NONLINEAR ELASTIC STATE OF THIN-WALLED TOROIDAL SHELLS MADE OF ORTHOTROPIC COMPOSITES

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A study is made of problems on the statics of circular toroidal shells made of nonlinear elastic orthotropic composites. The study is conducted on the basis of the method of successive approximation, the variational-difference method, and the method of Lagrangian multipliers. The parameters of the circular torus are varied within broad ranges of values in the calculations. Numerical results are presented in the form of tables and graphs and are analyzed.

The theoretical principles which frame research into the stress and strain states of the load-bearing elements of modern thin-walled structures were developed mainly for the design of elastic isotropic and anisotropic toroidal shells; simplified approaches and more rigorous efficient analytical and numerical methods have been used to perform numerous calculations for one-layered and multi-layered segments of such shells with circular and noncircular cross sections [1, 2, 5, 9, 10, 13, 14, etc.]. In recent years, increasing attention has been given to investigation of the stress–strain state and stability of shells of the given type with allowance for geometric and physical nonlinearities. Specific numerical results on the strain and stress fields created by moderate and severe bending have been obtained for circular isotropic [3, 8] and orthotropic [4, 6] toroidal shells subjected to axisymmetric loading by internal or external pressure. A few numerical studies have been conducted for isotropic shells of circular cross section with allowance for physical nonlinearity (plastic strains) of their materials [10, 11].

It is also interesting to examine the distribution of the displacements, strains, and stresses in orthotropic toroidal shells made of nonlinearly elastic composites. The main theoretical principles guiding the study of the stress–strain state of shallow thin-walled shells of arbitrary form with allowance for the physical nonlinearity were presented in [7, 12], which included the main nonlinear resolvent equations and a method for solving them. The equations and the method were used to examine nonlinear problems for circular toroidal shells, obtain specific numerical results, and analyze those results.

We examined thin nonshallow toroidal shells which in the general case have a cross section described by a closed smooth plane curve $F(x, y) = 0$. The case of such a shell with a circular cross section is depicted in Fig. 1, where a is the radius of the circle and c is the distance from the center of the cross section of the torus to the axis of revolution. In this case, the equation of the curve takes the form

$$F(x, y) = \left(\frac{x-c}{a}\right)^2 + \left(\frac{y}{a}\right)^2 - 1 = 0. \tag{1}$$

It is assumed that the shell is of variable (or constant) thickness h and is made of an orthotropic composite (glass- or organic-fiber-reinforced plastic) which has specified physico-mechanical characteristics and corresponding stress–strain curves [7]. The shell is subjected to a distributed surface load. The effect of edge forces and moments on the shell will also be considered when one of its segments is being studied.

We note that the main geometric and physical relations for thin-walled shells of revolution of arbitrary form were obtained in [7] on the basis of the moment theory of orthotropic shells and the nonlinear theory of elasticity and plasticity of anizotropic media. To obtain the system of resolvent equations, we used linear geometric and physically nonlinear relations from these theories and a variant of the variational-difference method in which the mixed functional has the form

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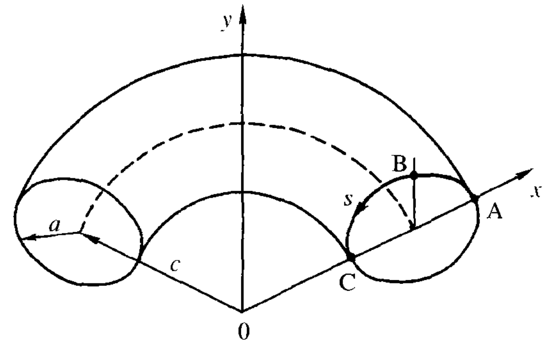

