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NONLINEAR PROBLEMS OF THE DYNAMICS OF ELASTIC SHELLS PARTIALIY FILLED WITH A LIQUID

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Theoretical and experimental investigations of the nonlinear vibrations and dynamic stability of thin shells partially filled with a liquid are reviewed. The paper deals with the basic laws governing the dynamic high-deflection deformation of carrying shell structures and the considerable vibrations of the free liquid surface due to the natural, forced, and parametrically excited vibrations of the combined system and also due to impulse loads acting on the carrying object. The nonlinear dynamic interaction of shells with a liquid filler is analyzed with allowance for the wave motions of the free liquid surface.

Introduction. Problems on the nonlinear dynamic interaction of elastic shells with liquids filling them pertain to scientific and technical problems of continuum mechanics that during a long time (more than four decades) have attracted considerable attention of researchers, engineers, and designers. This is, first of all, due to the needs of aeronautical and space-rocket engineering whose creation and reliable operation is impossible without a preliminary and profound (with the use of strict calculation models) study of the mobility of large liquid masses (fuel) on the stress–strain state (SSS) and the stability of carrying elastic tanks (compartments) and, consequently, on the stability and controllability of the structure as a whole. Nonlinear problems of the dynamics of shells carrying a liquid are also important for the strength and stability analysis of: (a) bulky water reservoirs under the action of intensive periodic or impulse (for example, wind or seismic) loads, (b) different piping systems, and (c) ground and water transport intended for carrying liquid cargo, etc.

An extremely important domain of application of the problems under consideration is also biomechanics, in particular, those of its parts where the hydroelasticity of blood systems is studied.

The necessity of studying nonlinear problems of the dynamics of elastic liquid objects became obvious in the 50–60s owing to several experimentally revealed unusual physical effects that could not be accounted for using linear mathematical models. At that time, however, an appropriate strict theoretical basis for solution of these problems was absent. Nonlinear problems have some specific features that severely hamper the derivation and analysis of solutions. One of them, in particular, is that the boundary conditions specified on the wet surface of a carrying elastic body and on the free liquid surface are generally essentially nonlinear. Moreover, the strain state of an elastic shell as well as the shape of the perturbed free liquid surface are not known beforehand (and evolve in time); therefore, the domain of the velocity potential of the liquid turns out to be unknown. One more, equally important feature is associated with the necessity of matching frames of references in the analytical formulation of the matching conditions on the shell–liquid interface. As is known, the Euler variables are usually used to describe the equations of liquid motion [92, 96], while it is expedient to study the dynamics of shells in the Lagrange variables. Matching both systems is not difficult if the displacements of the points of the shell are small, which allows us to neglect the spatial deformation of the boundaries and, thus, to consider the Euler and Lagrange variables coincident. A difficulty arises in the presence of finite deflections, since, in this case, Euler's technique does not allow us to automatically determine the coordinates of the deformed boundaries of the liquid (i.e., the liquid–shell interface) [27, 28, 49, 50, 120].

Thus, the case in point is complex nonclassical boundary-value problems of mathematical physics arising at the interfaces between different knowledge domains (the mechanics of deformable solids, fluid mechanics, and nonlinear mechanics). The nonlinear and nonclassical nature of these problems, which constitute the class of complex problems of

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mechanics [42], necessitated the development of special approaches to their solution. It is these approaches and the related results of studies that are briefly discussed in the present paper. It should be noted that this review does not claim to be an exhausting and complete review of all the aspects of the complex problem on nonlinear liquid–shell interaction and reflects only some, the most typical (in the authors' opinion), trends in the present-day studies on this matter. Here, emphasis (with the basic analytical dependences and plots) is on the investigations that have been carried out at the S. P. Timoshenko Institute of Mechanics of the National Academy of Sciences of Ukraine, where the well-known scientific schools on the mechanics of hydroelastic and aerohydroelastic systems were created and have been successfully functioning for a long time. A distinctive feature of these studies is the use of multiparametric models of shell–liquid systems (with several modes of the carrying body and the liquid filler taken into account simultaneously) in solving nonlinear problems, which allows us to obtain a more real pattern of the dynamic interaction between the shell and the liquid.

The Background of the Nonlinear Theory. A methodological basis in the development of the nonlinear theory of the dynamic interaction of elastic shells with a liquid partially filling them was the fundamental principles used for constructing: (a) the linear theory of vibrations of shell–liquid systems and (b) the nonlinear theory of vibrations of a bounded volume of a liquid contained in fixed (or mobile) cavities of solids. The linearity of problems of hydroelasticity (internal problems associated with the dynamic analysis of a deformable shell filled with a liquid will be considered hereafter) assumes that one uses the linear equations of motion (vibrations) of the carrying object and the liquid filler and linear boundary conditions at the interface and on the free liquid surface. Since the principle of superposition is true for linear systems (i.e., an additive action leads to an additive response), methods for solving problems on the small vibrations of a shell–liquid system have been thoroughly developed and applied to date. The most commonly encountered among them are the variable separation methods (Fourier's), variation methods, etc. The results obtained in these studies were described in many publications and reviewed in detail in [7, 13, 34, 35, 137, 164, etc.]. The fullest bibliography (consisting of more than a hundred sources) on linear problems of the dynamics of elastic bodies carrying a liquid is presented in the fundamental monograph [107] by G. N. Mikishev and B. I. Rabinovich. It and other known monographs [1, 25, 26, 74, 75, 93, 99, 105, 113, 122, 136, 139, 140, 152, 165] contain numerous specific solutions of linear problems on the vibrations of shell–liquid systems. The basic objective of these problems was to determine the fundamental frequencies and modes of vibrations of these systems, which allows us to predict dangerous resonance regimes that may occur under external periodic loads and to reveal the laws governing the deformation of both the carrying shell itself and the free liquid boundary under resonance conditions. Considerable attention was also focused on the substantiation of the truncated hierarchy, i.e., selection of the necessary number of degrees of freedom that correctly reflect the response characteristics of the initial system (within some specified frequency range), consideration of problems concerning the applicability domain of the linear theory, the effect of the wave movement of liquids on the fundamental frequencies, and other problems directly related to nonlinear problems.

A great many studies are also devoted to the nonlinear vibrations of a liquid filling rigid vessels. The most significant results obtained in this domain were described in the generalizing monographs [9, 10, 30, 47, 94, 95, 97, 98, 106, 110, 112, 113, 118, 153, 165, 167, etc.]. In solving nonlinear problems, the original technique developed by G. S. Narimanov and published in his papers [116, 117] is mainly used. By this method, the unknown perturbed free liquid surface Σ is represented by a Fourier series in terms of some complete system of functions that is orthogonal to the nonperturbed surface Σ_0 and has the generalized coordinates of the perturbed surface as coefficients. The special recursion scheme proposed by Narimanov allows us to reduce the initial nonlinear problem to some sequence of inhomogeneous boundary-value problems for domains with fixed boundary conditions (specified, in particular, on the surface Σ_0).

Many authors solve nonlinear problems of the dynamics of a liquid and bodies containing a liquid by using direct methods based on variation principles [21, 94, 95, 98, 106, 112, 113, 151, etc.].

Theoretical studies carried out on the basis of different approaches have accounted for many nonlinear effects revealed in the vibrations of solids containing a liquid. Among them are the anisochronism of the vibrations of the free liquid surface, quenching in the resonance regions with changing over to qualitatively new dynamic conditions (in particular, liquid whirling), the asymmetry of the wave profile (the hump height is greater than the trough depth), splashing of the liquid near the walls, etc. [4, 43, 105, 106, 165, 166, 169, 173, etc.]. It was established that, for cylindrical bodies, these effects are clearly manifested for the vibration amplitudes of the liquid $b > 0.25R$ (R is the radius of the body). For $b \approx 0.15R$, the linear theory yields satisfactory results [105, 106].

Afterwards, all these approaches were extended to the case where objects partially filled with a liquid are thin-walled shell structures.

Formulation of Nonlinear Problems. The "nonlinear" period in the development of studies of the vibrations of shells with a liquid originates from the works of the following American scientists: Lindholm, Chu, Kana, Abramson, Dodge, and Craig [165, 170, 171, 174, 175, 177, 178, etc.]. They described and discussed experimentally revealed specific effects that are due to the complex essentially nonlinear mechanisms of dynamic interaction and energy interchange between the elastic, usually flexural vibrations of a carrying solid and the wave motions of the liquid free surface. Some of the effects were manifested even for very small vibration amplitudes (having, for example, the order of the wall thickness) of the solid. Among the most abundant effects are:

(i) the steady-state low-frequency high-amplitude axisymmetric and nonaxisymmetric motions (of the plane-wave type) of the free liquid surface excited by the high-frequency forced vibrations of the elastic shell walls;

(ii) excitation of the axisymmetric and nonaxisymmetric subharmonic vibrations of the free liquid surface having the order of half the excitation frequency of the shell;

(iii) excitation of the low-frequency large-amplitude rotary movements of the free liquid surface (of an axisymmetric form) under the conditions of the high-frequency small-amplitude vibrations (of the circumferential-wave type, i.e., propagating in the circumferential direction) of a cylindrical shell, which, in turn, cause the low-frequency vibrations of the shell walls.

It is obvious that all of these effects can be accounted for only through the analysis of the corresponding nonlinear multiparameter mathematical models, since the linear levels of relations between the shell and the liquid, which are schematized by models, and simplified ones (in particular, one- or two-parameter models), do not allow us to describe the complex energy transfer in a bound elastic liquid system from its elastic component to the liquid and vice versa. Moreover, specific nonlinear effects were revealed in the studies carried out in the 70s and later [2, 22, 168, etc.]. Approximate theoretical justification of some nonlinear phenomena (mentioned above) using the nonlinear theory of the second and third orders was attempted in [39, 165, 168, 170, etc.]. The monographs [90, 118] present (in the introductions) a more detailed review of theoretical and experimental investigations (mainly conducted in the 60–80s) of the nonlinear vibrations of elastic shells filled with a liquid with a free surface. Hereafter, emphasis will be on the theoretical studies on the subjects being discussed. These investigations have been conducted during the last two decades. These years constitute a period of intensive development of effective analytical and numerical-analytical approaches to the solution of multiparameter problems of the nonlinear dynamics of a shell–liquid system. A number of interesting scientific and practically important results have been obtained using those approaches.

