THERMAL INSTABILITY IN PLANE COUETTE FLOW HEATED FROM BELOW

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Abstract—The conditions marking the onset of natural convection due to buoyant forces are investigated in the thermal entrance region of horizontal plane Couette flow. The base temperature profile produced by pure forced convection is approximated as a fourth order polynomial using the integral method. With this approximate base-temperature profile stability criteria are obtained by means of the local stability analysis, its modification, and the natural-amplification analysis. The last one takes into consideration the axial amplification rate of disturbances at the onset of thermal instability. This new concept is tested here for the first time. The consideration of axial amplification of disturbances makes the system more stable. It is shown that the results of the natural-amplification analysis agree well with the existing experimental data.

INTRODUCTION

When a horizontal layer of fluid is heated from below with a high heating rate, the system becomes unstable and finally natural convection sets in. In the laminar forced-convection flow over a heated flat plate the occurrence of longitudinal vortex rolls is well known. Once such a secondary flow appears, the heat transfer characteristics based on the primary forced convection are no longer applicable. Thus, the determination of critical conditions of the onset of natural convection has been of great interest from both theoretical and practical viewpoints.

In plane Couette flow Choi[1], and Davis and Choi[2] suggested the modified local stability analysis that for large Prandtl numbers temperature disturbances at the axial position of the onset of natural convection are confined within the local thermal boundary layer. Using the Galerkin method, they solved this extended Rayleigh-Bénard problem. They concluded that their theory produces the most reasonable stability criteria in comparison with the experimental data for water. Recently, Choi and Kim [3] transformed the base nonlinear temperature profile to the approximate one in a power form and used the rapidly converging power-series solution technique in plane Couette flow. Then, they critically re-examined the results of Davis and Choi [2].

In the present study the onset of buoyancy-driven convection in the plane Couette flow over a uniformly heated horizontal flat plate is analysed by means of linear stability theory. The base temperature profile is approximated as a fourth order polynomial, by using the integral method. With this approximate base-temperature profile, the stability condition will be found by means of the local stability analysis, its modificaiton, and the natural-amplification analysis. The last one takes into consideration the axial rate of change of disturbances. This new concept is introduced here for the first time. Therefore, the purpose of this study is to critically examine various concepts in order to test their validity.

METHOD OF ANALYSIS

Base Temperature

The system considered here is the thermal entrance region of horizontal plane Couette flow of an incompressible Newtonian fluid with the free-right boundaries. The fluid is heated with constant heat flux through the bottom plate. A schematic diagram of the flow system is shown in Figure. 1. See Nomenclature for the notations of all variables and parameters that appear in this paper.

It is essential in the stability analysis to describe the base temperature profile caused by both conduction and pure forced convection. The base temperature profile is obtained by solving the following partical differential equation:

$$2z\frac{\partial \theta_0}{\partial x} = \frac{\partial^2 \theta_0}{\partial z^2} \tag{1}$$

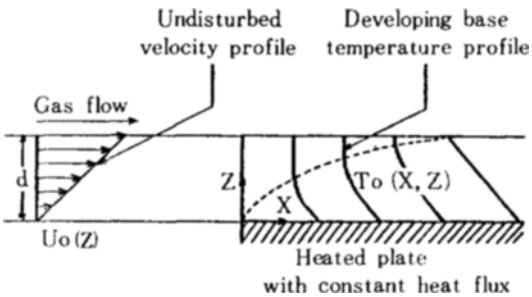

Fig. 1. Schematic Diagram of the System

with the boundary conditions:

$$\theta_o(0, z) = \theta_o(x, 1) = 0 \qquad (2)$$

$$\frac{\partial \theta_o}{\partial z}(x, 0) = -1 \qquad (3)$$

The above equation is valid, wherein the Peclet number is larger than 100.

