SOLUTION OF NONAXISYMMETRIC THREE-DIMENSIONAL THERMOPLASTICITY PROBLEM BY THE SECONDARY-STRESS METHOD

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The nonaxisymmetric thermoplasticity problem for a laminar solid of revolution is solved by successive approximation. The theory of deformation along slightly curved trajectories linearized by the secondary-stress method is employed. Numerical examples show that the proposed procedure reduces the number of successive approximations by 20 % relative to the traditional approach.

As a development of [3–8, 10], we solve the nonaxisymmetric three-dimensional thermoplasticity problem for laminar solids of revolution by successive approximation, and evaluate the effectiveness of this approach.

Experience shows [4–8, 10] that numerical solution of three-dimensional nonaxisymmetric plasticity boundary problems for stepwise uniform solids of revolution in nonisothermal loading is a very laborious procedure, which involves reducing the dimensionality of the problem, linearizing it, dividing the loading process into stages, and applying a grid to the meridional cross section. An important step in this procedure is organizing the successive-approximation process in accordance with the defining equations employed and the linearization method. In [1, 2], a method of constructing the successive-approximation process that converges more rapidly than the algorithms in [4–8, 10] was proposed and tested for representative problems of thin-shell theory. In the present work, the method in [1, 2] is applied to a nonaxisymmetric three-dimensional boundary problem, and its convergence is compared with that of the traditional approach. The relations of the theory of small-curvature processes, linearized by the secondary-stress method, are employed here.

1. Consider a nonaxisymmetric thermoelastoplastic stress state of a laminar solid of revolution made of various isotropic materials. The cylindrical coordinate system z, r, φ is used; z is directed along the axis of revolution of the body, r is radial, and φ is the azimuthal coordinate. Suppose that the layers of the body are attached without tension, and ideal contact is maintained at their common boundary during deformation. The body, which is initially in an unstressed and undeformed state at temperature T_0, is subjected to nonaxisymmetric bulk $\vec{K}(K_z, K_r, K_\varphi)$ and surface $\vec{t_n}(t_{nz}, t_{nr}, t_{n\varphi})$ forces and a nonuniform heating. Under the action of these loads, the materials of the body are deformed within and beyond the elastic limits over trajectories in the form of straight and slightly curved lines. The loads and their duration are such that creep deformation is negligible in comparison with the elastic and plastic deformation. Suppose that the temperature field of the body at any time is known — for example, determined from the heat-conduction problem by the method of [3, 10]. The stress-strain state of the body is determined in a quasi-static formulation at small strain. The loading of the body is divided into a number of small stages over time. Within each stage, the defining equations of the theory of deformation over small-curvature trajectories, linearized by the secondary-stress method, are used to describe the elastoplastic deformation of the materials [10]. The temperature dependence of the shear modulus G and coefficient of linear thermal expansion of the material α_T is taken into account; Poisson's ratio ν does not depend on the temperature. The relations between the components of the stress and strain tensors in any element of the body are given in the form of Hooke's law for an isotropic body with additional terms

$$\sigma_{zz} = (2G_0 + \lambda_0)\varepsilon_{zz} + \lambda_0(\varepsilon_{rr} + \varepsilon_{\varphi\varphi}) - \sigma^*_{zz},$$

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$$\sigma_{zr} = 2\,G_0\,\varepsilon_{zr} - \sigma^*_{zr}, \quad (z, r, \varphi), \tag{1.1}$$

where

$$\sigma^*_{ij} = 2G\,\varepsilon^{(p)}_{ij} + 2\,G_0\,\omega\,\varepsilon_{ij} + (K\,\varepsilon_T + 3\,\lambda_0\,\omega\,\varepsilon_0)\,\delta_{ij}, \quad \delta_{ij} = 1 \quad \text{when} \quad i = j \quad \text{and} \quad \delta_{ij} = 0 \quad \text{when} \quad i \neq j,$$

$$\omega = 1 - G/G_0, \quad \lambda_0 = (K_0 - 2\,G_0)/3, \quad K_0 = 2\,G_0\,(1 + \nu)/(1 - 2\,\nu),$$

$$\varepsilon_0 = (\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi})/3, \quad \varepsilon_T = \alpha_T\,(T - T_0). \tag{1.2}$$