It should be noted that nonlinear formulations of problems of the dynamics of thin elastic shells containing a liquid are abundunt. It is natural that each such formulation will dictate which method for solving these problems to choose. Of abundance are also studies of the mutual motion and interaction of the carrying shell and the carried liquid (which is usually assumed ideal and incompressible) under the action of some specified forces. Let us briefly formulate this problem as applied to circular cylindrical shells bearing in mind that it is precisely these shells that are most often the subject of inquiry in nonlinear problems of hydroelasticity and that possess many properties characteristic of shells of arbitrary type.

Assume that a shell partially filled (up to a level h_0) with a liquid is subject to external radial pressure nonuniformly distributed over the lateral surface of the liquid $q = q_0(x, y) S_1(t)$ and is compressed along the edges by dynamic forces of the form $N_x = N_0 + N_1(t)$ (Fig. 1) ($N_0 = \text{const}$, $q_0(x, y)$, $S_1(t)$, and $N_1(t)$ are some given functions of spatial coordinates and time). The direct axis of the shell x is oriented along the field of bulk forces (the origin 0 is at the center of the flat bottom).

The dynamic equations of the shells correspond to the well-known equations of the theory of shallow finite-deflection shells [23, 24, 39, 48, 115, 123, 172, etc.], i.e., they have the following form with allowance for the load hydrodynamic term:

$$\frac{D}{h}\nabla^4 w = \frac{1}{R}\frac{\partial^2\phi}{\partial x^2} - \rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2\phi}{\partial x^2}$$

$$-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \frac{q}{h} - \left.\frac{P_h}{h}\right|_S - \frac{N_x}{h}\frac{\partial^2 w}{\partial x^2} + \varepsilon_0\rho\frac{\partial w}{\partial t},$$

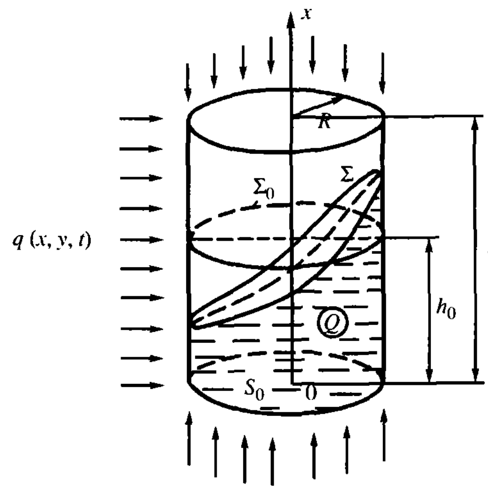

Fig. 1

$$\frac{1}{E}\nabla^4\phi = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{1}$$

Here, $w = w\,(x, y, t)$ is the transversal deflection (positive if directed toward the center of the curvature), D is the cylindrical rigidity, E is the elastic modulus, h is the thickness of the shell, ϕ is the function of stresses on the median surface, ε_0 is the integral damping factor, ρ is the density of the shell material, and P_h is the transversal hydrodynamic pressure acting on the wet lateral surface S of the shell and determined on the basis of the Lagrange–Cauchy integral [92, 96]

$$\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{P_h - P_0}{\rho_0} + U\right)\Bigg|_S = 0, \tag{2}$$

where φ is the velocity potential of the liquid, P_0 is the stress at the liquid–gas interface, ρ_0 is the density of the liquid, and U is the potential of the bulk forces.

The boundary-value problem for determination of the potential φ in cylindrical coordinates $x\,r\,\theta$ is formulated as follows [118, 136]:

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0 \quad \text{in } Q \tag{3}$$

(Q is the volume occupied by the liquid, $0 < r < R + w$; $0 < \theta < 2\pi$; $0 < x < h_0 + \xi$),

$$\frac{\partial\varphi}{\partial r}\bigg|_{r=0} < \infty, \quad \frac{\partial\varphi}{\partial x}\bigg|_{S_0} = 0, \quad \frac{\partial\varphi}{\partial r}\bigg|_S = -\frac{\partial w}{\partial t} - \nabla\varphi\,\nabla w,$$

$$\frac{\partial\varphi}{\partial x} - \nabla\xi\,\nabla\varphi = \frac{\partial\xi}{\partial t} \quad \text{on } \Sigma. \tag{4}$$

Here, S_0 is the bottom of the shell and ξ is the vertical displacements of the points of the free liquid surface relative to the nonperturbed plain Σ_0 ($x = h_0$).

Except for kinematic conditions (4), the potential φ must also satisfy the dynamic condition on the free surface Σ

$$\frac{\partial \varphi}{\partial t}+\frac{1}{2}(\nabla \varphi)^2+g\xi=0 \tag{5}$$

(g is the acceleration of gravity).

The mathematical complexity of system (1)–(5) necessitates using different hypotheses and assumptions to maximally simplify the formulation of the initial problem, keeping its practical importance. The simplifications concern both the equations of motion of the shell (or the liquid) and the boundary conditions on the boundary of the domain Q. However, these simplifications are not always justified and logically substantiated. In many cases, their selection is more likely to depend on the availability of methods for solving problems of the nonlinear dynamics of elastic liquid systems than on their practicability and importance.

Many authors believe that in order to reliably describe the dynamic interaction of shells and a liquid filling them, it is sufficient to use a calculation model that takes into account small (linear) vibrations of a carrying elastic body and nonlinear (with high amplitudes) vibrations of the free liquid surface. Such an approach is valid, since the domain of nonlinear perturbations is much wider for the liquid than for the shell [15, 17, 29, 121, etc.].

Other authors, in reverse, consider the nonlinear equations of deformation of a shell and the related linear equations of a liquid flow, i.e., they use the hypothesis on small wave motions of a liquid. Such an assumption might be considered valid for rather flexible carrying shells. At the same time, the wave motions of the liquid surface are sometimes neglected altogether in the calculation models [90, 124, 128, etc.]. The authors of some papers suppose that the energy of a shell–liquid system substantially exceeds the energy of the wave motions of the liquid, which, in G. N. Mikishev's and B. I. Rabinovich's opinion [107], is justified in problems on the longitudinal vibrations of shells containing a liquid (as opposed to lateral vibrations).

In other authors' opinion, this assumption may be considered acceptable for relatively high vibration modes of shells (with many nodal lines). In this case, the vibration modes of a dry shell and the same shell containing a liquid differ little (at the same time, for the lowest vibration modes, the kinetic energy of the liquid can considerably exceed that of the shell [161, 177], which, naturally, results in a considerable difference in the modes).

The boundary conditions were also simplified. So, in the overwhelming majority of cases, a linear variant of these conditions was used at the interface between the liquid and the shell walls, i.e.,

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=R}=-\frac{\partial w}{\partial t}. \tag{6}$$

The boundary conditions on the free liquid surface were usually as follows:

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=h_0}=\frac{\partial \xi}{\partial t}, \quad \left.\frac{\partial \varphi}{\partial t}\right|_{x=h_0+\xi}=-g\xi \tag{7}$$

or (if the waves in the liquid were not taken into account)

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=h_0}=0 \quad (\varphi|_{x=h_0}=0). \tag{8}$$

Despite the mentioned simplifications, the initial system (1)–(5) will possess nonlinear properties with all the ensuing consequences (since nonlinearity in one link, i.e., in an individual equation or boundary condition, will spread to the whole system during concurrent vibrations).

Approximate solutions of the above-mentioned system (with allowance for the simplifications formulated) are usually constructed by the following scheme.

The Bubnov–Galerkin variation method (or the Hamilton–Ostrogradskii principle of least action) is used to reduce a continuous system described by partial differential equations to a system with a countable number of degrees of freedom described by ordinary differential equations. To determine the velocity potential φ, different strict and approximate approaches, which are, in particular, stated in the establishing studies [38, 135, 155, 156, 161, 162] and also in [90, 104, etc.], are used. In this case, the well-known Lamé principle of superposition is mostly applied [176], namely, it is assumed that

$\varphi = \varphi_1 + \varphi_2$, where the function φ_1 characterizes the velocity field of the liquid with a still free surface in a deformable shell, and φ_2 describes the wave motions of the liquid in a rigid shell and, together with φ_1, provides satisfaction of dynamic condition (5) or (7) on Σ.

To construct the equations corresponding to the hydrodynamic part of the problem, the dynamic condition on the liquid surface (5) is used in some cases (which, in particular, is characteristic of Narimanov's technique), and, in other cases, the functional $L=T-V$ is determined first (T is the kinetic energy of the vibrating shell and liquid and V is the potential energy) and is then used to derive the Lagrange equations (the dynamic condition is satisfied by appropriate selection of the velocity potential φ of the liquid [179]).

If we represent the deflection of the shell w by the traditional two-parameter expansion [24, 88, 89]

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[f_1^{n,m} \cos s_n y + f_2^{n,m} \sin s_n y \right] X_m(x) \tag{9}$$

(here, X_m are axial coordinate functions, $s_n = \frac{n}{R}$, and n is the number of full circumferential waves) and use the corresponding mode to disturb the free liquid boundary [89, 118]

$$\xi = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \left[\xi_1^{n,k} \cos n\theta + \xi_2^{n,k} \sin n\theta \right] J_n\left(\frac{\mu_{nk} r}{R} \right) \tag{10}$$

(J_n is a Bessel function of the nth order of the real argument and μ_{nk} are the eigenvalues determined from the equations $J_n'(\mu) = 0$), then we obtain the following system of connected equations for determination of the unknown functions $f_i^{n,m}$ and $\xi_i^{n,k}$ [58, 89, 90]:

$$\ddot{f}_i^{n,m} + \omega_{n,m}^2 f_i^{n,m} + \sum_{\substack{m_1 \\ m_1 \neq m}} q_{m_1}^{n,m} \ddot{f}_i^{n,m_1} + \sum_{j=1} \delta_j^{n,m} \ddot{\xi}_i^{n,j} = \varepsilon F_i^{n,m}(\ldots),$$

$$\ddot{\xi}_i^{n,k} + \nu_{n,k}^2 \xi_i^{n,k} + \sum_{e} \gamma_e^{n,k} f_i^{n,e} = \varepsilon G_i^{n,k}(\ldots),$$

$$(n = 0, 1, 2, \ldots, \quad m, m_1, k, e = 1, 2, \ldots, \quad i = 1, 2). \tag{11}$$

Here, $\omega_{n,m}$ and $\nu_{n,k}$ are the fundamental frequencies of the shell (with the effect of the apparent additional masses of the liquid taken into account) and the free liquid surface (in a rigid tank), respectively, $q_{m_1}^{n,m}$, $\delta_j^{n,m}$, and $\gamma_e^{n,k}$ are constant coefficients depending on the physical and geometrical parameters of the shell and the liquid, $\varepsilon F_i^{n,m}$ and $\varepsilon G_i^{n,k}$ are time-dependent functions nonlinear with respect to the generalized displacements $\{f_i^{n,m}\}$ and $\{\xi_i^{n,k}\}$ and their derivatives, and ε is a small parameter ($\varepsilon > 0$).