Fig. 1

$$\Pi\left(u_1, u_2, u_3, \varphi_1, \varphi_2, T^f_{13}, T^f_{23}\right) = \iint\limits_{\Omega} A\,(u_1, u_2, u_3, \varphi_1, \varphi_2)\, d\,\Omega - A_n - A_k$$

$$+ \iint\limits_{\Omega} (T^f_{13}\, \varepsilon_{13} + T^f_{23}\, \varepsilon_{23})\, d\,\Omega\,, \tag{2}$$

where A is strain energy density such that

$$\delta A = T_{11}\,\delta\,\varepsilon_{11} + T_{22}\,\delta\,\varepsilon_{22} + T_{12}\,\delta\,\varepsilon_{12} + M_{11}\,\delta\,\kappa_{11} + M_{22}\,\delta\,\kappa_{22} + 2\,M_{12}\,\delta\,\kappa_{12}\,,$$

A_n and A_k represent the work of the external surface forces and edge forces; $\{U\} = \{u_1, u_2, u_3, \varphi_1, \varphi_2, T^f_{13}, T^f_{23}\}$ is the vector of the unknown functions; in the given case, the Kirchhoff–Love hypotheses are realized in (2) by the method of undetermined Lagrangian multipliers (T^f_{13}, T^f_{23}); in physical terms, the multipliers correspond to the longitudinal shearing forces.

In the case of a plane stress state, the physical relations are determined by the equations

$$e_{11} = \left(\frac{1}{E_{11}} + \Psi\, q_{1111}\right)\sigma_{11} + \left(-\frac{\nu_{12}}{E_{22}} + \Psi\, q_{1122}\right)\sigma_{22}\,, \quad (1 \leftrightarrow 2\,),$$

$$e_{12} = \left(\frac{1}{G_{12}} + 4\,\Psi\, q_{1212}\right)\sigma_{12}\,, \tag{3}$$

where E_{11}, E_{22}, G_{12}, and ν_{12} are the elastic constants of the composite; $\Psi\,(f\,)$ is the function describing the nonlinear deformation of the orthotropic material; q_{1111}, q_{2222}, q_{1122}, q_{1212} is the tensor accounting for the anisotropy of the nonlinear properties of the composite; f is the quadratic stress function [7, 12].

We have the following equations as the geometric relations linking the components of the strain tensor $e_{ij}\,(i = 1, 2\,;\; j = 1, 2, 3\,)$ and the displacement vector in the given system $(\alpha_1, \alpha_2, \alpha_3)$:

$$e_{11} = \varepsilon_{11} + \alpha_3\,\kappa_{11}, \quad e_{12} = \varepsilon_{12} + 2\,\alpha_3\,\kappa_{12}, \quad e_{13} = \varepsilon_{13} \quad (1 \leftrightarrow 2\,), \tag{4}$$

where the below equations are used to express the strains in the middle surface of the shell

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + k_1\, u_3, \quad \kappa_{11} = \frac{1}{A_1}\frac{\partial \varphi_1}{\partial \alpha_1} + \frac{\varphi_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\,,$$

$$\varepsilon_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right), \quad 2\,\kappa_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\varphi_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\varphi_2}{A_2}\right),$$

$$\varepsilon_{13} = \varphi_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} - k_1 u_1, \qquad (1 \leftrightarrow 2), \tag{5}$$

A_i ($i = 1, 2$) are the coefficients of the first quadratic form; k_i represents the curvatures of the middle surface.

Geometric relations (5) outwardly appear to be the same as the relations for hypotheses of the Timoshenko type. However, in the case of the Kirchhoff–Love hypotheses. the last expressions in (5) are used to determine the angles φ_1 and φ_2 from conditions specifying that the transverse shear strains are equal to zero

$$\varepsilon_{13} = \varepsilon_{23} = 0 \tag{6}$$

with the functional (2) being varied over T_{13}^f, T_{23}^f.

