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THERMAL CONVECTION IN A SATURATED POROUS MEDIUM SUBJECTED TO ISOTHERMAL HEATING

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Abstract—A theoretical study of thermal convection in a fluid-saturated horizontal porous layer is reported for the case of isothermal heating from below. The onset time of natural convection and convective heat transport for large Darcy-Rayleigh number systems are analyzed, based on the propagation theory involving temporal dependence of perturbed quantities. Also, an overall feature of heat transport is discussed in connection with the Forchheimer modification of the Darcy flow model.

INTRODUCTION

The phenomena of buoyancy-driven convection in porous media have recently received much attention. This is known to be important in a wide variety of engineering applications such as geothermal reservoirs, thermal insulation, packed-bed catalytic reactors, underground spreading of chemical pollutants and the cooling of rotating superconducting machinery. An excellent review of work in this field is given by Combarnous and Bories [1].

When an initially quiescent, fluid-saturated porous layer bounded between two horizontal plates is heated isothermally from below, it is well known that natural convection occurs with Darcy-Rayleigh numbers larger than its critical value $4\pi^2$. But in the case of rapid heating the basic temperature profile of pure conduction becomes nonlinear and also time-dependent. In this case natural convection sets in, before the basic temperature profile becomes fully developed to a vertically linear one. This stability problem for the large Darcy-Rayleigh number remains unsolved due to its inherent complexity.

For time-dependent temperature fields in a horizontal fluid layer several theoretical methods to analyse thermal instabilities have been suggested: (a) the amplification theory [2], (b) the energy method [3], (c) the nonlinear amplitude method [4], (d) the stochastic model [5], and (e) the propagation theory [6]. Of these methods the propagation theory does not need any empirical specification. This newly developed concept leads to the determination of the onset time of natural convection by applying the principle of exchange of stabilities under linearized theory. Under the propagation theory the disturbances would experience the instantaneous variations in their quantities upon their onset and therefore they are time-dependent. The resulting predictions have been consistent with most of the experimental evidences, even in laminar forced convection [6-11]. Therefore it seems evident that the propagation theory is a most powerful method in analysing stability criteria for deep-pool systems. Furthermore Choi et al. [6,12] predicted the heat transport of thermal convection with success by incorporating stability criteria into Howard's boundary layer instability model [13,14] and Long's model of turbulent heat transport [15,16].

In the following, we present an analytical solution of stability criteria in a horizontal fluid-saturated porous layer, using the propagation theory. The celebrated simplicity of the Darcian formulation is kept in the stability analysis, but its limitation is discussed in connection with predictions of heat transport. The present work allows verification of the analytical solutions and reveals details peculiar to the heat transport caused by thermal convection in a porous layer heated from below.

STABILITY ANALYSIS

1. Disturbance Equations

Consider a horizontally infinite layer of porous material saturated with a Newtonian fluid, as shown in Fig. 1. The porous medium is homogeneous and isotropic, and the layer of depth L is confined between two rigid boundaries. The layer is initially quiescent with uniform temperature T_1. For time $t \geq 0$ its bottom temperature is kept at a higher temperature T_2 with a fixed $\Delta T(=T_2-T_1)$. Then, natural convection will set in at a certain time with the condition of $Ra_D \geq 39.5$.

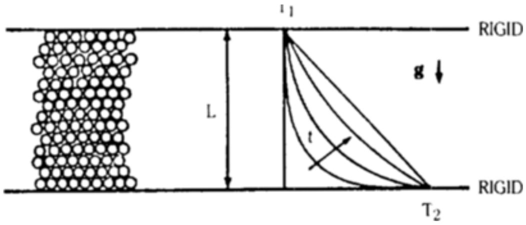

Fig. 1. Schematic diagram of system.

Ra_D is the Darcy-Rayleigh number defined by

$$Ra_D = Ra\ Da$$

$$Ra = \frac{g\beta L^3 \Delta T}{\alpha \nu}$$

where Ra is the Rayleigh number and Da is the Darcy number. g, β, α, ν and K denote the gravity acceleration, the coefficient of thermal expansion, the effective thermal diffusivity, the kinematic viscosity and the permeability of the saturated porous medium, respectively. With increasing ΔT the time of the onset of convection will become earlier.

