Appl Math Optim 35:311–330 (1997)

Applied Mathematics & Optimization

Hilbertian Convex Feasibility Problem: Convergence of Projection Methods*

P. L. Combettes

Department of Electrical Engineering, City College and Graduate School,
City University of New York, New York, NY 10031, USA
plc@ee-mail.engr.ccny.cuny.edu

Abstract. The classical problem of finding a point in the intersection of countably many closed and convex sets in a Hilbert space is considered. Extrapolated iterations of convex combinations of approximate projections onto subfamilies of sets are investigated to solve this problem. General hypotheses are made on the regularity of the sets and various strategies are considered to control the order in which the sets are selected. Weak and strong convergence results are established within this broad framework, which provides a unified view of projection methods for solving hilbertian convex feasibility problems.

Key Words. Alternating projections, Boundedly regular sets, Chaotic iterations, Convergence, Convex feasibility problem, Convex sets, Extrapolated projections, Fejér-monotone sequences, Hilbert spaces, Parallel projections, Relaxations, Successive projections.

AMS Classification. 90C25, 65J05, 52A41, 40A05.

1. Introduction

Numerous problems in applied mathematics, science, and engineering can be reduced to finding a common point of a family of closed and convex sets in a Hilbert space. This abstract formulation, known as the hilbertian convex feasibility problem, captures problems in disciplines as diverse as approximation theory, integral equations, control theory, signal and image processing, biomedical engineering, communications, and geophysics.

* This work was supported by the National Science Foundation under Grant MIP-9308609.

For detailed accounts of concrete applications, the reader is referred to [20], [22], and [29].

In Hilbert spaces, the use of projection methods to solve convex feasibility problems goes back at least to 1933. Let $P_i(a)$ denote the projection of a point a onto S_i, i.e., the unique point in S_i such that $\|a - P_i(a)\| = \inf\{\|a - b\| \mid b \in S_i\}$. In [54] Von Neumann showed that a point in the intersection of two closed vector subspaces (S_0, S_1) could be obtained as the strong limit of any sequence of iterates

$$(\forall n \in \mathbb{N}) \qquad a_{n+1} = P_{i(n)}(a_n), \tag{1.1}$$

where $i(n) = n$ modulo 2. This result was extended to finite families of closed subspaces $(S_i)_{0 \leq i \leq M-1}$ in [34] by considering the periodic control scheme $i(n) = n$ modulo M. For more general control strategies, weak convergence results were established in [12] and [49]. These efforts culminated with a result of Amemiya and Ando [5], who showed that under the chaotic control rule

$$(\forall i \in \{0, \ldots, M-1\}) \qquad i(n) = i \quad \text{infinitely often}, \tag{1.2}$$

the iterated projections (1.1) converge weakly to a point in the intersection of the M subspaces. For more restrictive control rules, nonlinear versions of this result were given in [10] and [13], where arbitrary convex sets were considered. Methods such as (1.1) are serial in the sense that a single set is selected at each iteration. Their counterparts are methods of parallel projections such as the barycentric method

$$(\forall n \in \mathbb{N}) \qquad a_{n+1} = (1/M) \sum_{i=0}^{M-1} P_i(a_n), \tag{1.3}$$

which was shown in [6] to converge weakly to a point in the intersection of the closed and convex sets $(S_i)_{0 \leq i \leq M-1}$. For both (1.1) and (1.3), strong convergence results have also been established under certain regularity conditions on the sets [33], [44].

The goal of this paper is to study the convergence of a broad class of projection methods for solving hilbertian convex feasibility problems with a countable number of sets. A general algorithm is proposed which provides a unifying formulation for projection-based methods. It proceeds by extrapolated iterations of convex combinations of approximate projections onto subfamilies of sets. This formulation includes in particular serial methods, simultaneous methods, extrapolated relaxation method, and, under suitable assumptions, subgradient methods. In addition, general regularity conditions on the sets are used and various strategies are considered to control the order in which they are selected. The results presented herein extend and improve most known results on the weak and strong convergence of projection methods.

The following two definitions describe the framework of this study.

Definition 1.1. Let Ξ be a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$, norm $\|\cdot\|$, and distance d. Let $(S_i)_{i \in I}$ be a countable (finite or countably infinite) family of closed and convex subsets of Ξ with nonempty intersection S and such that $(\forall i \in I) \;\; S_i \neq \Xi$. The *hilbertian convex feasibility problem* is to find a point in S.

Definition 1.2. Fix $a_0 \in \Xi$, $C \in \mathbb{N}^*$, $\delta \in]0, 1/C[$, and $(\varepsilon, \eta) \in]0, 1]^2$. Let

$$(\forall n \in \mathbb{N}) \qquad a_{n+1} = a_n + \lambda_n \left(\sum_{i \in I_n} w_{i,n} P_{i,n}(a_n) - a_n \right), \tag{1.4}$$

where at each iteration n:

(a) The family I_n of indices of selected sets satisfies

$$I_n \subset I \quad \text{and} \quad 1 \leq \operatorname{card} I_n \leq C. \tag{1.5}$$

(b) For every i in I_n, $P_{i,n}$ is the projection operator onto any closed and convex superset $S_{i,n}$ of S_i such that

$$d(a_n, S_{i,n}) \geq \eta d(a_n, S_i). \tag{1.6}$$

(c) The weights $(w_{i,n})_{i \in I_n}$ are convex and bounded away from zero on active sets, i.e.,

$$(\forall i \in I_n) \qquad \begin{cases} w_{i,n} \geq \delta & \text{if} \quad a_n \notin S_i \\ w_{i,n} \geq 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \sum_{i \in I_n} w_{i,n} = 1. \tag{1.7}$$

(d) The relaxation parameter λ_n varies over an iteration-dependent interval

$$\varepsilon \leq \lambda_n \leq (2 - \varepsilon) L_n, \tag{1.8}$$

with

$$L_n = \begin{cases} \dfrac{\sum_{i \in I_n} w_{i,n} \| P_{i,n}(a_n) - a_n \|^2}{\| \sum_{i \in I_n} w_{i,n} P_{i,n}(a_n) - a_n \|^2} & \text{if} \quad a_n \notin \bigcap_{i \in I_n} S_i, \\ 1 & \text{otherwise.} \end{cases} \tag{1.9}$$

The iterative scheme (1.4)–(1.9) is called the *extrapolated method of parallel projections (EMOPP)*.

EMOPP unifies and extends existing projection methods in several respects:

(a) The total number of sets may be countably infinite. In addition, the sets acted upon may vary at each iteration according to various control strategies defined by the sequence $(I_n)_{n \geq 0}$. Such flexibility is very valuable in practice as it allows us to match the computational load of each iteration to the power of the concurrent processors available. It also brings together serial methods, e.g., (1.1), and barycentric methods, e.g., (1.3).

(b) If exact projections are used, i.e., $P_{i,n} = P_i$ in (1.4), conventional projection methods are obtained. Otherwise, the approximate projection operator $P_{i,n}$ can be regarded as the projection onto an affine hyperplane $H_i(a_n)$ separating a_n from S_i, as in [2] and [32]. When $S^\circ \neq \varnothing$, this framework also includes the subgradient projection methods of [17], [28], and [36] where the sets take the form $S_i = \{a \in \Xi \mid g_i(a) \leq 0\}$ in the euclidean space Ξ, $g_i \colon \Xi \to \mathbb{R}$ being a convex functional. In this case, $H_i(a_n) = \{a \in \Xi \mid \langle a_n - a \mid t_{i,n} \rangle = g_i(a_n)\}$, where $t_{i,n}$ is a subgradient of g_i at a_n.

(c) The weights on the projections may vary at each iteration, unlike in the parallel projections methods of [6], [19], [26], [27], [44], [45], and [50]. Note that if the current iterate a_n belongs to a selected set S_i, the corresponding weight $w_{i,n}$ can be set to zero.

(d) In the vast majority of projection methods, the sequence of relaxations parameters must satisfy

$$(\forall n \in \mathbb{N}) \qquad \varepsilon \leq \lambda_n \leq 2 - \varepsilon. \tag{1.10}$$

The exceptions are the extrapolated projection methods presented in [43]–[46] where

$$(\forall n \in \mathbb{N}) \qquad \varepsilon \leq \lambda_n \leq L_n. \tag{1.11}$$

Since the extrapolation parameter L_n in (1.9) is never less than 1, the relaxation range (1.8) encompasses both (1.10) and (1.11). In numerical applications, the large overrelaxations allowed by (1.8) have been shown to accelerate significantly the convergence of parallel projection methods [22].