A Graetz-type solution, based on the method of the separation of variables, is obtained as follows:

$$\theta_o = 1 - z - \sum_{n=1}^{\infty} K_n R_n(z) S_n(x) \qquad (4)$$

where

$$K_n = \frac{3^{4/3}}{\Gamma(\frac{2}{3})\lambda_n^{7/3} J^2_{2/3}(\frac{2}{3}\lambda_n)}$$

$$R_n(z) = z^{1/2} J_{-1/3}(\tfrac{2}{3}\lambda_n z^{3/2})$$

$$S_n(x) = \exp\left(-\frac{\lambda_n^2}{2}x\right)$$

$$J_{-1/3}(\frac{2}{3}\lambda_n) = 0$$

At small axial positions (say $x<0.05$) this solution converges very slowly and thus the following approximation (Leveque type) based on the fluid having an infinite depth and the similarity variable $\eta = z/(4.5x)^{1/3}$ is known to be more useful:

$$\theta_o = \frac{(4.5x)^{1/3}}{\Gamma(\frac{2}{3})}[e^{-\eta^3} - \eta\Gamma(\frac{2}{3}, \eta^3)] \qquad (5)$$

The derivation of Equations (1) to (5) are described in the work of Choi[1], and Davis and Choi[2].

Choi and Kim[3] approximated the base temperature profile by the following form:

$$\theta_o = \theta_{o,w}(1 - z/\delta)^{\delta/\theta_{o,w}} \quad \text{for } 0 \leqq z \leqq \delta \qquad (6)$$

$$\theta_o = 0 \quad \text{for } \delta \leqq z \leqq 1 \qquad (7)$$

They showed that the above simulated base temperature profile represents the system well.

But the simulated base temperature profile is still complicated in mathematical treatment for the stability analysis. Therefore in the present study the base temperature profile is approximated as a fourth order polynomial using the integral method which was developed by von Kármán and Pohlhausen. According to their procedure the following equations are derived:

$$\theta_o = \delta/2 - z + z^3/\delta^2 - z^4/2\delta^3 \quad \text{for } 0 \leqq z \leqq \delta \qquad (8)$$

$$\theta_o = 0 \quad \text{for } \delta \leqq z \leqq 1 \qquad (9)$$

where $\delta = (15x)^{1/3}$

In Figure 2 a comparison is made between the exact base temperature profile and the approximate ones at $x=0.05$. It is shown that the base temperature profile obtained by the integral method agrees very well with the exact one on the whole. The base temperature profile from the integral method is almost exact before the thermal boundary layer approaches the upper surface ($x<0.06$).

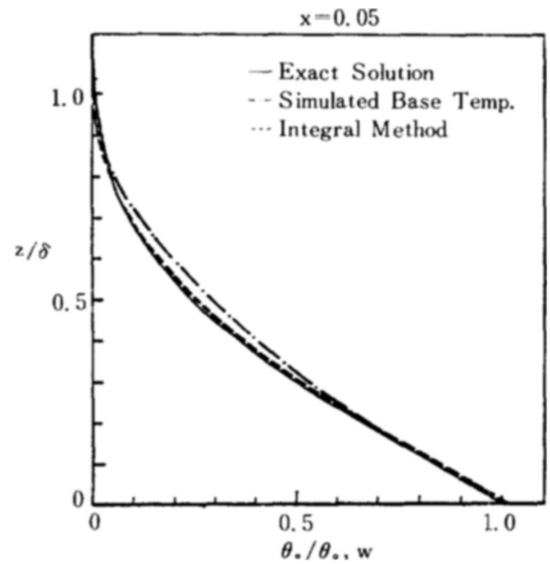

Fig. 2. Comparison of the Exact Base Temperature Profile and the Approximate Ones at x=0.05

Stability Analysis

At the onset position of natural convection the proper governing equations are constructed from the equations of continuity, motion, and energy under the Boussinesq approximation. To perform the stability analysis, we apply the usual method of introducing infinitesimal perturbations on the undisturbed components.

On the basis of the conventional stability theory, the following assumptions are made:

1. The principle of exchange of stabilities holds locally at the axial position marking the onset of natural convection. Thus, the onset of natural convection will be marked by a regular 2-dimensional vortex roll with a unique standing wave number 'a' at each axial positon.

2. The Prandtl number is infinite. It is believed that this simplification provides at least a qualitatively correct desctiption of fluidus in a range $Pr \geqq 7$[4].

