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SPLINE-APPROXIMATION SOLUTION OF PROBLEMS OF THE STATICS OF ORTHOTROPIC SHALLOW SHELLS WITH VARIABLE PARAMETERS

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An approach to the solution of problems of the statics of shallow orthotropic shells is proposed. It is based on reducing a two-dimensional boundary value problem to a one-dimensional one using the spline-collocation method and solution of the problem by the stable numerical method of discrete orthogonalization. Solutions are presented for problems on the stress state of orthotropic shells of double curvature for several values of the elastic constants of the material.

Along with isotropic shallow shells [1, 9, 12], orthotropic shells [2, 5, 13, 14] are widely used as structural elements due to the use of composite materials. Moreover, ribbed shallow shells are reduced to a model of orthotropic shells upon smearing of ribs, i.e., as structurally orthotropic shells.

In the present paper, an approach to the solution of two-dimensional problems of the statics of shallow shells made of orthotropic materials is described. The approach involves spline-collocation to reduce a two-dimensional problem to one-dimensional and the numerical solution of the problems by the method of discrete orthogonalization. Such an approach to solution of problems of the theory of plates and shells is proposed in [3, 4, 6–8]. The results of solution of some problems and estimates of their accuracy are presented in [2–4, 8].

Let us present the input equations of deformation of shallow orthotropic shells with a rectangular planform on the basis of the equations of the Mushtari–Donnell–Vlasov theory of shells [1, 5, 9, 11, 12]:

the expressions for deformations

$$\varepsilon_1=\frac{\partial u}{\partial x}+\frac{w}{R_1},\quad \varepsilon_2=\frac{\partial v}{\partial y}+\frac{w}{R_2},\quad \varepsilon_{12}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},$$

$$\kappa_1=-\frac{\partial^2 w}{\partial x^2},\quad \kappa_2=-\frac{\partial^2 w}{\partial y^2},\quad \kappa_{12}=-\frac{\partial^2 w}{\partial x\partial y},\tag{1}$$

the equilibrium equations

$$\frac{\partial N_1}{\partial x}+\frac{\partial S}{\partial y}=0,\quad \frac{\partial N_2}{\partial y}+\frac{\partial S}{\partial x}=0,$$

$$\frac{\partial M_1}{\partial x}+\frac{\partial H}{\partial y}=Q_1,\quad \frac{\partial M_2}{\partial y}+\frac{\partial H}{\partial x}=Q_2,$$

$$\frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}-\frac{1}{R_1}N_1-\frac{1}{R_2}N_2+q_z=0,\tag{2}$$

the elastic relations

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$$N_1 = C_{11}\,\varepsilon_1 + C_{12}\,\varepsilon_2, \quad N_2 = C_{12}\,\varepsilon_1 + C_{22}\,\varepsilon_2, \quad S = C_{66}\,\varepsilon_{12},$$

$$M_1 = D_{11}\,\kappa_1 + D_{12}\,\kappa_2, \quad M_2 = D_{12}\,\kappa_1 + D_{22}\,\kappa_2, \quad H = 2\,D_{66}\,\kappa_{12}, \tag{3}$$

where

$$C_{11} = \frac{E_1 h}{1 - \nu_1 \nu_2}, \quad C_{12} = \frac{\nu_1 E_2 h}{1 - \nu_1 \nu_2} = \frac{\nu_2 E_1 h}{1 - \nu_1 \nu_2},$$

$$C_{22} = \frac{E_2 h}{1 - \nu_1 \nu_2}, \quad C_{66} = G_{12} h,$$

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_1 \nu_2)}, \quad D_{12} = \frac{\nu_1 E_2 h^3}{12(1 - \nu_1 \nu_2)} = \frac{\nu_2 E_1 h^3}{12(1 - \nu_1 \nu_2)},$$

$$D_{22} = \frac{E_2 h^3}{12(1 - \nu_1 \nu_2)}, \quad D_{66} = G_{12}\frac{h^3}{12}. \tag{4}$$

In relations (1)–(4), x and y are coordinates ($0 \le x \le a$, $0 \le y \le b$), u, v, and w are displacements in the plane and in the direction of the coordinate axes and along the normal to this plane, ε_1, ε_2, ε_{12} and κ_1, κ_2, κ_{12} are the tangential and bending deformations, N_1, N_2, S, Q_1, Q_2, M_1, M_2, and H are forces and moments, E_1, E_2, G_{12}, ν_1, and ν_2 are the elastic and shear moduli and Poisson's ratios, R_1 and R_2 are the radii of curvature in two directions, and $h = h(x, y)$ is the shell thickness.

