a function of the dimensionless separation distance d for three values of the dimensionless permeability $k/a^2 = 0.0$, 0.1 and 1.0; —, for $\lambda\to\infty$; ---, for $\lambda = 0.0$. Markers are the corresponding exact solution results of Goren and O'Neill [3] for $\lambda\to\infty$ and $k/a^2 = 0$.

are defined in (41) and (42).

Consequently, for the linear shear flow past a porous sphere, the singularities required at the center of the sphere through $O(\varepsilon^3)$ are:

Stokeslet : $-\frac{3\Gamma_{13}}{64}\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}$

$$C(\sigma)\varepsilon^2\mathbf{u}_S(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)\left[1-\frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\varepsilon\right] \quad (46)$$

Potential Dipole : $\frac{\Gamma_{13}}{64}\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}$

$$D(\sigma)\varepsilon^2\mathbf{u}_D(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x)\left[1-\frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\varepsilon\right] \quad (47)$$

Stresslet : $-\frac{5\Gamma_{13}}{6}A(\sigma)\mathbf{u}_{SS}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z)$

$$\left[1+\frac{5(1+3\lambda)A(\sigma)-3(1+\lambda)H(\sigma)}{16(1+\lambda)}\varepsilon^3\right] \quad (48)$$

Rotlet : $-\frac{\Gamma_{13}}{2}H(\sigma)\mathbf{u}_R(\mathbf{x},\mathbf{x}_p;\mathbf{e}_y)$

$$\left[1-\frac{5(1+\lambda)A(\sigma)-(5\lambda-1)H(\sigma)}{16(1+\lambda)}\varepsilon^3\right] \quad (49)$$

Potential Quadrupole : $-\frac{\Gamma_{13}}{6}G(\sigma)$

$$\Big[\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_x,\mathbf{e}_z)\left\{1+\frac{5\lambda A(\sigma)-(2-\lambda)H(\sigma)}{16(1+\lambda)}\varepsilon^3\right\}$$

$$+\mathbf{u}_{PQ}(\mathbf{x},\mathbf{x}_p;\mathbf{e}_z,\mathbf{e}_x)\varepsilon^3\frac{5(1+2\lambda)A(\sigma)-(1+4\lambda)H(\sigma)}{16(1+\lambda)}\Big]. \quad (50)$$

From this solution and equations (8) and (9), we can easily determine the hydrodynamic force and torque exerted on a *porous* sphere located at an arbitrary point $\mathbf{x}_p$, in the simple shear flow $\mathbf{U}_i^\infty = \Gamma^{(i)}\cdot\mathbf{x}$ with stagnation point at the interface. The result is

$$\mathbf{F} = \mathbf{K}_T\cdot\Gamma^{(2)}\cdot\mathbf{x}_p + \mathbf{K}_{SF}:\Gamma^{(2)} \quad (51)$$

$$\mathbf{T} = \mathbf{K}_C\cdot\Gamma^{(2)}\cdot\mathbf{x}_p + \mathbf{K}_{ST}:\Gamma^{(2)} \quad (52)$$

in which the nonzero components of the third-order hydrodynamic tensors $\mathbf{K}_{SF}$ and $\mathbf{K}_{ST}$ are given by

$$K_{SF}^{113} = K_{SF}^{223} = \frac{3\pi}{8}\cdot\frac{5\lambda A(\sigma)-2H(\sigma)}{1+\lambda}\cdot C(\sigma)\cdot\varepsilon^2$$

$$\left[1-\frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\varepsilon\right]+O(\varepsilon^4) \quad (53)$$

and

$$K_{ST}^{123} = -K_{ST}^{213} = -4\pi H(\sigma)$$

$$\left[1-\frac{5(1+\lambda)A(\sigma)-(5\lambda-1)H(\sigma)}{16(1+\lambda)}\varepsilon^3\right]+O(\varepsilon^4). \quad (54)$$

