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To the Beginning of the Third Millennium

COMPRESSIBLE, VISCOUS FLUID DYNAMICS (REVIEW). PART I

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The article is the first part of a survey of problems in compressible, viscous fluid dynamics as related to the dynamics of rigid and elastic bodies in a compressible, viscous fluid in the linearized formulation. The formulation of basic problems is discussed, along with a method of solution based on general solutions of the Navier–Stokes equations in vector and scalar form in dynamical problems. Forced harmonic vibrations of rigid bodies in rest and moving compressible, viscous fluids are discussed. Publications relevant to the stated problems are analyzed.

Brief Historical Summary. Fluid dynamics together with rigid and deformable bodies interacting with fluids constitutes one of the fundamental classical problems of mechanics, physics, and applied mathematics; the development of this problem has a long history and is identified with classical names in the natural sciences. Specific results obtained in the investigation of the problem are of major practical significance in regard to various aspects of science and engineering, including the latest technological processes. Initial research efforts addressed fluid dynamics in interaction with rigid bodies; at a much later date research began to be extended to the dynamics of elastic bodies in a fluid, motivated by the interests of related trends in science and various engineering applications. It is also important to mention the significance of results on the dynamics of rigid and elastic bodies in a fluid as applied to biology, bearing in mind the development of biomechanics (problems in the dynamics of bodies in the bloodstream, the propagation of disturbances in blood vessels, etc.). The development of topics in fluid dynamics in conjunction with fluid-interacting rigid and elastic bodies therefore poses a timely problem in science and engineering by virtue of the heightened concern of specialists in various scientific disciplines.

To adequately describe the dynamics of rigid and elastic bodies in a fluid, it is necessary to devise suitable fluid models, because rigid or elastic solid models are predetermined by the initial statement of the specific problem. Quite a broad range of possibilities for describing the indicated dynamical processes is afforded by classical fluid models (ideal incompressible, ideal compressible, viscous incompressible, and compressible viscous fluids), each of which is limited to the capability of describing only certain dynamical processes. The earliest research progressed using the simplest model: an incompressible, ideal fluid; as the investigation probed more deeply into dynamical processes and explored new physical phenomena, and also with the refinement of research techniques, more sophisticated fluid models came into play. The transition to the analysis of problems requiring the incorporation of damping of dynamical processes (vibrations) made it necessary to discard the ideal incompressible fluid model and go over to an ideal compressible fluid model. In this sense the model of a compressible, viscous fluid is the most general of the classical fluid models, because its mission is to combine the property of compressibility, which permits the wave character of the propagation of disturbances to be described within the framework of an ideal compressible fluid, and the property of viscosity, which permits the damping of dynamical processes to be described on the basis of the viscous incompressible fluid model.

When the viscous fluid model is used to describe the dynamics of rigid and elastic bodies in a fluid, the most important information can be obtained by means of the Navier–Stokes equations. In this case, however, major mathematical difficulties are encountered in solving the corresponding problems, and at the present time they are overcome in each specific situation by the application of modern numerical methods and computers. Nonetheless, simplifications of the kind usually found in mechanics are used in the analysis of individual problems (involving the viscous fluid model to describe the dynamics of

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rigid and elastic bodies in a fluid). For example, the Stokes [96] or Oseen [95] approximations are customarily used in the case of low Reynolds numbers [74, 76, 78, 83]; an extensive literature has been devoted to this scientific approach, including publications in periodicals and books, for example, [85]. Boundary-layer theory is normally used at higher Reynolds numbers [74, 76, 78, 83].

It is essential to note that the Stokes and Oseen approximations are essentially obtained by linearization (which may or may not be sufficiently consistent) with the Navier–Stokes equations for steady (time-invariant) fluid motion, whereupon the results obtained by means of these approximations can be used to describe low-frequency processes. Since the Stokes [96] and Oseen [95] approximations (when applied consistently) refer to steady fluid motions, their context is such that only the convection derivative in the total derivative of the velocity vector is retained in the Navier–Stokes equations. The Stokes approximation in this case completely rejects the convection derivative of the velocity vector (as a nonlinear term in the linearization operation); the Oseen approximation, on the other hand, linearizes the convection derivative of the velocity vector (as a nonlinear term in the linearization operation), taking into account the constant-velocity uniformity of the flow "at infinity" (i.e., the freestream flow).

There is another class of viscous fluid problems in which the Navier–Stokes equations can be greatly simplified. This class encompasses dynamical processes in which the disturbances are small, and the Navier–Stokes equations can be consistently linearized using the exact expression for the total derivative of the velocity vector; these cases are also fully consistent with the theory of small vibrations of mechanical systems in the customary terminology used in mechanics. In Landau and Lifshitz's well-known course in theoretical physics [76] (p. 125) this situation is treated as the motion of bodies in a fluid when the amplitude of the vibrations of the body is much smaller than its dimensions. Consequently, the above-mentioned class of problems includes problems in the dynamics (small vibrations or motions) of rigid bodies in a compressible, viscous fluid and in the propagation of small disturbances (small-amplitude waves) in elastic bodies interacting with a compressible, viscous fluid.

We note that problems in the dynamics of rigid and elastic bodies in a viscous fluid are currently investigated primarily for the model of an incompressible, viscous fluid; the results of such investigations can be found in numerous publications in the periodical and monograph literature, for example, [84] and [77] (in part). The compressible, viscous fluid model, on the other hand, appears only in isolated publications, for example, [87]; only in the last two decades, beginning in 1980 [24], members of the S. P. Timoshenko Institute of Mechanics of the National Academy of Sciences of Ukraine have published the results of studies in compressible, viscous fluid dynamics [1–22, 24–43, 45–68, 70, 73, 86, 89–91, 97] (formulated in the linearized version for fluids at rest and in motion); the present author has published a book on the same subject [44].

In the above-cited studies, the author and his students have developed the basic principles of the dynamics of rigid and elastic bodies in a compressible, viscous fluid in application to the theory of small vibrations or motions of rigid bodies in a compressible, viscous fluid, and also to the theory of the propagation of small disturbances (small-amplitude waves) in elastic bodies interacting with a compressible, viscous fluid. These studies have been carried out on the basis of the linearized Navier–Stokes equations resulting from their linearization for the cases of unsteady (time-dependent) and harmonic motions. The cited investigations cover the following topics: statement of the problems, general questions, and the representation of general solutions of the linearized Navier–Stokes equations for compressible, viscous fluids at rest and in motion; forced harmonic vibrations of rigid bodies in moving and rest compressible, viscous fluids; unsteady motions of rigid bodies in a rest compressible, viscous fluid; the dynamics of rigid bodies in a compressible, viscous fluid under the influence of radiation forces generated in the interaction of acoustic waves in the fluid and in the rigid bodies; the dynamics of thin-walled shells interacting with a compressible, viscous fluid; hydroelasticity problems for initially stressed elastic bodies and a compressible, viscous fluid. These problems define the subsequent structure of the present survey, which is devoted to a brief analysis of the main results obtained in the indicated scientific directions. It is important to note that the mathematical apparatus used in studying the specific problems stated above rests on the application of general solutions constructed for the linearized Navier–Stokes equations for moving and nonmoving (at rest) compressible, viscous fluids. The scientific results analyzed in the article have the following characteristic features in common: rigorous and consistent application of the linearized theory of compressible, viscous fluids, which provides a means for describing the wave nature of the propagation of disturbances and their damping; the derivation of fundamental results in the model of a piecewise-homogeneous medium in a three-dimensional setting, thereby ensuring that specific results will be obtained in general form without restrictions on the wavelength relative to the linear dimensions of the bodies or on the degree of viscosity of the fluid; systematic passage to limits for simpler fluid models in the majority of the specific problems considered; an analysis of mechanical phenomena associated with the wave nature of the propagation and damping of disturbances.