As a special case, based on system (11), we can derive the equations of nonlinear vibrations of shells carrying a liquid that correspond to the simplified formulations mentioned above. Equations (11) may be used to analyze both steady-state (periodic) processes and transients. In this case, periodic solutions can be constructed and analyzed for stability using different approximate methods of nonlinear mechanics [8, 101–103, 108, 111, 182, etc.].

It should be noted that many problems associated with the creation of reliable nonlinear dynamic models of shell–liquid systems are still open. Their solution necessitates special fundamental investigations that would allow us to work out certain scientifically justified recommendations as for the use in calculations of one simplification and hypothesis or another. Note that even general scientific discussions of this extremely important problem have not been organized yet.

Free Vibrations. The first theoretical studies of the nonlinear vibrations of shells partially filled with a liquid were carried out in the middle 60s by Yu. S. Shkenev [154] and E. T. Grigor'ev [40]. In the former case [154], the calculation model was based on the linear equations of vibrations of a liquid and the nonlinear dynamic equations of a shell, and in the latter case [40], vice versa, the vibration amplitudes of the liquid surface were assumed considerable, and the elastic displacements of the shell (longitudinal and radial) and the bottom were assumed small (axisymmetric vibrations were considered). The approach described in [40] was then developed in [41] and other studies. The solution of the problem was usually constructed using Narimanov's technique in combination with the Bubnov–Galerkin variation method. The potential φ was selected in the form [161]. Among the important results of this period is revealing the asymmetry of the vibrations of the free liquid surface relative to the surface Σ_0 (i.e., the dynamic-equilibrium position about which vibrations occur somewhat shifts).

The refined technique for calculation of the free vibrations of a liquid in a cylindrical tank with an elastic spherical bottom (a shallow model) was proposed by M. P. Petrenko [132–134, etc.]. This method allows us to examine the dynamics of an elastic liquid system for arbitrary values of the parameter $\bar{h}_0 = \frac{h_0}{R}$. This is due to the fact that the boundary conditions on the free liquid surface were satisfied exactly, as opposed to the previous studies.

It should be noted that one of the central problems in studying the free nonlinear vibrations of a shell–liquid system is determination of its fundamental frequencies as functions of the vibration amplitudes. Such problems have been studied in detail for special cases where the wave motions of a liquid in rigid cylindrical tanks are considered or the vibrations of empty cylindrical shells are analyzed. In the former case, as shown in many studies [4, 46, 47, 87, 118, etc.], the fundamental frequencies of the liquid increase with increase in the amplitudes if the tanks are filled to relatively small depths $\left(\frac{h_0}{R} < h_0^* \approx 0.45 - 0.49\right)$ and decrease for large depths $\left(\frac{h_0}{R} > h_0^*\right)$. Both weak and strong amplitude–frequency dependences are also characteristic of dry elastic shells (without a liquid) [86].

A similar situation will naturally arise upon the mutual influence of both factors — the elasticity of a carrying body and the mobility of a liquid filling it. Let us consider an illustrative example where the wave motions of a liquid in a freely vibrating hinged cylindrical shell are not taken into account. Approximating the deflection w by the binomial expression [89]

$$w = (f_1 \cos s_n y + f_2 \sin s_n y) \sin \lambda_m x + f_3 \sin^2 \lambda_m x$$

$$\left(\lambda_m = \frac{m\pi}{l}, \quad m \text{ is the number of longitudinal half-waves}\right) \tag{12}$$

we obtain the following system to determine the dominant generalized displacements f_1 and f_2 (the function f_3 was found from the quasistatic variant of the problem [24]):

$$\ddot{f}_1 + \omega_1^2 f_1 + \gamma_1 (f_1^2 + f_2^2) f_1 + g_1 (f_1^2 + f_2^2)^2 f_1 = 0,$$

$$\ddot{f}_2 + \omega_1^2 f_2 + \gamma_1 (f_1^2 + f_2^2) f_2 + g_1 (f_1^2 + f_2^2)^2 f_2 = 0. \tag{13}$$

Here,

$$\omega_1^2 = \frac{\omega_{nm}^{(0)2}}{1 + m_{01}}, \quad \gamma_1 = \frac{\gamma_{01}}{1 + m_{01}}, \quad g_1 = \frac{g_{01}}{1 + m_{01}}, \tag{14}$$

$\omega_{nm}^{(0)}$ are the fundamental frequencies of the small (linear) vibrations of the empty shell, γ_{01} and g_{01} are constant parameters describing the nonlinear (up to the fifth order) elastic properties of this shell, and m_{01} are the parameters of the reduced mass ($m_{01} \geq 0$). As is seen from (14), the presence of the liquid, apart from decreasing the fundamental frequencies of the shell, somewhat smooths out the effect of geometrical nonlinearity (the amplitude–frequency responses (AFR) become more steep and come nearer to the corresponding linear curves). However, the qualitative nature of the geometrical nonlinearity of a shell (weak or strong) does not vary in this case. If we represent the solution of Eqs. (13) as [8] $f_i = a_i \cos(\omega_1 t + \vartheta_i)$ $(i = 1, 2)$, then it is easy to obtain from the averaged equations the two first integrals [87, 91, 182]

$$a_1^2 + a_2^2 = C_1, \quad \zeta(1 - \zeta)(1 - \cos 2\bar{\theta}) = C_2 \tag{15}$$

that completely describe the nonlinear relationship between the amplitude and the phase characteristics of a freely vibrating shell containing a liquid. In (15), $\zeta = \frac{a_1^2}{C_1}$, $\bar{\theta} = \vartheta_2 - \vartheta_1$, and $C_i = \text{const}$.

The functions ζ and $\bar{\theta}$ are determined from the equations

$$\frac{d\zeta}{dt} = \frac{2\chi}{\omega_1}\zeta(1 - \zeta)\sin 2\bar{\theta}, \quad \frac{d\bar{\theta}}{dt} = \frac{\chi}{\omega_1}(2\zeta - 1)(1 - \cos 2\bar{\theta}),$$

$$\chi = \frac{(\gamma_1 + g_1 C_1) C_1}{8}. \tag{16}$$

Integrals similar to (15) were also obtained in studying the nonlinear energy exchange between the natural modes of a liquid contained in a cylindrical vessel, which may be fixed on elastic supports [56, 73]. Integrals of the same type will be characteristic of the free vibrations of an elastic shell containing a liquid with its wave motions taken into account. To construct them, it is first necessary to transform a system of resolvent equations of the form (11) to normal (or, more precisely, quasinormal) coordinates and then to apply the methods of nonlinear mechanics. It is important to emphasize that integrals of the type (15) are due to the natural resonances in a shell–liquid system. In the general case, they cannot serve as a strict basis (as is sometimes affirmed) for reducing the number of degrees of freedom of this system, since they do not answer the basic question on the hierarchy of the influence of each of the degrees of freedom on the total energy of the system.

Note that the free (due to the initial conditions) vibration modes of shells and their filler gradually damp in time because of inevitable energy losses due to the dissipative forces. In theoretical analysis, the damping properties of such systems are not usually considered. Occasionally, they are conventionally allowed for by using the well-known hypothesis of viscous friction [12, 87, 90, 105, 107, etc.]. In this case, the damping factor is quite often assumed the same for modes with different wave parameters and for vibrations with arbitrary (including considerable) amplitudes, which cannot always be justified. As for experimental investigations of the damping properties of full-scale structures with a liquid, very few results were obtained in this subject area. Moreover, the experimental techniques developed to date allow us to determine only the general (integral) characteristics of damping, whereas the actual laws governing the formation of dissipative forces, which could be allowed for directly in the nonlinear dynamic equations of a shell and a liquid, remain, as a rule, unknown.

Some decrement characteristics obtained in vibration tests by the method of free vibrations of cylindrical and spherical shells filled with a liquid are presented in [33, 145–147]. Yu. A. Gorbunov [33] was the first to notice that with increase in the vibration amplitudes of a carrying shell the logarithmic decrements not only can increase (which seems to be natural), but also decrease. It was also established that different structural and force factors (reinforcing elements, axial compression loads, etc.) contribute considerably to the dissipation of energy. V. F. Sivak and A. I. Telalov [145–147] showed, in particular, that the damping capacity of longitudinally reinforced cylindrical shells carrying a liquid heavily depends on both the amplitudes of their flexural vibrations (in the case of small amplitudes, this dependence is nearly linear) and the presence of contact of the liquid with the stringers. The logarithmic decrements tend to increase upon approaching any resonance state of the system. It was also established that, as a first approximation, there exists a proportional dependence between the level (depth h_0) of the filling of the shell and the corresponding logarithmic decrement of vibrations (as h_0 increases, the decrement increases too).

Periodically Excited Vibrations. In parallel with the studies of free vibrations, nonlinear problems of the dynamics of shell–liquid objects under external periodic forces were solved. The early stage is characterized by using simplified models of the system (basically, the unimodal vibrations of shells were studied, the wave motions of the liquid–gas interface were neglected, the geometrical nonlinearities in the dynamic equations of the shell or the free liquid surface were not taken into account, the nonlinearities in the boundary conditions were neglected, etc.). Principal attention was drawn to the analysis of the resonance modes of vibrations usually corresponding to the principal harmonic or parametric resonances. The basic studies in this subject area are briefly reviewed in the monographs [90, 118].