The drag ratio (the drag divided by the Stokes drag $-6\pi\mu_2\Gamma_{13}da$) is simply given as

$$-\frac{\mathbf{F}}{6\pi\mu_2\Gamma_{13}da} = C(\sigma)\Big[1+\sum_{n=1}^{3}\left\{\frac{3}{16}C(\sigma)\frac{3\lambda-2}{1+\lambda}\varepsilon\right\}^n$$

$$+\Big\{\frac{15}{128}C(\sigma)\frac{2-5\lambda}{1+\lambda}-\frac{1}{32}\cdot D(\sigma)\frac{1+2\lambda}{1+\lambda}-\frac{1}{128}$$

$$E(\sigma)\frac{34-67\lambda}{1+\lambda}\Big\}\varepsilon^3-\frac{5\lambda A(\sigma)-2H(\sigma)}{16d(1+\lambda)}\varepsilon^2$$

$$\left\{1-\frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\cdot\varepsilon\right\}\Big]\mathbf{e}_x+O(\varepsilon^4) \quad (55)$$

where we have again considered the shear component Γ_{13} with no loss of generality. When $k\to 0$(or $\sigma\to\infty$), the equation (55) reduces to the drag ratio for the case of an *impermeable* sphere, and is identical with the results of Yang and Leal [15] to $O(\varepsilon^3)$. For a linear shear flow parallel to a *rigid* plane boundary, Goren and O'Neill [3] calculated the hydrodynamic force and torque on a sphere, using the eigenfunctions of Laplace's equation in bipolar coordinates. In Figure 3 the drag ratio (55) is plotted as a function of d, the separation distance between the sphere and the interface for the same set of parameters as in Figure 2. Also shown for comparison are the corresponding drag ratios determined by Goren and O'Neill. In many respects, the results are similar to those for parallel translation of a porous permeable sphere obtained by Yang and Leal [2]. As mentioned previously, we presume $\varepsilon \ll 1$ in the derivation of (55). Thus for $\varepsilon \ll 1$ (i.e., $d \gg 1$) the asymptotic solution (55) coincides almost exactly with Goren and O'Neill's result, which is the exact solution for the simple shear flow parallel to a *solid* wall. Even for $d \simeq 1.5$, the approximation solution shows reasonably good agreement with the exact solution. Indeed, for an *impermeable* sphere the relative error is within 2.6% for $d > 1.5$.

The hydrodynamic torque, $\mathbf{T}$, on a sphere in the flow $\mathbf{U}_i^\infty = \Gamma^{(i)} \cdot \mathbf{x}$ can be evaluated from (52), and is equal to

$$\frac{\mathbf{T}}{4\pi\mu_2\Gamma_{13}a^3} = H(\sigma)\left[1 + \frac{3}{8}C(\sigma)\frac{1}{1+\lambda}\varepsilon\left\{1 - \frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\varepsilon\right\} - \frac{5(1+\lambda)A(\sigma) - (5\lambda-1)H(\sigma)}{16(1+\lambda)}\varepsilon^3\right]\mathbf{e}_y + O(\varepsilon^4). \quad (56)$$

This is the negative of the torque that is required to keep the sphere from rotating.

We have now a complete set of solutions for a stationary porous sphere located at arbitrary point $\mathbf{x}_p$ in either an axisymmetric extensional flow or in a simple shear flow field. These solutions provide the necessary relationship between the flow parameters (i.e., strain rate or shear rate) and the hydrodynamic force and torque for calculation of particle trajectories, which we shall consider in section 5.