Neglecting the squares and products of the velocity and temperature perturbations by the linear stability theory, the following linearized perturbation equations are obtained [1,2]:

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)^2 w^* - a^2 Ra\,\theta^* = 0 \tag{10}$$

$$\left[2z\frac{\partial}{\partial x}-\left(\frac{\partial^2}{\partial z^2}-a^2\right)\right]\theta^* + w^*\frac{\partial\theta_0}{\partial z} = 0 \tag{11}$$

B. C.'s : $w^*(x,0)=w^*(x,1)=\frac{\partial w^*}{\partial z}(x,0)=\frac{\partial^2 w^*}{\partial z^2}(x,1)=0$

and $\theta^*(x,1)=\frac{\partial\theta^*}{\partial z}(x,0)=0$ (12)

Local Stability Analysis

Since $\frac{\partial\theta_0}{\partial Z}$ is a function of x as well as z, the variables z and x in Equations (10) and (11) may not be separable. In the local stability analysis [1,2] the nonlinear base-temperature profile is fixed at each axial position so that $\frac{\partial\theta_0}{\partial Z}$ becomes a function of z only and x is considered as a parameter. It is then possible to separate the variables z and x in the differential Equations (10) and (11) so that the disturbances can be expressed as the following relationship:

$$[w^*(x,z),\theta^*(x,z)] = [\hat{w}(z),\hat{\theta}(z)]\exp(\delta, x) \tag{13}$$

where δ is the spatial growth rate. Then, from the nuetral stability concept δ can be set to be zero. Thus Equations (10) and (11) can be written as follows:

$$\left(\frac{d^2}{dz^2}-a^2\right)^2\hat{w} - a^2 Ra\,\hat{\theta} = 0 \tag{14}$$

$$-\left(\frac{d^2}{dz^2}-a^2\right)\hat{\theta} + \hat{w}\frac{\partial\theta_0}{\partial z} = 0 \tag{15}$$

Substituting Equations (8) and (9), and eliminating $\hat{\theta}$ from the system of Equations (14) and (15), we can reformulate the perturbation equations as follows:

$$[(D^2-a^{*2})^3 + a^{*2}Ra^*(1-3\zeta^2+2\zeta^3)]\hat{w}_a = 0 \quad \text{for} \quad 0\leq\zeta\leq 1 \tag{16}$$

$$(D^2-a^{*2})^3\hat{w}_b = 0 \quad \text{for} \quad 1\leq\zeta\leq 1/\delta \tag{17}$$

The appropriate boundary conditions are:

$$\hat{w}_a = D\hat{w}_a = D(D^2-a^{*2})^2\hat{w}_a = 0 \quad \text{at} \quad \zeta=0 \tag{18}$$

$$\hat{w}_b = D^2\hat{w}_b = (D^2-a^{*2})^2\hat{w}_b = 0 \quad \text{at} \quad \zeta=1/\delta \tag{19}$$

At $\zeta=1$, the conditions can be obtained from the assumption that disturbance velocities, stresses, temperature, and heat transfer are all continuous:

$$\hat{w}_a-\hat{w}_b = D^n\hat{w}_a - D^n\hat{w}_b = 0 \quad (n=1,2,3,4,5) \quad \text{at} \quad \zeta=1 \tag{20}$$

A general solution of the above problem can be constructed in the form:

$$\hat{w}_a = \sum_{i=0}^{5} H_i f^{(i)}(\zeta) \tag{21}$$

$$\hat{w}_b = (H_6+H_7\zeta+H_8\zeta^2)e^{-a^*\zeta} + (H_9+H_{10}\zeta+H_{11}\zeta^2)e^{a^*\zeta} \tag{22}$$

where H_i (i=0,1,2, ..., 11) are arbitrary constants and $f^{(i)}(\zeta)$ are rapidly convergent power series similar to those developed by Sparrow et al. [5] as follows:

$$f^{(i)}(\zeta) = \sum_{n=0}^{\infty} b_n^{(i)}\zeta^n \tag{23}$$

where $b_{-3}^{(i)} = b_{-2}^{(i)} = b_{-1}^{(i)} = 0$

$b_n^{(i)} = \delta_{ni}$ for $n=0,1,\cdots,5$

$$b_n^{(i)} = 3a^{*2}\frac{(n-2)!}{n!}b_{n-2}^{(i)} - 3a^{*4}\frac{(n-4)!}{n!}b_{n-4}^{(i)} + a^{*6}\frac{(n-6)!}{n!}b_{n-6}^{(i)} - a^{*2}Ra^*\frac{(n-6)!}{n!}(b_{n-6}^{(i)} - 3b_{n-8}^{(i)} + 2b_{n-9}^{(i)}) \quad \text{for } n\geq 6$$