An essential undercurrent in the present survey, as in the author's book [44], is frequent mention and emphasis of the fact that the linearized theory of compressible, viscous fluids can be used to describe the wave nature of the propagation and damping of disturbances. This remark is a reflection of the understanding that in simpler models the wave nature of the propagation of disturbances can be described only if compressibility is taken into account (ideal compressible fluid model), and the damping of disturbances can be described, of course, only if viscosity is taken into account (incompressible, viscous fluid model); the compressible, viscous fluid model, on the other hand, treats the properties of compressibility and viscosity concurrently.

The survey is divided more or less conditionally into two parts to accommodate the character of the problems analyzed in each part. Thus, the first part covers information with a historical slant and results obtained mainly by methods of an analytical nature, along with a bibliography that covers both parts of the survey. The second part is dedicated to results also obtained by analytical methods, but with the application of numerical methods in the final stage of analysis.

Within the guidelines of the foregoing brief historical and formulative outline we can now proceed with an analysis of the fundamental results.

1. Basic Relations and General Solutions. In this section, we analyze results pertaining to the basic relations and their linearization, the identification of Lagrangian and Eulerian coordinates, the representation of general solutions in vector and scalar forms in application to moving and rest fluids, certain analogies in continuum mechanics, and limiting transitions from the general solutions to simpler fluid models.

1.1. Basic Relations. We write the basic relations for a compressible, viscous fluid in Eulerian coordinates, using the notation of [35, 36, 44]. In this notation, the breve over a variable or differential operator indicates that the variable or operator is being used in the Eulerian description of motion based on Eulerian coordinates. In the Eulerian description, according to [74–76, 78, 83], the Navier–Stokes equations can be written in the form

$$\breve{\rho}\frac{D}{D\tau}\breve{\vec{v}} - \mu^{(1)}\breve{\Delta}\breve{\vec{v}} - (\lambda^{(1)} + \mu^{(1)})\breve{\vec{\nabla}}(\breve{\vec{\nabla}}\cdot\breve{\vec{v}}) + \breve{\vec{\nabla}}\breve{p} = 0. \tag{1.1}$$

The equation of continuity, also in Eulerian coordinates, has the form

$$\frac{\partial}{\partial\tau}\breve{\rho} + \breve{\vec{\nabla}}\cdot\breve{\rho}\,\breve{\vec{v}} = 0. \tag{1.2}$$

In Eqs. (1.1) and (1.2) and below, $\breve{p}$, $\breve{\rho}$, and $\breve{\vec{v}}$ denote the pressure, density, and velocity vector in the Eulerian description, $\nu^{(1)}$ is the kinematic viscosity coefficient, $\mu^{(1)}$ is the viscosity coefficient, and $\lambda^{(1)}$ is the second viscosity coefficient. The superscript "(1)" is attached to the quantities $\lambda^{(1)}$, $\mu^{(1)}$, and $\nu^{(1)}$ for the fluid, because the analogous quantities without the superscript refer to an elastic body. In Eqs. (1.1) and (1.2), we have also introduced the differential operators

$$\frac{D}{D\tau} = \frac{\partial}{\partial\tau} + \breve{\vec{v}}\cdot\breve{\vec{\nabla}}, \qquad \breve{\Delta} = \breve{\vec{\nabla}}\cdot\breve{\vec{\nabla}}. \tag{1.3}$$

Thermal effects will be ignored below, and we shall assume that the fluid is barotropic; accordingly, we write the equation of state for a compressible, viscous fluid in the form

$$\breve{p} = f(\breve{\rho}). \tag{1.4}$$

In the terminology of [78], a compressible, viscous fluid characterized by an equation of state in the form (1.4) is called an "elastic" fluid.

In the Eulerian description, a symmetric Euler or Cauchy stress tensor is defined for the fluid by the following relations for the covariant components:

$$\overset{\vee}{T}_{ij} = -\overset{\vee}{p}\, G_{ij} + \lambda^{(1)} G_{ij} \overset{\vee}{\nabla}_n \overset{\vee}{v}^n + \mu^{(1)} (\overset{\vee}{\nabla}_i \overset{\vee}{v}_j + \overset{\vee}{\nabla}_j \overset{\vee}{v}_i). \tag{1.5}$$

It is important to note that the bulk viscosity coefficient is often set equal to zero for rapidly evolving processes:

$$\lambda^{(1)} + \frac{2}{3}\mu^{(1)} = 0, \tag{1.6}$$

whereupon one of the viscosity coefficients can be eliminated. The above relations (1.1)–(1.6) are basic in compressible, viscous fluid dynamics as long as thermal processes are ignored. It is a foregone conclusion that in the formulation of specific problems these relations must be augmented with initial conditions; also, if the fluid is bounded or if rigid or elastic bodies are moving through it, the relations must be further augmented with boundary conditions at the surface of the fluid or at its interfaces with the inclusions.

1.2. Linearization and Remarks on the Identification of Lagrangian and Eulerian Coordinates in Continuum Mechanics. In the ensuing paragraphs we discuss dynamical processes in which the disturbances are small, and the basic relations (1.1), (1.2), (1.4), and (1.5) of compressible, viscous fluid dynamics can be successively linearized in application to the dynamics of rigid and elastic bodies interacting with a compressible, viscous fluid. As mentioned, in the terminology of Landau and Lifshitz's eminent course in theoretical physics ([76], p. 125), these phenomena are regarded as motions of bodies in a fluid (a compressible, viscous fluid in our situation) when the amplitudes of the vibrations of the body are much smaller than its dimensions.

A generally accepted principle in continuum mechanics is the practicality of using different coordinates in application to different models for deriving the basic relations in the simplest and most compact form in the description of motion of a continuous medium. To achieve this goal, it is customary to use Eulerian coordinates and the Eulerian description of motion in the mechanics of liquids and gases and to use Lagrangian coordinates and the Lagrangian description of motion or various generalizations (such as counting methods or method of comoving coordinates) in the mechanics of deformable solids. Accordingly, a major issue in the investigation of the combined motions of a liquid, a gas, and deformable and rigid bodies is the choice of coordinates, again so that the basic relations of the compound (general) problem can be written in a simple form. Naturally, the resolution of this issue is most successful when general coordinates can be introduced in application to the given class of problems with the property that they go over to Eulerian coordinates in application to a liquid or a gas and to Lagrangian coordinates in application to a deformable body. We thus encounter the problem of identifying Lagrangian and Eulerian coordinates in application to specific classes of problems in continuum mechanics. We note that the problem of identifying Lagrangian and Eulerian coordinates can only be considered when such coordinates fall within the same curvilinear coordinate system, i.e., when the same basis covariant and contravariant vectors and metric tensors are used. This approach has been used, for example, in three books [35, 36, 44]. In the general case, as in the book [83] for example, Lagrangian and Eulerian coordinates can be introduced with different covariant and contravariant vectors and metric tensors; it is probably meaningless to even consider the problem of identifying Lagrangian and Eulerian coordinates in this case.

Aspects of identification of specially chosen Lagrangian and Eulerian coordinates in application to fluid dynamics (including compressible, viscous fluids) and deformable bodies have been investigated [36, 43, 44] within the context of the foregoing considerations in a linearized setting for the following classes of problems: a fluid at rest and an elastic body without initial stresses; a moving fluid with uniform freestream ("at infinity") flow and an elastic body without initial stresses; a fluid at rest and an initially stressed elastic body; a moving fluid with uniform freestream flow and an initially stressed elastic body. These results are presented in the most compact form in [43]; it should be noted that in most publications the first of the indicated four classes of problems is usually analyzed in application to various models of a fluid and deformable bodies.