Since the longitudinal shearing forces T_{13}^f and T_{23}^f are considered to be independent functions, we can treat them in a manner similar to the displacements and assign them values before we construct the functional on the part of the boundary of the shell where they are known for certain. However, hypothesis (6) will already be violated on this boundary, since here $\varepsilon_{i3}\,\delta T_{i3}^f = 0$. Despite this, $T_{i3}^f\,\delta\,\varepsilon_{i3}$ still contributes to the virtual work and the energy $T_{i3}^f\varepsilon_{i3}$. It is best to assign $T_{i3}^f = 0$ on the line of symmetry $\alpha_i = \text{const}$, since it is always the case that $\varepsilon_{i3} = 0$ on that line.

We will use Lagrange's variational equation $\delta\,\Pi = 0$ with allowance for Eqs. (2)–(5). Then in accordance with the method being used to solve the given class of problems — which is based on the method of successive approximation in combination with the variational-difference approach — we assume that the linear parts of the main equations are known from the previous approximation and are not varied. We ultimately arrive at a system of linear algebraic equations [7] which we represent as follows at point (i, j) in the next approximation:

$$\sum_{k=i-1}^{i+1}\ \sum_{l=j-1}^{j+1}\ \sum_{n=1}^{7} A_{mn}(k,l)\,U_n(k,l) = \Delta V(i,j)\,q_m(i,j) + \Delta s(i,j)\,T_{\nu m}(i,j) + \Phi_m(i,j),$$

$$m = \overline{1,7}, \quad i = \overline{1,K^I}, \quad j = \overline{1,K^J}, \tag{7}$$

where $[A]$ is the symmetric band matrix of the variable coefficients; $\{\Phi\}$ is the vector of the nonlinear terms; $\{q\}$ and $\{T_\nu\}$ are the vectors of the components of the surface load and the edge load; ΔV and Δs are the discrete analogs of the differentials of the surface and the arc; K^I and K^J represent the number of nodes of the grid along the axes α_1 and α_2, respectively.

Negative numbers may appear on the diagonal in the case of mixed functionals. The system is then solved by Kholetskii's method in a variant of the square-root method, since this method is more stable. The problem of the possible appearance of $\sqrt{-1}$ on the diagonal is circumvented algorithmically by reducing the equation by $\sqrt{-1}$. Although the theorem on the stability of Kholetskii's method has been proven for positive-definite matrices, experience shows that the method also works in the case being discussed.

To calculate the nonlinear terms $\{\Phi\}$ in (7), it is necessary to solve nonlinear physical relations (2) for the stresses. We represent them in the form

$$F_i(\{\sigma\}, \{e\}) = 0, \quad (i = \overline{1,3}), \tag{8}$$

where $\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$, $\{e\} = \{e_{11}, e_{22}, e_{12}\}^T$.

We will solve nonlinear system (8) numerically by Newton's method

$$\{\sigma^{j+1}\} = \{\sigma^j\} + \{\Delta\,\sigma^j\}, \tag{9}$$

where the increments $\{\Delta\,\sigma^j\}$ for the jth step are found from the solution of the linear algebraic system of equations

TABLE 1

$\tilde{s}$	$\tilde{u}_s$ $\tilde{u}_\gamma$	$\tilde{\gamma}$	$e_{ss} \cdot 10^2$	$e_{\theta\theta} \cdot 10^2$	σ_{ss}	$\sigma_{\theta\theta}$
0	0.0 1.190	0.5 –0.5	0.566 0.573	0.396 0.397	82.8 83.7	49.8 49.9
0.1	–0.191 1.168	0.5 –0.5	0.571 0.578	0.396 0.397	83.4 84.3	49.8 49.9
0.2	–0.367 1.107	0.5 –0.5	0.585 0.592	0.395 0.396	85.1 86.0	49.7 49.8
0.3	–0.512 1.023	0.5 –0.5	0.601 0.627	0.393 0.392	87.1 90.1	49.4 49.5
0.4	–0.639 1.120	0.5 –0.5	0.624 0.683	0.417 0.411	89.7 96.5	51.8 51.2
0.5	–0.840 1.510	0.5 –0.5	0.743 0.674	0.426 0.418	103.3 95.6	52.7 52.0
0.6	–1.064 1.255	0.5 –0.5	0.807 0.773	0.371 0.376	110.3 106.6	47.3 47.9
0.7	–1.081 0.553	0.5 –0.5	0.885 0.917	0.391 0.394	118.5 121.8	49.2 49.5
0.8	–0.868 0.0513	0.5 –0.5	1.024 1.041	0.397 0.397	132.3 133.9	49.6 49.5
0.9	–0.484 –0.280	0.5 –0.5	1.144 1.163	0.400 0.397	143.4 145.1	49.6 49.3
1	0.0 –0.399	0.5 –0.5	1.196 1.218	0.401 0.397	148.1 150.0	49.6 49.2