The most significant results in studies of the nonlinear forced vibrations of cylindrical shells partially filled with a liquid were obtained by F. N. Shklyarchuk, E. I. Obraztsova, V. S. Pavlovskii and V. G. Filin, L. G. Boyarshina, and other authors [62, 87, 90, 124, 129, 160].

If the wave motions of a liquid are neglected and approximation of the deflection of a shell in the double-mode form (12) is used, then its forced bending high-deflection vibrations will be described by Eqs. (13) with the right-hand sides containing the periodic functions of time $\frac{Q_{01}}{1+m_{01}}\cos\Omega t$ and $\frac{Q_{02}}{1+m_{02}}\cos\Omega t$, where Ω is the frequency of transversal excitation and Q_{0i} are constant parameters determining the level of this excitation for each of the modes involved. Let us take into account nonlinearity of the third order in these equations and assume that $Q_{02}=0$. Therefore, we obtain the equations

$$\ddot{f}_1+\omega_1^2 f_1+\gamma_1(f_1^2+f_2^2)f_1=\frac{Q_{01}}{1+m_{01}}\cos\Omega t,$$

$$\ddot{f}_2+\omega_2^2 f_2+\gamma_1(f_1^2+f_2^2)f_2=0. \tag{17}$$

Thus, the model of the forced vibrations of a shell with a liquid is a system of two nonlinearly coupled oscillators, one of which is subject to a periodic action, and the other is excited indirectly [32] due to the nonlinear connection and the natural resonance ($\omega_1=\omega_2$).

In the case of single-mode vibrations (when $f_2=0$ and $f_1\neq 0$), the AFR will correspond to the equation

$$\Omega^2=\omega_1^2+\frac{3}{4}\gamma_1 a_{10}^2\pm\frac{Q_{01}}{(1+m_{01})a_{10}}. \tag{18}$$

From here, it follows that the presence of a liquid somewhat narrows down the resonance band (in comparison with the case of an empty shell), since both branches (cophased and antiphased) of the AFR approach each other.

An analysis of the equations in variations shows that the solutions $a_{10}=a_{10}(\Omega)$ of Eq. (18) are stable outside the frequency band $\Omega_1^2<\Omega^2<\Omega_2^2$, where

$$\Omega_1^2=\omega_1^2+\frac{9}{4}\sqrt[3]{\frac{4}{9}\frac{Q_{01}^2\gamma_1}{(1+m_{01})^2}},$$

$$\Omega_2^2=\omega_1^2+\frac{1}{4}\sqrt[3]{\frac{4Q_{01}^2\gamma_1}{(1+m_{01})^2}},\quad(\gamma_1<0). \tag{19}$$

The bimodal vibrations with amplitudes a_1 and a_2 that are determined from the following equations [89] are stable in that band:

$$\Omega^2=\omega_1^2+\gamma_1 a_1^2\pm\frac{1.5\,Q_{01}}{(1+m_{01})a_1},$$

$$a_2^2=\frac{4}{3\gamma_1}\left(\Omega^2-\omega_1^2-\frac{\gamma_1}{4}a_1^2\right). \tag{20}$$

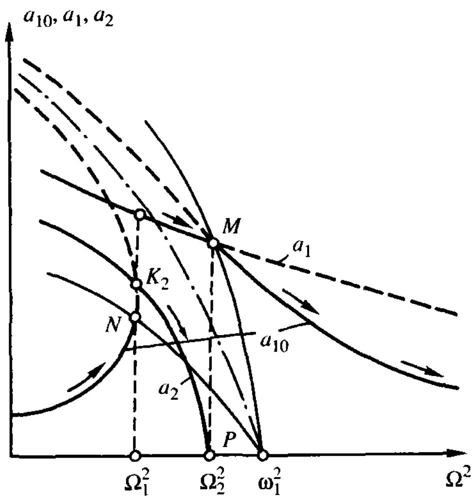

Fig. 2

Figure 2 demonstrates the positional relationship of the frequency curves $a_1(\Omega^2)$ and $a_2(\Omega^2)$ plotted based on system (20) with allowance for the conditions $\gamma_1 < 0$ and indication of stable sections (solid lines). The AFR for the single-mode regime $a_{10}(\Omega^2)$ is shown for comparison.

In [90, 129], the influence of the initial deflection w of a carrying shell on its forced nonlinear vibrations is examined with allowance for variation of the filling depth h_0. In this case, the fundamental frequencies of the shell ω_1 and ω_2 corresponding to the conjugate bending modes will differ slightly from each other, which will cause some transformation of the instability region of the steady-state regimes and will have an effect on the amplitudes of these regimes. Only single-mode resonance regimes of the vibrations of a standing-wave-type system were numerically analyzed in [90, 129].

T. S. Krasnopol'skaya and N. P. Podchasov [79, 80] studied the nonlinear forced vibrations of a liquid contained between two coaxial cylindrical shells, one of which (outer) is rigid and the other (inner) is elastic with a specified law of deformation in time, namely, corresponding to a bending wave propagating in the circumferential direction. They carried out the theoretical and experimental investigations in parallel. The dynamic equations of the free liquid surface were derived using the approach proposed by J. Miles [179]. These equations differ somewhat from those of Narimanov's method. It was shown that, apart from the traditional forced resonance vibrations of a liquid with a frequency Ω_w (Ω_w is the frequency of a traveling wave), transversal (corrugated) waves with the multiple frequency $\nu_1 \approx \frac{\Omega_w}{2}$ can be excited.

Problems on the stability and parametrically excited vibrations of liquid-containing shells subject to axial periodic excitation were first analyzed by Yu. S. Shkenev [154], B. N. Bublik and V. I. Merkulov [20], E. T. Grigor'ev [40, 41], Kana and Craig [174], and other authors. They reduced the stability problem to equations with periodic coefficients. Primary attention in these studies was drawn to the construction, usually using Bolotin's technique [11], of dynamic-instability regions (DIR) and to the analysis of the influence of the filling depth on those regions.

The cycle of studies performed by Shklyarchuk [157–160, etc.] was devoted to the effect of compressibility of a liquid on the parametric instability of cylindrical shells. The stability problem is solved in two stages. The axisymmetric vibrations of a shell filled with a liquid (the gravitation waves were not taken into account) are considered at the first stage, and the perturbed (nonaxisymmetric) motions caused by periodic forces N_θ acting in the median surface due to changes in the hydrodynamic pressure are studied at the second stage. It was established that allowance for the compressibility of the liquid decreases substantially the frequencies of the axisymmetric vibrations of shells. Simultaneously, the coefficient of apparent additional masses increases, which may result in some transformation of the DIR for the vibrations at the axisymmetric and nonaxisymmetric stages of deformation.

Problems on the parametric vibrations of a cylindrical shell with a liquid filled to different depths ($h_0 = h_0(t)$) were solved by G. E. Bagdasaryan and V. Ts. Gnuni [5] and G. M. Ulitin [150]. In [5], the problem was solved based on the quasistationary theory, which allowed the authors to restrict the study to small rates of change in the filling depth. Ulitin presents in [150] a technique for deriving the exact solution for the velocity potential of a liquid in a shell with a variable filling depth (the Green-function method and the apparatus of distributions were used [131]).

The next step in the studies of the parametric vibrations of cylindrical shells filled with a liquid was the nonlinear problems examined by E. I. Obraztsova and F. N. Shklyarchuk [125–127]. In vibration analysis, the deformation processes were divided into two stages — axisymmetric and nonaxisymmetric. The cases of both the rigid and elastic bottom of a shell were considered.

When the conjugate modes (12) (with an additional axisymmetric term $f_4(t)$ [24]) are allowed for, the nonlinear problem on the parametric vibrations of a cylindrical shell with a liquid is reduced to the system of equations with periodic coefficients [53, 64]

$$\ddot{f}_1 + (\omega_1^2 - H_1 \cos \nu t) f_1 + \gamma_1 (f_1^2 + f_2^2) f_1 = 0,$$

$$\ddot{f}_2 + (\omega_1^2 - H_1 \cos \nu t) f_2 + \gamma_1 (f_1^2 + f_2^2) f_2 = 0. \tag{21}$$

Here, ν is the frequency of parametric excitation, $H_1 = \dfrac{N_1 \lambda_m^2}{\rho(1 + m_{01})}$, and N_1 is the amplitude of this excitation.

The amplitudes of parametric vibrations for single-mode regimes are determined by the dependences

$$a_i^2 = -\frac{4}{3\gamma_1}\left(\omega_1^2 - \frac{\nu^2}{4} \pm \frac{H_1}{2}\right), \quad a_j = 0,$$

$$(i, j = 1, 2, \quad i \neq j). \tag{22}$$

In the instability regions of these regimes, a double-mode regime with the following amplitudes is realized:

$$a_1^2 = \frac{\Delta \mp H_1}{\gamma_1}, \quad a_2^2 = \frac{\Delta \pm H_1}{\gamma_1}, \quad \Delta = \frac{\nu^2}{4} - \omega_1^2 \tag{23}$$

(here, the subscripts and superscripts must agree). The domain of its existence is the frequency band $\nu < \nu^* = 2\sqrt{\omega_1^2 - H_1}$.

V. S. Pavlovskii and V. G. Filin [90, 130] investigated the nonlinear double-mode parametric vibrations of an imperfect (with a small initial deflection) cylindrical shell partially filled with a liquid (the gravitation waves were not taken into account). The AFRs corresponding to the principal parametric resonance were plotted, and their dependence on the filling depth of the shell and on the amplitude parameters of the initial deflection was studied. The resonance modes of vibrations were analyzed for stability.