Remark 1.1. Projection methods similar to (1.4) have already been studied in the literature under more restrictive assumptions than those of Definitions 1.1 and 1.2. Thus, (1.4) was proposed in [43] with exact projections and relaxation scheme (1.11). For relaxations as in (1.10) and I finite, (1.4) was proposed in [8] (and previously in [32] for euclidean spaces) in the equivalent form

$$(\forall n \in \mathbb{N}) \qquad a_{n+1} = \sum_{i \in I_n} w_{i,n}((1 - \lambda_{i,n})a_n + \lambda_{i,n} P_{i,n}(a_n)). \tag{1.12}$$

Finally, EMOPP was proposed in [23] for I finite and Ξ euclidean. Since the present paper was submitted (Spring 1994), it has come to our attention that a similar method was independently studied in that particular context in [38]. Relaxations of type (1.8) were apparently first proposed in the parallel projection method of [40] to solve systems of linear inequalities in $\mathbb{R}^n$.

Remark 1.2. In the special case when only one set is selected at each iteration, EMOPP is under *serial control* and reduces to

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} a_{n+1} = a_n + \lambda_n (P_{i(n),n}(a_n) - a_n), \\ \varepsilon \leq \lambda_n \leq 2 - \varepsilon, \\ i(n) \in I. \end{cases} \tag{1.13}$$

Such methods are also known as methods of successive projections or "row-action" methods [16].

Remark 1.3. Less general projection methods have been proposed to solve problems which extend the convex feasibility framework of Definition 1.1 in certain directions. Thus, problems with uncountably many sets are addressed in [15] and [42], while the inconsistent case, i.e., $S = \varnothing$, is discussed in [9], [24], and [33]. Feasibility problems outside Hilbert spaces are considered in [4], [25], and [50].

The remainder of the paper is organized as follows. In Section 2 some general properties of EMOPP are presented. In Section 3 several control schemes are introduced and preliminary convergence results are proved. The convergence of EMOPP to a feasible point in the weak topology is then studied in Section 4 for various control strategies. In Section 5 regularity conditions on the sets are discussed and convergence results are established in the strong topology. Unless otherwise stated, the notations and assumptions introduced in Definitions 1.1 and 1.2 are used throughout the paper.

2. General Propositions

Notations. $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, $\mathbb{R}_+$ is the set of nonnegative real numbers, and $\mathbb{R}_+^* = \mathbb{R}_+\backslash\{0\}$. The closed ball of center a and radius γ in Ξ is denoted by $B(a, \gamma)$. The cardinal of a set A is denoted by card A. The expressions $a_n \overset{n}{\rightharpoonup} a$ and $a_n \overset{n}{\to} a$ denote respectively the weak and strong convergence to a of a sequence $(a_n)_{n\geq 0}$. The sets of weak and strong cluster points of $(a_n)_{n\geq 0}$ are denoted by $\mathfrak{W}(a_n)_{n\geq 0}$ and $\mathfrak{S}(a_n)_{n\geq 0}$, respectively. ∂S_i is the boundary of S_i and S_i° its interior. If S_i is an affine subspace (a translation of a vector subspace), the vector space $S_i^\perp$ is its orthogonal complement. The expression $a \propto b$ indicates that the vectors a and b are collinear.

In this section $(a_n)_{n\geq 0}$ is a fixed, but otherwise arbitrary, orbit of EMOPP.

Remark 2.1. The convexity of $\|\cdot\|^2$ yields

$$\left\|\sum_{i\in I_n} w_{i,n} P_{i,n}(a_n) - a_n\right\|^2 \leq \sum_{i\in I_n} w_{i,n}\|P_{i,n}(a_n) - a_n\|^2. \tag{2.1}$$

Now, fix $(c, n) \in S \times \mathbb{N}$. Then we have [55]

$$(\forall i \in I_n) \qquad \langle P_{i,n}(a_n) - c \mid P_{i,n}(a_n) - a_n\rangle \leq 0. \tag{2.2}$$

Whence

$$\left\langle a_n - c \,\middle|\, \sum_{i\in I_n} w_{i,n} P_{i,n}(a_n) - a_n\right\rangle \leq -\sum_{i\in I_n} w_{i,n}\|P_{i,n}(a_n) - a_n\|^2, \tag{2.3}$$

and, thanks to Definition 1.2(b), we easily get

$$\begin{aligned} a_n \in \bigcap_{i\in I_n} S_i \quad &\Leftrightarrow \quad \sum_{i\in I_n} w_{i,n}\|P_{i,n}(a_n) - a_n\|^2 = 0 \\ &\Leftrightarrow \quad \left\|\sum_{i\in I_n} w_{i,n} P_{i,n}(a_n) - a_n\right\|^2 = 0. \end{aligned} \tag{2.4}$$

Therefore L_n is well defined in (1.9) and in view of (2.1) we always have $L_n \geq 1$.

Proposition 2.1. *For every c in S and every n in $\mathbb{N}$, we have*

$$\|a_{n+1} - c\|^2 \leq \|a_n - c\|^2 - \lambda_n(2 - \lambda_n/L_n) \sum_{i\in I_n} w_{i,n} \|P_{i,n}(a_n) - a_n\|^2.$$

Proof. Take any $(c, n) \in S \times \mathbb{N}$. Then (1.4), (1.9), and (2.3) give

$$\begin{aligned}\|a_{n+1} - c\|^2 &= \left\| a_n - c + \lambda_n \left(\sum_{i\in I_n} w_{i,n} P_{i,n}(a_n) - a_n \right) \right\|^2 \\ &= \|a_n - c\|^2 + 2\lambda_n \left\langle a_n - c \,\middle|\, \sum_{i\in I_n} w_{i,n} P_{i,n}(a_n) - a_n \right\rangle \\ &\quad + (\lambda_n^2/L_n) \sum_{i\in I_n} w_{i,n} \|P_{i,n}(a_n) - a_n\|^2 \\ &\leq \|a_n - c\|^2 - \lambda_n(2 - \lambda_n/L_n) \sum_{i\in I_n} w_{i,n} \|P_{i,n}(a_n) - a_n\|^2, \end{aligned} \tag{2.5}$$

which proves the assertion. □

Proposition 2.2. *The following results hold*:

(i) *Fejér-monotonicity* [41]: $(\forall (c, n) \in S \times \mathbb{N})\ \|a_{n+1} - c\| \leq \|a_n - c\|$;
(ii) $\operatorname{card} \mathfrak{W}(a_n)_{n\geq 0} \geq 1$;
(iii) $\operatorname{card} \mathfrak{W}(a_n)_{n\geq 0} \cap S \leq 1$;
(iv) *if* $\mathfrak{W}(a_n)_{n\geq 0} \subset S$, *then* $(a_n)_{n\geq 0}$ *converges weakly to a point in* S.

Proof. (i) follows from Proposition 2.1 and (1.8).

(ii) Fix $c \in S$. Then (i) $\Rightarrow$ $(a_n)_{n\geq 0} \subset B(c, \|a_0 - c\|)$, where $B(c, \|a_0 - c\|)$ is weakly compact.

(iii) The proof of (i) $\Rightarrow$ (iii) appears explicitly or implicitly in [8], [10], [13], and [24].

(iv) In this case, (iii) implies that $(a_n)_{n\geq 0}$ has a unique weak cluster point, which must therefore be its weak limit. □

We now fix an arbitrary point c in S and define

$$(\forall n \in \mathbb{N}) \qquad \beta_n = \|a_n - c\|^2 - \|a_{n+1} - c\|^2. \tag{2.6}$$

Proposition 2.3. *For every integer n, we have*:

(i) $\sum_{i\in I_n} w_{i,n} \|P_{i,n}(a_n) - a_n\|^2 \leq \varepsilon^{-2}\beta_n$;
(ii) $\max_{i\in I_n} d(a_n, S_i)^2 \leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\beta_n$;
(iii) $\|a_{n+1} - a_n\|^2 \leq (2\varepsilon^{-1} - 1)\beta_n$;
(iv) $\langle a_n - c \mid a_n - a_{n+1}\rangle \leq \varepsilon^{-1}\beta_n$.