To obtain a dimensionless description for the present system, we shall use $L, L^2/\alpha, \Delta T, \alpha\nu/(g\beta L^3)$ and α/L as units of length, time, basic temperature, perturbed temperature and velocity, respectively. For a deep-pool system the basic temperature of pure conduction θ_0 can be approximated [6]:

$$\theta_0 = [1 - 2\zeta + 2\zeta^3 - \zeta^4][1 - H_{\zeta-1}] \tag{1}$$

where $H_{\zeta-1}$ is the unit step function and $\zeta = z/\delta$ with $\delta = (40\tau/3)^{1/2}$. z, δ, and τ represent the dimensionless vertical distance, thermal penetration depth and time, respectively. This approximate solution agrees well with the exact one for $\tau \leq 0.01$. It is stressed that equation (1) is useful only in the case of pure conduction under deep-pool systems.

With increasing Ra_D the validity of equation (1) will be limited to a shorter time zone $\tau \leq \tau_c$. τ_c is the critical time marking the onset of convection which we wish to know. Upon the onset of convection it is assumed that the velocity field will follow the Darcy model:

$$\frac{\mu}{K}\mathbf{u} = -\nabla p + \rho \mathbf{g} \tag{2}$$

where μ denotes the viscosity, $\mathbf{u}$ the velocity caused by natural convection and p the pressure. This model cannot provide the no-slip boundary condition. For the instability analysis the instantaneous values of velocity, pressure, and temperature fields are perturbed by infinitesimally small disturbances. By linearizing equation (2) under the Boussinesq approximation and taking its double curl the disturbance equation in terms of its vertical velocity component w_1 is produced as follows:

$$\frac{1}{Da}\nabla^2 w_1 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta_1 \tag{3}$$

which is valid in the Darcy regime implying small Da. From the energy equation the temperature disturbance θ_1 is easily obtained:

$$\frac{\partial \theta_1}{\partial \tau} + Ra\ w_1 \frac{\partial \theta_0}{\partial z} = \nabla^2 \theta_1 \tag{4}$$

All this procedure can be seen in the work of Yoon, Choi and Yoo [23]. The proper boundary conditions are given by

$$\theta_1 = w_1 = 0 \quad \text{for } z=0 \text{ and } z=1 \tag{5}$$

This corresponds to boundaries on which the temperature is fixed and through which no flow occurs. Our goal is to find the critical time τ_c for given Ra_D by using equations (1) to (5).

2. Propagation Theory

In the conventional frozen-time model the time-dependent term in equation (4) is neglected and the minimum value of Ra_D is sought for a given τ_c. This method can be applied when the temperature gradient is almost linear. In the amplification theory the description of the initial conditions at $\tau = 0$ and the criterion for making the onset of motion, in terms of the amplification factor at $\tau = \tau_c$, are assumed empirically. Since both methods involve disadvantages, we employ the newly evolving propagation theory in the present study. This method follows the conventional linear stability theory involving the principle of exchange of stabilities. But the time-dependent property is maintained by introducing the similarity variable based on the proper length scale $\delta(\tau)$.

Since there are not lateral boundaries, the horizontal variations of disturbances with respect to x and y are respresented by the dimensionless wave number 'a' as usual [1]:

$$[w_1, \theta_1] = [w_1^*(\tau, z), \theta_1^*(\tau, z)]\exp[i(a_x x + a_y y)] \tag{6}$$

where a is described as $(a_x^2 + a_y^2)^{1/2}$ and i is the imaginary number. The resulting amplitude functions are transformed by using the relation:

$$[w_1^*(\tau, z), \theta_1^*(\tau, z)] = [\delta^2 w^*(\zeta), \theta^*(\zeta)] \tag{7}$$

Note that the new amplitude functions w^* and θ^* are dependent on ζ only. Finally, we get the following set of stability equations from equations (3) and (4) [23]:

$$(D^2 + \frac{20}{3}\zeta D - a^{*2})(D^2 - a^{*2})w^* = -a^{*2} Ra_D^* w^* D\theta_0 \tag{8}$$

where $a^* = a\delta$, $Ra_D^* = Ra_D\delta$ and $D = d/d\zeta$. If Ra_D^* and a* are eigenvalues, then the conventional method in stability analysis can be applied with ease. This postulate followed by the similarity transformation (7) is the essence of the propagation theory.