The constants H_i are chosen to satisfy the twelve boundary conditions (Equations (18) to (20)). From the boundary conditions at $\zeta=0$, we obtain:

$$H_0 = H_1 = 0 \quad \text{and} \quad H_5 = \frac{a^{*2}}{10}H_3 \tag{24}$$

From the above relationships (21) to (24) and the remaining boundary conditions the characteristic equations in the form of a(9×9)square matrix are obtained. For a nontrivial solution, its determinant must vanish. Thus, for a given x the minimum Ra is obtained by means of a plot of Ra vs. 'a' to satisfy the above relationship. That minimum value of Ra is the critical Rayleigh number and the corresponding 'a' is the critical wave number.

Modified Local Stability Analysis

There is some experimental indication that at the onset of thermal convection the disturbances are confined to a thin region near the heated surface. Although the velocity disturbances are controlled mainly by the boundary conditions at the fixed and free surfaces, it seems reasonable to assume that temperature disturbances are confined to the thermal boundary layer produced by the pure forced convection. So Choi [1], and Davis and Choi [2] suggested the modified local stability analysis using this premise.

Since in the modified local stability analysis temperature disturbances vanish outside the thermal boundary layer, the equations can be reformulated as follows:

$$[(D^2-a^{*2})^3 + a^{*2}Ra^*(1-3\zeta^2+2\zeta^3)]\hat{w}_a = 0 \quad \text{for} \quad 0\leq\zeta\leq 1$$

$$(D^2-a^{*2})^2\hat{w}_b = 0 \quad \text{for} \quad 1\leq\zeta\leq 1/\delta \tag{26}$$

$$\text{B. C.'s}: \hat{w}_a = D\hat{w}_a = D(D^2-a^{*2})^2\hat{w}_a = 0 \quad \text{at} \quad \zeta=0 \tag{27}$$

$$\hat{w}_b = D^2\hat{w}_b = 0 \quad \text{at} \quad \zeta=1/\delta \tag{28}$$

$$\hat{w}_a-\hat{w}_b = D^n\hat{w}_a - D^n\hat{w}_b = 0 \quad (n=1,2,3,4) \quad \text{at} \quad \zeta=1 \tag{29}$$

Following the same procedure in the local stability analysis as is described above, the critical conditions can be obtained.

Natural-Amplification Analysis

In the local stability analysis and its modification, the axial rate of change of the temperature disturbances in Equation(11) was neglected by the neutral stability concept. But it is in doubt whether the disturbances at the onset of thermal convection begin to grow, once the thermal instability has been generated, or not. Therefore, the axial amplification rate of disturbances at the onset of thermal instability is considered. Here, this concept is called "natural-amplification analysis". Nobody has ever introduced that concept. For the convenience of mathematical manipulation, the similarity variable is employed in the following form:

$$\zeta = z/\delta \quad \text{and} \quad \delta = (15x)^{1/3} \tag{30}$$

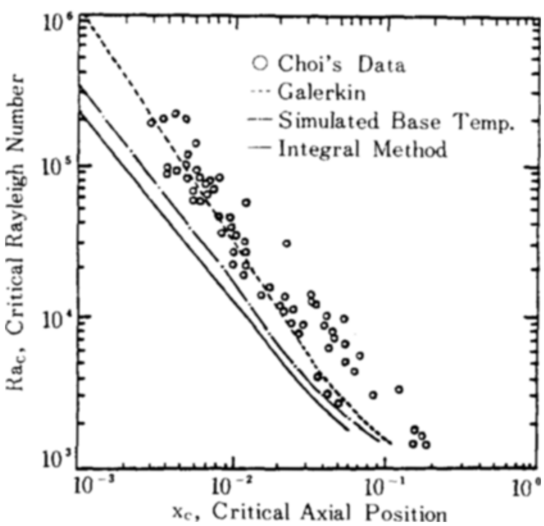

Fig. 3. Critical Axial Position versus Critical Rayleigh Number for the Modified Local Stability Analysis

Here ζ is the one from the integral method. Then, by the chain rule the axial rate of change is obtained as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x} = -\frac{\zeta}{3x}\frac{\partial}{\partial \zeta} \tag{31}$$