Eliminating the intersecting forces Q_1 and Q_2 in the equilibrium equations (2), expressing in relations of elasticity (3) the forces and moments in terms of displacements using expressions (1), and substituting them into the equilibrium equations (2), we obtain the three resolving differential equations in displacements:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + C_{66}\frac{\partial^2 u}{\partial y^2} + \left(C_{12} + C_{66}\right)\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial C_{11}}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial C_{66}}{\partial y}\frac{\partial u}{\partial y}$$

$$+ \frac{\partial C_{66}}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial C_{12}}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial x}\left(\frac{C_{11}}{R_1} + \frac{C_{12}}{R_2}\right)w + \left(\frac{C_{11}}{R_1} + \frac{C_{12}}{R_2}\right)\frac{\partial w}{\partial x} = 0,$$

$$C_{22}\frac{\partial^2 v}{\partial y^2} + C_{66}\frac{\partial^2 v}{\partial x^2} + \left(C_{12} + C_{66}\right)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial C_{22}}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial C_{66}}{\partial x}\frac{\partial v}{\partial x}$$

$$+ \frac{\partial C_{66}}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial C_{12}}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left(\frac{C_{12}}{R_1} + \frac{C_{22}}{R_2}\right)w + \left(\frac{C_{12}}{R_1} + \frac{C_{22}}{R_2}\right)\frac{\partial w}{\partial y} = 0,$$

$$D_{11}\frac{\partial^2 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + \left(2D_{12} + 4D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + 2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_{22}}{\partial y}\frac{\partial^3 w}{\partial y^3}$$

$$+ \left(2\frac{\partial D_{12}}{\partial y} + 4\frac{\partial D_{66}}{\partial y}\right)\frac{\partial^3 w}{\partial x^2 \partial y} + \left(2\frac{\partial D_{12}}{\partial x} + 4\frac{\partial D_{66}}{\partial x}\right)\frac{\partial^3 w}{\partial x \partial y^2} + \left(\frac{\partial^2 D_{11}}{\partial x^2} + \frac{\partial^2 D_{12}}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2}$$

$$+ \left(\frac{\partial^2 D_{12}}{\partial x^2} + \frac{\partial^2 D_{22}}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 D_{66}}{\partial x \partial y}\cdot\frac{\partial^2 w}{\partial x \partial y}\cdot\frac{C_{11}}{R_1} + \frac{\partial u}{\partial x} + \frac{C_{12}}{R_1}\frac{\partial v}{\partial y} + \frac{1}{R_1}\left(\frac{C_{11}}{R_1} + \frac{C_{12}}{R_2}\right)w$$

$$+ \frac{C_{12}}{R_2}\cdot\frac{\partial u}{\partial x} + \frac{C_{22}}{R_2}\frac{\partial v}{\partial y} + \frac{1}{R_2}\left(\frac{C_{12}}{R_1} + \frac{C_{22}}{R_2}\right)w - q_z = 0. \tag{5}$$

Boundary conditions, which can be expressed in terms of displacements, are specified on the shell contours x = const and y = const.

We will solve the system of differential equations (5) for the derivatives $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 v}{\partial x^2}$, and $\frac{\partial^2 w}{\partial x^4}$.

The solution of the boundary-value problem for the system of differential equations (5) will be searched for in the form of the expansions

$$u(x, y) = \sum_{i=0}^{N} u_i(x)\,\varphi_i(y),$$

$$v(x, y) = \sum_{i=0}^{N} v_i(x)\,\varphi_i(y), \quad w(x, y) = \sum_{i=0}^{N} w_i(x)\,\psi_i(y), \tag{6}$$

where u_i, v_i, and w_i ($i = \overline{0, N}$) are unknown functions of the variable x, and φ_i and ψ_i ($i = \overline{0, N}$, $N \geq 6$) are functions constructed with the help of B-splines of the third and fifth degrees, respectively, which allow us to construct their linear combinations so as to satisfy different conditions on the shell contours $y = 0$ and $y = b$ [3, 6, 10].