Krasnopol'skaya and Lavrov attempted in [78] to allow for the wave motions of liquid in studying the nonlinear parametric vibrations of an elastic cylindrical shell. The case of kinematic excitation of the system with an excitation source of limited intensity is considered [76]. To approximate the dynamic deflection w, the beam Krylov functions obeying the conditions of cantilever fixation of the shell are used [149] in combination with conjugate (circumferential) modes. As a result of approximate solutions of the problem (taking into account two (out of four) principal coordinates of the system directly resonating with the external excitation), the effects of braking the shaft of an electromotor and changes in the characteristics of steady-state vibration processes in a hydroelastic system (as compared to the case of ideal excitation) were revealed.

The interaction of the wave motions of the free liquid boundary with the bending bimodal vibrations of a cylindrical shell under resonance condition was examined by L. G. Boyarshin in [14] (nonlinearity was allowed for in the hydrodynamic part of the problem, and the equations of the shell corresponded to a linear model).

In several studies [54, 64, 70, etc.], features of the nonlinear dynamic deformation of cylindrical liquid-containing shells subject to combined (transversally longitudinal) periodic loading (the waves on the liquid surface are neglected) were

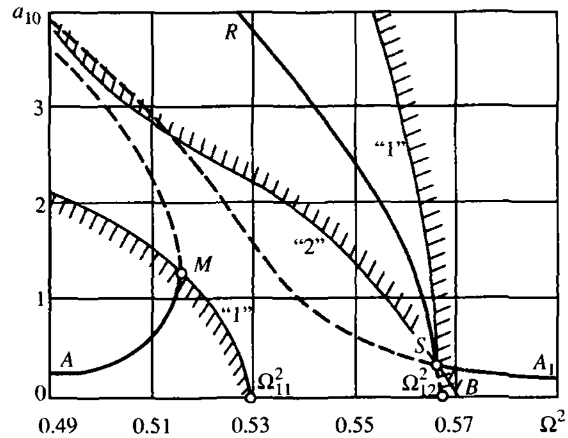

Fig. 3

considered in a simplified formulation. Figure 3 illustrates a typical AFR of a carrying shell [64] corresponding to its deformation in the form of a standing wave (when $\omega_1 \approx \Omega = \nu/2$) under specified loading. Here, boundaries 1 limit the stability of the frequency curves *AM* and $A_1 S$ characteristic of purely forced vibrations (Fig. 2), and $\Omega_{11}^2 = \omega_1^2 - H_1$, $\Omega_{12}^2 = \omega_1^2 + H_1$; and boundaries 2 limit the stability of the curve *RB* characteristic of parametric vibrations. In the instability regions of the standing wave, bimodal vibrations will be realized, whose amplitudes a_1 and a_2 are determined from equations similar to (20)

$$\Omega^2 = \omega_1^2 + \gamma_1 a_1^2 - 2 H_1 \pm \frac{1.5\, Q_{01}}{(1 + m_{01})\, a_1},$$

$$a_2^2 = \frac{4}{3\gamma_1} \left(\Omega^2 - \omega_1^2 + H_1 + \frac{\gamma_1 a_1^2}{4} \right). \tag{24}$$

It was shown in [64] that the presence of the initial deflection w_0 changes a little the amplitudes of resonance vibrations and influences substantially the stability region of steady-state regimes.

Nonlinear Wave Processes. All the above-discussed theoretical approaches to the investigation of the nonlinear vibrations of a shell–liquid system are traditional and based on the classical representation of the sought-for deflection of the shell and the profile of the free liquid surface. In this representation, the unknown functions of time $\xi_k(t)$ and $f_k(t)$ (generalized coordinates) and the functions of spatial coordinates (natural modes) are separated termwise. Meanwhile, to calculate wave processes in the mentioned systems, it is expedient to use the mixed (space–time) mode (in the class of traveling waves), since it allows us to reduce the determination of the deformation parameters of the shell and the liquid surface to the solution of a system of nonlinear equations without natural-resonance-type singularities [58, 87], which considerably simplifies the investigations. This is achieved by using special wave coordinates (amplitude–phase) that have different physical dimensions (as is known, the conventional coordinates ξ_k and f_k are equidimensional). It should be noted that the wave processes (mainly unsteady) in shells with a liquid considered within the framework of linear and nonlinear models were studied earlier by many authors [25, 26, 36, 37, 44–46, 49, 84, 109, 143, 144, 148, etc.]. A. M. Bagno and A. N. Guz' present in [6] a rather detailed review of studies of wave propagation in prestressed elastic bodies interacting with a viscous compressible liquid.

A special technique for calculation of periodic nonlinear waves in shells carrying a liquid was developed and has been employed during the last decade mainly by the experts of the S. P. Timoshenko Institute of Mechanics of the Academy of Sciences of Ukraine [15, 16, 19, 55, 57, 58, 62, 63, 68, 69, 71, 85, 87, 89, etc.]. At the initial stage, it was used in examining the nonlinear free waves in a liquid contained in rigid cylindrical vessels [18, 87]. The equation of a perturbed surface had the following form, characteristic of wave decompositions:

$$\xi = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} b_{nk}(t) \sin\left[n\theta + \beta_{nk}(t) \right] J_n\left(\frac{\mu_{nk} r}{R} \right). \tag{25}$$

Here, b_{nk} and β_{nk} are the amplitude and phase characteristics of a wave process. To determine them, based on dynamic condition (5), the following system of equations was obtained:

$$\ddot{b}_{nk} + (\nu_{nk}^2 - \dot{\beta}_{nk}^2)\, b_{nk} = \varepsilon G_{nk}^{(1)}(\bar{b}, \dot{\bar{b}}, \bar{\beta}, \dot{\bar{\beta}}),$$

$$b_{nk}\ddot{\beta} + 2\dot{b}_{nk}\dot{\beta}_{nk} = \varepsilon G_{nk}^{(2)}(\bar{b}, \dot{\bar{b}}, \bar{\beta}, \dot{\bar{\beta}})$$

$$(n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots), \tag{26}$$

where ν_{nk} are the natural frequencies of the liquid, $G_{nk}^{(i)}$ are nonlinear analytical functions of the components $\{b_{nk}\}$, $\{\dot{b}_{nk}\}$, $\{\beta_{nk}\}$, and $\{\dot{\beta}_{nk}\}$ of the vectors $\bar{b}$, $\dot{\bar{b}}$, $\bar{\beta}$, and $\dot{\bar{\beta}}$, and ε is a small parameter ($\varepsilon > 0$). The special change of variables [58, 66, 71, 87]

$$b_{nk} = \sqrt{u_{nk} + v_{nk} \sin\psi_{nk}}, \quad \dot{b}_{nk} = \frac{\nu_{nk} v_{nk}}{b_{nk}} \cos\psi_{nk},$$

$$\beta_{nk} = \varphi_{nk} + \arctan \frac{u_{nk}\tan\frac{\psi_{nk}}{2} + v_{nk}}{M_{nk}}, \quad \dot{\beta}_{nk} = \frac{M_{nk}\nu_{nk}}{b_{nk}^2},$$

$$M_{nk} = \sqrt{u_{nk}^2 - v_{nk}^2}, \quad \psi_{nk} = 2(\nu_{nk} t + \vartheta_{nk}) \tag{27}$$

has allowed transforming system (26) to a standard form relative to the functions u_{nk}, v_{nk}, φ_{nk}, and ϑ_{nk}

$$\frac{du_{nk}}{dt} = \varepsilon\left\{ \frac{G_{nk}^{(1)} v_{nk}}{b_{nk}\nu_{nk}} \cos\psi_{nk} + \frac{G_{nk}^{(2)} M_{nk}}{b_{nk}\nu_{nk}} \right\},$$

$$\frac{dv_{nk}}{dt} = \varepsilon\left\{ \frac{G_{nk}^{(1)} u_{nk}}{b_{nk}\nu_{nk}} \cos\psi_{nk} + \frac{G_{nk}^{(2)} M_{nk}}{b_{nk}} \sin\psi_{nk} \right\},$$

$$\frac{d\varphi_{nk}}{dt} = \varepsilon\left\{ \frac{G_{nk}^{(1)} v_{nk}}{2 b_{nk}\nu_{nk}} \left(\frac{\sin\psi_{nk}}{v_{nk}} - \frac{\cos\psi_{nk}}{u_{nk}} \right) \right.$$

$$\left. + \frac{G_{nk}^{(2)}}{2 b_{nk}\nu_{nk}} \left(\sin\psi_{nk} + \frac{u_{nk}}{v_{nk}} \cos\psi_{nk} + \frac{v_{nk}}{u_{nk}} \right) \right\},$$

$$\frac{d\vartheta_{\nu\kappa}}{dt} = -\varepsilon\left\{ \frac{G_{nk}^{(1)} (u_{nk}\sin\psi_{nk} + v_{nk})}{2 b_{nk}\nu_{nk} v_{nk}} + \frac{G_{nk}^{(2)} M_{nk}}{2 b_{nk}\nu_{nk} v_{nk}} \cos\psi_{nk} \right\}. \tag{28}$$

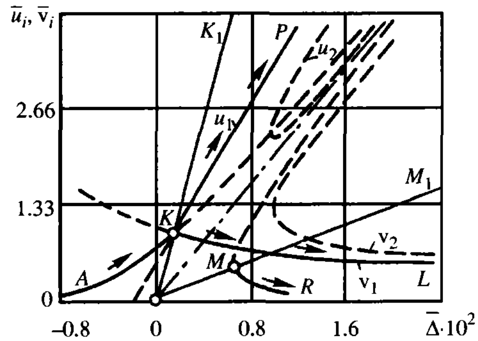

Fig. 4

From the averaged equations (28), the phase velocities v_{ph} of a circumferential wave in a liquid (in the case of approximation (25)) were determined depending on the amplitudes of this wave. It was shown that with increase in the amplitudes b_{nk} those velocities can both increase (for $h_0/R > 0.49$) and decrease (for $h_0/R < 0.49$).