3. Method of Solution

With equation (8) the minimum value of Ra_D^* and its corresponding wave number a* must be found subject to boundary conditions (5). For the deep-pool system the proper boundary conditions are constructed as

$$w^* = (D^2 - a^{*2})w^* = 0 \quad \text{for } \zeta = 0 \text{ and } \zeta \to \infty \qquad (9)$$

Since the right-side term in equation (8) can be neglected for $\zeta \geq 1$, the outer solution of θ^* can be approximated by using the WKB method and then that of w* is obtained by using the differential operator technique. The detailed procedure will be easily understood by referring to Choi et al.'s work [8,10,11,23]. The inner solutions for $\zeta \leq 1$ are obtained in the form of power series, based on the Frobenius method. The individual solutions are patched at the interface $\zeta = 1$ by considering that the velocity, stresses and temperature are all continuous. The proper interface conditions are

$$D^n w^*_{inner} = D^n w^*_{outer}; \; n = 0, 1, 2, 3 \quad \text{at } \zeta = 1 \qquad (10)$$

From conditions (9) and (10) the characteristic equations in the form of a (4×4) square matrix are generated. In order to produce a nontrivial solution the determinant of the matrix must be zero. The value of Ra_D^* is obtained for a given a* and then the minimum value of Ra_D^* is found from the plot of Ra_D^* versus a* by following the procedure suggested by Yoon, Choi and Yoo [23].

4. Results and Discussion

In the Darcy regime the stability criteria for $Ra_D \geq 125$ are obtained for the present system, based on the propagation theory:

$$\tau_c = 154.5\, Ra_D^{-2} \qquad (11)$$

$$a_c = 0.0736\, Ra_D \qquad (12)$$

The predicted values of τ_c are plotted in Fig. 2. In comparison with the experimental results of Elder [17] the present predictions look reasonable. Once buoyancy-driven convection sets in, the growth period to manifest convection will be required. Conjecturing from the work of Choi et al. [6], the initiated disturbances at $\tau = \tau_c$ will lead to manifest convection at $4\tau_c$. Elder [17] noticed that there is a rapid increase of amplification to a maximum at $\tau = 4\tau_c$. In this matter more refined work is required.

The amplitude functions are featured in Fig. 3 by normalizing in terms of maximum magnitude. It is shown that disturbances are mainly confined within the thermal penetration depth. Therefore it is justified to a certain degree that δ is the proper length scale in

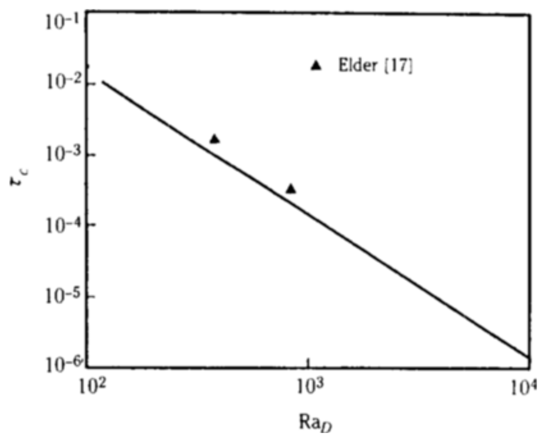

Fig. 2. Onset time of natural convection.

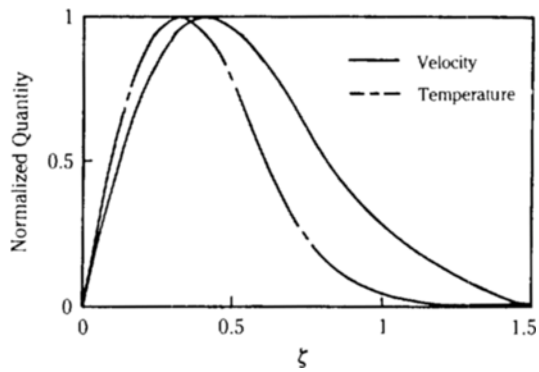

Fig. 3. Distribution of normalized amplitude functions.

deep pools. From equations (7) and (11) the temporal growth rates of disturbances at $\tau = \tau_c$ are obtained as follows:

$$\frac{1}{w_1^*}\frac{\partial w_1^*}{\partial \tau}\bigg|_{\tau=\tau_c} = \frac{Ra_D^2}{154.5}\left(1 - \frac{\zeta}{2w^*}Dw^*\right) \qquad (13)$$

$$\frac{1}{\theta_1^*}\frac{\partial \theta_1^*}{\partial \tau}\bigg|_{\tau=\tau_c} = -\frac{Ra_D^2}{154.5}\left(\frac{\zeta}{2\theta^*}D\theta^*\right) \qquad (14)$$