PARTICLE TRAJECTORIES

At sufficiently small Reynolds number, equations of motion for a rigid body of arbitrary shape can be expressed in general terms, provided the interface remains flat, by defining the so-called translation tensor $\mathbf{K}_T$, the rotation tensor $\mathbf{K}_R$, and the coupling tensor $\mathbf{K}_C$. Two fundamental relations exist between the translational and angular velocities and the force and torque in terms of these tensors,

$$\mathbf{F} = \mathbf{K}_T \cdot \mathbf{U} + \mathbf{K}_C^t \cdot \Omega \quad (57)$$

$$\mathbf{T} = \mathbf{K}_C \cdot \mathbf{U} - \mathbf{K}_R \cdot \Omega \quad (58)$$

where $\mathbf{F}$ and $\mathbf{T}$ are the total hydrodynamic force and torque, and $\mathbf{U}$ and Ω are the translational and angular velocities, respectively. The nonzero components of these tensors for a *porous* permeable particle were evaluated through terms $O(\varepsilon^3)$ by Yang and Leal [2]:

$$K_T^{11} = K_T^{22} = 6\pi C(\sigma)\left[1 + \sum_{n=1}^{3}\left\{\frac{3}{16}C(\sigma)\frac{3\lambda-2}{1+\lambda}\cdot\varepsilon\right\}^n + \left\{\frac{5}{128}C(\sigma)\frac{2-5\lambda}{1+\lambda} - \frac{1}{32}D(\sigma)\frac{1-2\lambda}{1-\lambda} - \frac{1}{128}E(\sigma)\frac{34-67\lambda}{1+\lambda}\right\}\varepsilon^3\right] + O(\varepsilon^4)$$

$$K_T^{33} = 6\pi C(\sigma)\left[1 + \sum_{n=1}^{3}\left\{\frac{3}{8}C(\sigma)\frac{2+3\lambda}{1+\lambda}\varepsilon\right\}^n + \left\{\frac{15}{16}C(\sigma)\frac{2+5\lambda}{1+\lambda} - \frac{1}{16}D(\sigma)\frac{1+4\lambda}{1+\lambda} - \frac{1}{16}E(\sigma)\frac{31+79\lambda}{1+\lambda}\right\}\varepsilon^3\right] + O(\varepsilon^4)$$

$$K_C^{12} = -K_C^{21} = \frac{3\pi}{2}C(\sigma)H(\sigma)\cdot\frac{1}{1+\lambda}\left[\varepsilon^2 - \frac{3}{16}C(\sigma)\frac{2-3\lambda}{1+\lambda}\varepsilon^3\right] + O(\varepsilon^4)$$

$$K_R^{11} = K_R^{22} = 8\pi H(\sigma)\left[1 - \frac{1}{16}H(\sigma)\frac{1-5\lambda}{1+\lambda}\varepsilon^3\right] + O(\varepsilon^4)$$

$$K_R^{33} = 8\pi H(\sigma)\left[1 - \frac{1}{8}H(\sigma)\frac{1-\lambda}{1+\lambda}\varepsilon^3\right] + O(\varepsilon^4).$$

With the preceding relationship established for the resistance tensors, the velocity vectors and the force and torque vectors, we can readily apply (57) and (58) to general trajectory calculations. In the present paper, we consider only the simplest case of a neutrally buoyant freely suspended body. In this case, an instantaneous solution for $\mathbf{U}$ and Ω is easily obtained from (57) and (58):

$$\mathbf{U} = \dot{\mathbf{x}}_p = (\mathbf{K}_T - \mathbf{K}_C^t \cdot \mathbf{K}_R^{-1} \cdot \mathbf{K}_C)^{-1} \cdot (\mathbf{F} - \mathbf{K}_C^t \cdot \mathbf{K}_R^{-1} \cdot \mathbf{T}) \quad (59)$$

$$\Omega = \mathbf{K}_R^{-1} \cdot (\mathbf{T} - \mathbf{K}_C \cdot \mathbf{U}). \quad (60)$$