The next step in the development of the wave approach was its use for calculation of periodically excited waves in cylindrical shells containing a liquid when the energy of wave motions of the liquid is low [58, 63, 87, 89]. The dynamic deflection w was represented in a form similar to (25)

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm}(t) \cos\left(n\frac{y}{R} - \alpha_{nm}(t) \right) \sin \lambda_m x . \tag{29}$$

In the case of one-wave approximation (29) ($n = m = 1$) (with allowance for a complementary axisymmetric mode), the AFRs were constructed for transversally and longitudinally excited waves. For radial excitation with frequency Ω, these characteristics correspond to the equations

$$\Delta = -\frac{3\gamma_1}{2} u_1 + \frac{Q_1^2}{2\gamma_1 v_1^2}, \quad u_1 = \mp v_1 + \frac{Q_1^2}{\gamma_1^2 v_1^2} \tag{30}$$

(($u_1 = u_{nm}$, $v_1 = v_{nm}$, $\Delta = \omega_1^2 - \Omega^2$, Q_1 is the amplitude parameter of an external force, $\omega_1 = \omega_{nm}$ is the fundamental frequency of a shell with a liquid, and $\gamma_1 = \gamma_{nm}$ is the parameter of nonlinear elasticity (14)). These AFRs are shown in Fig. 4 [63]. Here, $\bar{u}_1 = u_1/h^2$, $\bar{v}_1 = v_1/h^2$, $\bar{\Delta} = \Delta/\omega_1^2$, the superscripts correspond to u_1 and v_1, and the subscripts to u_2 and v_2. The following values of the parameters were used:

$$E = 2\cdot 10^{11}\ \text{Pa}, \quad \rho_0 = 1\cdot 10^3\ \text{kg/m}^3, \quad \rho = 7.8\,\rho_0,$$

$$h_0 = 0.25\,l, \quad \frac{h}{R} = 3.125\cdot 10^{-3}, \quad \frac{l}{R} = 2.5, \quad \mu = 0.3,$$

$$R = 0.16\ \text{m}, \quad m = 1, \quad n = 6, \quad Q_0 = 31.1\ \text{Pa}. \tag{31}$$

The frequency curves $u_0(\bar{\Delta})$ characterizing a standing wave in a shell (its stability is limited by the curves OK_1 and OM_1) are presented for comparison. The curves *KP* and *KL* correspond to the shell deformation in the form of a traveling wave.

When the shell is parametrically (axially) excited with frequency ν, the AFR of the wave process has the simpler form

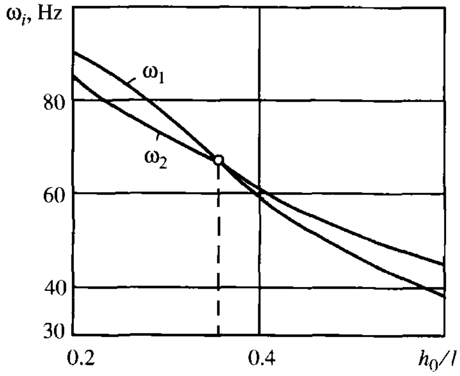

Fig. 5

$$\Delta_1 = -\frac{3\gamma_1}{2}u_1 \mp \frac{H_1 u_1}{2\sqrt{u_1^2 - C_1^2}},$$

$$C_1 = \frac{a_{nm}^4(0)\,\dot{\alpha}_{nm}^2(0)}{\omega_1^2} \neq 0, \quad \Delta_1 = \omega_1^2 - \frac{\nu^2}{4} \tag{32}$$

(H_1 is the level of longitudinal excitation).

If a shell is subject to the simultaneous action of both loads q and N_x, then the frequency curves for the case of a complex resonance $\omega_1 \approx \nu/2 = \Omega$ are determined by the system of equations

$$\Delta = \Delta_1 = -\frac{3\gamma_1}{2}u_1 \mp \frac{H_1 u_1}{2\nu_1} + \frac{Q_1^2}{2\nu_1^2\left(\gamma_1 \mp \frac{H_1}{\nu_1}\right)};$$

$$u_1 = \frac{Q_1^2}{\nu_1^2\left(\gamma_1 \mp \frac{H_1}{\nu_1}\right)^2} \mp \nu_1 \tag{33}$$

(either superscripts or subscripts should be used here).

Problems on the nonlinear interaction of several bending waves in shells carrying liquid masses are more complicated. They are encountered when the shells have close fundamental frequencies (practice testifies that the number of such frequencies may be two or three). Figure 5 illustrates, for example, the process of convergence of two fundamental frequencies of a simply supported orthotropic cylindrical shell when the depth h_0 of filling it with a liquid is varied [57] (an orthotropic shell with the parameters $E_1 = 2.12 \cdot 10^9$ Pa, $E_2 = 1.23 \cdot 10^9$ Pa, $\rho = 1.65 \cdot 10^3$ kg/m^3, $h = 3 \cdot 10^{-3}$ m was considered; it was filled with water, and the wave parameters n_i and m corresponded to the values $n_1 = 3$, $m = 1$; $n_2 = 4$, $m = 1$; $\omega_1 = \omega_{n_1 m}$, $\omega_2 = \omega_{n_2 m}$). Koval'chuk and Kruk [57] were the first to obtain a number of the first integrals that describe the interaction of two bending waves in cylindrical liquid-containing shells performing free vibrations (waves in the liquid were not taken into account). The first three integrals relate the amplitude parameters of waves

$$\frac{u_1}{\delta_1} + \frac{u_2}{\delta_2} = C_1, \quad u_1^2 - \nu_1^2 = C_2, \quad u_2^2 - \nu_2^2 = C_3,$$

$$(C_1 = \text{const}, \quad u_1 = u_{n_1 m}, \quad u_2 = u_{n_2 m}, \quad \nu_1 = \nu_{n_1 m}, \quad \nu_2 = \nu_{n_2 m}). \tag{34}$$

Here, the wave parameters n_i and m satisfy the resonance condition $\omega_1 \approx \omega_2$. The fourth integral determines the energy relationship between the amplitude and phase parameters of both waves

$$G_1 u_1 - \frac{G_2 u_1^2}{2} - \frac{\delta_1}{2\omega_1} L_1 L_2 \cos\bar{\theta} = C_4 = \text{const}. \tag{35}$$

Here, G_1 and G_2 are constant coefficients depending on the nonlinearity of the shell and frequency detuning $\Delta = \omega_2^2 - \omega_1^2$, $L_1 = \sqrt{(u_1/\delta_1 - C_1)^2 \delta_2^2 - C_3}$, $L_2 = \sqrt{u_1^2 - C_2}$, $\delta_i = \frac{\delta}{1 + m_{0i}}$, m_{0i} are the parameters of the apparent additional masses of the liquid, δ is the parameter of the nonlinear relationship between the interacting waves, and $\bar{\theta} = 2(\vartheta_{n_2 m} - \vartheta_{n_1 m})$.

The periodic functions u_1 and $\bar{\theta}$ are determined by the equations

$$\frac{du_1}{dt} = -\frac{\delta_1}{2\omega_1} K(u_1)\sin\bar{\theta}, \quad \frac{d\bar{\theta}}{dt} = G_1 - G_2 u_1 - \frac{\delta_1}{2\omega_1} S(u_1)\cos\bar{\theta}, \tag{36}$$

where

$$K(u_1) = L_1 L_2, \quad S(u_1) = \frac{dK}{du_1}.$$

If a shell with a liquid having close frequencies $\omega_1 \approx \omega_2$ is under radial harmonic pressure with frequency $\Omega \approx \omega_i$, then the averaged equations admit four qualitatively different groups of solutions for the amplitudes of the wave parameters u_i and v_i, namely [19, 68, 69]: (i) $u_1 = v_1, u_2 = v_2$, (ii) $u_1 \neq v_1, u_2 = v_2$, (iii) $u_1 = v_1, u_2 \neq v_2$, and (iv) $u_1 \neq v_1, u_2 \neq v_2$. The first group corresponds to the interaction of two standing waves, the second and third groups describe the more complex nature of energy redistribution in the shell with the liquid, when a standing (traveling) wave excited by an external force can, under some conditions (in particular, for particular filling depths), indirectly excite a traveling (standing) wave. Finally, the fourth group characterizes the interaction of two travelling bending waves with the parameters m, n_1 and m, n_2, respectively ($n_1 \neq n_2$).

Problems on the interaction of bending waves excited in cylindrical shells with circumferential waves in a liquid were considered in [15, 58, 87, 89, etc.]. The interaction processes were investigated in the cases of free vibrations of the system and its external periodic excitation. In [58, 89], these processes were analyzed in detail based on linear formulations of problems of hydrodynamics and hydroelasticity. A general system of connected wave equations was obtained, and the solutions of these equations were constructed. Not only do these solutions contain the traditional harmonics with principal frequencies λ_1 and λ_2 of the shell–liquid object, but also harmonics with multiple ($2\lambda_1, 2\lambda_2$) and combination ($\lambda_1 \pm \lambda_2$) frequencies. It was established that when a carrying shell is under an external harmonic (with frequency Ω) action, the regime of dynamic extinction of a bending wave realized at $\Omega = v_{nk}$ is possible. In this case, the energy introduced to the shell–liquid system is completely spent to realize a circumferential wave in the liquid, whose amplitudes reach maximum values. The carrying shell remains still (within the framework of the accepted model), though, under the conditions of the problem, an external load is applied to it.

In [15], the dependences of the phase velocities of nonlinear waves excited in a shell with a liquid on the amplitudes of these waves are constructed and the criteria of instability of forced traveling waves in a carrying shell (its dynamics is described by linear equations) are established. Figure 6 demonstrates the AFR $a = a(\Delta)$ (a is the amplitude of the wave process, $\Delta = \omega_1^2 - \Omega^2$, Ω is the frequency of the external excitation) with stable (A, B) and unstable (S) domains indicated. Here, $\Delta_{1,2} = \mp 4 Q_1 \gamma_{11}$ (Q_1 is the amplitude parameter of excitation and γ_{11} is a parameter characterizing the nonlinear relationship between the vibrations of the liquid and the shell). In the domains A and B, the standard forced bending waves in the carrying shell are realized (the liquid surface remains still). In the domain S, in Boyarshina's opinion [15], the circumferential-wave-type motion of the liquid excited by the circumferential waves traveling in the shell can be realized.