Proof. Since (1.8) implies that $\lambda_n(2 - \lambda_n/L_n) \geq \varepsilon^2$, (i) follows directly from Proposition 2.1.

(ii) Take any $i \in I_n$. If $a_n \in S_i$, $d(a_n, S_i) = 0$. Otherwise, using (1.6), (1.7), and (i), we get

$$\begin{aligned} d(a_n, S_i)^2 &\leq \eta^{-2}\|P_{i,n}(a_n) - a_n\|^2 \\ &\leq \eta^{-2}\sum_{j\in I_n} w_{j,n}\|P_{j,n}(a_n) - a_n\|^2/w_{i,n} \\ &\leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\beta_n, \end{aligned} \tag{2.7}$$

and obtain (ii).

To establish (iii), note that (1.4) and Proposition 2.1 entail

$$\begin{aligned} \|a_{n+1} - a_n\|^2 &= \frac{\lambda_n^2}{L_n}\sum_{i\in I_n} w_{i,n}\|P_{i,n}(a_n) - a_n\|^2 \\ &\leq \frac{\lambda_n^2}{L_n}\cdot\frac{\beta_n}{\lambda_n(2 - \lambda_n/L_n)} \\ &\leq (2\varepsilon^{-1} - 1)\beta_n, \end{aligned} \tag{2.8}$$

where we have used (1.8) to get $\lambda_n/L_n \leq 2 - \varepsilon$ and $1/(2 - \lambda_n/L_n) \leq \varepsilon^{-1}$.

(iv) Note that $\|a_{n+1} - c\|^2 = \|a_{n+1} - a_n\|^2 + 2\langle a_{n+1} - a_n \mid a_n - c\rangle + \|a_n - c\|^2$. Therefore, using (iii) and (2.6), we obtain the last assertion. □

Proposition 2.4. *$(\beta_n)_{n\geq 0}$ is summable.*

Proof. According to Proposition 2.2(i), $(\beta_n)_{n\geq 0} \subset \mathbb{R}_+$. Moreover, (2.6) implies $(\forall n \in \mathbb{N})\ \sum_{k=0}^{n} \beta_k = \|a_0 - c\|^2 - \|a_{n+1} - c\|^2 \leq \|a_0 - c\|^2$ and, therefore, $\sum_{n\geq 0} \beta_n \leq \|a_0 - c\|^2$. □

3. Control Schemes

Several control strategies will be considered for EMOPP. They constitute extensions to parallel projection methods of schemes which have been proposed for serial ones.

Definition 3.1. Assume that card $I < +\infty$. Then the control is:

- *Static* if all the sets are selected at each iteration, i.e.,

$$(\forall n \in \mathbb{N}) \qquad I_n = I. \tag{3.1}$$

  This control condition goes back to Cimmino's algorithm [19].
- *Cyclic* if there exists a positive integer M such that

$$(\forall n \in \mathbb{N}) \qquad I = \bigcup_{k=n}^{n+M-1} I_k. \tag{3.2}$$

  In words, if the control is M-cyclic, all the sets must be selected at least once within any M consecutive iterations. This condition was utilized in [49] for the serial case and in [43] for the parallel case. In the serial case with M sets, say $(S_i)_{0\leq i\leq M-1}$, an important example of cyclic control is the *periodic* control scheme

$$(\forall n \in \mathbb{N}) \qquad i(n) = n \text{ (modulo } M), \tag{3.3}$$

that was used in Kaczmarz' algorithm [37]. For two vector subspaces, it yields the alternating projection scheme of [54], which has been rediscovered in many places [29].

- *Quasi-cyclic* if there exists an increasing sequence of integers $(M_m)_{m\geq 0}$ such that

$$\begin{cases} M_0 = 0, \\ \sum_{m\geq 0} (M_{m+1} - M_m)^{-1} = +\infty, \\ (\forall m \in \mathbb{N}) \qquad I = \bigcup_{k=M_m}^{M_{m+1}-1} I_k. \end{cases} \tag{3.4}$$

Thus, under $(M_m)_{m\geq 0}$-quasi-cyclic control, all the sets are selected at least once within each variable cycle of iterations $\{M_m, \dots, M_{m+1}-1\}$. The nonsummability condition ensures that the lengths $(M_{m+1}-M_m)_{m\geq 0}$ of the cycles do not eventually increase too fast. This type of control was introduced in [53] for a serial method.

Remark 3.1. The above control modes are applicable only when $(S_i)_{i\in I}$ is a finite family because they impose that all the sets be selected over a finite number of iterations. Henceforth, any statement pertaining to static, cyclic, or quasi-cyclic control will implicitly carry the assumption card $I < +\infty$.

We now introduce control modes applicable to countable families.

Definition 3.2. The control is:

- *Admissible* if there exist positive integers $(M_i)_{i\in I}$ such that

$$(\forall (i, n) \in I \times \mathbb{N}) \qquad i \in \bigcup_{k=n}^{n+M_i-1} I_k. \tag{3.5}$$

Hence, the set S_i is selected at least once within any M_i consecutive iterations. Of course, if card $I < +\infty$, this control mode coincides with the cyclic mode (3.2). The admissible control condition was introduced in [12] for the serial method (1.1) (we adopt the terminology of [13] here).

- *Chaotic* if each set is selected infinitely often in the iteration process, i.e.,

$$I = \limsup_{n\to+\infty} I_n. \tag{3.6}$$

This condition is an extension of (1.2), which goes back to Poincaré's *balayage* (sweeping) method [47]. It was used in the serial method of [49] and in the parallel method of [43]. Note that (3.6) generalizes (3.1)–(3.5).

- *Coercive* if

$$\left(\exists (i(n))_{n\geq 0} \in \mathop{\times}_{n\geq 0} I_n\right) \qquad d(a_n, S_{i(n)}) \xrightarrow{n} 0 \quad \Rightarrow \quad \sup_{i\in I} d(a_n, S_i) \xrightarrow{n} 0. \tag{3.7}$$

In the serial case, this control mode was proposed in [33] as a generalization of the *most-remote set* control scheme

$$(\forall n \in \mathbb{N})(\exists i(n) \in I_n) \qquad d(a_n, S_{i(n)}) = \sup_{i\in I} d(a_n, S_i), \tag{3.8}$$

which is not always applicable when card $I = +\infty$. The most-remote set control strategy was introduced in [1] and [41].

- *Chaotically coercive* if $(I_n)_{n\geq 0}$ contains a subsequence $(I_{n_k})_{k\geq 0}$ such that

$$\left(\exists (i(k))_{k\geq 0} \in \mathop{\times}_{k\geq 0} I_{n_k}\right) \qquad d(a_{n_k}, S_{i(k)}) \xrightarrow{k} 0 \quad \Rightarrow \quad \sup_{i\in I} d(a_{n_k}, S_i) \xrightarrow{k} 0. \tag{3.9}$$

This condition generalizes (3.7) as well as the control strategy consisting in selecting one of the most remote sets infinitely often in the course of the iterations.

The results of Section 2 have been obtained without making any assumption on the control sequence $(I_n)_{n\geq 0}$. We now establish convergence properties that depend on the control.