Here $\mathbf{F}$ and $\mathbf{T}$ are the hydrodynamic force and torque acting on a stationary particle due to the existence of a mean flow at large distance from the particle. Thus, given the initial position of the particle, these equations provide its complete trajectory. We consider trajectories for the special cases of a porous sphere freely suspended in a uniaxial extensional and linear shear flows. The purpose of these two calculations is primarily illustrative. However, these two elementary problems are relevant to the processes of particle capture at the surface of a larger bubble or drop which may be viewed as locally planar in the limit where the particle is very much smaller than the collector [4]. First, we begin with the case of a neutrally buoyant porous sphere freely suspended in the extensional flow $\mathbf{U}_i^\infty(\mathbf{x}) = \mathbf{E} \cdot \mathbf{x}$ with stagnation point at the interface. The results for the force and torque $\mathbf{F}$ and $\mathbf{T}$ in this case are given in (31) and (32). Substituting for **F** and **T** in (59) and (60), respectively, it is a simple matter to show that the translational and angular velocities of the particle are

$$\mathbf{U}=\mathbf{U}_z^\infty(\mathbf{x}_p)-\frac{5}{8}\varepsilon^2 A(\sigma)\frac{2+3\lambda}{1+\lambda}\mathbf{e}_z+O(\varepsilon^4) \tag{61}$$

and

$$\Omega=O(\varepsilon^4). \tag{62}$$

Thus the particle does not rotate at all, at the level of approximation represented by (61) and (62), and it is only the z-component of **U** (i.e., U_z) that is altered from the undisturbed velocity of the fluid by the presence of an interface. It can be noted from (61) that the particle velocity U_z is always decreased in magnitude by the presence of an interface, independently of the viscosity ratio λ and the particle permeability k. Further, the difference between the undisturbed velocity of the fluid and U_z (i.e., $\mathbf{E}\cdot\mathbf{x}_p\cdot\mathbf{e}_z-U_z$) is monotonically increased as the separation distance between the interface and the sphere is decreased, but is independent of the distance from the axis of symmetry of the uniaxial extensional flow.

The other problem considered here is the motion of a freely suspended porous sphere in a linear shear flow $\mathbf{U}_i^\infty(\mathbf{x})=\Gamma^{(i)}\cdot\mathbf{x}$ parallel to the interface. Since the hydrodynamic force on the particle is oriented parallel to the undisturbed flow, the path followed by the sphere in the (x, z)-plane is exactly coincident with a streamline of the undisturbed flow. However, the translational velocity of the sphere, $\mathbf{U}=U_x\mathbf{e}_x$, is altered considerably from the undisturbed velocity, $\mathbf{U}_i^\infty(\mathbf{x}_p)=-d\mathbf{e}_x$, of the fluid by interaction with the interface:

$$U_x-(-d)=\frac{5\lambda}{16(1+\lambda)}A(\sigma)\varepsilon^2 + \frac{3}{64(1+\lambda)^2}H(\sigma)C(\sigma)\varepsilon^3+O(\varepsilon^4). \tag{63}$$

The magnitude of translational velocity is decreased relative to the unbounded case, independently of the viscosity ratio λ, and this effect is enhanced strongly as the body is placed closer to the interface. This is illustrated in Figure 4, where the difference between the velocity of the sphere and the undisturbed velocity of the fluid U_x-($-d$) is given as a function of the separation distance d between the sphere and the interface for $\lambda=0$ and ∞ and $\frac{k}{a^2}=0.0$, 0.1 and 1.0. Also included for comparison are the corresponding results of Goldman, Mason and Brenner [16], who obtained an exact solution of the Stokes' equation, using bipolar

