DETERMINATION OF THE NATURAL FREQUENCIES OF VIBRATION OF NONUNIFORM SLABS

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The natural frequencies of vibrations of laminated plates are determined in a three-dimensional formulation by analytical separation of the sought functions for plate thickness. The system of differential equations which describes the natural vibrations of the plates is solved analytically. The solution makes it possible to study plates with a large number of layers, including orthotropic plates with elastic characteristics that vary through the thickness. Numerical experiments show that a step approximation can be used to approximate the variable elastic modulus.

Introduction. Natural frequencies of vibration of nonuniform slabs were examined in [1–3, 6, 8, 9] in a three-dimensional formulation based on analytical separation of the variables. It was determined in [6, 8] that three-layer isotropic slabs undergo vibration in modes which are skew-symmetric relative to the middle plane. The general solution for laminated isotropic slabs was presented in [9], where a numerical calculation was performed to determine the fundamental frequency for symmetric three-layer slabs. Vibrations of a uniform slab were examined in [2]. The most complete solution to the problem of determining the natural frequencies of vibration of laminated structures in a three-dimensional formulation was obtained in [1] by the method of separation of variables. Orthotropic cylindrical shells and slabs comprise one area of current research. Boundary-value problems for systems of ordinary differential equations were solved in [1] by the highly accurate numerical method of orthogonal trial run. Results of the calculations were reported for bending vibrations of thin slabs. The spectrum of frequencies of natural vibration for three-layer slabs was determined in [3] by an approach similar to that used in [1].

In this investigation, we obtain a three-dimensional solution to determine the natural frequencies of vibration of pinned shells. The solution is obtained on the basis of analytical separation of the sought functions. In contrast to [1], the system of differential equations that describes the natural vibrations is solved analytically. The proposed solution makes it possible to examine slabs with a large number of layers, including orthotropic slabs with variable elastic characteristics. The bottom surface of the slab can be rigidly fastened to the base.

1. Initial Conditions. We use a rectangular Cartesian coordinate system to examine a flat laminated structure in which the interfaces between the layers are parallel to the faces of the structure and the surface Oxy. The z axis is directed downward. The material of the layers may be an orthotropic material. For such a material, the column matrices $\{\bar{e}\}=\{e_{11}, e_{22}, \sigma_{33}, 2e_{12}, 2e_{23}, 2e_{13}\}^T$ and $\{\bar{\sigma}\}=\{\sigma_{11}, \sigma_{22}, e_{33}, 2\sigma_{12}, 2\sigma_{23}, 2\sigma_{13}\}^T$ are connected to one another by well-known relations: $\{\bar{\sigma}\}=[B]\{\bar{e}\}$. We are examining a special case of nonuniformity of the layers: $E_1(z)=\bar{E}_1 e^{\gamma z}, E_2(z)=\bar{E}_2 e^{\gamma z}, E_3(z)=\bar{E}_3 e^{\gamma z}$, $G_{12}(z)=\bar{G}_{12} e^{\gamma z}, G_{13}(z)=\bar{G}_{13} e^{\gamma z}, G_{23}(z)=\bar{G}_{23} e^{\gamma z}$, $\rho(z)=\bar{\rho} e^{z\gamma}$ (γ is an assigned number). The Poisson ratios are constant within each layer. The number of the layer will be denoted by superscripts in parentheses. The structure has n layers. The coordinates of the kth layer along the z axis are designated as c_{k-1} and c_k.

Ukrainian Transportation University, Kiev, Ukraine. Translated from Prikladnaya Mekhanika, Vol. 35, No. 2, pp. 47–53, February, 1999. Original article submitted October 27, 1997.

 1063-7095/99/3502-0152$22.00

Summation is performed over dummy indices. The indices 1, 2, and 3 after a comma denote differentiation with respect to the x, y, and z axes. In the matrix expressions, the matrix operations are performed before summation over the dummy indices.

2. Construction of the Solution. We represent the displacements and the stresses in the form:

$$U_i^{(k)}(x,y,z,t) = V_i(x,y) f_i^{(k)}(z) r_i(t);$$

$$\sigma_{i3}^{(k)}(x,y,z,t) = \tau_{i3}(x,y) f_{i+3}^{(k)}(z) r_{i+3}(t) \quad (i = 1, 2, 3). \tag{2.1}$$

We use the Cauchy relations to obtain the strain tensor

$$e_{ii} = V_{i,i}(x,y) f_i^{(k)}(z) r_i(i);$$

$$e_{33} = V_3(x,y) f_{3,3}^{(k)}(z) r_3(t); \tag{2.2}$$

$$2e_{12} = V_{1,2}(x,y) f_1^{(k)}(z) r_1(t) + V_{2,1}(x,y) f_2^{(k)}(z) r_2(t);$$

$$2e_{i3} = V_i(x,y) f_{i,3}^{(k)}(z) r_i(t) + V_{3,i}(x,y) f_3^{(k)}(z) r_3(t) \quad (i = 1, 2).$$

We find the missing components of the stress tensor from Hooke's law.