Assuming that the temperature disturbances be confined to the thermal boundary layer, the equations can be formulated as follows:

$$[10\zeta^2 D(D^2-a^{*2})^2 + (D^2-a^{*2})^3 + a^{*2}Ra^*(1-3\zeta^2+2\zeta^3)]\, w_a^* = 0 \quad \text{for} \quad 0 \leq \zeta \leq 1 \tag{32}$$

$$(D^2-a^{*2})^2 w_b^* = 0 \quad \text{for} \quad 1 \leq \zeta \leq 1/\delta \tag{33}$$

$$\text{B. C.'s}: w_a^* = Dw_a^* = D(D^2-a^{*2})^2 w_a^* = 0 \quad \text{at} \quad \zeta = 0 \tag{34}$$

$$w_b^* = D^2 w_b^* = 0 \quad \text{at} \quad \zeta = 1/\delta \tag{35}$$

$$w_a^* - w_b^* = D^n w_a^* - D^n w_b^* = 0 \ (n=1,2,3,4) \quad \text{at} \quad \zeta = 1 \tag{36}$$

By following the same procedure in the previous sections the critical conditions can be obtained.

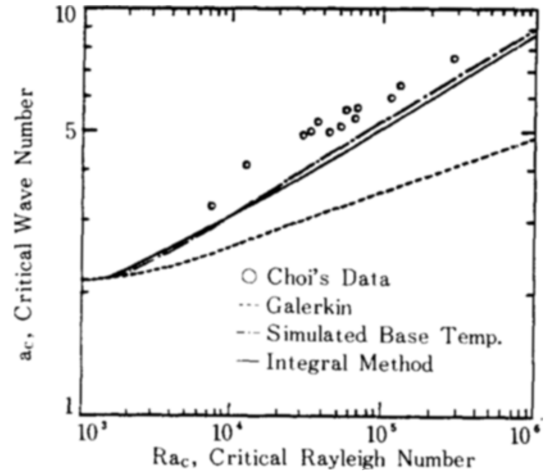

Fig. 4. Critical Rayleigh Number versus Critical Wave Number for the Modified Local Stability Analysis

RESULTS AND DISCUSSION

In order to test the validity of the approximate base-temperature profile obtained by the integral method, we present Figure 3 which shows the relationship between the critical Rayleigh number and the critical axial position for the modified local stability analysis. The experimental points were obtained with water having Pr in the range from 6 to 8. The curve "Galerkin" represents the one from the Galerkin method. All these experiments and Galerkin procedures are described in the

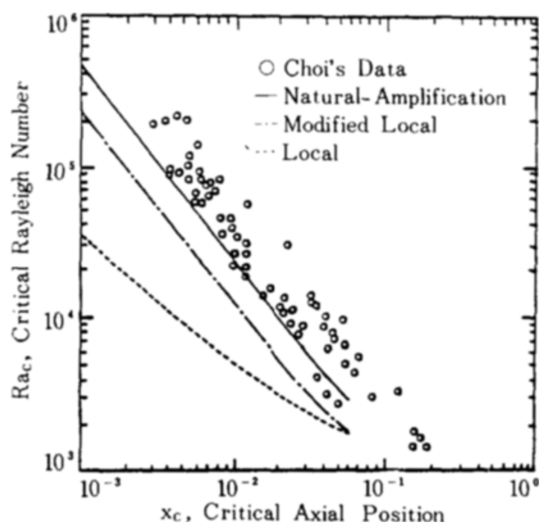

Fig. 5. Critical Axial Position versus Critical Rayleigh Number for the Local Stability Analysis, its Modification, and the Natural-Amplification Analysis

work of Choi [1], and Davis and Choi [2]. The curve "Simulated Base Temp." represents the one obtained by Choi and Kim[3]. The curve "Integral Method" indicates the one using the approximate base-temperature profile obtained by the integral method. Figure 4 shows the relationship between the critical wave number and the critical Rayleigh number for the modified local stability analysis.