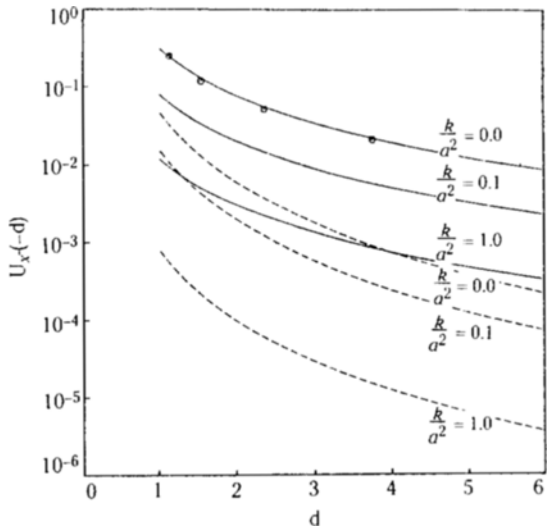

Fig. 4. Dimensionless disturbed translational velocity, U_x-($-d$), as a function of the dimensionless separation distance d for three values of the dimensionless permeability $k/a^2=0.0$, 0.1 and 1.0; —, for $\lambda\to\infty$; ----, for $\lambda=0.0$. Markers are the corresponding exact solution results of Goldman et al. [16] for $\lambda\to\infty$ and $k/a^2=0$.

coordinates, for translational and angular velocities of an *impermeable* (i.e., $k\to 0$) sphere moving in a linear shear flow in proximity to a single plane wall (i.e., $\lambda\to\infty$). It can be seen from Figure 4 that the present result for translational velocity is in reasonable agreement with the exact solution in the entire region of $d>1$.

The angular velocity Ω, (60), for motion of a freely suspended sphere in the linear shear flow can be obtained by substituting the results for the force and torque given in (51) and (52):

$$\Omega_y-\frac{1}{2}=-\frac{5}{32}A(\sigma)\varepsilon^3+O(\varepsilon^4). \tag{64}$$

In an unbounded fluid-domain, a freely suspended particle will rotate with an angular velocity $\Omega=\frac{1}{2}\mathbf{e}_y$ which is $\frac{1}{2}$ of the vorticity vector in the primary flow irrespective of the permeability of the particle. However, owing to the presence of the interface, the magnitude of Ω_y is decreased for any arbitrary λ and σ and this effect is a strong function of the particle position relative to the interface. In particular, the angular velocity is independent of the viscosity ratio at the level of approximation represented by (64). The disturbed angular velocity, $\Omega_y-\frac{1}{2}$, (64), for motion of a freely suspended

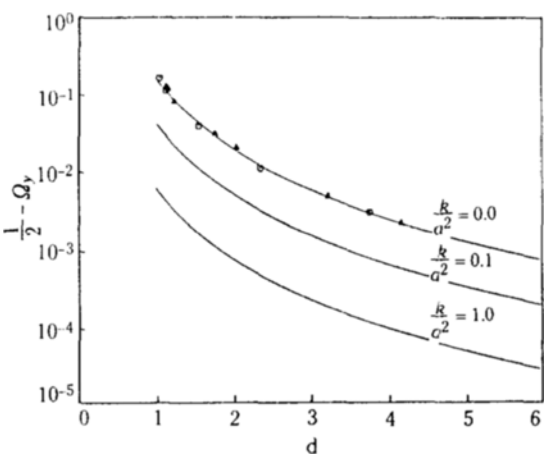

Fig. 5. Dimensionless disturbed angular velocity, $1/2-\Omega_y$, as a function of the dimensionless separation distance d for three values of the dimensionless permeability $k/a^2=0.0$, 0.1 and 1.0; O, exact solution results of Goldman et al. [16]; △, experimental data of Darabaner and Mason [17].

sphere in the simple shearing flow is plotted in Figure 5 as a function of d for three values of $\frac{k}{a^2}=0.0$, 0.1 and 1.0. Darabaner & Mason [17] experimentally measured the angular velocity of a neutrally buoyant *impermeable* sphere in a Couette viscometer as a function of the separation distance between the sphere and the wall of the viscometer. Their results are included in the figure. In addition, the exact solution of Goldman et al. [16] for $\frac{k}{a^2}=0.0$ is also compared with our approximate solution in this figure.The present asymptotic solution is qualitatively consistent both with the experimental data and the exact solution over the whole range of d, and is quantitatively accurate except in the region $d \sim 1$. Considering that the experimental data have neither been corrected for wall curvature nor for the presence of a second wall at a larger distance, and in view of the difficulties of maintaining and measuring the separation distance from the wall, the agreement is quite good.