$$F_i + \frac{\partial F_i}{\partial\, {}^{l}\sigma^{j}{}_{l}}\, {}^{l}\Delta \sigma^{j}{}_{l} = 0, \tag{10}$$

with the stresses ${}_{l}\sigma^{1} = {}_{l}\sigma^{L}{}_{l}$ for a linear elastic body being chosen as the initial approximation. Iteration process (9)–(10) is continued until we obtain agreement between the maximum absolute values of the stress components in two successive iterations with a prescribed relative error.

Thus, studies of the nonlinear elastic deformation of a shell of the specified form reduces to the solution of a sequence of systems of equations of type (7) with the corresponding edge (boundary) conditions and the initial geometric parameters of the shell expressed in terms of the physico-mechanical characteristics of its material. The algorithm that has been developed for numerically solving nonlinear problems and the program that was written to calculate the stress–strain state of shells of the given type make it possible to evaluate the components of the displacement vector, the angles of rotation and shearing forces ${}_{l}U_{l}$, the strain tensors e_{ij} $(i, j = 1, 2)$, and the stresses σ_{ij} with allowance for changes in the parameters of the shell, the magnitude and nature of the load, the properties of the material, and the type of boundary conditions.

We will present specific numerical results for a toroidal shell (Fig. 1) referred to a curvilinear coordinate system (s, θ, γ) in which the orthotropy axis of the material coincides with the coordinate lines of the chosen system. The shell has

TABLE 2

$\tilde{s}$	$\tilde{u}_s$ $\tilde{u}_\gamma$	$\tilde{\gamma}$	$e_{ss} \cdot 10^2$	$e_{\theta\theta} \cdot 10^2$	σ_{ss}	$\sigma_{\theta\theta}$
0	0.0 1.048	0.5 –0.5	0.513 0.519	0.349 0.350	82.8 83.8	49.7 50.0
0.1	–0.164 1.029	0.5 –0.5	0.516 0.523	0.348 0.349	83.4 84.4	49.7 50.0
0.2	–0.314 0.975	0.5 –0.5	0.528 0.534	0.346 0.347	85.1 86.1	49.7 49.9
0.3	–0.439 0.902	0.5 –0.5	0.541 0.562	0.342 0.342	87.0 90.3	49.4 49.7
0.4	–0.547 0.968	0.5 –0.5	0.559 0.602	0.357 0.353	89.9 96.5	51.5 51.6
0.5	–0.707 1.238	0.5 –0.5	0.649 0.595	0.358 0.352	103.7 95.4	52.9 51.4
0.6	–0.871 0.997	0.5 –0.5	0.697 0.675	0.309 0.313	110.3 106.9	47.6 47.8
0.7	–0.867 0.440	0.5 –0.5	0.750 0.776	0.315 0.318	118.4 122.4	49.2 49.8
0.8	–0.687 0.488	0.5 –0.5	0.844 0.857	0.309 0.308	132.5 134.5	49.7 49.9
0.9	–0.380 –0.205	0.5 –0.5	0.919 0.934	0.301 0.299	143.9 146.1	49.9 49.8
1	0.0 –0.296	0.5 –0.5	0.951 0.967	0.297 0.294	148.7 151.1	49.9 49.7