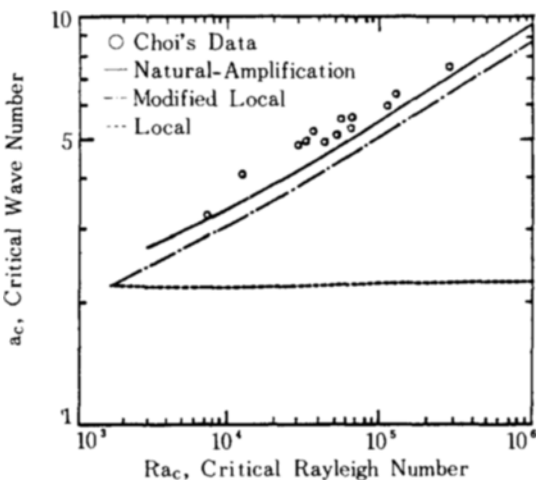

Fig. 6. Critical Rayleigh Number versus Critical Wave Number for the Local Stability Analysis, its Modification, and the Natural-Amplification Analysis

In Figure 3 the curve "Galerkin" locates among the experimental points. It is reasonable for the critical Rayleigh numbers to fall below those in experiments, for the enough amplification of disturbances to be observed is required. In Figure 4 the curve "Galerkin" deviates so much from the data points. It seems that the deviation of the curve "Galerkin" is due to the inadequacy of the Galerkin scheme. In Figure 3 the curve "Simulated Base Temp." gives a little higher values of Ra_c than the curve "Integral Method". The difference between thsese two curves seems to be caused by the fact that the simulated base temperature profile brings larger deviation from the exact values as is shown in Figure 2.

With the base temperature profile obtained by the integral method, in Figure 5 the critical Rayleigh number with respect to the critical axial position is shown and in Figure 6 the critical Rayleigh number versus the critical wave number for the local stability analysis, its modification, and the natural-amplification analysis.

From Figures 5 and 6, the general trend of the curve "Local" does not agree well with that of the experiments. Based on these results, it seems clear that the local stability analysis brings too conservative stability criteria. The curve "Natural-Amplification" gives higher values of the critical Rayleigh number than the curve "Modified Local" as shown in Figure 5, and it locates just below the experimental points. So it is found that the amplification of disturbances (the heat transfer in the x-direction) makes the system more stable.

In Figures 5 and 6, it is shown that x_c vs. Ra_c or Ra_c vs. a_c becomes linear on logarithmic coordinates as the onset position approaches the leading edge of heating section. Based on the modified local stability analysis and the natural-amplification analysis, the following asymptotic relationships are obtained for small x_c's:

(Modified Local Stability Analysis)

$$Ra_c = 24.18 x_c^{-4/3} \text{ and } a_c = 0.2798\, Ra_c^{1/4}\ ; \quad x_c < 10^{-2} \tag{37}$$

(Natural-Amplification Analysis)

$$Ra_c = 48.60 x_c^{-4/3} \text{ and } a_c = 0.3071\, Ra_c^{1/4}\ ; \quad x_c < 10^{-2} \tag{38}$$

In Figures 7 and 8 the distributions of temperature and velocity disturbances are shown for the modified local stability analysis and the natural-amplification analysis. The points of the maximum magnitude of disturbances of the natural-amplification analysis locate at the lower axial position than those of the modified local analysis. The distribution of temperature disturbances of the natural-amplification analysis seems more natural than that of the modified local stability analysis. Therefore, it is concluded that as of now the natural-amplification analysis generates the most preferred stability criteria. And the modified local stability analysis could give good approximate solutions, although the local stability analysis produces unreasonable results in the thermal entrance region. Since we could get the most reasonable results by the natural-amplification analysis, the other existing stability analyses must be reexamined.

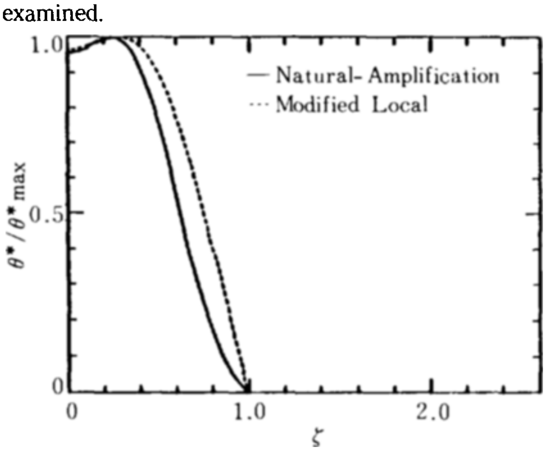

Fig. 7. Distribution of Temperature Disturbances for the Modified Local Stability Analysis and the Natural-Amplification Analysis