Proposition 3.1. *Let $(a_n)_{n\geq 0}$ be an arbitrary orbit of EMOPP. If the control is:*

(i) *quasi-cyclic, then $(a_n)_{n\geq 0}$ possesses a subsequence $(a_{n_k})_{k\geq 0}$ such that $\max_{i\in I} d(a_{n_k}, S_i) \xrightarrow{k} 0$;*
(ii) *admissible, then $(\forall i \in I)$ $d(a_n, S_i) \xrightarrow{n} 0$;*
(iii) *chaotic, then, for every i in I, $(a_n)_{n\geq 0}$ possesses a subsequence $(a_{n_k})_{k\geq 0}$ such that $d(a_{n_k}, S_i) \xrightarrow{k} 0$;*
(iv) *coercive, then $\sup_{i\in I} d(a_n, S_i) \xrightarrow{n} 0$;*
(v) *chaotically coercive, then $(a_n)_{n\geq 0}$ possesses a subsequence $(a_{n_k})_{k\geq 0}$ such that $\sup_{i\in I} d(a_{n_k}, S_i) \xrightarrow{k} 0$.*

Proof. To demonstrate (i) and (ii), fix (i, n) in $I \times \mathbb{N}$. Let $\mathbb{K}_{n,i} \subset \mathbb{N}$ be a set of $K_{n,i}$ consecutive integers containing n and some integer p such that $i \in I_p$. Define $\gamma_{n,i} = K_{n,i} \sum_{k\in\mathbb{K}_{n,i}} \beta_k$. Proposition 2.3(ii) yields

$$d(a_p, S_i)^2 \leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\beta_p \leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\gamma_{n,i}. \tag{3.10}$$

On the other hand, Proposition 2.3(iii) yields

$$\begin{aligned} \|a_p - a_n\|^2 &\leq \left(\sum_{k\in\mathbb{K}_{n,i}} \|a_{k+1} - a_k\|\right)^2 \\ &\leq K_{n,i} \sum_{k\in\mathbb{K}_{n,i}} \|a_{k+1} - a_k\|^2 \\ &\leq (2\varepsilon^{-1} - 1)\gamma_{n,i}. \end{aligned} \tag{3.11}$$

Let $\zeta = 2(\delta^{-1}\varepsilon^{-2}\eta^{-2} + 2\varepsilon^{-1} - 1)$. By combining (3.10) and (3.11), we get

$$\begin{aligned} d(a_n, S_i)^2 &\leq \|P_i(a_p) - a_n\|^2 \\ &\leq \big(d(a_p, S_i) + \|a_p - a_n\|\big)^2 \\ &\leq 2(d(a_p, S_i)^2 + \|a_p - a_n\|^2) \\ &\leq \zeta\gamma_{n,i}. \end{aligned} \tag{3.12}$$

(i) Suppose (3.4) holds and let $m = m(n)$ be the largest integer such that $n \geq M_m$. Then $\mathbb{K}_{n,i} = \{M_m, \ldots, M_{m+1} - 1\} \triangleq \mathbb{K}_m$ will work for every $i \in I$. From (3.12), we obtain

$$(\forall m \in \mathbb{N})(\forall n \in \{M_m, \ldots, M_{m+1} - 1\})$$
$$\max_{i\in I} d(a_n, S_i)^2 \leq \zeta(M_{m+1} - M_m) \sum_{k=M_m}^{M_{m+1}-1} \beta_k \triangleq \zeta\gamma_m. \tag{3.13}$$

Hence, to prove assertion (i), it suffices to show $0 \in \mathfrak{S}(\gamma_m)_{m\geq 0}$. Observe that, if we had $0 \notin \mathfrak{S}(\gamma_m)_{m\geq 0}$, there would exist $(\mu, N) \in \mathbb{R}_+^* \times \mathbb{N}$ such that $(\forall m \geq N)\ \ \gamma_m \geq \mu$. In view of (3.13), this would yield

$$\sum_{m\geq N}(M_{m+1} - M_m)^{-1} \leq \mu^{-1} \sum_{m\geq N} \sum_{k=M_m}^{M_{m+1}-1} \beta_k \leq \mu^{-1} \sum_{k\geq 0} \beta_k. \tag{3.14}$$

However, a contradiction would arise since the series in the left-hand side diverges by (3.4) while the series in the right-hand side converges by Proposition 2.4. This establishes (i).

(ii) If (3.5) holds, we can take $\mathbb{K}_{n,i} = \{n, \ldots, n + M_i - 1\}$ and (3.12) leads to

$$(\forall (n, i) \in \mathbb{N} \times I) \qquad d(a_n, S_i)^2 \leq \zeta M_i \sum_{k\geq n} \beta_k. \tag{3.15}$$

However, by Proposition 2.4, the right-hand side is the tail of a convergent series and it must converge to zero as n increases indefinitely. Thus, we obtain (ii).

(iii) Fix $i \in I$. If the control is chaotic, there exists an increasing sequence $(n_k)_{k\geq 0} \subset \mathbb{N}$ such that $(\forall k \in \mathbb{N})\ i \in I_{n_k}$. By Proposition 2.3(ii), we then get

$$(\forall k \in \mathbb{N}) \qquad d(a_{n_k}, S_i)^2 \leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\beta_{n_k}. \tag{3.16}$$

Since $\beta_{n_k} \overset{k}{\to} 0$, the proof is complete.

(iv) Consider the coercive control scheme and define $(i(n))_{n\geq 0}$ as in (3.7). Then Proposition 2.3(ii) gives

$$(\forall n \in \mathbb{N}) \qquad d(a_n, S_{i(n)})^2 \leq \max_{i\in I_n} d(a_n, S_i)^2 \leq \delta^{-1}\varepsilon^{-2}\eta^{-2}\beta_n. \tag{3.17}$$

However, since $\beta_n \overset{n}{\to} 0$, we have $d(a_n, S_{i(n)}) \overset{n}{\to} 0$ and therefore (3.7) completes the proof. Note that (v) in the chaotically coercive case is proven in an analogous manner. □

4. Weak Convergence

4.1. *Quasi-Cyclic and Chaotically Coercive Controls*

We start with the following facts.

Lemma 4.1. *For every sequence* $(a_n)_{n\geq 0} \subset \Xi$ *the following statements hold:*

(i) *Suppose* $(\exists i \in I)\ d(a_n, S_i) \overset{n}{\to} 0$. *Then* $a_n \overset{n}{\rightharpoonup} a \Leftrightarrow P_i(a_n) \overset{n}{\rightharpoonup} a$.
(ii) *Suppose* $(\exists i \in I)\ d(a_n, S_i) \overset{n}{\to} 0$. *Then* $a_n \overset{n}{\to} a \Leftrightarrow P_i(a_n) \overset{n}{\to} a$.
(iii) *Suppose* $(\exists i \in I)\ d(a_n, S_i) \overset{n}{\to} 0$, *where* S_i *is boundedly compact (its intersection with any closed ball is compact). Then* $a_n \overset{n}{\to} a \Leftrightarrow a_n \overset{n}{\rightharpoonup} a$.
(iv) *Suppose* $(\forall i \in I)\ d(a_n, S_i) \overset{n}{\to} 0$. *Then* $\mathfrak{W}(a_n)_{n\geq 0} \subset S$.

Proof. (i) and (ii) are trivial.

(iii) The forward implication is obvious. To prove the reverse implication, suppose $a_n \overset{n}{\rightharpoonup} a$. Then (i) $\Rightarrow P_i(a_n) \overset{n}{\rightharpoonup} a \Rightarrow (P_i(a_n))_{n\geq 0}$ is bounded. However, since $(P_i(a_n))_{n\geq 0}$ lies in the boundedly compact set S_i, we must have $\mathfrak{S}(P_i(a_n))_{n\geq 0} = \{a\}$. Therefore $P_i(a_n) \overset{n}{\to} a$ and (ii) $\Rightarrow a_n \overset{n}{\to} a$.

(iv) If $\mathfrak{W}(a_n)_{n\geq 0} = \varnothing$, (iv) holds trivially. Otherwise, take any $a \in \mathfrak{W}(a_n)_{n\geq 0}$, say $a_{n_k} \overset{k}{\rightharpoonup} a$, and any $i \in I$. Then (i) $\Rightarrow P_i(a_{n_k}) \overset{k}{\rightharpoonup} a$, but $(P_i(a_{n_k}))_{k\geq 0} \subset S_i$ and S_i is closed in the weak topology. Whence, $a \in S_i$. Since i was arbitrary, we conclude $a \in S$. □

Theorem 4.1. *Under quasi-cyclic or chaotically coercive control, every orbit of EMOPP possesses one and only one weak cluster point in* S.