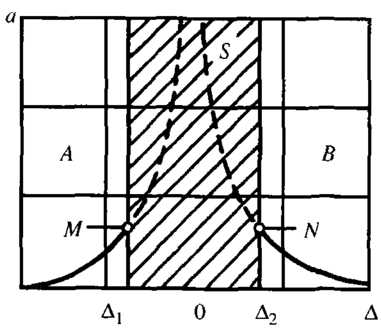

Fig. 6

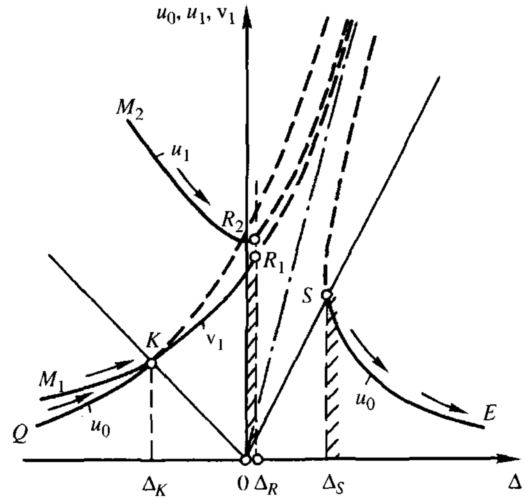

Fig. 7

A similar problem was considered in [16], where the interaction of circumferential waves in a liquid contained in a cylindrical vessel with bending waves excited in its elastic bottom was studied.

The last two decades are characterized by growth of interest in the study of nonlinear wave processes in shell–liquid systems in the instability regions of regular periodic regimes. As in-depth studies testify, the specific effect of chaotization of the wave motions can be observed in these regions. The case in point is excitation of nonregular (of chaotic nature) motions in a deterministic system.

It is well known that dynamic chaos, which recently seemed improbable, is a widespread phenomenon encountered in some domains of science and engineering [114, 119, 163] and, in the opinion of many specialists, can be classed as one of the most significant scientific discoveries in mathematics at the end of the 20th century. As applied to bodies with a liquid, this phenomenon has been studied theoretically in sufficient detail to date [81–83, 87, 179–181, etc.]. As for experimental investigations of chaotic regimes of liquid motion in a vibrating cylindrical body, they were described as early as 1955 by G. N. Mikishev [105]. The author called these unsteady regimes beatings. They exist only in some narrow part of the resonance region, outside which regular plane or spatial waves take place.

The basic properties of the chaotic vibrations of a liquid in cylindrical bodies subject to transversal periodic excitation by an energy source of limited intensity are studied in [82, 83, 87]. The evolutionary equations were supplemented here by an equation describing the rotation of the shaft of an electromotor. The studies [55, 58, 87] were the first to theoretically obtain the domains of the parameters of a system consisting of a cylindrical vessel and a liquid moving chaotically.

The effect of dynamic chaos is also naturally observed when an elastic cylindrical shell carrying a liquid is periodically excited [58, 81]. Figure 7 presents typical AFRs $u_1(\Delta)$ and $v_1(\Delta)$ plotted for a hinged cylindrical shell with a liquid in the specific case where the theoretical model of the general system allows for the nonlinearities due to considerable displacements of the free liquid surface. The shell deformation corresponds to beam vibrations (n = 1), and the first antisymmetric modes can be excited on the liquid surface. The shell is under radial periodic pressure (with a period of $T = \frac{2\pi}{\Omega}$, $\Omega \approx \lambda_1$, where λ_1 is one of the principal frequencies of the shell–liquid system, which was determined from the characteristic equation of the linear parts of system (11)). The dynamic-chaos domain is the frequency domain $\Delta_R < \Delta < \Delta_S$, where

$$\Delta = \lambda_1^2 - \Omega^2, \quad \Delta_R = \frac{1}{2\sqrt[3]{4}}\left(\frac{Q}{\bar{\gamma}_2}\right)^{2/3}(\bar{\gamma}_2 - 9\bar{\gamma}_1),$$

$$\Delta_S = -\frac{3}{2}\sqrt[3]{\frac{3}{2}Q^2\bar{\gamma}_1}, \tag{37}$$

and

$$Q = \frac{Q_1(\gamma_1 + kA)}{(1+kA^2)\rho h}, \quad A = \frac{\lambda_1^2 - \omega_1^2}{\delta_1}, \quad \bar{\gamma}_1 = \frac{2b_1\lambda_1^2}{3(1+kA^2)},$$

$$\bar{\gamma}_2 = \frac{2\lambda_1^2}{1+kA^2}(b_1 - 2b_2), \quad k = \frac{\gamma_1\omega_1^2}{\delta_1}$$

($\gamma_1 = \gamma_1^{11}$ and $\delta_1 = \delta_1^{11}$ are the coefficients in the linear parts of Eqs. (11) and b_1 and b_2 are constant parameters of hydrodynamic nonlinearities [87]). It was assumed that $\bar{\gamma}_1 < 0$. The curves M_1R_1 and M_2R_2 characterize the traveling-wave regime both in the shell and on the liquid surface, and standing-wave regimes are realized on the sections QK and SE. The existence of chaotic regimes in the region (Δ_R, Δ_S) was proved by analyzing the system of evolutionary equations by the technique stated in [87, 179–181]. It was established, in particular, that local instability and global-compression conditions are characteristic of this system [119].

Note that in the case of longitudinal (axial) periodic excitation of a shell with a liquid, chaotic regimes are not observed. As studies testify, the evolutionary equations fall into the class of coarse dynamic systems [3] (the topological structure of their phase paths does not vary for small changes in the parameters of these equations). The mentioned feature was pointed out earlier in [80, 81].

In summary, note that the above-mentioned wave processes in shells are related to the form of the solution of Eqs. (1) (which, as is known, are not hyperbolic), i.e., these processes are some analog of dispersive waves [138, 141]. They differ from the classic harmonic waves characterized by constant amplitudes and phase velocities. The formation of these waves is due to the complex process of nonlinear interaction of conjugate bending modes, one of which is directly excited by an external periodic force, and the other is indirectly involved in the process due to the nonlinear relationships between the modes and the natural resonance.

Vibration Stability under Complex Resonance Conditions. A special class of problems on the dynamics of a shell–liquid system consists of problems on the stability and vibrations of these systems under the conditions of complex (in particular, combination) resonances. Among them also are problems on the spatial vibrations of shells carrying a liquid, where the above-mentioned resonances are frequently encountered, creating premises for radical energy redistribution between different generalized coordinates of the whole system.

Nonlinear problems on vibration stability of shells carrying a liquid under complex-resonance conditions were analyzed in [14, 29, 30, 87, 128, etc.]. General theoretical premises for solving such problems were worked out by V. O. Kononenko and R. F. Ganiev [29, 32].

V. S. Pavlovskii and V. G. Filin [90, 128, etc.], using Narimanov's technique in combination with the Bubnov–Galerkin method, performed a stability analysis of steady-state regimes of the axisymmetric vibrations of a liquid-containing cylindrical shell subject to longitudinal periodic excitation under complex-resonance conditions. The nonlinearity of the problem was taken into account only in its hydrodynamic part. Criteria establishing necessary and sufficient conditions of the stability of forced vibrations with excitation frequency were constructed.

L. G. Boyarshina [14, 87, etc.] also examined problems on the stability of shells with a liquid when complex resonance relations (linking the fundamental frequencies of the shell and the free liquid surface with the external excitation frequency) are realized.

As for investigations of the spatial vibrations of shells with a liquid under resonance conditions, special approaches to the solution of such problems were developed at the Institute of Mechanics of the Academy of Sciences of Ukraine by R. F. Ganiev and his colleagues [17, 29–32, etc.]. In constructing the dynamic equations of the general system, the concept of the theory of relative motion [100] is used in this case. According to this concept, the basic (translational or translationally angular) oscillating motion of a carrying rigid skeleton is considered with the elastic (bending and tangential) vibrations of the shell and the vibrations of the free liquid surface superimposed on it. The theoretical model takes into account both the

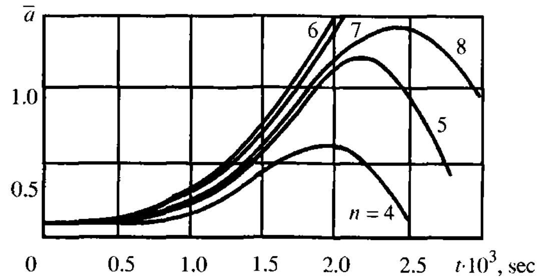

Fig. 8

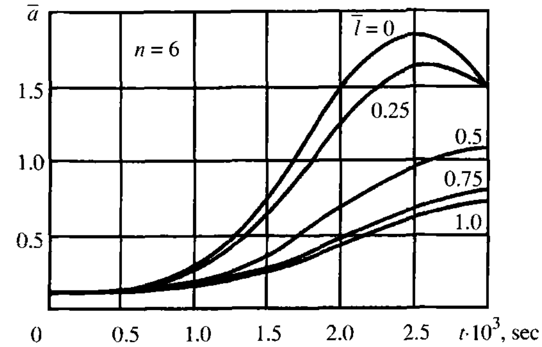

Fig. 9

considerable deviations of the free liquid surface from the nonperturbed state and the same (considerable) displacements of the rigid skeleton in space. The geometrical nonlinearities of the shell itself are not considered. On the basis of the Hamilton–Ostrogradskii variation principle, an infinite system of connected nonlinear equations was obtained, which is used to consider some mechanisms of spatial instability under the nonlinear-resonance conditions. The mechanism determining the low-frequency vibrations of the free liquid surface under high-frequency excitation of the carrying shell (this phenomenon was mentioned earlier in [171]) is of primary interest.

It should be noted that problems of the theory of spatial resonance vibrations of deformed bodies with a liquid have not been fully investigated yet, and, therefore, further development is needed. This undoubtedly important domain of nonlinear mechanics is characterized by a significant gap between the general initial statement of the problem and numerical solution of specific problems. On the other hand, it is inexpedient that the same model allows for nonlinearities of different orders that are due to, for example, the considerable amplitudes of the flexural and tangential vibrations of a carrying shell, or the flexural vibrations of the shell and the wave motions of the free liquid boundary, or the tangential vibrations of the shell and the angular motion of the rigid skeleton, etc. Urgent questions that may be the subject of future studies are as follows: which nonlinearities and when should be taken into consideration in constructing models of spatial motions of shell–liquid objects and what is the degree of influence of these nonlinearities on the resonance situations in these systems?