In Eqs. (1.1) and (1.2), $\sigma_{zz}, \ldots, \sigma_{r\varphi}$ and $\varepsilon_{zz}, \ldots, \varepsilon_{r\varphi}$ are components of the stress and strain tensors, respectively; G_0 and K_0 are the shear modulus and bulk expansion modulus at the initial temperature T_0; the function ω reflects the temperature dependence of these moduli; $G = G_0\,(1 - \omega)$; $K = K_0\,(1 - \omega)$; $\varepsilon^{(p)}_{ij}$ are the plastic strain components. The plastic strain components in Eq. (1.2) take the following form at an arbitrary mth loading stage:

$$(\varepsilon^{(p)}_{ij})_m = \sum_{k=1}^{m} \Delta_k\,\varepsilon^{(p)}_{ij} = (\varepsilon^{(p)}_{ij})_{m-1} + \Delta_m\,\varepsilon^{(p)}_{ij}, \tag{1.3}$$

$$\Delta_k\,\varepsilon^{(p)}_{ij} = \Delta_k\,e^{(p)}_{ij} = \langle c_{ij} \rangle_k\,\Delta_k\,\Gamma^*_p, \tag{1.4}$$

where $\langle c_{ij} \rangle_k$ is the mean value at the kth stage of

$$c_{ij} = s_{ij}/S, \tag{1.5}$$

$$s_{ij} = \sigma_{ij} - \sigma_0\,\delta_{ij}, \quad \sigma_0 = (\sigma_{zz} + \sigma_{rr} + \sigma_{\varphi\varphi})/3, \tag{1.6}$$

$$S = (s_{ij}\,s_{ij}/2)^{1/2}$$

is the intensity of the tangential stress; $\Delta_k\,\Gamma^*_p$ is the increment in the intensity of the accumulated plastic shear deformation Γ^*_p at the kth stage

$$(\Gamma^*_p)_m = \sum_{k=1}^{m} \Delta_k\,\Gamma^*_p = (\Gamma^*_p)_{m-1} + \Delta_k\,\Gamma^*_p. \tag{1.7}$$

The functional relation between the intensity of the tangential stress S and the accumulated plastic shear strain Γ^*_p and the temperature T is assumed not to depend on the type of stress state [9, 10]. In the case of negligible creep deformation, this dependence reduces to a function. To obtain a specific form of this dependence, we use the equation of the instantaneous thermomechanical surface, which is a geometric locus of the tensional diagrams of the cylindrical samples obtained at various fixed temperatures and loading rates, within the range where the rate has no influence on the form of the tension diagram [9, 10]. The equation of the instantaneous thermomechanical surface takes the form

$$\sigma = f(\varepsilon, T), \tag{1.8}$$

where σ and ε are the stress and longitudinal strain of the sample.

The following formula converts the complex stress–strain state of an element of the body to uniaxial tension of the sample [9]

$$\varepsilon = \sqrt{3}\,S/[\,2\,G\,(1 + \nu)\,] + 2\,\Gamma^*_p/\sqrt{3}, \quad \sigma = \sqrt{3}\,S, \tag{1.9}$$

where S and Γ^*_p are given by Eqs. (1.6) and (1.7), respectively.

The expressions in Eq. (1.2) for the secondary terms σ^*_{ij} in Eq. (1.1) are of the same form for both active loading and unloading. In the latter case, the increment in the intensity of the accumulated plastic strain $\Delta_k \Gamma^*_p$ is assumed to be zero in the corresponding stage. Note that σ^*_{ij} is determined by successive approximation.

2. The problem is solved by the finite-element method, using the variational Lagrange equation. The basic unknowns are the components of the displacement, which are expressed as trigonometric series with respect to the azimuthal coordinate. The finite elements in the meridional cross section of the body are triangular finite elements with linear approximation of the displacement amplitudes. In constructing the solution, the secondary stresses σ^*_{ij} in Eq. (1.2) and the bulk and surface loads are expressed as trigonometric series in terms of the azimuthal coordinate. Then, to find each amplitude value of the displacement at the grid points, a system of $3N$ algebraic equations is obtained [10], where N is the number of grid points. The matrix elements of this system are calculated in terms of the coefficients of Eq. (1.1) and the vertex coordinates of the triangular elements in the meridional plane, while the right side of the system is expressed in terms of the amplitudes of the secondary terms σ^*_{ij} in Eq. (1.2) and the bulk and surface loads at the corresponding points of the meridional cross section. This system of algebraic equations must be solved in each approximation at an arbitrary stage of loading, refining the right side on the basis of the results in the preceding approximation. The modification of the successive-approximation method here proposed involves the calculation of the increment of the intensity of the accumulated plastic shear strain $\Delta_k \Gamma^*_p$, determining the plastic strain components in Eq. (1.3), which appear on the right side of the resolving algebraic equations.