Proof. Let $(a_n)_{n\geq 0}$ be an arbitrary orbit of EMOPP. By Proposition 3.1(i) and (v), there exists a subsequence $(a_{n_k})_{k\geq 0}$ of $(a_n)_{n\geq 0}$ such that $(\forall i \in I)\ d(a_{n_k}, S_i) \overset{k}{\to} 0$. Clearly, Proposition 2.2 applies to $(a_{n_k})_{k\geq 0}$. Thus, by Proposition 2.2(ii), we can find $a \in \mathfrak{W}(a_{n_k})_{k\geq 0}$. Lemma 4.1(iv) then gives $a \in S$. Uniqueness follows from Proposition 2.2(iii). □

Remark 4.1. Under quasi-cyclic control, Theorem 4.1 was obtained in Theorem 2 of [52] for a variant of the serial algorithm (1.13) in which exact firmly nonexpansive operators $(T_i)_{i\in I}$ with sets of fixed points $(S_i)_{i\in I}$ were considered in lieu of approximate projections (projection operators are special cases of firmly nonexpansive mappings [55]). As shown in [21], several of our results still hold true for the corresponding variant of EMOPP, which provides a proper extension of results of [52].

Corollary 4.1. *Under quasi-cyclic or chaotically coercive control, if an orbit of EMOPP possesses no weak cluster point outside of* S*, then it converges weakly to a point in* S.

4.2. *Admissible and Coercive Controls*

Theorem 4.2. *Under admissible or coercive control, every orbit of EMOPP converges weakly to a point in S.*

Proof. The claim follows from Proposition 3.1(ii) and (iv), Lemma 4.1(iv), and Proposition 2.2(iv). □

Remark 4.2. In the special case of algorithm (1.1), Theorem 4.2 coincides with Theorem 2 of [13] (Lemma 3 of [12] in the linear case) for admissible control and Theorem 2 of [10] for most-remote set control. Now suppose that card $I < +\infty$ and that the constant relaxation range (1.10) is in force. Theorem 4.2 is established in this context in Theorem 3.20(i) of [8] for cyclic control and in Theorem 4.26(ii) of [8] for most-remote set control. It should be noted that these results assumed more general approximate projections than those of Definition 1.2(b). For exact projections, Theorem 3.20(i) of [8] appears in Theorem 1 of [24], which contains results of [6], [10], [26], and [27], while Theorem 4.26(ii) of [8] contains the finite-dimensional results of [1], [31], and [41].

Remark 4.3. Theorem 4.2 also generalizes Theorem 1.1(i) of [45], which considered the static algorithm

$$(\forall n \in \mathbb{N}) \qquad a_{n+1} = a_n + \lambda_n \left(\sum_{i \in I} w_i P_i(a_n) - a_n \right), \tag{4.1}$$

with (1.11), $(\forall i \in I)\ w_i > 0$, and $\sum_{i \in I} w_i = 1$. It is worth pointing out that this result can also be deduced from Theorem 1 of [48], where the weak convergence of the convex minimization algorithm

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} a_{n+1} = a_n - (\alpha_n(\Phi(a_n) - \Phi_{\min})/\|\nabla\Phi(a_n)\|^2)\nabla\Phi(a_n), \\ \varepsilon \leq \alpha_n \leq 2 - \varepsilon \end{cases} \tag{4.2}$$

to a minimizer of Φ is demonstrated (take $\Phi\colon a \mapsto \sum_{i \in I} w_i d(a, S_i)^2$ and note that $(\forall i \in I)\ \nabla d(a, S_i)^2 = 2(a - P_i(a))$ [55], $\Phi_{\min} = 0$, and $S = \Phi^{-1}(\{\Phi_{\min}\})$).

4.3. *Chaotic Control*

As shown below, without further assumptions on $(S_i)_{i \in I}$, EMOPP may fail to converge weakly under chaotic control. However, some results are available for the special instance (1.1)–(1.2). Weak convergence to a point in S of every orbit of this algorithm is proved in [5] in the case of a finite family of closed vector subspaces. A nonlinear extension of this result is proposed in Theorem 5 of [30], where it is shown to remain true for finitely many closed and convex subsets sharing a "weak internal point" (WIP). It is also shown in Theorem 2 of [30] that, in the presence of three sets, the assumption of a WIP is not necessary to ensure weak convergence to a feasible point.

Example 4.1. Take $(\theta_i)_{i \geq 0} \subset \mathbb{R}_+$ with $\theta_0 = 0$ and $(\forall i \in \mathbb{N})\ 0 < \theta_{i+1} - \theta_i < 1$. In the euclidean plane, let S_i be the ray emanating from the origin at an angle θ_i with respect to S_0. As shown in [14], the iterative process $a_{n+1} = P_{n+1}(a_n)$ with $a_0 = (1, 0)$ leads to $\|a_{n+1}\| = \prod_{i=0}^{n} \cos(\theta_{i+1} - \theta_i) \geq \prod_{i=0}^{n}(1 - (\theta_{i+1} - \theta_i)^2/2) \triangleq \ell_n$. Now, to make this process chaotic, we choose the (modulo 2π) dyadic sequence

$$(\theta_i)_{i\geq 0} = (0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, 0, \pi/8, \pi/4, 3\pi/8, \pi/2, \ldots, 15\pi/8, 0, \pi/16, \pi/8, 3\pi/16, \ldots). \tag{4.3}$$

We obtain a countable family of distinct sets $(S_i)_{i\in I}$ with $\bigcap_{i\in I} S_i = \{0\}$. However, $\sum_{i\geq 0}(\theta_{i+1} - \theta_i)^2 = \pi^2$ and therefore $(\exists \ell \in \mathbb{R}_+^*)\ \ell_n \overset{n}{\to} \ell$. We conclude $\|a_n\| \overset{n}{\not\to} 0$.

5. Strong Convergence

In Hilbert spaces, strong convergence of projection algorithms requires some regularity conditions on $(S_i)_{i\in I}$. Thus, in the early serial-periodic projection methods, properties such as linearity [34], [54], compactness [18], [51], uniform convexity, or Slater condition [33] were imposed. In this section we establish strong convergence of EMOPP under general regularity conditions and various control schemes.

5.1. *Quasi-Cyclic and Chaotically Coercive Controls*

Definition 5.1. The family $(S_i)_{i\in I}$ is *boundedly regular* if, for every bounded sequence $(a_n)_{n\geq 0}$ in Ξ, $\sup_{i\in I} d(a_n, S_i) \overset{n}{\to} 0 \Rightarrow d(a_n, S) \overset{n}{\to} 0$.

Remark 5.1. The concept of bounded regularity was first used extensively in [33] to prove the strong convergence of serial projections algorithms. Conditions for bounded regularity were previously discussed in [39] in the case of two sets. We use the terminology of [7] here.

Lemma 5.1 [33]. *Let $(a_n)_{n\geq 0}$ be a Fejér-monotone sequence with respect to S. If $(\sup_{i\in I} d(a_n, S_i))_{n\geq 0}$ converges to zero and $(S_i)_{i\in I}$ is boundedly regular, then $(a_n)_{n\geq 0}$ converges strongly to a point in S.*

Theorem 5.1. *Under quasi-cyclic or chaotically coercive control, every orbit of EMOPP converges strongly to a point in S if $(S_i)_{i\in I}$ is boundedly regular.*

Proof. Take an arbitrary orbit $(a_n)_{n\geq 0}$. According to Proposition 3.1(i) and (v), it contains a subsequence $(a_{n_k})_{k\geq 0}$ such that $\sup_{i\in I} d(a_{n_k}, S_i) \overset{k}{\to} 0$. Moreover, Proposition 2.2(i) indicates that $(a_{n_k})_{k\geq 0}$ is Fejér-monotone with respect to S. Lemma 5.1 implies that there exists a point $a \in S$ such that $a_{n_k} \overset{k}{\to} a$. Proposition 2.2(i) then yields $a_n \overset{n}{\to} a$. □

Remark 5.2. The notion of bounded regularity appears explicitly or implicitly in the proofs of strong convergence of several projection algorithms. Thus, for the serial algorithm (1.13) with exact projections and either periodic or coercive control, Theorem 5.1

was obtained in Theorem 1 of [33]. For the static algorithm (4.1), Theorem 5.1 is found as Theorem 1.1(ii) of [45]. Finally, for card $I < +\infty$ and relaxation rule (1.10), related results are Theorem 5.2 of [8] and Theorem 2 of [24] for cyclic control, and Theorem 5.3 of [8] for most-remote set control.

We now give more specific and conventional conditions for the strong convergence of EMOPP under quasi-cyclic and chaotically coercive controls.