Further on in the article we analyze results obtained with [43] taken into account in specially chosen coordinates, which go over to Eulerian coordinates in application to a compressible, viscous fluid and to Lagrangian coordinates in application to an elastic body.

1.3. Linearized Relations for a Compressible Viscous Fluid (at Rest and Moving). For a *rest compressible, viscous fluid*, as a result of linearization, from Eq. (1.1) we obtain linearized Navier–Stokes equations in the form

$$\rho_0 \frac{\partial}{\partial \tau}\vec{v} - \mu^{(1)} \Delta \vec{v} - (\lambda^{(1)} + \mu^{(1)}) \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \vec{\nabla} p^{(1)} = 0\,. \tag{1.7}$$

From Eq. (1.2), in this case we obtain linearized continuity condition for a homogeneous fluid

$$\frac{\partial}{\partial \tau} \rho^{(1)} + \rho_0 \vec{\nabla} \cdot \vec{v} = 0 . \tag{1.8}$$

Analogously, from Eq. (1.5) we obtain linearized relations for the covariant components of a symmetric stress tensor

$$T_{ij} = - p^{(1)} g_{ij} + \lambda^{(1)} g_{ij} \nabla_n v^{n} + \mu^{(1)} (\nabla_i v_j + \nabla_j v_i) . \tag{1.9}$$

Again, as a result of linearization, from Eq. (1.4) we obtain a closing equation in the form

$$\frac{\partial p^{(1)}}{\partial \rho^{(1)}} = a_0^2 . \tag{1.10}$$

In Eqs. (1.7)–(1.10), $\rho^{(1)}, p^{(1)}, \vec{v}$ and T_{ij} are perturbations of the density, pressure, velocity vector, and covariant components of the stress tensor, respectively, ρ_0, p_0, and a_0 are the density, pressure, and sound velocity for the fluid at rest. Equations (1.7)–(1.10) are standard.

In the case of a *moving fluid*, we assume that the compressible, viscous fluid occupies unbounded space. We then consider unperturbed motion of the fluid along the x_3 axis, corresponding to uniform flow with a constant velocity U = const. In the vicinity of an elastic or rigid body in the fluid, we can write the following expression for the components of the velocity vector:

$$U \delta_n^3 + v_n (x_m , \tau), \tag{1.11}$$

where $v_n (x_m , \tau)$ denotes the components of the perturbation of the velocity vector. From now on we label all quantities pertaining to unperturbed motion with a constant velocity U by the subscript "0". For a moving compressible, viscous fluid, owing to linearization, from Eqs. (1.1) and (1.11) we obtain a linearized Navier–Stokes equation in the form

$$\rho_0 \frac{\partial}{\partial \tau} \vec{v} + \rho_0 U \frac{\partial}{\partial x_3} \vec{v} - \mu^{(1)} \Delta \vec{v} - (\lambda^{(1)} + \mu^{(1)}) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) + \vec{\nabla} p^{(1)} = 0. \tag{1.12}$$

From Eqs. (1.2) and (1.11), in this case we obtain a linearized equation of continuity, which appears as follows for a homogeneous fluid:

$$\frac{\partial}{\partial \tau} \rho^{(1)} + U \frac{\partial}{\partial x_3} \rho^{(1)} + \rho_0 \vec{\nabla} \cdot \vec{v} = 0. \tag{1.13}$$

As a result of linearization, from Eq. (1.4) we obtain a closing equation in the form (1.10). It is important to note that for the given situation of a moving fluid, Eqs (1.10) and (1.12)–(1.14) already incorporate the notation: ρ_0, p_0, and a_0 for the density, pressure, and sound velocity in a fluid moving with a constant velocity U along the x_3 axis; $\rho^{(1)}, p^{(1)}, \vec{v}$ and T_{ij} are perturbations of the density, pressure, velocity vector, and covariant components of the stress tensor. It should be noted that the boundary conditions for the fluid or the conditions at the fluid–solid interfaces can involve components of the total stress tensor (not just perturbations); in this case the following relations must be used instead of Eq. (1.9):

$$T_{ij}^0 + T_{ij} = - (p_0 + p^{(1)}) g_{ij} + \lambda^{(1)} g_{ij} \nabla_n v^{n} + \mu^{(1)} (\nabla_i v_j + \nabla_j v_i). \tag{1.14}$$

Equations (1.12)–(1.14) and (1.10) are sufficiently rigorous in the setting of the linearized theory for a moving compressible, viscous fluid when thermal processes are ignored.

If Eq. (1.12) is used for steady motions (with the partial time derivatives equal to zero), we obtain the classical Oseen approximation [95]. If we also set $U = 0$ in this equation, we obtain the classical Stokes approximation [96].

1.4. An Analogy in Continuum Mechanics. In the solution of specific problems in continuum mechanics, a major simplification is achieved when analogies between dissimilar problems are known, including analogies between different models. In this event, solving methods developed for a particular model can be transferred to other models with the indicated analogies taken into account. In application to the linearized theory for a compressible, viscous fluid at rest without regard for thermal processes (Eqs. (1.7)–(1.10)), an analogy has been established [33] with a specific rheological model of a deformable solid; the same analogy has been investigated and exploited in later papers [65–68]. Its details are described in the book [44]; consequently, only brief information will be covered below.

It has been proved [33, 44] that for the case in question (linearized theory for a compressible, viscous fluid at rest) the governing equations in continuum mechanics (relations between the stress and strain tensors) can be characterized as follows: *The governing equations for the spherical part correspond to a Voigt body; the governing relations for the deviator part correspond to a Newtonian body.* With regard for the above-indicated property, along with the appropriate boundary and initial conditions, an analogy has been established [33, 44] between dynamical problems for the above-indicated rheological body (with the spherical part characterized by a Voigt body, and the deviator part by a Newtonian body) and problems in compressible, viscous fluid dynamics (in the context of the linearized theory).

1.5. General Solution for a Fluid at Rest; Vector and Scalar Potentials. The solutions of the basic equations (1.7), (1.8), and (1.10) for a compressible, viscous fluid at rest in terms of the vector potential $\vec{\Psi}^{(1)}$ and the scalar potential $\Phi^{(1)}$ can be written in the form

$$\vec{v} = \vec{\nabla}\Phi^{(1)} + \vec{\nabla}\times\vec{\Psi}^{(1)}, \quad \vec{\nabla}\cdot\vec{\Psi}^{(1)} \equiv \operatorname{div}\vec{\Psi}^{(1)} = 0,$$

$$p^{(1)} = \rho_0\left(\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \frac{\partial}{\partial\tau}\right)\Phi^{(1)},$$

$$\rho^{(1)} = \frac{\rho_0}{a_0^2}\left(\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \frac{\partial}{\partial\tau}\right)\Phi^{(1)}. \tag{1.15}$$

We note that the second equation in (1.15) is standard and applies solely to the vector potential. In the representation (1.15), the scalar and vector potentials are determined from the equations

$$\left[\left(1 + \frac{\lambda^{(1)} + 2\mu^{(1)}}{a_0^2\rho_0}\frac{\partial}{\partial\tau}\right)\Delta - \frac{1}{a_0^2}\frac{\partial^2}{\partial\tau^2}\right]\Phi^{(1)} = 0,$$

$$\left(\nu^{(1)}\Delta - \frac{\partial}{\partial\tau}\right)\vec{\Psi}^{(1)} = 0, \quad \mu^{(1)} = \rho_0\nu^{(1)}. \tag{1.16}$$

The representation of the general solution for a viscous fluid at rest in the form (1.15) and (1.16) (in the linearized theory) has been obtained previously in [25, 27] with a slightly different notation; the above results are written in the notation of the book [44] without the additional condition (1.6).