We write the Reissner functional as follows:

$$R = \iiint_V \{U\}^T [d]^T [D] [d] \{U\} dv, \tag{2.3}$$

where

$$\{U\} = \{U_1, U_2, U_3, \sigma_{13}, \sigma_{23}, \sigma_{33}\}^T;$$

the nontrivial terms of the matrix $[d]$ are

$$d(1,1) = \partial/\partial x, \quad d(2,2) = \partial/\partial y, \quad d(3,3) = \partial/\partial z, \quad d(4,6) = 1, \quad d(5,1) = \partial/\partial y,$$

$$d(6,2) = \partial/\partial x, \quad d(7,2) = \partial/\partial z, \quad d(8,3) = \partial/\partial y, \quad d(9,5) = 1, \quad d(10,1) = \partial/\partial z,$$

$$d(11,3) = \partial/\partial x, \quad d(12,4) = 1;$$

the nontrivial terms of the symmetric matrix $[D]$ are

$$D(1,1) = B_{11}; \quad D(1,2) = B_{12}; \quad D(1,4) = B_{13}; \quad D(2,2) = B_{22}; \quad D(2,4) = B_{23};$$

$$D(3,4) = 1; \quad D(4,4) = -B_{33}; \quad D(5,5) = B_{44}; \quad D(5,6) = B_{44}; \quad D(6,6) = B_{44};$$

$$D(7,9) = 1; \quad D(8,9) = 1; \quad D(9,9) = -1/B_{55}; \quad D(10,12) = 1; \quad D(11,12) = 1;$$

$$D(12,12) = -1/B_{66}.$$

The kinetic energy has the form

$$T = \iiint_V \{U\}^T [\underline{d}]^T [\underline{D}] [\underline{d}] \{U\} dv, \tag{2.4}$$

where the nontrivial terms of the matrix $[d]$: $\underline{D}(1,1)=\partial/\partial t$, $\underline{d}(2,2)=\partial/\partial t$, $\underline{d}(3,3)=\partial/\partial t$, $\underline{d}(4,4)=1$, $\underline{d}(5,5)=1$, $\underline{d}(6,6)=1$;

the nontrivial terms of the matrix $[D]$

$$\underline{D}(1,1)=\rho(z), \quad \underline{D}(2,2)=\rho(z), \quad \underline{D}(3,3)=\rho(z).$$

Inserting Eqs. (2.3) and (2.4) into Reissner's variational equation for dynamic problems with the condition that external loads are absent

$$\delta\int_{t_1}^{t_2}(R-T)\,dt=0, \tag{2.5}$$

we obtain the equations of free vibration of laminated structures. If the functions

$$V_1=\cos(\pi m x/a)\sin(\pi n y/b), \quad V_2=\sin(\pi m x/a)\cos(\pi n y/b),$$

$$V_3=\sin(\pi m x/a)\sin(\pi n y/b), \quad \tau_{13}=\cos(\pi m x/a)\sin(\pi n y/b),$$

$$\tau_{23}=\sin(\pi m x/a)\cos(\pi n y/b), \quad \tau_{33}=\sin(\pi m x/a)\sin(\pi n y/b),$$

which corresponds to pinned support, $r_s=e^{-i\omega t}$ $(s=1,\ldots,6)$, then by analogy with [4], these equations are transformed into the following system:

$$\int_{c_{k-1}}^{c_k}[-f_{1,3}^{(k)}-f_3^{(k)}\bar{a}+f_4^{(k)}(1/(\bar{G}_{13}^{(k)}e^{z\gamma^{(k)}}))]\,dz=0;$$

$$\int_{c_{k-1}}^{c_k}[-f_{2,3}^{(k)}-f_3^{(k)}\bar{b}+f_5^{(k)}(1/(\bar{G}_{23}^{(k)}e^{z\gamma^{(k)}}))]\,dz=0;$$

$$\int_{c_{k-1}}^{c_k}[f_1^{(k)}\bar{B}_{13}^{(k)}\bar{a}+f_2^{(k)}\bar{B}_{23}^{(k)}\bar{b}-f_{3,3}^{(k)}+f_6^{(k)}\bar{B}_{33}^{(k)}/e^{z\gamma^{(k)}}]\,dz=0;$$