Definition 5.2 [39]. Let $\mathcal{F}$ be the class of all nondecreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ that vanish only at zero. Then S_i is *f-uniformly convex* if $(\exists f \in \mathcal{F})(\forall (a,b) \in S_i^2)$ $B((a+b)/2, f(\|a-b\|)) \subset S_i$ and *locally uniformly convex* if $(\forall a \in S_i)(\exists f \in \mathcal{F})$ $(\forall b \in S_i) B((a+b)/2, f(\|a-b\|)) \subset S_i$.

Remark 5.3. Since we assume $S_i \neq \Xi$, if S_i is uniformly convex, then it is necessarily bounded [39]. However, locally uniformly convex sets need not be bounded.

The following definition is motivated by [39].

Definition 5.3. S_i is a *Levitin–Polyak set* if, for every sequence $(a_n)_{n\geq 0} \subset \Xi$ such that $d(a_n, S_i) \overset{n}{\to} 0$, we have $a_n \overset{n}{\rightharpoonup} a \in \partial S_i \Rightarrow a_n \overset{n}{\to} a$.

Corollary 5.1. *Under quasi-cyclic or chaotically coercive control, every orbit of EMOPP converges strongly to a point in S if any of the following conditions is satisfied*:

(i) $(\exists j \in I)\ S_j \cap (\bigcap_{i\in I\setminus\{j\}} S_i)^\circ \neq \varnothing$.
(ii) *All, except possibly one, of the sets in $(S_i)_{i\in I}$ are f-uniformly convex.*
(iii) *One of the sets in $(S_i)_{i\in I}$ is boundedly compact (in particular compact or contained in a finite-dimensional affine subspace).*
(iv) *Ξ has finite dimension.*
(v) *$(S_i)_{i\in I}$ is a finite family and all, except possibly one, of its sets are Levitin–Polyak sets.*
(vi) *$(S_i)_{i\in I}$ is a finite family and all, except possibly one, of its sets are locally uniformly convex.*
(vii) *$(S_i)_{i\in I}$ is a finite family of closed affine subspaces such that $\sum_{i\in I} S_i^\perp$ is closed.*
(viii) *$(S_i)_{i\in I}$ is a finite family of closed affine subspaces, all of which, except possibly one, have finite codimension.*
(ix) *$(S_i)_{i\in I}$ is a finite family of closed affine subspaces, all of which, except possibly one, are affine hyperplanes.*
(x) *$(S_i)_{i\in I}$ is a finite family of closed polyhedrons (finite intersections of closed affine half-spaces).*

Proof. According to Theorem 5.1, it is enough to show that the families described in (i)–(x) are boundedly regular. This was done in [33] for cases (i), (ii), and (iv), in [8] and [11] for case (vii), and in [8] for case (x). Note that (iv) is a particular case of (iii), (vi) is a particular case of (v) [39], and (viii) and (ix) are particular cases of (vii). It therefore remains to prove (iii) and (v).

Take any bounded sequence $(a_n)_{n\geq 0} \subset \Xi$ such that $\sup_{i\in I} d(a_n, S_i) \overset{n}{\to} 0$ and take any $\ell \in \mathfrak{S}(d(a_n, S))_{n\geq 0}$, say $d(a_{n_k}, S) \overset{k}{\to} \ell$. Thanks to the boundedness assumption and Lemma 4.1(iv), we can find $a \in \mathfrak{W}(a_{n_k})_{k\geq 0} \cap S$. It is sufficient to show that $a \in \mathfrak{S}(a_{n_k})_{k\geq 0}$ for this will yield $\ell = 0$.

(iii) Suppose that, for some $i \in I$, S_i is boundedly compact. Then Lemma 4.1(iii) entails $a \in \mathfrak{S}(a_{n_k})_{k\geq 0}$, as desired.

(v) Select $j \in I$ such that $(S_i)_{i\in I\setminus\{j\}}$ are Levitin–Polyak sets and note that $S = S_j \cap (\bigcap_{i\in I\setminus\{j\}} S_i^\circ \cup \partial S_i)$. Now define $A = S_j \cap (\bigcap_{i\in I\setminus\{j\}} S_i^\circ)$. If $a \in A$, then (i) holds and $(S_i)_{i\in I}$ is boundedly regular [33]. Otherwise, $a \in S\setminus A$ and, for some $i \in I\setminus\{j\}$, $a \in \partial S_i$. Therefore $a \in \mathfrak{W}(a_{n_k})_{k\geq 0} \cap \partial S_i$. Since $d(a_{n_k}, S_i) \overset{k}{\to} 0$ and S_i is a Levitin–Polyak set, we conclude $a \in \mathfrak{S}(a_{n_k})_{k\geq 0}$. □

Remark 5.4. Under cyclic or coercive control with exact projections and relaxation rule (1.11), Corollary 5.1(i) and (ii) follows from Corollary 5.1(i) of [43]. Special cases of Corollary 5.1 can also be found in [18], [27], [35], and [51].

5.2. *Admissible Control*

Theorem 5.2. *Under admissible control, every orbit of EMOPP converges strongly to a point in S if $(S_i)_{i\in I}$ contains a boundedly compact set.*

Proof. A direct consequence of Proposition 3.1(ii), Theorem 4.2, and Lemma 4.1(iii). □

Corollary 5.2. *If Ξ has finite dimension, every orbit of EMOPP converges to a point in S under admissible control.*

5.3. *Chaotic Control*

Proposition 5.1. *Let $(a_n)_{n\geq 0}$ be an arbitrary orbit of EMOPP under chaotic control. Then $(a_n)_{n\geq 0}$ converges strongly to a point in S if either of the following conditions holds*:

(i) *$(a_n)_{n\geq 0}$ converges strongly*;
(ii) $\mathfrak{S}(a_n)_{n\geq 0} \neq \varnothing$ *and* $\operatorname{card} I < +\infty$.

Proof. (i) Suppose $(\exists\, a \in \Xi)$ $a_n \overset{n}{\to} a$ and fix $i \in I$. By Proposition 3.1(iii), there exists a subsequence $(a_{n_k})_{k\geq 0}$ of $(a_n)_{n\geq 0}$ such that $d(a_{n_k}, S_i) \overset{n}{\to} 0$. Therefore $P_i(a_{n_k}) \overset{k}{\to} a$ and, since S_i is (strongly) closed, $a \in S_i$. Since this argument is valid for any $i \in I$, $a \in S$.

(ii) Fix $a \in \mathfrak{S}(a_n)_{n\geq 0}$. According to Proposition 2.2(i) it suffices to show that $a \in S$. Suppose to the contrary that $a \notin S$ and define $I^+ = \{i \in I \mid a \in S_i\}$, $I^- = I\setminus I^+$, $\mu = \min_{i\in I^-} d(a, S_i)$, and $\nu = \delta\varepsilon^2\eta^2$. A slight extension of Proposition 2.3(ii) yields

$$(\forall n \in \mathbb{N}) \left(\forall e \in \bigcap_{i\in I_n} S_i\right) \qquad \|a_{n+1} - e\|^2 \leq \|a_n - e\|^2 - \nu \max_{j\in I_n} d(a_n, S_j)^2. \tag{5.1}$$

Now fix $j \in I^-$, $c \in S$, and $\gamma \in]0, \mu[$. As $a \in \mathfrak{S}(a_n)_{n\geq 0}$, there exists an integer p such that $a_p \in B(a, \gamma)$. Note that $\|a_p - c\| \leq \gamma + \|a - c\|$ and $d(a_p, S_j) \geq d(a, P_j(a_p)) - d(a, a_p) \geq d(a, S_j) - d(a, a_p) \geq \mu - \gamma$. Consequently, if we had $j \in I_p$, (5.1) would imply

$$\|a_{p+1} - c\|^2 \leq (\gamma + \|a - c\|)^2 - \nu(\mu - \gamma)^2 \tag{5.2}$$

and, for γ sufficiently small, we would obtain $\|a_{p+1} - c\| < \|a - c\|$. However, this would contradict Proposition 2.2(i) which implies $(\forall n \in \mathbb{N})\ \|a - c\| \leq \|a_n - c\|$. Therefore $j \notin I_p$. Since j is arbitrary, it follows that $I_p \cap I^- = \varnothing$ and $I_p \subset I^+$. Hence, $a \in \bigcap_{i\in I_p} S_i$ and (5.1) $\Rightarrow \|a_{p+1} - a\| \leq \|a_p - a\| \Rightarrow a_{p+1} \in B(a, \gamma)$. Reiterating the same argument for index $p + 1$, gives $j \notin I_{p+1}$ and $a_{p+2} \in B(a, \gamma)$. Thus, by induction, we obtain $(\forall k \in \mathbb{N})\ j \notin I_{p+k}$, which violates (3.6). We conclude that $a \in S$. □

