Appl Math Optim 35:69–76 (1997)

Applied Mathematics & Optimization

A Class of Projection and Contraction Methods for Monotone Variational Inequalities*

Bingsheng He

Department of Mathematics, Nanjing University,
Nanjing 210008, People's Republic of China

Abstract. In this paper we introduce a new class of iterative methods for solving the monotone variational inequalities

$$u^* \in \Omega, \qquad (u - u^*)^T F(u^*) \geq 0, \qquad \forall u \in \Omega.$$

Each iteration of the methods presented consists essentially only of the computation of $F(u)$, a projection to Ω, $v := P_\Omega[u - F(u)]$, and the mapping $F(v)$. The distance of the iterates to the solution set monotonically converges to zero. Both the methods and the convergence proof are quite simple.

Key Words. Variational inequality, Monotone operator, Projection, Contraction.

AMS Classification. 90C30, 90C33, 65K05.

1. Introduction

Let Ω be a nonempty subset of R^n and let F be a mapping from R^n into itself. The variational inequality problem, denoted by VI(Ω, F), is to find a vector $u^* \in \Omega$ such that

$$\mathrm{VI}(\Omega, F) \quad F(u^*)^T(u - u^*) \geq 0, \qquad \forall u \in \Omega. \tag{1}$$

Variational inequalities play a significant role in mathematical programming and this subject has been studied by many researchers [1], [2], [9]–[11]. The interested reader

* This research is supported by the National Natural Science Foundation of the People's Republic of China and the Natural Science Foundation of Province Jiangsu.

may consult the survey paper by Harker and Pang [4] and the papers cited therein. In the last several years we have developed some projection and contraction methods for solving monotone linear variational inequalities (see [5]–[7]). Our objective in this paper is to offer a new class of projection and contraction methods for solving monotone variational inequalities, in which Ω is a closed convex set and the mapping F is continuous and monotone, i.e.,

$$[F(u) - F(v)]^T(u - v) \geq 0, \qquad \forall u, v \in R^n. \tag{2}$$

Throughout this paper we assume that the solution set, denoted by Ω^*, is nonempty and the projection on Ω, denoted by $P_\Omega(\cdot)$, is simple to carry out. In the following the Euclidean norm is denoted by $\|\cdot\|$, G denotes a symmetric positive definite matrix, and $\|u\|_G$ denotes $(u^T Gu)^{1/2}$.

2. Some Fundamental Inequalities

Let $P_\Omega(\cdot)$ denote the projection to Ω. A basic property of the projection mapping is

$$(v - P_\Omega(v))^T(P_\Omega(v) - u) \geq 0, \qquad \forall v \in R^n, \quad \forall u \in \Omega. \tag{3}$$

It is well known [3] that the variational inequality VI(Ω, F) is equivalent to the following projection equation:

$$\text{(PE)} \quad u = P_\Omega[u - F(u)], \tag{4}$$

i.e., to solve VI(Ω, F) is equivalent to finding a zero point of the residue function

$$e(u) := u - P_\Omega[u - F(u)]. \tag{5}$$

Let $u^* \in \Omega^*$ be a solution. For any $u \in R^n$, $P_\Omega[u - F(u)] \in \Omega$. It follows from (1) that

$$F(u^*)^T\{P_\Omega[u - F(u)] - u^*\} \geq 0, \qquad \forall u \in R^n. \tag{6}$$

Setting $v = u - F(u)$ in inequality (3), we obtain

$$\{e(u) - F(u)\}^T\{P_\Omega[u - F(u)] - u^*\} \geq 0, \qquad \forall u \in R^n. \tag{7}$$

Under the assumption that F is monotone we have

$$\{F(P_\Omega[u - F(u)]) - F(u^*)\}^T\{P_\Omega[u - F(u)] - u^*\} \geq 0, \qquad \forall u \in R^n. \tag{8}$$

Inequalities (6)–(8) play an important role in projection and contraction methods.