The above equations indicate that disturbances does not grow in the form of an exponential function with respect to τ. Considering the distribution of disturbances shown in Fig. 3, the amplitude of temperature disturbances near the bottom boundary are damped, but the outer ones experience amplification. It is interesting that for $\zeta \to 0$ the absolute values of growth rates represented by equations (13) and (14) approach the same value $Ra_D^2/309$. The trend that the growth rate is proportional to Ra_D^2 is the same as Elder's [17]. It seems clear that a so-called conduction layer exists near the bottom boundary and plays an important role in heat transport.

HEAT TRANSPORT

1. Darcy's Regime

The possibility of connecting the stability criteria to heat transport of fully-developed, turbulent thermal convection has been discussed by Howard [13] and Busse [14]. By employing both their boundary layer instability model and the models of Long [15] and Cheung [16], Choi et al. [6,12] has derived the correlations of the Nusselt number Nu with success in Benard convection. Therefore the similar method will be applied to the present porous layer system.

For very large Ra_D the generation of thermals is assumed to be the result of thermal instability in the conduction layers near boundaries. This assumption is closely related to equations (1), (11) and (14). According to the boundary layer instability model and Choi et al.'s concept Nu at the fully-developed state is obtained as

$$Nu = Nu_c/(2\cdot 2\cdot 2) \quad \text{for } Ra_D \to \infty \tag{15}$$

in Darcy's regime, as sketched in Fig. 4. Nu_c denotes the Nusselt number of conduction at τ_c represented by equation (11). By using equations (1) and (11), the above equation is transformed to

$$Nu = 0.00551\, Ra_D \quad \text{for } Ra_D \to \infty \tag{16}$$

For the present system Busse and Joseph [18] reported the following relation:

$$\frac{dNu}{dRa_D} = \frac{1}{2\pi^2} \quad \text{as } Ra_D \to 39.5 \tag{17}$$

A new correlation is generated according to the procedure similar to the work of Long [15] and Cheung [16]. By incorporating equations (16) and (17) into the resulting correlation and following Choi et al.'s procedure, the correlation of heat transport is obtained as follows:

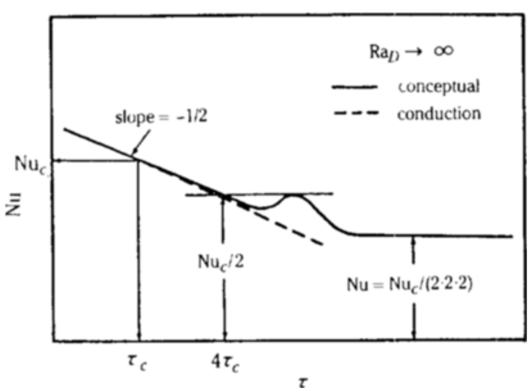

Fig. 4. Conceptual diagram of heat transport.

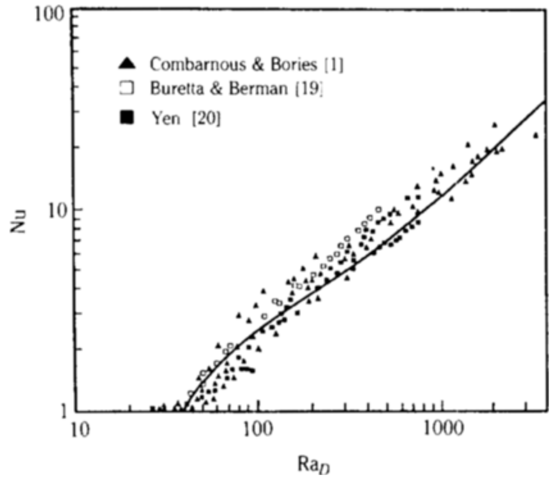

Fig. 5. Correlation of heat transport in Darcy's regime.

$$Nu = 1 + \frac{0.00551\,(Ra_D - 39.5)}{[1 - 1.237\,(NuRa_D)^{-1/6}]^2} \tag{18}$$

The above prediction agrees reasonably well with most of experimental data points as shown in Fig. 5. But the use of equation (18) is limited to Darcy's regime.