1.6. General Solution for a Moving Fluid; Vector and Scalar Potentials. The solutions of the basic equations (1.12), (1.13), and (1.10) for a moving compressible, viscous fluid (in uniform flow with a constant velocity U along the x_3 axis) in terms of the vector potential $\vec{\Psi}^{(1)}$ and the scalar potential $\Phi^{(1)}$ can be written in the form

$$\vec{v} = \vec{\nabla}\Phi^{(1)} + \vec{\nabla}\times\vec{\Psi}^{(1)}, \quad \vec{\nabla}\cdot\vec{\Psi}^{(1)} \equiv \operatorname{div}\vec{\Psi}^{(1)} = 0,$$

$$p^{(1)} = \rho_0\left[\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \left(\frac{\partial}{\partial\tau} + U\frac{\partial}{\partial x_3}\right)\right]\Phi^{(1)},$$

$$\rho^{(1)} = \frac{\rho_0}{a_0^2}\left[\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \left(\frac{\partial}{\partial\tau} + U\frac{\partial}{\partial x_3}\right)\right]\Phi^{(1)}. \tag{1.17}$$

The second equation in (1.17) is a standard condition and applies solely to the vector potential. In the representation (1.17) the scalar and vector potentials are determined from the equations

$$\left\{\left[1+\frac{\lambda^{(1)}+2\mu^{(1)}}{a_0^2\rho_0}\left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\Delta-\frac{1}{a_0^2}\left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)^2\right\}\Phi^{(1)}=0,$$

$$\left[\nu^{(1)}\Delta-\left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\vec{\Psi}^{(1)}=0. \qquad (1.18)$$

A representation of the general solution for a moving compressible, viscous fluid in the form (1.17) and (1.18) (in the linearized theory) has been obtained previously [26, 32] with a slightly different notation; the above results are written in the notation of the book [44] without the additional condition (1.6).

It is important to note that for a compressible, viscous fluid at rest the representation of the general solution in terms of the vector potential $\vec{\Psi}^{(1)}$ and the scalar potential $\Phi^{(1)}$ in the form (1.15) and (1.16) is valid for arbitrary coordinate systems. In the case of a moving compressible, viscous fluid, on the other hand, the representation of the general solution in terms of the vector potential $\vec{\Psi}^{(1)}$ and the scalar potential $\Phi^{(1)}$ in the form (1.17) and (1.18) is confined to arbitrary cylindrical coordinates whose axis coincides with the x_3 axis; this situation is discussed in detail in [44].

1.7. Representation of the General Solution for a Fluid at Rest in Terms of Scalar Potentials. We now consider the representation of the general solution (1.15), (1.16) for a compressible, viscous fluid at rest in terms of scalar potentials in rectangular, circular cylindrical, and spherical coordinates; these results have been discussed in various forms in several papers [25–27, 29, 31, 37, 89]; they are partially covered in books [36, 58, 60] and in complete form with corresponding proofs in the book [44].

Rectangular Cartesian Coordinates y_m with Unit Vectors $\vec{y}_m$. In this case, the vector potential $\vec{\Psi}^{(1)}$ (1.15) is written in the form

$$\vec{\Psi}^{(1)}=\vec{y}_m\Psi_m^{(1)}(y_1,y_2,y_3,\tau),\qquad m=1,2,3 \qquad (1.19)$$

and the second condition (1.15) now assumes the form

$$\frac{\partial}{\partial y_1}\Psi_1^{(1)}+\frac{\partial}{\partial y_2}\Psi_2^{(1)}+\frac{\partial}{\partial y_3}\Psi_3^{(1)}=0. \qquad (1.20)$$

Taking Eqs. (1.19) and (1.20) into account, from Eqs. (1.15) we obtain a general solution in the form

$$v_1=\frac{\partial}{\partial y_1}\Phi^{(1)}+\frac{\partial}{\partial y_2}\Psi_1^{(1)}-\frac{\partial}{\partial y_3}\Psi_2^{(1)},$$

$$v_2=\frac{\partial}{\partial y_2}\Phi^{(1)}-\frac{\partial}{\partial y_1}\Psi_3^{(1)}+\frac{\partial}{\partial y_3}\Psi_1^{(1)},$$

$$v_3=\frac{\partial}{\partial y_3}\Phi^{(1)}+\frac{\partial}{\partial y_1}\Psi_2^{(1)}-\frac{\partial}{\partial y_2}\Psi_1^{(1)},$$

$$p^{(1)}=\rho_0\left(\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta-\frac{\partial}{\partial\tau}\right)\Phi^{(1)},$$

$$\rho^{(1)}=\frac{\rho_0}{a_0^2}\left(\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta-\frac{\partial}{\partial\tau}\right)\Phi^{(1)}. \qquad (1.21)$$

The general solution (1.21) is written in terms of scalar potentials which, according to Eqs. (1.19), (1.15), and (1.16), are determined from separate equations

$$\left[\left(1+\frac{\lambda^{(1)}+2\mu^{(1)}}{a_0^2\rho_0}\frac{\partial}{\partial\tau}\right)\Delta-\frac{1}{a_0^2}\frac{\partial^2}{\partial\tau^2}\right]\Phi^{(1)}=0,$$

$$\left(\nu^{(1)}\Delta-\frac{\partial}{\partial\tau}\right)\Psi_n^{(1)}=0,\quad \Delta=\frac{\partial}{\partial y_m}\frac{\partial}{\partial y_n},\quad n, m=1, 2, 3. \tag{1.22}$$

Simplifications of the representation (1.21), (1.22) are discussed in [44] in application to plane and antiplane problems.

Circular Cylindrical Coordinates r, φ, and y_3 with Unit Vectors $\vec{e_r}$, $\vec{e_\varphi}$, and $\vec{e_3}$. In this case, the vector potential $\vec{\Psi}^{(1)}$ (1.15) is written in the form

$$\vec{\Psi}^{(1)}=\vec{e_3}\Psi_1^{(1)}+\vec{\nabla}\times\vec{e_3}\Psi_2^{(1)} \tag{1.23}$$

in terms of the scalar potentials $\Psi_1^{(1)}$ and $\Psi_2^{(1)}$; this result together with the first equation in (1.15) yields the representation of the velocity vector

$$\vec{v}=\vec{\nabla}\Phi^{(1)}+\vec{\nabla}\times\vec{e_3}\Psi_1^{(1)}+\vec{\nabla}\times\vec{\nabla}\times\vec{e_3}\Psi_2^{(1)}. \tag{1.24}$$

Consequently, taking Eqs. (1.19) and (1.23) into account, we obtain the representation of the general solution

$$v_r=\frac{\partial}{\partial r}\Phi^{(1)}+\frac{1}{r}\frac{\partial}{\partial\varphi}\Psi_1^{(1)}+\frac{\partial^2}{\partial r\,\partial y_3}\Psi_2^{(1)},$$

$$v_\varphi=\frac{1}{r}\frac{\partial}{\partial\varphi}\Phi^{(1)}-\frac{\partial}{\partial r}\Psi_1^{(1)}+\frac{1}{r}\frac{\partial^2}{\partial\varphi\,\partial y_3}\Psi_2^{(1)},$$

$$v_3=\frac{\partial}{\partial y_3}\Phi^{(1)}-\left(\Delta-\frac{\partial^2}{\partial y_3^2}\right)\Psi_2^{(1)},$$

$$p^{(1)}=\rho_0\left(\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta-\frac{\partial}{\partial\tau}\right)\Phi^{(1)},$$

$$\rho^{(1)}=\frac{\rho_0}{a_0^2}\left(\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta-\frac{\partial}{\partial\tau}\right)\Phi^{(1)} \tag{1.25}$$

in terms of the scalar potentials $\Phi^{(1)}$, $\Psi_1^{(1)}$, and $\Psi_2^{(1)}$, which, according to Eqs. (1.16) and (1.23), are determined from separate equations

$$\left[\left(1+\frac{\lambda^{(1)}+2\mu^{(1)}}{a_0^2\rho_0}\frac{\partial}{\partial\tau}\right)\Delta-\frac{1}{a_0^2}\frac{\partial^2}{\partial\tau^2}\right]\Phi^{(1)}=0,$$

$$\left(\nu^{(1)}\Delta-\frac{\partial}{\partial\tau}\right)\Psi_j^{(1)}=0,\quad j=1, 2,$$