$$\int_{c_{k-1}}^{c_k}\left[(f_1^{(k)}(\bar{B}_{11}^{(k)}\bar{a}^2+\bar{G}_{12}^{(k)}\bar{b}^2-\bar{\rho}\omega^2)+f_2^{(k)}(\bar{B}_{12}^{(k)}+\bar{G}_{12}^{(k)})\bar{a}\bar{b})\,e^{z\gamma^{(k)}}-f_{4,3}^{(k)}-f_6^{(k)}\bar{B}_{13}^{(k)}\bar{a}\right]dz=0;$$

$$\int_{c_{k-1}}^{c_k}\left[(f_1^{(k)}(\bar{B}_{12}^{(k)}+\bar{G}_{12}^{(k)})\bar{a}\bar{b}+f_2^{(k)}(\bar{B}_{22}^{(k)}\bar{b}^2+\bar{G}_{12}^{(k)}\bar{a}^2-\bar{\rho}\omega^2))\,e^{z\gamma^{(k)}}-f_{5,3}^{(k)}-f_6^{(k)}\bar{B}_{23}^{(k)}\bar{b}\right]dz=0; \tag{2.6}$$

$$\int_{c_{k-1}}^{c_k}[-f_1^{(k)}\bar{\rho}\omega^2e^{z\gamma^{(k)}}+f_4^{(k)}\bar{a}+f_5^{(k)}\bar{b}-f_{6,3}^{(k)}]\,dz=0 \quad (\bar{a}=\pi m/a,\ \bar{b}=\pi m/b).$$

Representing the solution of system (2.6) in the form:

$$f_1=\mu_1e^{(\beta-\gamma)z}, \quad f_2=\mu_2e^{(\beta-\gamma)z}, \quad f_3=\mu_3e^{(\beta-\gamma)z}, \quad f_4=\mu_4e^{\beta z}, \quad f_5=\mu_5e^{\beta z}, \quad f_6=\mu_6e^{\beta z},$$

we arrive at a system of homogeneous algebraic equations (the superscript which indicates the layer has been omitted)

$$-\mu_1(\beta-\gamma)-\mu_3\bar{a}+\mu_4(1/\bar{G}_{13})=0;$$

$$-\mu_2(\beta-\gamma)-\mu_3\bar{b}+\mu_5(1/\bar{G}_{23})=0;$$

$$\mu_1\bar{B}_{13}\bar{a}+\mu_2\bar{B}_{23}\bar{b}-\mu_3(\beta-\gamma)+\mu_6\bar{B}_{33}=0; \tag{2.7}$$

$$\mu_1(\bar{B}_{11}\bar{a}^2+\bar{G}_{12}\bar{b}^2-\bar{\rho}\omega^2)+\mu_2(\bar{B}_{12}+\bar{G}_{12})\bar{a}\bar{b}-\mu_4\beta-\mu_6\bar{B}_{13}\bar{a}=0;$$

$$\mu_1(\bar{B}_{12}+\bar{G}_{12})\bar{a}\bar{b}+\mu_2(\bar{B}_{22}\bar{b}^2+\bar{G}_{12}\bar{a}^2-\bar{\rho}\omega^2)-\mu_5\beta-\mu_6\bar{B}_{23}\bar{b}=0;$$

$$-\mu_3\bar{\rho}\omega^2+\mu_4\bar{a}+\mu_5\bar{b}-\mu_6\beta=0.$$

Equating the determinant to zero, we find the relationship between the parameters β and ω. In the general case, the sought functions are represented in the form:

$$f_1=C_l e^{(\beta_l-\gamma)z};\quad f_2=C_1\mu_{2l}e^{(\beta_l-\gamma)z};\quad f_3=C_l\mu_{3l}e^{(\beta_l-\gamma)z};$$

$$f_4=C_l\mu_{4l}e^{\beta_l z};\quad f_5=C_l\mu_{5l}e^{\beta_l z};\quad f_6=C_l\mu_{6l}e^{\beta_l z}\quad (l=1,\ldots 6).$$

In the special case when $\gamma=0$ and the layer is isotropic,

$$\beta_1^2=(\pi m/a)^2+(\pi n/b)^2-\rho\omega^2/G_{12};\quad \beta_2^2=(\pi m/a)^2+(\pi n/b)^2-\rho\omega^2/C_{11}.$$