3. Methods for Monotone Variational Inequalities

In this section we consider how to construct some projection and contraction methods for monotone variational inequalities. Adding (6), (7), and (8), we obtain

$$\{e(u) - [F(u) - F(P_\Omega[u - F(u)])]\}^T\{(u - u^*) - e(u)\} \geq 0, \qquad \forall u \in R^n. \tag{9}$$

Denote

$$d(u) := e(u) - \{F(u) - F(P_\Omega[u - F(u)])\}. \tag{10}$$

It follows from (9) that

$$(u - u^*)^T d(u) \geq e(u)^T d(u). \tag{11}$$

For convenience, first, we assume that

$$[F(u) - F(v)]^T (u - v) \leq (1 - \delta)\|u - v\|^2, \qquad \forall u, v \in R^n, \tag{12}$$

with $\delta \in (0, 1)$. Under this assumption we have

$$\begin{aligned} e(u)^T d(u) &= \|e(u)\|^2 - e(u)^T \{F(u) - F(P_\Omega[u - F(u)])\} \\ &\geq \delta \|e(u)\|^2 \end{aligned} \tag{13}$$

and via (11) it follows that

$$(u - u^*)^T d(u) \geq \delta \|e(u)\|^2, \qquad \forall u \in R^n. \tag{14}$$

Based on inequality (11), we can construct a class of projection and contraction (PC) methods as follows.

PC Methods for Monotone VI (under assumption (12)).
Let $\gamma \in (0, 2)$ be a constant and let G be a symmetric and positive definite matrix. Given an arbitrary u^0. For $k = 0, 1, \ldots$, if $u^k \notin \Omega^*$, then

$$u^{k+1} = u^k - \gamma \rho(u^k) g(u^k), \tag{15}$$

where

$$g(u^k) = G^{-1} d(u^k) \tag{16}$$

and

$$\rho(u^k) = \frac{e(u^k)^T d(u^k)}{\|g(u^k)\|_G^2}. \tag{17}$$

If we take $G = I$, then each iteration of the method consists essentially of only the computation of $F(u)$, a projection $v := P_\Omega[u - F(u)]$, and the mapping $F(v)$. We call it a projection and contraction method because in each iteration a projection has to be carried out and the distance of the iterates to the solution set monotonically converges to zero.

Theorem 1. *The sequence $\{u^k\}$ generated by the PC methods for monotone variational inequality satisfies*

$$\|u^{k+1} - u^*\|_G^2 \leq \|u^k - u^*\|_G^2 - \gamma(2 - \gamma)\rho(u^k) e(u^k)^T d(u^k), \qquad \forall u^* \in \Omega^*. \tag{18}$$

Proof. Using (11), (13), and (14)–(17) by a simple computation. □

Note that, for fixed G, it is possible to prove that the steplength ρ is bounded below. Therefore, there is a constant $\tau > 0$ (depend on γ, G, and δ), so that the sequence $\{u^k\}$ generated by each projection and contraction method satisfies

$$\|u^{k+1} - u^*\|_G^2 \le \|u^k - u^*\|_G^2 - \tau \cdot \|e(u^k)\|^2, \qquad \forall u^* \in \Omega^*. \tag{19}$$

As in [6], from inequality (19), it is easy to prove that the PC methods are globally convergent if the solution set is nonempty.

For a general continuous monotone mapping F, assumption (12) may not be satisfied. Note that the variational inequality $\mathrm{VI}(\Omega, F)$ is invariant under multiplication F by some positive scalar β. We denote

$$e(u, \beta) = u - P_\Omega[u - \beta F(u)] \tag{20}$$

and

$$d(u, \beta) = e(u, \beta) - \beta[F(u) - F(P_\Omega(u - \beta F(u)))]. \tag{21}$$

It follows that (see (11))

$$(u - u^*)^T d(u, \beta) \ge e(u, \beta)^T d(u, \beta). \tag{22}$$

Because the mapping F is continuous, we can use Armijo's rule to find a $\beta_k > 0$, such that

$$\beta_k\{F(u^k) - F(P_\Omega[u^k - \beta_k F(u^k)])\}^T e(u^k, \beta_k) \le (1 - \delta)\|e(u^k, \beta_k)\|^2. \tag{23}$$

An equivalent expression of (23) is

$$e(u^k, \beta_k)^T d(u^k, \beta_k) \ge \delta\|e(u^k, \beta_k)\|^2. \tag{24}$$

In practice, we use the following methods.

PC Methods with Armijo's Linesearch (without assumption (12)).