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2}\frac{\partial^2}{\partial y_3^2}. \tag{1.26}$$

Simplifications of the representation (1.25), (1.26) are discussed in [44] in application to plane and antiplane problems and also for the axisymmetric problem and the rotation problem. However, when the vector potential is represented in the form (1.23), it may appear at first glance that the second condition in (1.15) (div $\vec{\Psi}^{(1)} = 0$) is not satisfied; this problem is solved in [44] using the approaches discussed on pp. 705–707 of the eminent course in theoretical physics [79]. It should be mentioned that the methods in [79] refer to a vector potential that satisfies the wave equation (a hyperbolic equation); in [44], the methods of [79] are applied to a vector potential that satisfies the second equation in (1.16) (a parabolic equation).

Spherical Coordinates r, θ, and φ with Unit Vectors $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\varphi$, where the angle θ is measured from the y_3 axis, and the angle φ is measured from the plane $y_2 = 0$. In this case, the vector potential (1.15) is written in the form

$$\vec{\Psi}^{(1)} = \vec{e}_r r\,\Psi_1^{(1)} + \vec{\nabla} \times \vec{e}_r r\,\Psi_2^{(1)} \tag{1.27}$$

in terms of the scalar potentials $\Psi_1^{(1)}$ and $\Psi_2^{(1)}$; this result together with the first equation in (1.15) yields the representation of the velocity vector

$$\vec{v} = \vec{\nabla}\Phi^{(1)} + \vec{\nabla} \times \vec{e}_r r\,\psi_1^{(1)} + \vec{\nabla} \times \vec{\nabla} \times \vec{e}_r r\,\Psi_2^{(1)}. \tag{1.28}$$

Consequently, from Eqs. (1.15) and (1.27) we obtain the representation of the general solution

$$v_r = \frac{\partial}{\partial r}\Phi^{(1)} - \left(r\Delta - \frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)\Psi_2^{(1)},$$

$$v_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}\Phi^{(1)} + \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\Psi_1^{(1)} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial \theta}\Psi_2^{(1)},$$

$$v_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\Phi^{(1)} + \frac{\partial}{\partial \theta}\Psi_1^{(1)} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r} r \frac{\partial}{\partial \varphi}\Psi_2^{(1)},$$

$$p^{(1)} = \rho_0\left(\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \frac{\partial}{\partial \tau}\right)\Phi^{(1)},$$

$$\rho^{(1)} = \frac{\rho_0}{a_0^2}\left(\frac{\lambda^{(1)} + 2\mu^{(1)}}{\rho_0}\Delta - \frac{\partial}{\partial \tau}\right)\Phi^{(1)} \tag{1.29}$$

in terms of the scalar potentials $\Phi^{(1)}$, $\Psi_1^{(1)}$, and $\Psi_2^{(1)}$, which, according to Eqs. (1.16) and (1.27), are determined from separate equations

$$\left[\left(1 + \frac{\lambda^{(1)} + 2\mu^{(1)}}{a_0^2 \rho_0}\frac{\partial}{\partial \tau}\right)\Delta - \frac{1}{a_0^2}\frac{\partial^2}{\partial \tau^2}\right]\Phi^{(1)} = 0,$$

$$\left(\nu^{(1)}\Delta - \frac{\partial}{\partial \tau}\right)\Psi_j^{(1)} = 0, \quad j = 1, 2,$$

$$\Delta = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r} r^2 \sin\theta \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi}\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right). \tag{1.30}$$

Simplifications of the representation (1.29), (1.30) are discussed in [44] in application to the axisymmetric problem and the rotation problem. However, when the vector potential is represented in the form (1.27), it may appear at first glance that the second condition in (1.15) (div $\vec{\Psi}^{(1)} = 0$) is not satisfied; the solution of this problem is given in [44] using the approaches discussed on pp. 705–707 of Landau and Lifshitz [79]. It should be mentioned that the methods in [79] refer to a vector potential that satisfies the wave equation (a hyperbolic equation); in [44], the methods of [79] are applied to a vector potential that satisfies the second equation in (1.16) (a parabolic equation). Another important consideration is the fact that in spherical coordinates investigations can also be carried out in vector form; in application to a compressible, viscous fluid at rest the representation of the general solution in vector form is discussed in several papers [29, 31, 89] and in comparative detail in the book [44]. In the vector formulation [29, 31, 36, 44, 89] of problems for a compressible, viscous fluid at rest in spherical coordinates, following [79], an orthogonal system of vector functions $\{\vec{P}_{mn}; \vec{B}_{mn}; \vec{C}_{mn}\}$, which is complete for r = const, is constructed on the basis of spherical harmonics.

1.8. Representation of the General Solution for a Moving Fluid in Terms of Scalar Potentials. As mentioned after Eqs. (1.18), the general solution formulated for a moving compressible, viscous fluid in the form (1.17), (1.18) refers to arbitrary cylindrical coordinates whose axis coincides with the $x_3 \equiv y_3$ axis (the direction of the unperturbed flow velocity); of course, this general solution also applies to an arbitrary corresponding rectangular coordinate system. A general solution in terms of scalar potentials has been obtained for the above-indicated cases; the main results are briefly summarized in two papers [26, 32] and in more complete form in the book [44]. In the present article, we give these results in their final form. Invoking the representation (1.23) for the longitudinal and transverse components of the vector potential in the above-indicated arbitrary cylindrical coordinate system, along with Eqs. (1.18), and making a number of transformations by analogy with [79], we obtain the representation of the general solution for the given situation in the form

$$\vec{v} = \vec{\nabla}\Phi^{(1)} + \vec{\nabla}\times\vec{e_3}\Psi_1^{(1)} + \vec{\nabla}\times\vec{\nabla}\times\vec{e_3}\Psi_2^{(1)},$$

$$p^{(1)} = \rho_0\left[\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta - \left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\Phi^{(1)},$$

$$\rho^{(1)} = \frac{\rho_0}{a_0^2}\left[\frac{\lambda^{(1)}+2\mu^{(1)}}{\rho_0}\Delta - \left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\Phi^{(1)} \tag{1.31}$$

in terms of the scalar potentials $\Phi^{(1)}$, $\Psi_1^{(1)}$, and $\Psi_2^{(1)}$, which are determined from separate equations

$$\left\{\left[1+\frac{\lambda^{(1)}+2\mu^{(1)}}{a_0^2\rho_0}\left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\Delta - \frac{1}{a_0^2}\left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)^2\right\}\Phi^{(1)} = 0,$$

$$\left[\nu^{(1)}\Delta - \left(\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial x_3}\right)\right]\Psi_j^{(1)} = 0, \quad j = 1, 2. \tag{1.32}$$