the following parameters: $\tilde{a} = a/h = 100$; $\tilde{c} = c/h = 125 \ldots 2000$. The shell is made of a composite with the following characteristics: $E_{ss} = 15$ GPa; $E_{\theta\theta} = 12$ GPa; $\nu_{s\theta} = 0.12$. The stress–strain curves, the function $\Psi(f)$, and the other quantities in (3) were presented in [7]. The shell is subjected to a surface load (internal pressure) of specified intensity ($q_3 = \text{const}$).

To perform the calculations, the shell was divided into a series of nodal points along its meridian ($K^s = 200$) and through its thickness ($K^\gamma = 9$). Symmetry conditions were adopted at the points $\tilde{s} = s/\pi a = 0$ and $\tilde{s} = 1$ with allowance for $T^f_{s\gamma} = 0$.

Below, we present specific numerical data from calculations of the stress–strain state of a toroidal shell with the following parameters: $\tilde{c} = 200$; ($\tilde{a} = 100$); $q_3 = 1$ MPa. The nonlinear problems were solved with the use of the main equations (2)–(7), in addition to the corresponding stress–strain curves and the strain-hardening function of the material. Table 1 shows numerical results on the distribution of components of the displacements ($\tilde{u}_\rho = u_\rho/h$, $\rho = s, \gamma$), strains ($e_{ss}, e_{\theta\theta}$), and stresses $\sigma_{ss}, \sigma_{\theta\theta}$ (MPa) along the meridian of the shell $\tilde{s}\,(0 \le \tilde{s} \le 1)$ with allowance for the nonlinear properties of its material. The values of the displacements $\tilde{u}_s$ and $\tilde{u}_\gamma$ correspond to points of the middle surface of the shell, while the values of the strains and stresses are shown for points of the outside and inside surfaces ($\tilde{\gamma} = \gamma/h = \pm 0.5$). Table 2 shows the results of the solution of the linearly elastic problem for the given orthotropic shell.

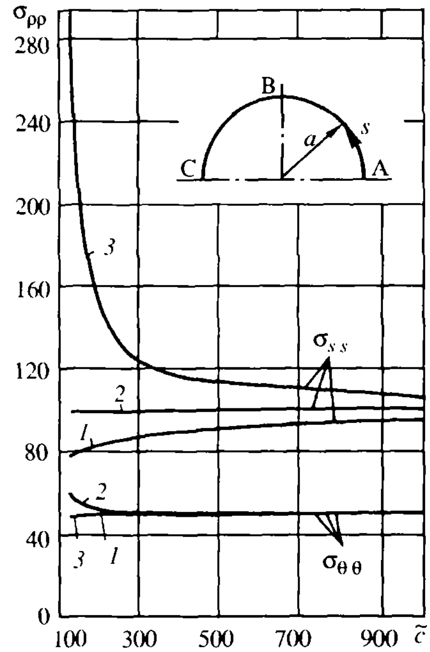

Fig. 2

TABLE 3

q_3	$\tilde{s}$	$\tilde{u}_\gamma$	$e_{ss} \cdot 10^2$	$e_{\theta\theta} \cdot 10^2$	σ_{ss}	$\sigma_{\theta\theta}$
1.0	0	1.19	0.57	0.39	83.3	49.8
	0.5	1.51	0.71	0.42	95.5	52.3
	1	–0.39	1.21	0.39	149.1	49.4
1.5	0	2.06	0.96	0.68	125.0	74.7
	0.5	2.76	1.23	0.76	149.0	78.7
	1	–0.78	2.26	0.78	223.0	73.8
2.0	0	3.26	1.47	1.09	166.6	99.6
	0.5	4.56	1.94	1.25	198.8	105.2
	1	-1.38	3.82	1.38	295.8	97.8