Let $\gamma \in (0, 2)$, $\alpha, \delta \in (0, 1)$, and $\beta > 0$ be constant.
Given an arbitrary u^0. For $k = 0, 1, \ldots$, if $u^k \notin \Omega^*$, then
$\beta_k := \beta$,
While $e(u^k, \beta_k)^T d(u^k, \beta_k) < \delta\|e(u^k, \beta_k)\|^2$ **do** $\beta_k := \alpha\beta_k$ **end**,
$\beta := \beta_k$,
Set

$$u^{k+1} = u^k - \gamma\rho(u^k, \beta)g(u^k, \beta), \tag{25}$$

where

$$g(u^k, \beta) = G^{-1}d(u^k, \beta) \tag{26}$$

and

$$\rho(u^k, \beta) = \frac{e(u^k, \beta)^T d(u^k, \beta)}{\|g(u^k, \beta)\|_G^2}. \tag{27}$$

Corollary 1. *The sequence $\{u^k\}$ generated by the PC methods with linesearch for monotone variational inequality satisfies*

$$\|u^{k+1} - u^*\|_G^2 \le \|u^k - u^*\|_G^2 - \gamma(2-\gamma)\delta \cdot \rho(u^k, \beta_k)\|e(u^k, \beta_k)\|^2, \qquad \forall u^* \in \Omega^*. \tag{28}$$

Proof. Using (22) and (24)–(27) by a simple computation. □

Because the sequence $\{u^k\}$ generated by any contraction method is bounded and the mapping F is continuous, it is possible to prove that there is a $\beta_{\min} > 0$ such that, for all k,

$$\beta_k \ge \beta_{\min}$$

and the PC method with Armijo's linesearch is well defined. Based on Corollary 1 we can prove that the methods are globally convergent.

4. Relationship to Some Existing PC Methods

In the last several years we have developed some projection and contraction methods for monotone linear variational inequalities (see [5]–[7]). If F is a monotone affine mapping, then $F(u) = Mu + q$, $q \in R^n$, and $M \in R^{n\times n}$ is a positive semidefinite matrix.

The method in [5] is based on using inequality (6), which can be rewritten as

$$\{(Mu+q) - M(u-u^*)\}^T\{u - u^* - e(u)\} \ge 0. \tag{29}$$

It follows that

$$(u-u^*)^T\{M^Te(u) + (Mu+q)\} \ge e(u)^T(Mu+q). \tag{30}$$

Because

$$e(u)^T(Mu+q) \ge \|e(u)\|^2, \qquad \forall u \in \Omega,$$

the search direction of the method in [5] is based on

$$d(u) := M^Te(u) + (Mu+q).$$

The methods in [6] and [7] are based on adding inequality (6) and (7), which yields

$$\{e(u) - M(u-u^*)\}^T\{(u-u^*) - e(u)\} \ge 0. \tag{31}$$

From (31) it follows that

$$(u-u^*)^T(I+M^T)e(u) \ge \|e(u)\|^2 + (u-u^*)^TM(u-u^*), \qquad \forall u \in R^n. \tag{32}$$

Based on inequality (32) we constructed a class of projection and contraction methods [6], [7]. The search directions of these methods are

$$g_l(u) = G^{-1}(I+M^T)e(u), \tag{33}$$

which can be viewed as straightforward extensions of the directions in traditional methods for unconstrained optimization (see [7]). The recursion

$$u^{k+1} = u^k - \rho_l(u^k) g_l(u^k) \tag{34}$$

with

$$\rho_l(u) = \frac{\|e(u)\|^2}{\|g_l(u)\|_G^2} \tag{35}$$

produces a sequence $\{u^k\}$, which is *not necessarily* contained in the feasible set Ω, but satisfies

$$\|u^{k+1} - u^*\|_G^2 \le \|u^k - u^*\|_G^2 - \rho_l(u^k)\|e(u^k)\|^2. \tag{36}$$

All projection and contraction methods for monotone linear variational inequalities in [5]–[7] are minimization methods without linesearch and their implementations are very simple. In general, for monotone linear variational inequalities, instead of the methods in Section 3 of this paper, we prefer to use the methods presented in [5]–[7], which do not need linesearch. However, it seems that the methods in [5]–[7] are not applicable for general monotone variational inequalities.

The extra gradient method, which was proposed by Korpelevich [8], is applicable for solving monotone variational inequalities. Under the assumption that

$$\|F(u) - F(v)\| \le L\|u - v\|, \tag{37}$$

his iterative scheme is

$$\hat{u}^k = P_\Omega[u^k - \beta F(u^k)],$$

$$u^{k+1} = P_\Omega[u^k - \beta F(\hat{u}^k)]$$

with a constant $0 < \beta < 1/L$. For convenience, we can assume that $L < 1$ and then take $\beta = 1$. In this case Korpelevich's scheme may be written as

$$u^{k+1} = P_\Omega[u^k - g_K(u^k)] \tag{38}$$

with

$$g_K(u) = F(P_\Omega[u - F(u)]). \tag{39}$$

Although the convergence analysis of the extra gradient method in [8] is different from the one in our papers, we can see that Korpelevich's search direction is based on adding inequalities (6) and (8), which yields