Simplifications of the representation (1.31), (1.32) are discussed in [44] for the axisymmetric problem and the rotation problem in circular cylindrical coordinates; in addition, the representation (1.31), (1.32) is also considered in [44] for the axisymmetric problem in spherical coordinates (with the angle θ measured from the direction of the unperturbed flow velocity). For this case, in spherical coordinates it is assumed that

$$\Phi^{(1)} = \Phi^{(1)}(r, \theta, \tau), \quad \psi_1^{(1)} \equiv 0, \quad \Psi_2^{(1)} = \Psi_2^{(1)}(r, \theta, \tau). \tag{1.33}$$

Taking Eqs. (1.33) into account, from (1.31) we obtain the following equations for the components of the velocity vector:

$$v_r = \frac{\partial}{\partial r}\Phi^{(1)} + \frac{\partial}{\partial r}\left(\frac{\partial}{\partial x_3}\Psi_2^{(1)}\right) - \cos\theta\left(\frac{U}{\nu^{(1)}}\frac{\partial}{\partial x_3}+\frac{1}{\nu^{(1)}}\frac{\partial}{\partial\tau}\right)\Psi_2^{(1)},$$

$$v_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\Phi^{(1)} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x_3}\Psi_2^{(1)}\right) + \sin\theta\left(\frac{U}{\nu^{(1)}}\frac{\partial}{\partial x_3}+\frac{1}{\nu^{(1)}}\frac{\partial}{\partial\tau}\right)\Psi_2^{(1)}. \tag{1.34}$$

The representation of the general solution in the given spherical coordinate system for the axisymmetric problem involves, together with Eqs. (1.33) and (1.34), the expressions for $p^{(1)}$ and $\rho^{(1)}$ and Eqs. (1.32). For the rotation problem [44] in the indicated spherical coordinates it is assumed that

$$\Phi^{(1)} \equiv 0, \quad \Psi_1^{(1)} = \Psi_1^{(1)}(r, \theta, \tau), \quad \Psi_2^{(1)} \equiv 0, \tag{1.35}$$

where the following relations for the components of the velocity vector are obtained from Eqs. (1.31):

$$v_r = 0, \quad v_\theta \equiv 0, \quad v_\varphi = -\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\Psi_1^{(1)}, \tag{1.36}$$

where the scalar potential $\Psi_1^{(1)}$ is determined from the corresponding equation (1.32).

Equations (1.31)–(1.36) cannot be applied to problems involving flow around a vibrating cylinder or sphere in an arbitrary direction. Representations of the general solutions for these cases have been discussed in brief form in several papers [34, 38, 40] in several cases taking Eq. (1.6) into account; a fairly complete representation of these general solutions are given in general form in [44] without regard for Eq. (1.6).

1.9. Limiting Cases. The representations of the general solution in the form (1.15)–(1.36) apply to the general case of a compressible, viscous fluid (without thermal effects) in the linearized theory and subsume a number of limiting cases for simpler fluid models (an incompressible, viscous fluid; a compressible, ideal fluid; and an incompressible, ideal fluid). These limiting transitions in the general solutions for a compressible, viscous fluid (at rest or in motion) are considered briefly in two papers [27, 37] and in greater detail in the book [44]; here we give information about these results in very concise form. Inasmuch as Eqs. (1.17), (1.18), and (1.31)–(1.36) represent general solutions for a moving fluid, we begin with the limiting transition to general solutions for a fluid at rest.

Compressible, Viscous Fluid at Rest. To make the transition to general solutions for this case of a compressible, viscous fluid at rest, it must be assumed that the following condition holds in the expressions for a moving compressible, viscous fluid (1.17), (1.18), and (1.31)–(1.36):

$$U \to 0. \tag{1.37}$$

In this case, for example, from Eqs. (1.17) and (1.18) we obtain a general solution in the form (1.15), (1.16) for a compressible, viscous fluid at rest. Accordingly, in the ensuing discussion we consider only limiting transitions from the general solutions for a compressible, viscous fluid at rest.

Incompressible, Viscous Fluid. To make the transition to general solutions for an incompressible, viscous fluid at rest, it must be assumed that the following condition holds in the expressions (1.15), (1.16), and (1.21)–(1.30) for a compressible, viscous fluid at rest:

$$a_0 \to \infty. \tag{1.38}$$

In this case, taking condition (1.38) into account, from (1.16) we obtain equations for the scalar potential $\Phi^{(1)}$ and the vector potential $\vec{\Psi}^{(1)}$

$$\Delta\Phi^{(1)} = 0, \quad \left(\nu^{(1)}\Delta - \frac{\partial}{\partial\tau}\right)\vec{\Psi}^{(1)} = 0. \tag{1.39}$$

Taking Eq. (1.38) and the first equation in (1.39) into account, from Eqs. (1.15) we obtain representations for the velocity vector and the pressure in terms of the vector and scalar potentials in the form

$$\vec{v} = \vec{\nabla}\Phi^{(1)} + \vec{\nabla}\times\vec{\Psi}^{(1)}, \quad \vec{\nabla}\cdot\vec{\Psi}^{(1)} \equiv \operatorname{div}\vec{\Psi}^{(1)} = 0, \quad p^{(1)} = -\rho_0\frac{\partial}{\partial\tau}\Phi^{(1)}. \tag{1.40}$$

Also, from Eqs. (1.15) in conjunction with (1.38) we obtain $\rho^{(1)} = 0$, which characterizes the incompressibility of the fluid. We note that the representation of the general solution in the form (1.39), (1.40) is well-known in the mechanics of incompressible, viscous fluids.

Compressible, Ideal Fluid. To make the transition to general solutions for a compressible, ideal fluid at rest, the following conditions must be assumed:

$$\mu^{(1)}, \lambda^{(1)}, \nu^{(1)} \to 0 . \tag{1.41}$$

Substituting expression (1.41) into Eqs (1.16), we obtain equations for determining the scalar and vector potentials

$$\left(\Delta - \frac{1}{a_0^2} \frac{\partial^2}{\partial \tau^2}\right) \Phi^{(1)} = 0, \quad \frac{\partial}{\partial \tau} \vec{\Psi}^{(1)} = 0 . \tag{1.42}$$

The second equation in (1.42) is satisfied if the vector potential is assumed to be

$$\vec{\Psi}^{(1)} = 0. \tag{1.43}$$

Taking Eqs. (1.41)–(1.43) into account, from Eqs. (1.15) and (1.16) we obtain a representation of the general solution for a compressible, ideal fluid at rest in the form

$$\vec{v} = \vec{\nabla} \Phi^{(1)} \equiv \operatorname{grad} \Phi^{(1)}, \quad p^{(1)} = -\rho_0 \frac{\partial}{\partial \tau} \Phi^{(1)}, \quad \left(\Delta - \frac{1}{a_0^2} \frac{\partial^2}{\partial \tau^2}\right) \Phi^{(1)} = 0 . \tag{1.44}$$

The representation of the solution in the form (1.44) is classical and well known in the mechanics of compressible, ideal fluids in application to the linearized theory describing the propagation of small disturbances in a compressible, ideal fluid at rest (small vibrations, small motions).

Incompressible, Ideal Fluid. To make the transition to general solutions for an incompressible, ideal fluid at rest, it must be assumed that conditions (1.38) and (1.41) hold in the expressions (1.15), (1.16), and (1.21)–(1.30) for a compressible, viscous fluid at rest; as a result, we obtain the well-known relations

$$\vec{v} = \vec{\nabla} \Phi^{(1)} \equiv \operatorname{grad} \Phi^{(1)}, \quad \Delta \Phi^{(1)} = 0 . \tag{1.45}$$

Mention should be made of partial results published in [41, 42] on the construction of general solutions in the incompressible, viscous fluid model. To summarize, all previously known general solutions for simpler fluid models (an incompressible, viscous fluid; a compressible, ideal fluid; and an incompressible, ideal fluid) are obtained as the above-noted limiting cases from the general solutions for compressible, viscous fluids at rest and in motion. It must be noted that additional considerations arise in application to the limiting transitions in the solutions of specific problems obtained in the linearized theory for a compressible, viscous fluid; they are discussed, for example, in [39] and in the book [44] and will be mentioned below in part in analyzing the results for individual classes of problems.