Consider the determination of $\Delta_k \Gamma^*_p$ in an arbitrary Lth approximation of the mth loading stage. Suppose that, at the beginning of the Lth approximation of the mth stage, the plastic strain components $(\varepsilon^{(p)}_{ij})_{L-1}$ and the intensity of the accumulated plastic shear strain $(\Gamma^*_p)_{L-1}$ and hence $\varepsilon^{(p)}_{L-1} = 2(\Gamma^*_p)_{L-1}/\sqrt{3}$ are known. In addition, $(\sigma^*_{ij})_{L-1}$ and the stress components $(\sigma_{ij})_{m-1}$ at the end of the preceding stage are known. In the first approximation of the mth stage, $(\varepsilon^{(p)}_{ij})_{L-1}$, $(\Gamma^*_p)_{L-1}$, and $(\sigma^*_{ij})_{L-1}$ are assumed to be equal to their values at the end of the preceding stage. The secondary terms $(\sigma^*_{ij})_{L-1}$ are used to calculate the right side of the resolving equations, which are then solved in the Lth approximation with load and temperature values corresponding to the end of the mth loading stage. After obtaining the displacement values, we determine the components of the strain $(\varepsilon_{ij})_L$ and stress $(\sigma_{ij})_L$. The components of the stress deviator $(s_{ij})_L$ and their intensity S_L in Eq. (1.6) are calculated. Using S_L and $(\Gamma^*_p)_{L-1}$, we find the strain ε_L in Eq. (1.9) (Fig. 1). Using Eq. (1.8), we determine the pair of values σ^*_L, ε^*_L, which (according to [2]) must satisfy the equation of the instantaneous thermomechanical surface and the condition of equal areas of the triangles ABC and ADE in Fig. 1

$$\sigma^*_L = f(\varepsilon^*_L, T_m), \tag{2.1}$$

$$3(S_L)^2/[2G(1+\nu)] = \sigma^*_L(\varepsilon^*_L - \varepsilon^{(p)}_{L-1}). \tag{2.2}$$

Calculating σ^*_L and ε^*_L from Eqs. (2.1) and (2.2), we find the corresponding plastic strain (Fig. 1)

$$(\varepsilon^*_p)_L = \varepsilon^*_L - \sigma^*_L/[2G(1+\nu)], \tag{2.3}$$

which we use to calculate the difference

$$\Delta_L \varepsilon^{(p)} = (\varepsilon^*_p)_L - \varepsilon^{(p)}_{L-1}. \tag{2.4}$$

This result is used to find the increment in the intensity of plastic shear strain

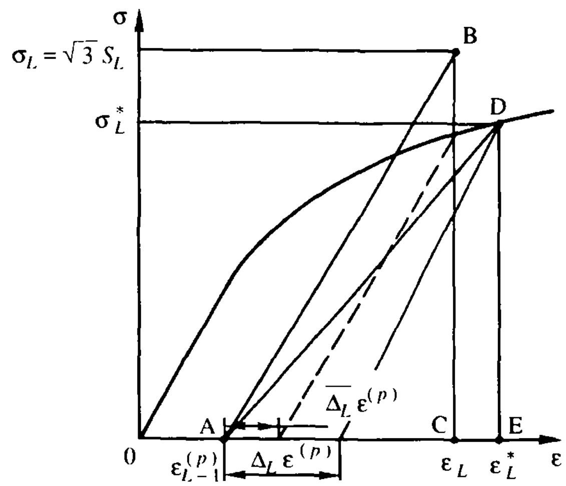

Fig. 1

$$\Delta_L \Gamma_p^* = \sqrt{3}\, \Delta_L \varepsilon^{(p)}/2 . \tag{2.5}$$

Equation (2.5) refines the increment in the intensity of the accumulated plastic shear strain at the mth stage

$$\Delta_m \Gamma_p^* = \left(\sum_{k=2}^{L} \Delta_k \Gamma_p^* \right)_m . \tag{2.6}$$

Calculating the coefficients in Eq. (1.5) and their means

$$< c_{ij} >_m = \sqrt{3}\ (s_{ij}^{(m-1)}/\sigma_{m-1} + s_{ij}^{(L)}/\sigma_L^*)/2 ,$$

we must determine the increments in the plastic strain components and the corresponding values of these components in Eq. (1.3) in the next approximation of the given stage. Then the new values of σ_{ij}^* in Eq. (1.2) may be determined and used to calculate the new value of the right side in the resolving system of equations, which may hence be solved in the new approximation. In the mth stage, successive approximation ends when

$$| \sqrt{3}\ S_L - \sigma_L^* | /\sigma_L^* \le \kappa , \tag{2.7}$$

where S_L is given by Eq. (1.6), and σ_L^* by Eqs. (2.1) and (2.2); κ is a specified number.