A comparative analysis of the numerical data (Tables 1 and 2) shows that accounting for the physical nonlinearity of the orthotropic material of the circular toroidal shell has a negligible effect on the stress distribution for the given type of loading. The largest nonlinear displacements $\tilde{u}_\gamma$ in the sections ($\tilde{s}$ = 0; 0.5; 1) change by 13, 22, and 33 % respectively, while the maximum values of the strains e_{ss} and $e_{\theta\theta}$ in the same sections on the inside surface of the shell ($\tilde{\gamma} = -0.5$) increase by 11, 13, and 36 % (e_{ss}) and 13, 18, and 34 % ($e_{\theta\theta}$) compared to the solution of the problems in the linear formulation.

The stress–strain state of the shell ($\tilde{a}$ = 100; $\tilde{c}$ = 200) when there is a change in the acting load (q_3) was studied on the basis of the nonlinear formulation of the problems; the results are shown in Table 3 in the form of values of the displacements $\tilde{u}_\gamma$, strains e_{ij}, and stresses σ_{ij} ($i, j = s, \theta$) calculated at three characteristic points of the meridian ($\tilde{s}$ = 0; 0.5; 1) of the shell on its middle surface ($\tilde{\gamma}$ = 0). Here, the solutions of the nonlinear problems with the indicated values of the loads (q_3 = 1.0; 1.5; 2.0) were obtained in 5, 8, and 11 approximations, respectively, for a relative error of the strains $\varepsilon = 10^{-2}$. It is apparent that the cross section of the shell changes disproportionately with an increase in the load; the deflection $\tilde{u}_\gamma$ at the point $\tilde{s}$ = 0.5 increases roughly threefold with a twofold change in the load.

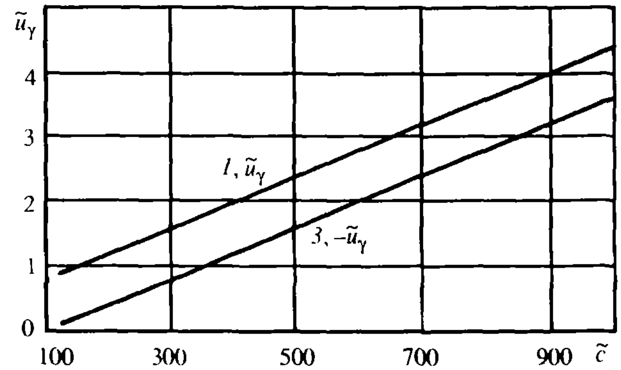

Fig. 3

It is interesting to study the laws which govern the distribution of the components of the displacement vector and the stress and strain tensors in a nonlinearly elastic shell with a change in the parameter $\tilde{c}$, which is connected with an increase in the radius of the circular axis of the torus ($100 < \tilde{c} \le 2000$). The results of solution of the nonlinear problems are shown in Figs. 2 and 3 for a specified load (q_3 = 1 MPa). Data from calculations of the maximum stresses is presented in Fig. 2, which shows the changes in the meridional and hoop stresses ($\sigma_{\rho\rho}$, MPa, $\rho = s, \theta$) at three characteristic points of the shell ($\tilde{\gamma} = 0$) with an increase in the parameter c. Curves *1*, *2*, and *3* correspond to the values of the stresses at the points A ($\tilde{s}$ = 0), B ($\tilde{s}$ = 0.5), and C ($\tilde{s}$ = 1). Figure 3 shows graphs of the distribution of the displacements $\tilde{u}_\gamma$ at points A and C (curves *1*, *3*).