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NOMENCLATURE

a	:	sphere radius
$A(\sigma)$, $G(\sigma)$	:	coefficient function for a stresslet
$B(\sigma)$	:	coefficient function for a potential quadrupole
$C(\sigma)$, $E(\sigma)$	:	coefficient function for a Stokeslet
d	:	separation distance between the sphere and the interface
$D(\sigma)$	:	coefficient function for a potential doublet
$\mathbf{E}$	:	strain rate tensor
$H(\sigma)$	:	coefficient function for a rotlet
$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$	:	base vectors in the Cartesian coordinate system (x, y, z)
$\mathbf{F}$	:	hydrodynamic force
$I_{n+\frac{1}{2}}(\sigma)$	:	modified Bessel function of the first kind of order $n+\frac{1}{2}$
k	:	permeability
$\mathbf{K}_C$	:	coupling tensor
$\mathbf{K}_R$	:	rotation tensor
$\mathbf{K}_{SF}$	:	third order tensor for hydrodynamic force in shear field
$\mathbf{K}_{ST}$	:	third order tensor for hydrodynamic torque in shear field
$\mathbf{K}_T$	:	translation tensor
l_c	:	characteristic length scale
$\mathbf{L}^{(i)}(\mathbf{E}$ or $\Gamma^{(i)})$	:	2nd order tensor (strain or shear rate tensor) in fluid i (= 1 and 2)
p	:	pressure field
p_c	:	characteristic stress scale
S	:	sphere surface
$\mathbf{T}$	:	hydrodynamic torque
$\mathbf{u}$	:	velocity field
$\mathbf{u}_2^{(j)}$	:	jth interface-reflected velocity field in fluid 2
$\mathbf{u}_{2,SH}^{(j)}$	:	jth interface-reflected shear field in fluid 2
$\mathbf{u}_{2,ST}^{(j)}$	:	jth interface-reflected streaming flow field in fluid 2
u_c	:	characteristic velocity
$\mathbf{u}_D$	:	velocity for a potential dipole in an unbounded fluid
$\mathbf{u}_{PQ}$	:	velocity for a potential quadrupole in an unbounded fluid
$\mathbf{u}_R$	:	velocity for a rotlet in an unbounded fluid
$\mathbf{u}_S$	:	velocity for a Stokeslet in an unbounded fluid
$\mathbf{u}_{SS}$	:	velocity for a stresslet in an unbounded fluid
$\mathbf{u}_{2,PQ}$	:	velocity for a potential quadrupole in fluid 2
$\mathbf{u}_{2,R}$	:	velocity for a rotlet in fluid 2
$\mathbf{u}_{2,SS}$	:	velocity for a stresslet in fluid 2

U(or U_x)	:	translational velocity of a particle
$\mathbf{U}_i^\infty$	:	undisturbed velocity in fluid i
x	:	position vector measured from the origin at the interface
$\mathbf{x}_B$	:	position vector placed on the sphere surface
$\mathbf{x}_p$	:	position vector of the sphere center
$\Gamma^{(t)}$(or Γ_{lm})	:	shear rate tensor
Γ^*(or Γ_{lm}^*)	:	reflected shear rate tensor
δ	:	Kronecker delta
ε	:	small parameter ($1/d$)
λ	:	viscosity ratio of fluids 1 and 2
μ_i	:	viscosity of fluid i
ρ_i	:	density of fluid i
ψ_n	:	coefficient function defined by (18)
Ω(or Ω_y)	:	angular velocity of the particle
τ	:	stress field

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