The same expressions were obtained in [6, 8, 9]. Here, the sought functions appear as:

$$f_1=C_1e^{\beta_1 z}+C_3e^{\beta_2 z}+C_4e^{-\beta_1 z}+C_6e^{-\beta_2 z};$$

$$f_2=C_2e^{\beta_1 z}+C_3e^{\beta_2 z}+C_5e^{-\beta_1 z}+C_6e^{-\beta_2 z};$$

$$f_3=C_1\mu_{31}e^{\beta_1 z}+C_2\mu_{32}e^{\beta_1 z}+C_3\mu_{33}e^{\beta_2 z}+C_4\mu_{34}e^{-\beta_1 z}+C_5\mu_{35}e^{-\beta_1 z}+C_6\mu_{36}e^{-\beta_2 z};$$

$$f_4=C_1\mu_{41}e^{\beta_1 z}+C_2\mu_{42}e^{\beta_1 z}+C_3\mu_{43}e^{\beta_2 z}+C_4\mu_{44}e^{-\beta_1 z}+C_5\mu_{45}e^{-\beta_1 z}+C_6\mu_{46}e^{-\beta_2 z};$$

$$f_5=C_1\mu_{51}e^{\beta_1 z}+C_2\mu_{52}e^{\beta_1 z}+C_3\mu_{53}e^{\beta_2 z}+C_4\mu_{54}e^{-\beta_1 z}+C_5\mu_{55}e^{-\beta_1 z}+C_6\mu_{56}e^{-\beta_2 z};$$

$$f_6=C_1\mu_{61}e^{\beta_1 z}+C_2\mu_{62}e^{\beta_1 z}+C_3\mu_{63}e^{\beta_2 z}+C_4\mu_{64}e^{-\beta_1 z}+C_5\mu_{65}e^{-\beta_1 z}+C_6\mu_{66}e^{-\beta_2 z}.$$

The parameters β may be real or complex. It is more convenient to represent them in complex form and regard the real numbers as a special case of the complex values.

We form a resolvent system of equations of the order $6n$ by satisfying the interlaminar contact conditions and the conditions at the surface. Equating the determinant of this system to zero, we obtain an equation to determine the natural frequencies of vibration of the structure.

3. Examples. As test problems, we compared our results with the analytical solution in [9] and the solution in [1]. It was found that the results agree fully with one another. Our solution differs only slightly from the three-dimensional numerical solution obtained in [7] for laminated orthotropic slabs.

1. We calculated the natural frequencies of vibration of a thin square ($a/h=40$) one-layer orthotropic slab having the following physico-mechanical characteristics: $\bar{E}_1e^{\gamma z}=10\bar{E}_2e^{\gamma z}=10\bar{E}_3e^{\gamma z}$; $\rho(z)=\bar{\rho}e^{\gamma z}$; $\gamma=1/h$; $\nu_{21}=0.03$; $\nu_{31}=0.03$; $\nu_{32}=0.3$; $G_{12}=G_{13}=0.48544E_1$; $G_{23}=0.48544E_2/2$; $c_0=-c_1$. Table 1 shows the frequencies of natural

TABLE 1

λ_1		λ_2		λ_3	
T	K	T	K	T	K
0.030467	0.030643	0.46683	0.46683	1.0322	1.0323

TABLE 2

$\bar{z}$	$T1$			$T2$		
	$\lambda_1 = 1.5359$			$\lambda_1 = 1.5357$		
	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$
−0.5	0.3365	0.0906	1.0000	0.3361	0.902	1.0000
0.0	0.0485	0.0140	1.0232	0.0482	0.0144	1.0232
0.5	−0.2462	−0.0872	1.0202	−0.2466	−0.0875	1.0201
$\bar{z}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$
0.0	4.1592	1.6249	0.1399	4.1571	1.6238	0.1419
$\bar{z}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$
− 0.5	−13.043	−0.7366	7.9008	−13.581	−0.7657	8.2206
0.0	−3.0465	−0.1375	1.9070	−3.1601	−0.1467	1.9917
0.5	26.011	1.6833	− 16.765	24.985	1.6187	−16.112

TABLE 3

$\bar{z}$	$T1$			$T2$		
	$\lambda_2 = 3.7337$			$\lambda_2 = 3.7337$		
	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$
− 0.5	8.2655	−17.909	1.0000	8.2718	−17.924	1.0000
0.0	8.0865	−18.036	0.1545	8.0931	−18.051	0.1533
0.5	8.1734	−17.947	0.6859	8.1806	−17.962	− 0.6874
$\bar{z}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$
0.0	0.4265	0.2029	−2.8136	0.4258	0.2029	− 2.8152
$\bar{z}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$
− 0.5	− 297.19	59.335	− 178.41	−310.07	61.913	− 186.17
0.0	− 479.50	98.097	− 303.48	−500.27	102.39	− 316.66
0.5	− 798.11	161.98	− 491.52	−766.22	155.49	− 471.82