On construction of linear combinations of B-splines as functions $\varphi_i(y)$ and $\psi_i(y)$ satisfying certain boundary conditions on the contours $y = 0$ and $y = b$, we substitute expressions (6) into resolved equations (5) and require that they be satisfied at the collocation points y_k ($k = \overline{0, N}$). Then we obtain a system of $3(N + 1)$ linear equations. We act similarly with the boundary conditions on the contours $x = 0$ and $x = a$. The obtained system of ordinary differential equations together with the boundary conditions will form a two-point boundary-value problem on the interval $0 \leq x \leq a$.

If we introduce the notation

$$\bar{Y} = \left\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_8\right\}^T = \left\{\bar{u}, \bar{u}', \bar{v}, \bar{v}', \bar{w}, \bar{w}', \bar{w}'', \bar{w}'''\right\}^T, \tag{7}$$

where $\bar{y}_m = \left\{y_{m_0}, y_{m_1}, \ldots, y_{m_N}\right\}^T$ ($m = \overline{1, 8}$), then the boundary-value problem obtained can be written as

$$\frac{d\bar{Y}}{dx} = A(x)\,\bar{Y} + \bar{f}(x) \quad (0 \leq x \leq a), \tag{8}$$

$$B_1\,\bar{Y}(0) = \bar{b}_1, \quad B_2\,\bar{Y}(0) = \bar{b}_2. \tag{9}$$

We solve the boundary-value problem for the system of equations (8) with the boundary conditions (9) by the stable numerical method of discrete orthogonalization [2]. Substituting the found values of the functions $u_i(x)$, $v_i(x)$, and $w_i(x)$ ($i = \overline{0, N}$) into expressions (6), we obtain the solution of the initial problem for displacements and calculate all the factors of the stress–strain state of the shell.

Let us present the results of solution of some problems obtained on the basis of the given approach. First, we will consider the problem on the stress–strain state of a shallow orthotropic shell of double curvature with radii R_1 and R_2. The shell has a rectangular planform, a constant thickness h, and legs a and b and is subject to the transversal surface load $q = q_0 \sin\frac{\pi x}{a}$ ($0 \leq x \leq a$, $0 \leq y \leq b$). Moreover, the shell is rigidly fixed along the legs $y = 0$ and $y = b$ and is hinged along the legs $x = 0$ and $x = a$, i.e., the following boundary conditions are given:

$$u = v = w = \frac{\partial w}{\partial y} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b,$$

$$\frac{\partial u}{\partial x} = v = w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a. \tag{10}$$

Then, in expressions (6), the functions φ_i and ψ_i ($i=\overline{0,N}$) are chosen in the form

$$\varphi_0=B_3^0-4B_3^{-1},\quad \varphi_1=B_3^1-\frac{1}{2}B_3^0+B_3^{-1},$$

$$\varphi_i=B_3^i\ (i=\overline{2,N-2}),$$

$$\varphi_{N-1}=B_3^{N-1}-\frac{1}{2}B_3^N+B_3^{N+1},\quad \varphi_N=B_3^N-4B_3^{N+1},$$

$$\psi_0=\frac{165}{4}B_5^{-2}-\frac{33}{8}B_5^{-1}+B_5^0,\quad \psi_1=B_5^1-\frac{26}{33}B_5^0+B_5^{-1},$$

$$\psi_2=B_5^2-\frac{1}{33}B_5^0+B_5^2,\quad \varphi_i=B_5^i\quad (i=\overline{3,N-3}),$$

$$\varphi_{N-2}=B_3^{N+2}-\frac{1}{33}B_5^N+B_3^{N-2},$$

$$\varphi_{N-1}=B_3^{N+1}-\frac{26}{33}B_5^N+B_3^{N-1},$$

$$\psi_N=\frac{165}{4}B_5^{N+2}-\frac{33}{8}B_5^{N+1}+B_5^{N-2},\tag{11}$$

where $B_3^k\,(k=\overline{-1,N+1})$ and $B_5^l\,(l=\overline{-2,N+2})$ are B-splines of the third and fifth degrees, respectively, [8, 10]. Such a choice allows us to exactly satisfy the boundary conditions on the legs $y=\text{const}$. Let us consider orthotropic shells for five variants of the elastic constants of the material. We assume that the elastic modulus E_x preserves the constant value $E_x=E=\text{const}$, and the elastic modulus $E_y=\mu E$, the shear modulus $G_{xy}=\lambda E$, and Poisson's ratio ν_x vary. The following values of the elastic constants of the shell material were considered [5]:

$$\text{(i) } \mu=2,\quad \lambda=0.3,\quad \nu_x=0.075,$$

$$\text{(ii) } \mu=1.35,\quad \lambda=0.215,\quad \nu_x=0.122,$$

$$\text{(iii) } \mu=1,\quad \lambda=0.385,\quad \nu_x=0.3,$$

$$\text{(iv) } \mu=0.741,\quad \lambda=0.159,\quad \nu_x=0.165,$$

$$\text{(v) } \mu=0.5,\quad \lambda=0.125,\quad \nu_x=0.15.\tag{12}$$

The values of the elastic constants of variant (iii) correspond to the isotropic case. The problem was solved for the following initial data:

$$a=12,\quad b=10,\quad h=0.4,\quad R_1=22.9,\quad R_2=13,\quad q=q_0=\text{const}.$$

The results of solution of the problem for a deflection are presented in Table 1.

By virtue of the symmetry of the problem, the table contains the values for $x=6$ and $0\le y\le 5$ and for $y=5$ and $0\le x\le 6$. The results obtained for the third variant coincide with the exact analytical solution.

From the table, it is seen that as the elastic modulus E_y decreases, the deflection at the center of the shell increases 1.4, 1.5, 2.4, and 3.5 times, respectively, as compared to the deflection of a shell made of a material corresponding to the first variant of orthotropy.

TABLE 1

y	$w/\frac{E}{q_0}$				
	1	2	3	4	5
	$x = 6$				
0	0	0	0	0	0
1	42.1	61.6	72.1	111.2	166.0
2	125.5	182.3	205.0	321.8	472.4
3	208.2	300.4	327.6	520.4	753.6
4	265.6	381.4	407.8	652.7	936.5
5	285.8	409.8	435.1	698.3	998.7
x	$y = 5$				
0.0	0	0	0	0	0
1.2	88.3	126.6	134.5	215.8	308.6
2.4	168.0	240.9	255.8	410.4	587.0
3.6	231.2	331.5	352.0	564.9	808.0
4.8	271.8	389.7	413.8	664.1	949.8
6.0	285.8	409.8	435.1	698.3	998.7

Figures 1 and 2 show the distributions of the stresses σ_x^+ and σ_y^+ on the external surface of the shell σ_x^- and σ_y^- on the internal surface for $y = 5$ along the axis OX and for $x = 6$ along the axis OY for the fifth variant of orthotropy. It is seen that σ_y^+ on the fastened contours of the shell are the greatest stresses. The influence of orthotropy on the distribution and values of stresses is not great.

The problem on an orthotropic shell with the same geometrical parameters deformed under the action of the uniform transversal surface load $q_z = q_0$ is also considered. All the four legs of the shell are rigidly fastened, i.e., the following boundary conditions are given:

$$u = v = w = \frac{\partial w}{\partial x} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a,$$

$$u = v = w = \frac{\partial w}{\partial y} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b. \tag{13}$$

Table 2 presents the deflection distributions for $x = 6$ and $0 \le y \le 5$ and for $y = 5$ and $0 \le x \le 6$ for five variants of orthotropy (12). It is seen that as the elastic modulus E_y decreases, the deflection at the center of the shell ($x = 6$, $y = 5$) increases 1.3, 1.4, 2.1, and 2.8 times as compared to the deflection of a shell made of a material corresponding to the first variant of orthotropy.