$$F(P_\Omega[u - F(u)])^T\{(u - u^*) - e(u)\} \ge 0 \tag{40}$$

and it follows that

$$(u - u^*)^T F(P_\Omega[u - F(u)]) \ge e(u)^T F(P_\Omega[u - F(u)]). \tag{41}$$

For $u \in \Omega$, it follows from (3) that $e(u)^T F(u) \geq \|e(u)\|^2$ and

$$\begin{aligned} e(u)^T F(P_\Omega[u - F(u)]) &= e(u)^T F(u) - e(u)^T \{F(u) - F(P_\Omega[u - F(u)])\} \\ &\geq e(u)^T F(u) - \|e(u)\| \cdot \|F(u) - F(P_\Omega[u - F(u)])\| \\ &\geq (1 - L)\|e(u)\|^2. \end{aligned} \tag{42}$$

Therefore, under the assumption $L < 1$, the direction $-g_{\kappa}(u)$ is a descent direction of the function $\|u - u^*\|^2$ for $u \in \Omega$.

It is clear that the efficiency of Korpelevich's method depends on the estimation of the Lipschitz constant. Since a suitable estimation of the Lipschitz constant even in the linear case is expensive, Sun's modified method in [12], using Armijo's one-dimensional research, was a contribution to making Korpelevich's method applicable in practice. However, for ill-conditioned problems, the direction based on extragradient may lead to very slow convergence, because we cannot expect the extragradient method to be better than a method of the steepest descent type.

In the methods presented in this paper, we use the direction $g(u, \beta) = G^{-1}d(u, \beta)$, which is based on adding the fundamental inequalities (6), (7), and (8). We would like to point out that the computational amount of

$$g_{\kappa}(u) = F(P_\Omega[u - F(u)]) \qquad \text{(in Korpelevich's method)}$$

and

$$d(u) = u - P_\Omega[u - F(u)] - F(u) + F(P_\Omega[u - F(u)]) \qquad \text{(in our method)}$$

is almost equal. Under assumption (37), the inequality

$$(u - u^*)^T g_{\kappa}(u) \geq e(u)^T F(P_\Omega[u - F(u)]) \geq (1 - L)\|e(u)\|^2$$

is true only for $u \in \Omega$, and the sequence $\{u^k\}$ generated by Korpelevich's method (and the modified method by Sun [12]) must be contained in Ω. However, the inequality

$$(u - u^*)^T d(u) \geq e(u)^T d(u) \geq (1 - L)\|e(u)\|^2$$

is true for all $u \in R^n$, and the sequence $\{u^k\}$ generated by our methods is *not necessarily* contained in Ω. The direction $-g(u, \beta) = -G^{-1}d(u, \beta)$ is a descent direction of the function $\|u - u^*\|_G^2$ for all $u \in R^n$. This offers us more possibilities (by choosing different G) of constructing better search directions and more efficient methods (see [7] for example).

References

1. Bruck, R. E. (1975), An iterative solution of a variational inequality for certain monotone operators in Hilbert space, Bulletin of the American Mathematical Society, 81:890–892.
2. Dafermos, S. (1983), An iterative scheme for variational inequalities, Mathematical Programming, 26:40–47.
3. Eaves, B. C. (1971), On the basic theorem of complementarity, Mathematical Programming, 1:68–75.
4. Harker, P. T., Pang, J. S. (1990), Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications, Mathematical Programming, 48:161–220.

5. He, B. S. (1992), A projection and contraction method for a class of linear complementarity problems and its application in convex quadratic programming, Applied Mathematics and Optimization, 25:247–262.
6. He, B. S. (1994), A new method for a class of linear variational inequalities, Mathematical Programming, 66:137–144.
7. He, B. S. (1994), Solving a class of linear projection equations, Numerische Mathematik, 68:71–80.
8. Korpelevich, G. M. (1976), The extragradient method for finding saddle points and other problems, Matekon, 12:747–756.
9. Pang, J. S. (1985), Asymmetric variational inequality problems over product sets: applications and iterative methods, Mathematical Programming, 31:206–219.
10. Pang, J. S. (1986), Inexact Newton methods for the nonlinear complementarity problem, Mathematical Programming, 36:54–71.
11. Pang, J. S., Chan, D. (1982), Iterative methods for variational and complementarity problems, Mathematical Programming, 24:284–313.
12. Sun, D. F. (1994), A projection and contraction method for the nonlinear complementarity problem and its extensions, Mathematica Numerica Sinica, 16:183–194.

Accepted 25 *July* 1995