Proposition 5.2. *Suppose that the control is chaotic and that $(S_i)_{i\in I}$ is a finite family. Then every orbit $(a_n)_{n\geq 0}$ of EMOPP such that $(a_n - a_0)_{n\geq 0} \subset W$, where W is a boundedly compact subset of Ξ, converges strongly to a point in S.*

Proof. By Proposition 2.2(i), $(a_n)_{n\geq 0} \subset B(c, \|a_0 - c\|) \cap (\{a_0\} + W) \triangleq K$. Since K is compact, Proposition 5.1(ii) provides the announced result. □

Definition 5.4 [43]. A point $c \in S$ is a *strongly regular point* of $(S_i)_{i\in I}$ if

$$(\forall(\rho_1, \rho_2) \in \mathbb{R}_+^{*2})(\exists \rho \in \mathbb{R}_+)(\forall(i, a, b) \in I \times \Xi \times \Xi)$$
$$\begin{cases} \|P_i(a) - c\| \geq \rho_1 \\ \|b - c\| \leq \rho_2 \end{cases} \Rightarrow \quad d(b, H_i(a)) \leq \rho d(c, H_i(a)), \tag{5.3}$$

where $H_i(a) = \{h \in \Xi \mid \langle h - P_i(a) \mid a - P_i(a)\rangle = 0\}$.

Our main result on the strong convergence of chaotic projection methods can now be stated.

Theorem 5.3. *Under chaotic control, every orbit of EMOPP converges strongly to a point in S if any of the following conditions is satisfied:*

(i) *$(S_i)_{i\in I}$ is a Slater family: $(\bigcap_{i\in I} S_i)^\circ \neq \varnothing$.*
(ii) *$(S_i)_{i\in I}$ has a strongly regular point and exact projections are used.*
(iii) *$(S_i)_{i\in I}$ is a family of f-uniformly convex sets and exact projections are used.*
(iv) *$(S_i)_{i\in I}$ is a finite family and one of its sets is boundedly compact (in particular compact or contained in a finite-dimensional affine subspace).*
(v) *$(S_i)_{i\in I}$ is a finite family and Ξ has finite dimension.*
(vi) *$(S_i)_{i\in I}$ is a finite family of closed affine subspaces with finite codimensions (in particular affine hyperplanes).*
(vii) *$(S_i)_{i\in I}$ is a finite family of closed affine half-spaces.*
(viii) *$(S_i)_{i\in I}$ is a finite family of closed polyhedrons and exact projections are used.*

Proof. Let $(a_n)_{n\geq 0}$ be an arbitrary orbit of EMOPP. (i) In Ξ, any sequence which is Fejér-monotone with respect to a closed and convex set with nonempty interior converges strongly [8, Theorem 2.16(iii)]. Hence, the result follows from Propositions 2.2(i) and 5.1(i).

(ii) From Propositions 2.3(ii) and 2.4, $\max_{i\in I_n} d(a_n, S_i) \overset{n}{\to} 0$. Therefore, following the proof of Theorem 4.1(i) of [43], (5.3) implies that, for n large enough, we can find $\rho \in \mathbb{R}_+$ such that $(\forall i \in I_n)\ d(a_n, S_i) \leq (\rho + 1)\langle a_n - c \mid a_n - P_i(a_n)\rangle$. Hence, by invoking Proposition 2.3(iv), we get, for n large enough,

$$\begin{aligned}\|a_{n+1} - a_n\| &\leq \lambda_n \sum_{i\in I_n} w_{i,n} d(a_n, S_i)\\ &\leq (\rho + 1)\langle a_n - c \mid a_n - a_{n+1}\rangle \leq (\rho + 1)\varepsilon^{-1}\beta_n.\end{aligned} \tag{5.4}$$

It then follows from Proposition 2.4 that $(\|a_{n+1} - a_n\|)_{n\geq 0}$ is summable. Whence, $(a_n)_{n\geq 0}$ is a Cauchy sequence and Proposition 5.1(i) gives the result.

(iii) is a special case of (ii) [43, Theorem 5.1(iii)].

(iv) Suppose that S_j is boundedly compact. By Proposition 3.1(iii), there exists a subsequence $(a_{n_k})_{k\geq 0}$ of $(a_n)_{n\geq 0}$ such that $d(a_{n_k}, S_j) \overset{k}{\to} 0$ and, according to Proposition 5.1(ii) and Lemma 4.1(ii), it is enough to show that $\mathfrak{S}(P_j(a_{n_k}))_{k\geq 0} \neq \varnothing$. To this end, take $c \in S$. Then c is a fixed point of the nonexpansive operator P_j and Proposition 2.2(i) entails $(\forall k \in \mathbb{N})\ \|P_j(a_{n_k}) - c\| \leq \|a_{n_k} - c\| \leq \|a_0 - c\|$. Hence $(P_j(a_{n_k}))_{k\geq 0} \subset B(c, \|a_0 - c\|) \cap S_j \triangleq K_j$. Since K_j is compact, $\mathfrak{S}(P_j(a_{n_k}))_{k\geq 0} \neq \varnothing$.

(v) is a special case of (iv).

(vi) Consider the finite-dimensional vector subspace $W = \sum_{i\in I} S_i^{\perp}$ and define

$$(\forall n \in \mathbb{N}) \qquad p_n = \lambda_n \sum_{i\in I_n} w_{i,n}(P_{i,n}(a_n) - a_n). \tag{5.5}$$

At every iteration n, the sets $(S_{i,n})_{i\in I_n}$ are supersets of the affine subspaces $(S_i)_{i\in I_n}$. Whence

$$(\forall n \in \mathbb{N})(\forall i \in I_n) \qquad P_{i,n}(a_n) - a_n \in S_i^{\perp}. \tag{5.6}$$

Consequently, $(p_n)_{n\geq 0} \subset W$. Clearly, $a_0 - a_0 \in W$. Now suppose that, for some $n \in \mathbb{N}$, $a_n - a_0 \in W$. Then, since $a_{n+1} - a_0 = (a_n - a_0) + p_n$, we obtain $a_{n+1} - a_0 \in W$. Thus, we have proved by induction that

$$(a_n - a_0)_{n\geq 0} \subset W \tag{5.7}$$

and Proposition 5.2 ends the proof since W is boundedly compact.

(vii) Let $(\forall i \in I)\ S_i = \{a \in \Xi \mid \langle a \mid b_i\rangle \leq \kappa_i\}$ and define W as the vector subspace spanned by the finite family $(b_i)_{i\in I}$. Notice that

$$(\forall n \in \mathbb{N})(\forall i \in I_n) \qquad \begin{cases} S_{i,n} = \{a \in \Xi \mid \langle a \mid b_i\rangle \leq \kappa_{i,n}\}, \\ P_{i,n}(a_n) - a_n \propto b_i. \end{cases} \tag{5.8}$$

Therefore, repeating the same argument as in (vi), we observe that (5.7) holds. Proposition 5.2 then gives the announced result.

(viii) Let $(\forall i \in I)\ S_i = \bigcap_{j=1}^{J_i}\{a \in \Xi \mid \langle a \mid b_{i,j}\rangle \leq \kappa_{i,j}\}$ where $(J_i)_{i\in I} \subset \mathbb{N}$. Then the proof is similar to that of (vii) since, with exact projections, we can take W to be the vector subspace spanned by the finite family $((b_{i,j})_{1\leq j\leq J_i})_{i\in I}$. □

Remark 5.5. For the relaxation rule (1.11) and exact projections, parts (i)–(iii) of Theorem 5.3 were given in Corollary 5.1(iii) of [43]. Particular cases of Theorem 5.3(iv) appear in Example 6.1 of [8], which considered the relaxation rule (1.10), and in Corollary 1.2 of [14], which considered (1.1)–(1.2) with a compact set. Theorem 5.3(v) improves upon results of [2], [3], and [32].