2. Forced Harmonic Vibrations of Rigid Bodies in a Compressible, Viscous Fluid. In this section, we analyze results pertaining to the formulation and special aspects of the theory of forced harmonic vibrations of spherical and circular cylindrical rigid bodies in a compressible fluid at rest and in motion; the main results refer to the determination of the reaction of moving and rest compressible, viscous fluids. The analysis is based on results published in several papers [30–32, 34, 39–41, 90] and in the book [44].

2.1. Formulation of Problems. In the interest of brevity we shall assume that a compressible, viscous fluid occupies "infinite" space; when the fluid occupies a bounded volume, boundary conditions must be additionally supplied at the boundary of the fluid volume. We consider the case of K perfectly rigid bodies situated in the fluid. The relations given below are equally applicable to transient and harmonic motions of perfectly rigid bodies in a compressible, viscous fluid; these results have been published previously in abbreviated form in [30–32] and in more complete form in the book [44].

We now introduce notation associated with the kth perfectly rigid body ($k \approx 1, 2, ..., K$): $\vec{u}^{(k)}$ denotes the displacement vector of the center of inertia of the body; $\vec{\omega}^{(k)}$ is the instantaneous angular velocity vector of the body; $\vec{L}^{(k)}$

is the angular momentum vector in relative motion about the center of inertia; $V^{(k)}$, $S^{(k)}$, and $\rho_{(k)}$ are the volume, surface, and density of the body; $\vec{r}^{(k)}$ is the radius vector drawn from the center of inertia to an arbitrary point of the body; $\vec{v}^{(k)}$ is the velocity of an arbitrary point of the body; $\vec{v}_S^{(k)}$ is the velocity of an arbitrary point on the surface of the body; $\vec{F}^{(k)}$ is the principal vector of forces acting on the body; $\vec{M}^{(k)}$ is the principal moment (about the center of inertia) of forces acting on the body; $\vec{R}_S^{(k)}$ is the force exerted on the fluid by the body at the point on its surface defined by the radius vector $\vec{r}_S^{(k)}$. In this notation, the velocity of an arbitrary point of the body is given by the expression

$$\vec{v}^{(k)} = \dot{\vec{u}}^{(k)} + \vec{\omega}^{(k)} \times \vec{r}^{(k)}. \tag{2.1}$$

Analogously, the velocity of an arbitrary point on the surface of the body is given by the expression

$$\vec{v}_S^{(k)} = \dot{\vec{u}}^{(k)} + \vec{\omega}^{(k)} \times \vec{r}_S^{(k)}. \tag{2.2}$$

The other quantities listed above are also given by conventional equations

$$\vec{L}^{(k)} = \rho_{(k)} \int_{V^{(k)}} \vec{r}^{(k)} \times (\vec{\omega}^{(k)} \times \vec{r}^{(k)})\, d\,V^{(k)}, \tag{2.3}$$

$$\vec{F}^{(k)} = \int_{S^{(k)}} \vec{R}_S^{(k)}\, d\,S^{(k)}, \quad \vec{M}^{(k)} = \int_{S^{(k)}} \vec{r}_S^{(k)} \times \vec{R}_S^{(k)}\, d\,S^{(k)}. \tag{2.4}$$

In the given notation, the equations of motion of the kth perfectly rigid body can be written in the form

$$(\rho_{(k)} V^{(k)})\, \ddot{\vec{u}} = \vec{F}^{(k)}, \quad \dot{\vec{L}}^{(k)} = \vec{M}^{(k)}, \quad k = 1, \dots, K. \tag{2.5}$$

It is important to note that interaction between the kth perfectly rigid body and the compressible, viscous fluid and mutual interaction of the perfectly rigid bodies are driven by the reaction $\vec{R}_s^{(k)}$ of the fluid to the motion of the kth body in Eq. (2.4). The reaction $\vec{R}_s^{(k)}$ is determined at the interface of the perfectly rigid body with the compressible, viscous fluid; we therefore formulate the force and kinematic conditions at this interface. To do so, in the fluid we investigate an arbitrary surface S, denoting its outward unit normal by $\vec{N}$. On this surface S (with outward normal $\vec{N}$) in the fluid we can define the stress vector $\vec{T}_N$, whose components are expressed in the linearized theory in terms of the covariant components (1.9) or (1.14) of the stress tensor in the compressible, viscous fluid. In addition to the one side of the surface S (with its outward normal $\vec{N}$), we can also investigate its other side (the surface S with outward normal $-\vec{N}$); in the latter case we can introduce the stress tensor $\vec{T}_{-N}$ in application to the compressible, viscous fluid and write the following equation on the basis of the equilibrium condition:

$$\vec{T}_N + \vec{T}_{-N} = 0. \tag{2.6}$$

Using the notation introduced above along with Eq. (2.6), we can determine the force exerted on the body by the fluid, specifically the force $\vec{R}_S^{(k)}$ acting at the point on the surface defined by the radius vector $\vec{r}_S^{(k)}$ for the kth body, in the form

$$\vec{R}_S^{(k)} = -\vec{T}_{-N^{(k)}} = \vec{T}_{N^{(k)}}. \tag{2.7}$$

In Eq. (2.7) and below, we denote by $\vec{N}^{(k)}$ the outward unit normal to the surface of the kth perfectly rigid body. Substituting Eq. (2.7) into (2.4), we obtain expressions for the principal vector of forces and the principal moment (about the center of inertia) of forces acting on the kth perfectly rigid body in the form

$$\vec{F}^{(k)} = \int_{S^{(k)}} \vec{T}_{N^{(k)}} \, dS^{(k)}, \quad \vec{M}^{(k)} = \int_{S^{(k)}} \vec{r}_S^{(k)} \times \vec{T}_{N^{(k)}} \, dS^{(k)}. \tag{2.8}$$

Equations (2.8) completely characterize the force conditions at the interface between the compressible, viscous fluid and the perfectly rigid body, their right hand sides containing the stress vector in the compressible, viscous fluid.

The kinematic conditions at an interface between media, one of which is a compressible, viscous fluid, can vary: complete no-slip conditions with continuity of the velocity vector; partial no-slip conditions with continuity of individual components of the velocity vector; partial no-slip conditions on selective parts of the interface, or so-called mixed boundary conditions. The conditions of continuity of the velocity vector (complete no-slip) on the surface $S^{(k)}$ of the kth perfectly rigid body can be written in the form

$$\vec{v}\,|_{S^{(k)}} = \vec{v}_S^{(k)}, \tag{2.9}$$

where $\vec{v}$ is the velocity vector of the compressible, viscous fluid as defined by the equations in the preceding section, and $\vec{v}_S^{(k)}$ is the velocity vector (2.2) of an arbitrary point on the surface of the kth body. Other kinematic conditions at the interface (for different problems) can be formulated analogously.