Note that, when the increment in the intensity of the plastic shear strain in Eq. (2.6) is nonnegative, active loading is assumed. If this increment is negative, however, unloading occurs in the corresponding element, and zero increment is assumed.

Note that, in the proposed successive-approximation procedure, the increment in the intensity of plastic shear strain in Eq. (2.5) is determined on the basis of Eq. (2.4), rather than the expression $\overline{\Delta_L}\, \varepsilon^{(p)} = (\sqrt{3}\ S - \sigma)/[\, 2G (1 + \nu)\,]$ adopted in the traditional procedure (Fig. 1).

3. To evaluate the effectiveness of the method proposed in specific nonaxisymmetric three-dimensional thermoplasticity problems, consider the following example. We will determine the thermoelastoplastic stress–strain state of a three-layer solid of revolution on heating from an initial temperature $T = T_0 = 20$ °C on account of convective heat transfer with an atmosphere at a temperature $\Theta = \Theta_0 (1 + 0.1 \cos \varphi)$, $\Theta_0 = 3000$ °C. Half the meridional cross section of the body is shown in Fig. 2. The part of the body within the covering shell (Fig. 2) is made of a material whose tensional diagram at various temperatures is shown in Table 1.

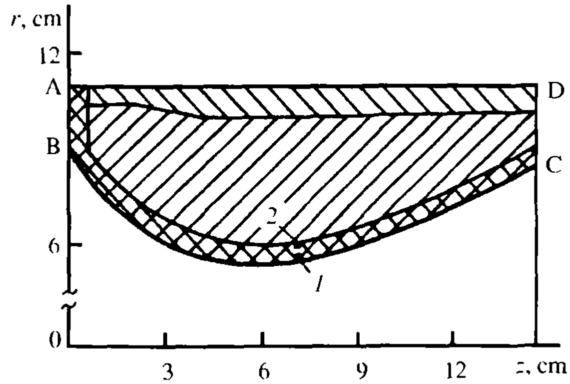

Fig. 2

TABLE 1

T, °C	σ, MPa						
	ε = 0 %	1.0	1.5	2.0	2.5	3.0	10.0
0	0	30	41	48	50	51.5	58
500	0	32	43	50	54	56	64.5
1000	0	35	49.5	56	60	62	72.5
1500	0	36	51	60	66	70	90

TABLE 2

T, °C	σ, MPa					
	ε = 0 %	0.04	0.1	0.2	2.0	10
0	0	156	330	580	2080	8747
2500	0	156	200	220	260	4378
3300	0	156	159	162	170	205.6

The coefficient of linear thermal expansion of the material $\alpha_T = 0.95 \cdot 10^{-5}$ $1/K$; Poisson's ratio $\nu = 0.2$; the thermal conductivity $\lambda = 1.49$ W/cm·K; the product of the specific heat c and the density of the material ρ is $c\,\rho = 2.93$ J/cm^3·K.

The internal part of the body is covered by a shell of material whose $\sigma \sim \varepsilon$ tension diagram is shown for various temperatures T in Table 2. The coefficient of linear thermal expansion of the material $\alpha_T = 0.12 \cdot 10^{-4}$ $1/K$; the Poisson's ratio $\nu = 0.17$; the thermal conductivity $\lambda = 0.222$ W/cm·K; $c\,\rho = 2.575$ J/cm^3·K.