Remark 5.6. Suppose that $(S_i)_{i\in I}$ is a finite family whose nonvoid subfamilies are all boundedly regular. Then strong convergence is achieved in the case of the chaotic iteration process (1.1)–(1.2) [7].

Remark 5.7. Suppose that Ξ is a euclidean space. According to Corollary 5.1(iv) and Corollary 5.2, EMOPP converges to a feasible point for any countable family of sets under chaotically coercive and admissible controls. Theorem 5.3(v) states that under chaotic control convergence holds for finite families of sets, while Example 4.1 shows that the condition card $I < +\infty$ cannot be eliminated.

References

1. Agmon S (1954) The relaxation method for linear inequalities. Canad J Math 6:382–392
2. Aharoni R, Berman A, Censor Y (1983) An interior points algorithm for the convex feasibility problem. Adv in Appl Math 4:479–489
3. Aharoni R, Censor Y (1989) Block-iterative methods for parallel computation of solutions to convex feasibility problems. Linear Algebra Appl 120:165–175
4. Alber Y, Butnariu D (1997) Convergence of Brègman-projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. J Optim Theory Appl 92:33–61
5. Amemiya I, Ando T (1965) Convergence of random products of contractions in Hilbert space. Acta Sci Math (Szeged) 26:239–244
6. Auslender A (1969) Méthodes Numériques pour la Résolution des Problèmes d'Optimisation avec Contraintes. Thèse, Faculté des Sciences, Grenoble
7. Bauschke HH (1995) A norm convergence result on random products of relaxed projections in Hilbert space. Trans Amer Math Soc 347:1365–1373
8. Bauschke HH, Borwein JM (1996) On projection algorithms for solving convex feasibility problems. SIAM Rev 38:367–426
9. Bauschke HH, Borwein JM, Lewis AS (1996) On the method of cyclic projections for closed convex sets in Hilbert space. To appear in Contemporary Mathematics: Optimization and Nonlinear Analysis (Censor Y, Reich S, editors). American Mathematical Society, Providence, RI
10. Brègman LM (1965) The method of successive projection for finding a common point of convex sets. Soviet Math Dokl 6:688–692
11. Brézis H (1983) Analyse Fonctionnelle. Masson, Paris
12. Browder FE (1958) On some approximation methods for solutions of the Dirichlet problem for linear elliptic equations of arbitrary order. J Math Mech 7:69–80
13. Browder FE (1967) Convergence theorems for sequences of nonlinear operators in Banach spaces. Math Z 100:201–225
14. Bruck RE (1982) Random products of contractions in metric and Banach spaces. J Math Anal Appl 88:319–332
15. Butnariu D, Flåm SD (1995) Strong convergence of expected-projection methods in Hilbert spaces. Numer Funct Anal Optim 16:601–636
16. Censor Y (1981) Row-action methods for huge and sparse systems and their applications. SIAM Rev 23:444–466
17. Censor Y, Lent A (1982) Cyclic subgradient projections. Math Programming 24:233–235
18. Cheney W, Goldstein AA (1959) Proximity maps for convex sets. Proc Amer Math Soc 10:448–450

19. Cimmino G (1938) Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari. Ricerca Sci (Roma) 1:326–333
20. Combettes PL (1993) The foundations of set theoretic estimation. Proc IEEE 81:182–208
21. Combettes PL (1995) Construction d'un point fixe commun à une famille de contractions fermes. C R Acad Sci Paris Sér I Math 320:1385–1390
22. Combettes PL (1996) The convex feasibility problem in image recovery. Advances in Imaging and Electron Physics (Hawkes P, editor), vol 95, pp 155–270. Academic Press, New York
23. Combettes PL, Puh H (1993) Extrapolated projection method for the euclidean convex feasibility problem. Research report, City University of New York
24. Combettes PL, Puh H (1994) Iterations of parallel convex projections in Hilbert spaces. Numer Funct Anal Optim 15:225–243
25. Combettes PL, Trussell HJ (1990) Method of successive projections for finding a common point of sets in metric spaces. J Optim Theory Appl 67:487–507
26. Crombez G (1991) Image recovery by convex combinations of projections. J Math Anal Appl 155:413–419
27. De Pierro AR, Iusem AN (1985) A parallel projection method for finding a common point of a family of convex sets. Pesqui Oper 5:1–20
28. De Pierro AR, Iusem AN (1988) A finitely convergent "row-action" method for the convex feasibility problem. Appl Math Optim 17:225–235
29. Deutsch F (1992) The method of alternating orthogonal projections. In Approximation Theory, Spline Functions and Application (Singh SP, editor), pp 105–121. Kluwer, The Netherlands
30. Dye JM, Reich S (1992) Unrestricted iterations of nonexpansive mappings in Hilbert space. Nonlinear Anal 18:199-207
31. Eremin II (1965) Generalization of the relaxation method of Motzkin–Agmon. Uspekhi Mat Nauk 20:183–187
32. Flåm SD, Zowe J (1990) Relaxed outer projections, weighted averages, and convex feasibility. BIT 30:289–300
33. Gurin (Gubin) LG, Polyak BT, Raik EV (1967) The method of projections for finding the common point of convex sets. USSR Comput Math and Math Phys 7:1–24
34. Halperin I (1962) The product of projection operators. Acta Sci Math (Szeged) 23:96–99
35. Iusem AN, De Pierro AR (1986) Convergence results for an accelerated nonlinear Cimmino algorithm. Numer Math 49:367–378
36. Iusem AN, Moledo L (1986) A finitely convergent method of simultaneous subgradient projections for the convex feasibility problem. Mat Apl Comput 5:169–184
37. Kaczmarz S (1937) Angenäherte Auflösung von Systemen linearer Gleichungen. Bull Acad Sci Pologne A35:355–357.
38. Kiwiel KC (1995) Block-iterative surrogate projection methods for convex feasibility problems. Linear Algebra Appl 215:225–259
39. Levitin ES, Polyak BT (1966) Convergence of minimizing sequences in conditional extremum problems. Soviet Math Dokl 7:764–767
40. Merzlyakov YI (1963) On a relaxation method of solving systems of linear inequalities. USSR Comput Math and Math Phys 2:504–510
41. Motzkin TS, Schoenberg IJ (1954) The relaxation method for linear inequalities. Canad J Math 6:393–404
42. Nashed MZ (1981) Continuous and semicontinuous analogues of iterative methods of Cimmino and Kaczmarz with applications to the inverse Radon transform. Lectures Notes in Medical Informatics (Herman GT, Natterer F, editors), vol 8, pp 160–178. Springer-Verlag, New York
43. Ottavy N (1988) Strong convergence of projection-like methods in Hilbert spaces. J Optim Theory Appl 56:433–461
44. Pierra G (1975) Méthodes de projections parallèles extrapolées relatives à une intersection de convexes. Rapport de Recherche, INPG, Grenoble
45. Pierra G (1984) Decomposition through formalization in a product space. Math Programming 28:96–115
46. Pierra G, Ottavy N (1988) New parallel projection methods for linear varieties and applications. Presented at the XIIth International Symposium on Mathematical Programming, Tokyo
47. Poincaré H (1890) Sur les équations aux dérivées partielles de la physique mathématique. Amer J Math 12:211–294

48. Polyak BT (1969) Minimization of unsmooth functionals. USSR Comput Math and Math Phys 9:14–29
49. Prâger M (1960) On a principle of convergence in Hilbert space. Czechoslovak Math J 10:271–282
50. Reich S (1983) A limit theorem for projections. Linear and Multilinear Algebra 13:281–290
51. Stiles WJ (1965) Closest point maps and their product II. Nieuw Arch Wisk 13:212–225
52. Tseng P (1992) On the convergence of products of firmly nonexpansive mappings. SIAM J Optim 2:425–434
53. Tseng P, Bertsekas DP (1987) Relaxation methods for problems with strictly convex separable costs and linear constraints. Math Programming 38:303–321
54. Von Neumann J (1949) On rings of operators. Reduction theory. Ann of Math 50:401–485 (the result of interest first appeared in 1933 in lecture notes)
55. Zarantonello EH (1971) Projections on convex sets in Hilbert space and spectral theory. In Contributions to Nonlinear Functional Analysis (Zarantonello EH, editor), pp 237–424. Academic Press, New York

Accepted 2 *October* 1995