The data shows that the meridional stresses (σ_{ss}) in the shell are the largest stresses; their values are considerably greater than the values of $\sigma_{\theta\theta}$; the value of σ_{ss} decreases sharply ($100 < \tilde{c} \le 250$) with an increase in the parameter $\tilde{c}$, while these stresses change more slowly in the interval $250 < \tilde{c} \le 2000$ and approach the values characteristic of the circular cylindrical shell. At the same time, the stresses σ_{ss} at point B do not change with an increase in the parameter $\tilde{c}$, while they increase continuously at point A; $\sigma_{\theta\theta}$ = (76–97) MPa. The dependence of the displacements $\tilde{u}_\gamma$ on the parameter $\tilde{c}$ is linear in character.

REFERENCES

1. V. E. Bazanov, "Axial shear strain of toroidal shells," in: *Three-Dimensional Structures in the Krasnoyarsk Region* [in Russian] (1987), pp. 95–102.
2. S. P. Gavelya, P. F. Kul'bashnyi, Yu. A. Sysoev, and V. S. Lityuga, *Comparison of Results of Calculation of the Strength of Toroidal Shells and Experimental Data*. Zaporozh'e (1989). Submitted to UkrNIINTI 15.12.89. No. 990-UK89.
3. V. V. Gaidachuk, V. E. Gotsulyak, and V. I. Gulyaev, "Branching of the solutions of nonlinear equations of toroidal shells under external pressure," *Prikl. Mekh.*, **14**, No. 9, 38–46 (1978).
4. M. S. Ganeeva and L. V. Kosolapova, "Nonlinear bending and stability of orthotropic toroidal shells under uniform external pressure," *Strength and Stability of Shells: Transactions of a Seminar*, Kazan (1986), Vol. 19, Part 1, pp. 23–36.
5. Ya. M. Grigorenko and A. T. Vasilenko, *Theory of Shells of Variable Stiffness*, Vol. 4 of the five-volume *Methods of Shell Design* [in Russian], Nauk. Dumka, Kiev (1981).
6. N. Ya. Kar'yagdyev and A. P. Mukoed, "Axisymmetric deformation of flexible orthotropic toroidal shells," *Vychisl. Prikl. Mat.*, **65**, 17–22 (1988).
7. A. N. Guz', A. S. Kosmodamianskii, V. P. Shevchenko, et al., *Stress Concentration*, Vol. 7 of the 12-volume *Mechanics of Composites* [in Russian], A.S.K., Kiev (1988).
8. A. V. Korovaitsev and A. Yu. Evkin, "Axisymmetric deformation of a toroidal shell during severe bending," *Prikl. Mekh.*, **28**, No. 4, 16–23 (1992).

9. Yu. B. Nedeshev and A. Yu. Popov, "Natural and modified dimensions of closed toroidal shells," *Issled. Teor. Plastin. Obolochek*, No. 24, 96–102 (1992).
10. A. N. Guz', I. S. Cheryshenko, V. N. Chekhov, et al., *Theory of Thin Shells Weakened by Holes*, Vol. 1 of five-volume *Methods of Shell Design* [in Russian], Nauk. Dumka, Kiev (1980).
11. V. A. Khoroshavin, T. V. Ul'yanova, and L. V. Endzhievskii, "Design of a toroidal shell by the variational-difference method with allowance for physical nonlinearity," in: *Study of the Reliability of Structures* [in Russian], Krasnoyarsk (1989), pp. 53–63.
12. I. S. Chernyshenko and V. A. Maksymyuk, "Physical nonlinearity of the deformation of orthotropic shells of revolution," *Prikl. Mekh.*, **22**, No. 1, 52–56 (1986).
13. Siliang Zhenk, Zhenhua Sun, and Yong Wang, "A general solution for orthotropic toroidal shells subjected to axisymmetric loads," *Lisyue Syuebao, Acta. Mech. Sin.*, **22**, No. 2, 246–251 (1990).
14. O. S. Zienkiewich, J. Bauer, K. Morgan, and E. Onate, "A simple and efficient element for axisymmetric shells," *Int. J. Num. Meth. Eng.*, **11**, 1545–1558 (1977).