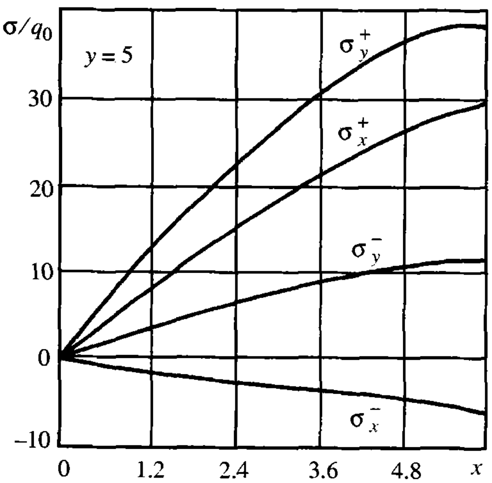

Fig. 1

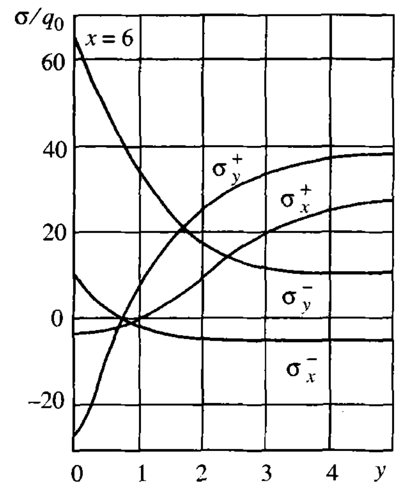

Fig. 2

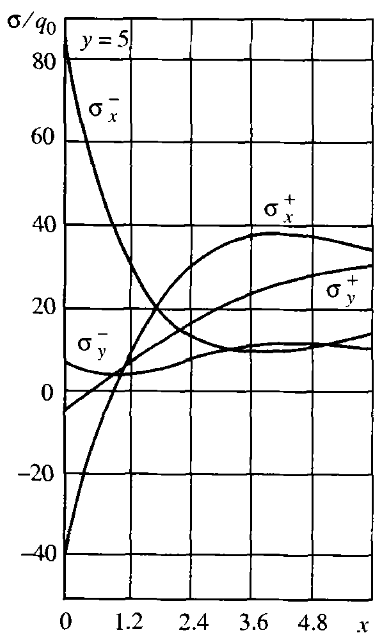

Fig. 3

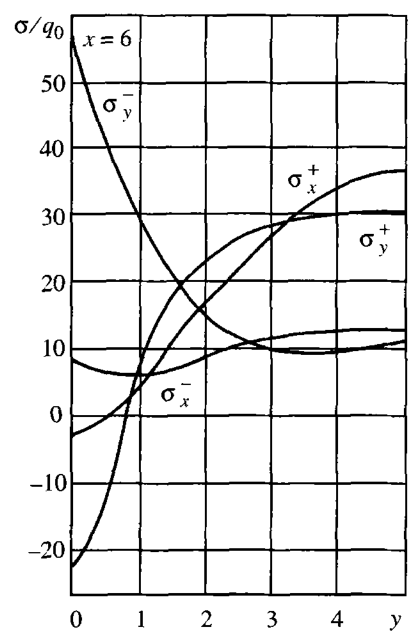

Fig. 4

Figures 3 and 4 show the distributions of the stresses σ_x^+, σ_x^-, σ_y^+, and σ_y^- along the axis OX for $y = 5$ and along the axis OY for $x = 6$, respectively, for the fifth variant of the orthotropy of the shell material. The stresses σ_x^- on the short legs of the shell are the greatest. The influence of the orthotropy on the distribution and values of the greatest stress σ_x^- is shown in Fig. 5.

The numbers designate orthotropy variants. As the elastic modulus E_y decreases, the maximum value of the stress σ_x^- increases 1.16, 1.24, 1.44, and 1. 64 times, respectively.

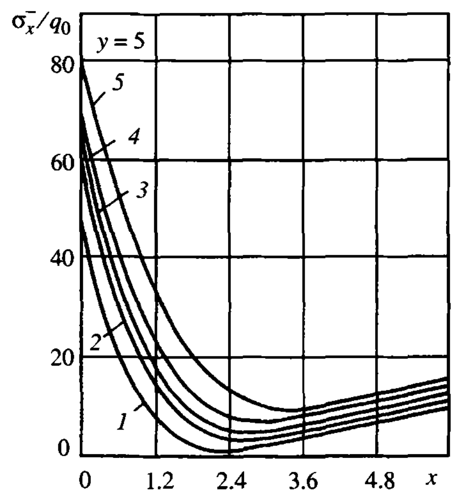

Fig. 5

TABLE 2

y	$w/\frac{E}{q_0}$				
	1	2	3	4	5
	$x = 6$				
0	0	0	0	0	0
1	40.0	56.7	65.3	98.2	141.7
2	120.9	169.0	188.0	282.0	394.4
3	202.6	279.7	301.9	452.0	614.9
4	259.9	353.9	376.3	562.7	751.1
5	280.2	382.6	401.6	600.4	795.8
x	$y = 5$				
0.0	0	0	0	0	0
1.2	89.5	105.1	102.4	133.8	156.6
2.4	201.9	251.2	250.4	344.9	422.4
3.6	259.7	339.3	346.7	398.5	635.6
4.8	277.5	374.6	390.2	577.5	757.6
6.0	280.2	382.6	401.6	600.4	795.8

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