The external cylindrical shell is made of a strong material operating within the elastic range. The elastic modulus of this material $E = 2 \cdot 10^5$ MPa; the coefficient of linear thermal expansion of the material $\alpha_T = 0.4 \cdot 10^{-5}$ $1/K$; the Poisson ratio $\nu = 0.23$; the thermal conductivity $\lambda = 0.02$ W/cm·K; $c\,\rho = 2$ J/cm^3·K. Suppose that the body surface is heat-insulated in

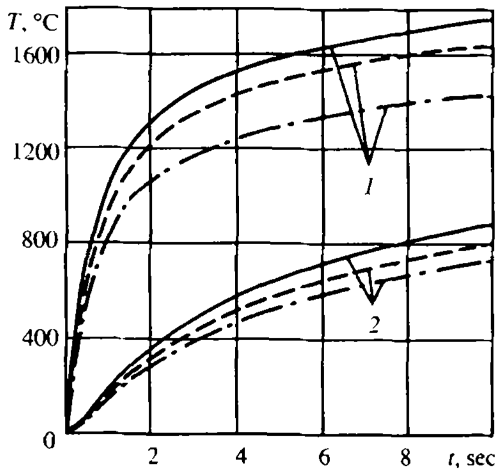

Fig. 3

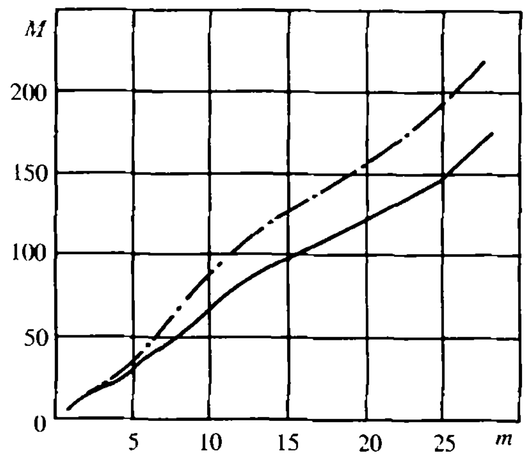

Fig. 4

section ADC (Fig. 2); the heat-transfer coefficient between the medium and the surface of the body at section AB varies linearly from 0 at point A to 0.35 W/cm^2 at point B and remains constant at 0.35 W/cm^2 over section BC.

The temperature field of the body on heating is determined by solution of the heat-conduction problem using the method in [3, 10]. The meridional cross section of the body is divided into 1508 triangular finite elements with 797 grid points. Five terms, including those with $\sin\varphi$, $\cos\varphi$, $\cos 2\varphi$, and $\cos 3\varphi$, are retained in the trigonometric series, i.e., in each approximation at any loading stage, the stress–strain state of the body is determined by solving five linear systems of algebraic equations of order 2391, using the Gauss method. The problem is solved both by the method here proposed and by the traditional approach [3, 7, 9]. In Eq. (2.7), $\kappa = 0.01$ is specified. The stress–strain state of the body is determined in 28 loading stages with a variable time step: 0.05 sec in stages 1–4, 0.1 sec in stages 5–7, 0.25 sec in stages 8–17, 0.5 sec in stages 18–25, and 1 sec in the last three stages. The very small time step at the beginning of the process is chosen because the rate of temperature increase differs sharply in adjacent elements (for example, at points *1* and *2* in Fig. 2) at the internal surface of the body on account of contact with a very hot medium; this indicates considerable temperature gradients. Evidence of this is provided by the time dependence of the temperature at several points of the body in Fig. 3, where continuous curves *1* and *2* correspond to points *1* and *2* of the meridional cross section (Fig. 2) at $\varphi = 0$; the dashed curves to $\varphi = \pi/2$; and the dash-dot curves to $\varphi = \pi$. Analysis of the results for the stress–strain state shows that considerable plastic strain develops in the shell covering the interior of the body; the intensity of the accumulated plastic shear strain in the vicinity of point *1* (Fig. 2) is 1.5 % at the end of the process, and its increase keeps pace with the temperature rise. In the part of the body adjacent to the internal shell, the intensity of the accumulated plastic strain is lower (0.5 %). In Fig. 4, the number of successive approximations M required to solve the problem by the two methods with specified accuracy is shown, as a function of the stage number m. The continuous curve corresponds to the modified approach, and the dash-dot curve to the traditional approach [10]. As is evident from Fig. 4, the modified approach reduces the total number of approximations in the 28 stages from 219 to 173, i.e., by more than 20 %. Calculations of the thermoelastoplastic stress–strain state of solids of revolution with anisotropic layers of the form in [4–7] indicate that the convergence is better for the modified method than for the traditional approach.

Thus, comparison of the results given by the two methods suggests that the modified secondary-stress method may expediently be used to solve nonaxisymmetric three-dimensional thermoplasticity problems for laminar solids of revolution.

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