Nonstationary Processes. One of the important aspects of the problems being discussed is investigation of transients of the dynamic interaction of liquid-containing shells when the carrying object (or the free liquid surface) is acted upon by aperiodic (impulse and other nonstationary) loads. To effectively solve such problems, it is expedient in the overwhelming majority of cases to use numerical–analytical approaches: at the first stage, variation methods are used to construct a nonlinear discrete (of certain dimension) dynamic model of a continual shell–liquid system, and at the second stage, the equations obtained are numerically integrated. The modes of the natural vibrations of this system obtained by solving the respective linear boundary-value eigenvalue problems are usually selected as basis functions.

The first problems on the stability and the nonstationary vibrations of liquid-containing shells subject to aperiodic loading were analyzed by A. V. Sevast'yanov [142], N. A. Kil'chevskii and his disciples [104], and other authors. In approximating the deflection w, one dominating mode in combination with some multiple harmonics was allowed for. The liquid surface waves were neglected.

Afterwards, nonstationary problems of the dynamics of elastic shells interacting with a liquid were solved by many authors [25–28, 36, 37, 109, 120, 143, 144, etc.] using computational approaches.

Nikitin [121] has undertaken extended studies of the nonstationary wave processes in differently shaped shells of revolution partially filled with a liquid. These wave processes were due to the action of intensive loads promptly varying both in time and in space. In calculating the dynamic variable fields of stresses and strains in a carrying body, Nikitin used nonlinear models. The behavior of the liquid medium was described only in a nonlinear formulation (the deformation of the shell was determined by linear equations of hyperbolic type).

Among other studies on this matter, we may point out those by A. V. Kochetkov [77], where he discussed different nonlinear models (with both geometrical and physical nonlinearities taken into account) of the nonstationary interaction of shell structures with a compressible liquid and proposed effective numerical methods to analyze them.

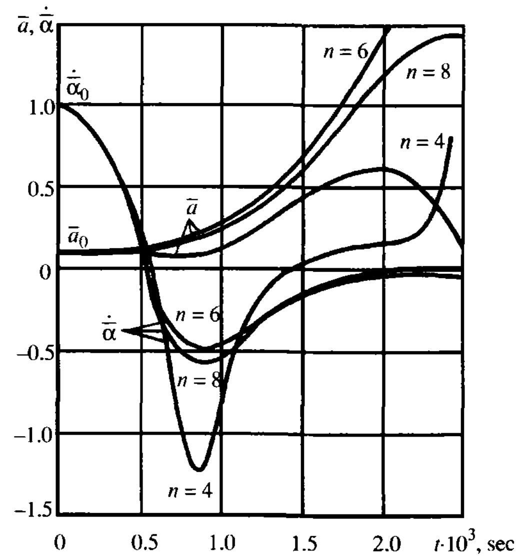

Fig. 10

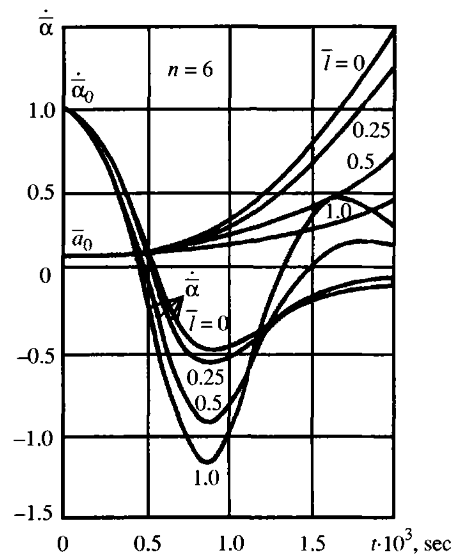

Fig. 11

The laws governing the nonlinear wave (as traveling bending waves) deformation of cylindrical liquid-containing shells under transversal and longitudinal impulse loads were studied in [59–62, 65, etc.]. One-wave models of shells were considered, and the energy of vibrations of the liquid surface was assumed low. Figures 8–11 present some results of the integration of the equations derived by the Bubnov–Galerkin method and composed in the amplitude a and phase velocity $\dot{\alpha}$ of the waves.

The calculations were carried out for shells with the following parameters: $E = 2 \cdot 10^{11}$ Pa, $\rho = 7.8\,\rho_0$, $\rho_0 = 1 \cdot 10^3$ kg/m^3, $\mu = 0.1$, $R = 0.16$ m, $R/h = 0.32 \cdot 10^3$, $l/R = 2.5$, and $m = 1$. Figures 8 and 9 illustrate amplitude–time curves ($\bar{a} = a/h$) plotted for a given shell with different parameters n of circumferential waves (Fig. 8) and depths $\bar{l} = h_0/l$ of filling with a liquid (for $n = 6$) (9) and subject to the action of a transversal triangular impulse having a duration of $t_1 = 2 \cdot 10^{-3}$ sec and an amplitude of $S_m = 7.8 \cdot 10^3$ kg/m $\cdot$ c^2. In both cases, it was assumed that $a(0) = h/10$ and $\dot{\alpha} = 0$. If $\dot{\alpha}(0) \neq 0$, in particular, $\dot{\alpha}(0) = \omega_1$, then the results of the studies correspond to Figs. 10 and 11 ($\dot{\bar{\alpha}} = \dot{\alpha}/\omega_1$) .

Some nonstationary problems on the dynamic interaction of shells of revolution with a viscous and incompressible liquid filling them were considered by Ya. F. Kayuk and his colleagues [51, 52]. To describe the deformation processes of both shells and the free liquid surface, the unified Lagrange coordinates were used. Methods of numerical integration were used in [67, 72] to investigate the nonstationary processes of passing the basic harmonic ($\omega_1 \approx \Omega$) and parametric ($\omega_1 \approx \nu/2$) resonances of the free boundary of an ideal incompressible liquid contained in oscillating cylindrical bodies. It was assumed that the frequencies of radial Ω and axial ν excitations vary slowly in time by a linear law.

It should be noted that in analyzing nonstationary processes in shell–liquid systems, a principal problem is to select a system of basis functions that approximate quite well the anticipated deflection of the shell and the perturbed surface of the liquid. In the overwhelming majority of cases, the modes of the natural linear vibrations of the dry shell and the modes of the liquid calculated regardless of the effect of the elastic walls of the carrying body are usually used as basis functions. In this case, one or two modes are considered, which may lead to significant errors in the determination of the actual SSS of shell–liquid structures and in the prediction of resonance situations that can arise when a system is subject to external periodic forces.

A new (multimodal), apparently promising, approach to the study of the nonlinear vibrations of shells carrying a liquid was proposed in [58, 89]. As the first stage of this approach, the natural modes of the flexural vibrations of a shell are determined with allowance for the effect of hydrodynamic pressure. For example, in the case of a hinged cylindrical shell, these modes have the form

$$w_1^{nk} = C_1^{nk} \cos \frac{n y}{R} \sum_{m=1}^{N} \chi_{km} \sin \lambda_m x,$$

$$w_2^{nk} = C_2^{nk} \sin \frac{n y}{R} \sum_{m=1}^{N} \chi_{km} \sin \lambda_m x, \qquad (38)$$

where C_i^{nk} are constants and χ_{km} are the components of the eigenvectors of the linear part of system (11), which are used to make the general system (11) quasinormal (N is the number of axial modes of the shell that are taken into account). At the second stage, the modes found are allowed for in the function approximating the nonlinear deflection and the Bubnov–Galerkin method and methods of nonlinear mechanics are applied.

Conclusion. The present review allows us to conclude that wide experience in the approximate solution of complex nonlinear problems of the dynamics of shell–liquid systems has been accumulated to date. The numerous and most significant results in this subject area have been obtained for simplified formulations of the mentioned problems corresponding, in particular, to the following two characteristic cases: (i) the elastic displacements of a carrying object are small (corresponding to the linear theory of shells) and (ii) the energy of the dynamic deformation of a shell with a liquid considerably exceeds that of the wave motions of the liquid filler. In the former case, the dominating factor is the incommensurability of the smallness concept for the vibrations of the free liquid surface and the elastic (bending and tangential) vibrations of the shell. The latter case is characterized by neglecting the effect of the liquid filler on the natural modes of the carrying shell. The liquid is usually assumed to be ideal and incompressible.

Other, more complex formulations of the problems, despite their urgency and practical importance, were considered in publications that are not numerous in number. This fact is indicative of fundamental mathematical difficulties in solving such problems. Note that to date there is no generalizing monograph that is devoted to nonlinear problems of the dynamics of shell–liquid objects (at least, the authors of the review are not aware of such monographs). Compare: about two dozen such monographs [95] concern problems of the nonlinear vibrations of solids partially or completely filled with a liquid.

Publications devoted to problems on the nonlinear dynamic interaction of shells with a viscous liquid are very limited in number. Such problems, as is known, are specific and considerably more complicated than the corresponding problems stated for an ideal liquid [9].

Among little-investigated problems in the scientific area being discussed are problems on the nonlinear vibrations of liquid-containing shells in a gentle gravitational field or in a field of random forces, on the interaction of liquid-containing shells with separation of the liquid from the elastic walls and formation of cavitational caverns, on the nonlinear vibrations of shells carrying a bubble (with gas inclusions) or a stratified liquid, on the interaction of rotating shells with a liquid filler, etc. Nonlinear problems of the dynamics of inclined, including horizontal, cylindrical shells partially filled with a liquid have not been sufficiently examined. Methods for solving nonlinear problems on the vibrations of shell–liquid objects with allowance for different design features such as damping baffles, rigid or elastic covers (restricting the motion of the free liquid boundaries), displacing flexible diaphragms, etc. have not been developed in sufficient detail. Still urgent are general problems of the dynamics of an elastic liquid system associated with the selection and justification of expedient techniques for description of the nonlinear motion of a continuous system consisting of an elastic body and a liquid.

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