2. Forchheimer's Extension

In many practical problems the porous medium has a high permeability, making Darcy's law inapplicable. Therefore inertia effects are often included in equations of motion through the so-called Forchheimer's extension [21] as follows:

$$\mathbf{u} + \frac{bK}{\nu}\,|\mathbf{u}|\,\mathbf{u} = \frac{K}{\mu}\,(-\nabla p + \rho \mathbf{g}) \tag{19}$$

where b is Forchheimer's constant. The condition of b = 0 corresponds to the Darcy flow model.

Based on equation (19), the following correlation is formulated according to the aforementioned procedure:

$$Nu = \frac{0.0202\, Ra_D^{1/2}}{\{C_1 (bK/L)^{1/3} - [C_2 (bK/L)^{1/3} - 1]\,(NuRa_D)^{-1/6}\}^{3/2}} \tag{20}$$

where C_1 and C_2 are constants decided by experiments or theory. They are dependent on the Prandtl number. For b = 0 this equation is reduced to equation (16). Therefore equation (20) is applicable for large Ra_D, while equation (18) is useful for small Ra_D. In this connection the general trend is featured in Fig. 6. Even though the methodologies employed are different, Bejan's work [22] in this field is the same in spirit as the present one.

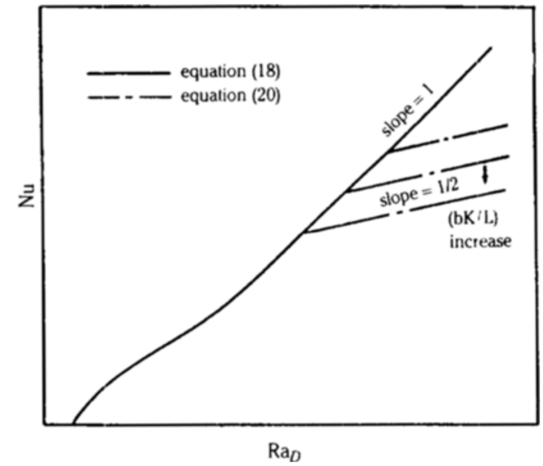

Fig. 6. Trend of heat transport in non-Darcy's regime.

CONCLUSIONS

The onset of thermal convection in a fluid-saturated porous medium heated from below is analysed, based on the propagation theory. The stability criteria are obtained analytically and amplification of disturbances are discussed qualitatively. The present results make the propagation theory look very promising. Based on stability criteria, a new correlation of the Nusselt number versus the Darcy-Rayleigh number is obtained theoretically in Darcy's regime. This is in excellent agreement with existing experimental results. Also, by using Forchheimer's extension a new form of correlation is proposed for large Ra_D system.

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NOMENCLATURE

a : dimensionless horizontal wave number
a* : modified wave number, $a\delta$
b : Forchheimer's constant $[m^{-1}]$
C_i : arbitrary constants in equation (20)
D : differential operator with respect to ζ
Da : Darcy number, K/L^2
g : gravity acceleration $[m/s^2]$
$H_{\zeta-1}$: unit step function
K : permeability $[m^2]$
L : depth of porous layer [m]
Nu : Nusselt number
p : pressure $[N/m^2]$
Ra : Rayleigh number, $g\beta L^3 \Delta T/(\alpha\nu)$
Ra_D : Darcy-Rayleigh number, $Kg\beta L \Delta T/(\alpha\nu)$
Ra_D^* : modified Darcy-Rayleigh number, $Ra_D\,\delta$
T : temperature [K]
t : time [s]
u : velocity [m/s]
w_1 : dimensionless vertical velocity
w_1^* : perturbed amplitude of w_1
w* : transformed amplitude function, w_1^*/δ^2
x,y,z : dimensionless Cartesian coordinates

Greek Letters

α : effective thermal diffusivity $[m^2/s]$
β : coefficient of thermal expansion [1/K]
δ : dimensionless thermal penetration depth, $(40\tau/3)^{1/2}$
ζ : dimensionless vertical distance, z/δ
θ : dimensionless temperature
μ : dynamic viscosity [Kg/(ms)]
ν : kinematic viscosity $[m^2/s]$
ρ : density $[Kg/m^3]$
τ : dimensionless time

Subscripts

c : critical state
0 : basic state
1 : perturbed state

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