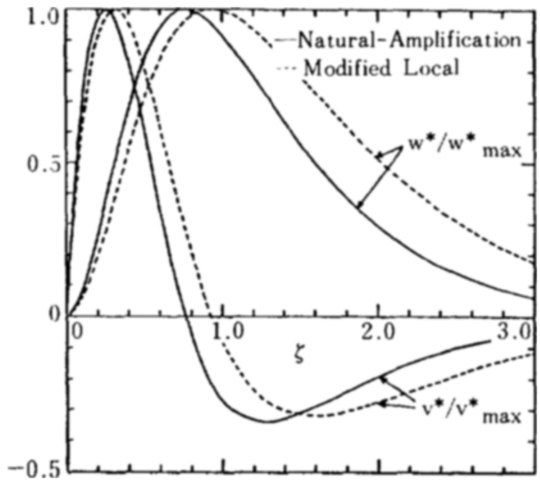

Fig. 8. Distribution of Velocity Disturbances for the Modified Local Stability Analysis and the Natural-Amplification Analysis

CONCLUSIONS

The onset of thermal instability in plane Couette flow heated from below with constant heat flux has been analysed. The present results are summarized as:

(a) The local stability analysis produces too conservative results in the thermal entrance region.

(b) The following asymptotic relationships of the critical conditions are obtained:

(Modified Local Stability Analysis)

$Ra_c = 24.18\, x_c^{-4/3}$ and $a_c = 0.2798\, Ra_c^{1/4}$; $x_c < 10^{-2}$

(Natural-Amplification Analysis)

$Ra_c = 48.60\, x_c^{-4/3}$ and $a_c = 0.3071\, Ra_c^{1/4}$; $x_c < 10^{-2}$

(c) The natural-amplification analysis, which includes the axial amplification of disturbances, seems more reasonable than the conventional stability analyses.

Acknowledgement

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NOMENCLATURE

a: wave number
a^*: modified wave number [$= a\delta$]
$b_n^{(i)}$: coefficient defined in Equation (23)
d: liquid-layer thickness (m)
$f^{(i)}$: z-dependent series function in Equation (23)
g: gravitational acceleration constant (m/sec^2)
H_i: constants in Equations (21) and (22)
$J_m(\zeta)$: Bessel function of order m
K_n: eigen constant in Equation (4)
Pe: Peclet number [$= \langle u \rangle d/\alpha$]
Pr: Prandtl number [$= \nu/\alpha$]
q_w: wall heat flux (J/m^2-sec)
R_n: z-dependent eigenfunction in Equation (4)
Ra: Rayleigh number [$= g\beta q_w d^4/\alpha\kappa\nu$]
Ra*: modified Rayleigh number [$= Ra\,\delta^4$]
S_n: x-dependent function defined in Equation (4)
T: temperature (K)
T_i: inlet temperature (K)
t: time (sec)
U,V,W: velocities in rectangular coordinates (m/sec)
u,v,w: dimensionless velocities [$= \frac{d}{\alpha}(\frac{U}{Pe}, V, W)$]
$\hat{u}, \hat{v}, \hat{w}$: z-dependent perturbations in Equation (13)
X,Y,X: positions in rectangular coordinates (m)
x,y,z: dimensionless positions [$= \frac{1}{d}(\frac{X}{Pe}, Y, Z)$]

Greek Letters

α : thermal diffusivities (m^2/sec)
β : thermal expansion coefficient (1/K)
$\Gamma(\zeta)$: Gamma function
$\Gamma(a,b)$: incomplete Gamma function
δ : effective thermal thickness
δ_{ij}: Kronecker delta
ζ : dimensionless reduced vertical distance [$= z/\delta$]
η : similarity variable in Equation (5)
θ: dimensionless temperature [$= \kappa(T-T_i)/q_w d$]
κ: thermal conductivity (J/m-sec-K)
λ_n: eigen values in Equation (4)
ν : kinematic viscosity (m^2/sec)
Γ: dimensionless spatial growth rate

Subscripts

a: refers to the disturbance within the thermal boundary layer
b: refers to the disturbance outside the thermal boundary layer
c: refers to the critical state
w: refers to the lower boundary
0: refers to the unperturbed state

Superscripts

* refers to x-and z-dependent perturbations for θ,v,w

Brackets

< > refers to average

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