TABLE 4

$\bar{z}$	$T1$			$T2$		
	$\lambda_2 = 8.1827$			$\lambda_2 = 8.1827$		
	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$
−0.5	−2.0733	−0.7807	1.0000	−2.0727	−0.7804	1.0000
0.0	−2.2707	−1.0876	0.1394	−2.2698	−1.0873	0.1374
0.5	−2.1675	−0.8843	−0.6604	−2.1663	−0.8836	−0.6610
$\bar{z}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$
0.0	0.2201	0.0716	−15.802	0.2225	0.0733	−15.804
$\bar{z}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$
−0.5	80.629	5.3939	−52.797	84.035	5.6218	−55.027
0.0	139.82	6.2875	−102.43	145.97	6.7602	−106.75
0.5	229.34	16.041	−153.47	219.86	15.375	−147.11

TABLE 5

$\bar{z}$	$T1$			$T2$		
	$\lambda_1 = 6.6303$			$\lambda_1 = 6.6238$		
	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$	$\bar{U}_1$	$\bar{U}_2$	$\bar{U}_3$
−0.5	0.3064	0.5811	1.0000	0.3060	0.5823	1.0000
0.0	0.0548	0.1406	0.6682	0.0547	0.1414	0.6673
0.5	0.0	0.0	0.0	0.0	0.0	0.0
$\bar{z}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$	$\bar{\sigma}_{13}$	$\bar{\sigma}_{23}$	$\bar{\sigma}_{33}$
0.0	7.3772	0.8352	−21.386	7.3991	0.8242	−21.359
$\bar{z}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$	$\bar{\sigma}_{11}$	$\bar{\sigma}_{22}$	$\bar{\sigma}_{12}$
−0.5	−12.452	−2.5881	16.418	−12.968	−2.7024	17.132
0.0	−9.5198	−5.5723	5.9610	−9.6607	−5.6091	6.2374
0.5	−8.4164	−6.6684	0.0	−8.4057	−6.6599	0.0

TABLE 6

$h^{(1)}/h$,	10^{-2}	10^{-4}	10^{-6}	10^{-8}	0
T	0.48610	0.25827	0.21271	0.21214	0.21214
[5]	0.48593	0.25696	0.21113	0.21056	0.21056

vibration $\lambda = 10\,\omega\,(\bar{\rho}\,h^2/\bar{E}_1)^{1/2}$ obtained on the basis of the proposed solution (*T*) and the classical mode (*K*). The results agree well with one another.

2. We used two design variants to obtain the same slab but with $a/h = 5$. In the first case (*T*1), we examined a one-layer slab with a variable elastic modulus. In the second case (*T*2), we examined a 12-layer slab with layers of equal thickness. The elastic characteristics of the second slab, which were constant within each layer, were approximated by a step function. Tables 2, 3, and 4 show the results of calculations of the natural frequencies of vibration $\lambda = 10\,\omega\,(\bar{\rho}\,h^2/\bar{E}_1)^{1/2}$ with $m = n = 1$. The modes corresponding to those frequencies are also shown. Nearly complete agreement between the frequencies and the displacements obtained by the first two design variants is seen for the first three types of vibrations. The errors are somewhat higher for the stresses but are still negligible. Table 5 shows the fundamental mode of vibration for the same slab when its bottom surface is rigidly fastened. As in the case of a freely sagging slab, the results obtained by the two variants are found to agree with each other.

3. Now we perform calculations for a three-layer slab that is symmetric through its thickness and has the following physico-mechanical characteristics: $E^{(1)}/E^{(2)} = 1000$; $\nu^{(1)} = \nu^{(2)} = 0.3$; $\rho^{(1)}/\rho^{(2)} = 10$; $a/h = 5$. Table 6 shows the fundamental frequencies of free vibration of such a slab ($\lambda = \omega\,(\rho^{(2)}\,h^2/E^{(2)})^{1/2}$) with variation of $h^{(1)}/h$. The results shown were obtained by using the method proposed here and by adapting the model in [5] to the solution of dynamic problems. The proposed method makes it possible to reduce the thickness of the outer layers to zero without loss of accuracy in the calculations. The result obtained with $h^{(1)}/h = 0$ is for a one-layer slab.

Thus, a method has been developed for determining the natural frequencies of vibration in a three-dimensional formulation on the basis of analytical separation of the variables. In contrast to [1], analytical means are used to solve boundary-value problems for systems of differential equations describing free vibrations of a slab. Using an analytical method makes it possible to accurately determine the natural frequencies and modes for slabs with layers differing significantly in thickness. The examples of slab design presented here add to the literature data and can be used to test applied models.

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