Equations (2.1)–(2.8) apply in equal measure to a compressible, viscous fluid at rest and to a moving compressible, viscous fluid; in the latter case, Eqs. (1.14) must be used instead of Eqs. (1.9) to determine the covariant components of the stress tensor. The kinematic conditions at the interface for a moving compressible, viscous fluid, taking (1.11) into account, now assumes the following form instead of Eq. (2.9):

$$\vec{e}_3 U + \vec{v}\,|_{S^{(k)}} = \vec{v}_S^{(k)}. \tag{2.10}$$

Equations (2.1)–(2.10) completely define the statement of the problems of the dynamics of a system of K perfectly rigid bodies in compressible, viscous fluids at rest and in motion within the framework of the linearized relations; for transient (time-dependent) problems, on the other hand, it is required to tie in with appropriate initial conditions.

Forced vibrations of, say, the kth perfectly rigid body can be characterized by specifying the velocity of the center of inertia in Eq. (2.2) in the form

$$\dot{\vec{u}}^{(k)} = \vec{v}_0^{(k)} \exp(-i\,\Omega_1\,\tau), \quad \vec{v}_0^{(k)} = \text{const}, \quad \Omega_1 = \text{const} \tag{2.11}$$

or by specifying the instantaneous angular velocity in Eq. (2.2) in the form

$$\vec{\omega}^{(k)} = \vec{\omega}_0^{(k)} \exp(-i\,\Omega_2\,\tau), \quad \vec{\omega}_0^{(k)} = \text{const}, \quad \Omega_2 = \text{const}\,. \tag{2.12}$$

In the cases of forced motions of the kth body in the forms (2.11) and (2.12), the velocity of an arbitrary point of the body on the right-hand sides of the kinematic conditions (2.9) and (2.10) can be determined from Eqs. (2.2), (2.11), and (2.12). The kinematic conditions (2.9) and (2.10) therefore acquire the significance of boundary conditions for the compressible, viscous fluid dynamics equations treated in the preceding section. Once the problems formulated by the above-described method have been solved, the reaction of the compressible, viscous fluid to forced vibrations of the rigid bodies can be determined in the form (2.11) and (2.12). Investigations have been carried out in the above-described formulation, with brief results published in several papers [30–32, 34, 39–41, 90] and in greater detail in the book [44].

2.2. Limiting Cases in the Exact Solutions. Exact solutions of problems involving the vibrations of a circular cylinder (of radius a) and a sphere (of radius a) in a compressible, viscous fluid have been obtained [30–32, 34, 39–41, 90] on the basis of the formulation given in this section using the general solutions set forth in the preceding section. These exact solutions, in turn, have been used to find exact expressions for the resisting (drag) forces as functions of cylindrical and spherical Hankel functions, whose arguments are the parameters

$$r_1 = \frac{a\,\Omega}{a_0}\left(1 - i\,\Omega\,\frac{\lambda^{(1)} + 2\,\mu^{(1)}}{a_0^2\,\rho_0}\right)^{-1/2}, \quad r_2 = \sqrt{i\,\Omega\,a^2/\nu^{(1)}}\,. \tag{2.13}$$

Accordingly, the transition to various special cases in the exact expressions for the resisting forces can be made by letting the quantities (2.13) tend to various limiting values. To facilitate the analysis of the limiting cases in question, we introduce the auxiliary parameters

$$\bar{c} = a\,\frac{\Omega}{a_0}, \quad \varepsilon = \sqrt{\nu^{(1)}/(a^2\,\Omega)}\,, \quad \bar{R} = \frac{\Omega\,a^2}{\nu^{(1)}}, \quad \varepsilon = \frac{1}{\sqrt{\bar{R}}}\,. \tag{2.14}$$

In Eqs. (2.14), $\bar{c}$ denotes the dimensionless velocity, ε is a parameter characterizing the influence of viscosity, where in the case of a low-viscosity fluid or in the limiting case attained by transition to an ideal fluid with condition (1.41) taken into account we have $\varepsilon \to 0$, and $\bar{R}$ is a parameter of the Reynolds number type. It is important to note that $\bar{R}$ resembles the Reynolds number in structure, but is not the same; consequently, following [44], from now on we refer to $\bar{R}$ as a parameter of the Reynolds number type, but sometimes for brevity we shall also call it the Reynolds number. Indeed, in [76], p. 123, the Reynolds number R for the investigated vibratory motions is defined as

$$R = \frac{u_0\,\Omega\,a}{\nu^{(1)}}, \tag{2.15}$$

where u_0 is the amplitude of the vibrations, and a represents a characteristic dimension of the body or, specifically in application to a sphere or a cylinder, a is the radius. Inasmuch as the velocity for vibratory motions is of the order of $u_0\,\Omega$, the parameter (2.15) has the conventional significance. From Eqs. (2.14) and (2.15), we obtain the functional relations

$$\bar{R} = R\,\frac{a}{u_0}, \quad R = \frac{u_0}{a}\,\bar{R}\,, \quad \bar{R} = \frac{\Omega\,a^2}{\nu^{(1)}}\,. \tag{2.16}$$

Since u_0 and a are bounded, nonzero quantities, R and $\bar{R}$ can tend to zero simultaneously. Consequently, if we consider cases involving low Reynolds numbers ($R << 1$) and calculate the first terms of the expansion in the limit $R \to 0$, this procedure is equivalent to the one used when small values of $\bar{R}$ are considered, and the first terms of the expansion are calculated in the limit $\bar{R} \to 0$. This special attribute must be taken into account in the ensuing analysis. In the notation (2.14), we can write the parameters (2.13) in the form

$$r_1 = \frac{a\,\Omega}{a_0}\left(1 - i\,\Omega\,\frac{\lambda^{(1)} + 2\,\mu^{(1)}}{a_0^2\,\rho_0}\right)^{-1/2} \equiv r_2\left(\frac{\mu^{(1)}}{\lambda^{(1)} + 2\,\mu^{(1)}}\right)^{1/2}\left(1 - \frac{r_2^2}{\bar{c}^2}\,\frac{\mu^{(1)}}{\lambda^{(1)} + 2\,\mu^{(1)}}\right)^{-1/2},$$

$$r_2 = \sqrt{i\,\bar{R}} \equiv \frac{\sqrt{i}}{\varepsilon}\,, \quad \bar{R} = \frac{\Omega\,a^2}{\nu^{(1)}}\,. \tag{2.17}$$

We now consider the distinctive characteristics of limiting transitions in the exact equations for the resisting forces, bearing all the foregoing information in mind.

1. Transition to a Low-Viscosity Fluid (and to an Ideal Fluid in the Limit). Equation (1.41) must be used in this case, where for finite values of Ω and a_0 we obtain the following from Eqs. (2.17):

$$\varepsilon \to 0, \quad \bar{R} \to \infty\,, \quad r_2 \to \sqrt{i} \cdot \infty\,, \quad r_1 \to \frac{a\,\Omega}{a_0}\,. \tag{2.18}$$

If the short-wavelength (high-frequency) approximation ($\Omega \to \infty$) is considered here, the last expression in (2.18) becomes invalid by virtue of the first expression in (2.17). If the long-wavelength (low-frequency) approximation ($\Omega \to 0$) is considered, the first three expressions in (2.18) become invalid by virtue of Eqs. (2.17). Consequently, if the transition is made to a low-viscosity fluid (and to an ideal fluid in the limit) by means of Eqs. (2.18), it is impossible to obtain results that also apply to both the high-frequency and the low-frequency approximation; the results obtained by means of Eqs. (2.18) are therefore applicable to a low-viscosity fluid only for $\Omega \neq 0$ and for finite values of Ω. From Eqs. (2.17) in conjunction with conditions (2.18) we deduce the conditions

$$\operatorname{Im} r_1 > 0, \quad \operatorname{Im} r_2 > 0, \tag{2.19}$$

which are characteristic of an ideal fluid.

2. Incompressible, Viscous Fluid. Here it is required to use Eq. (1.38), where the following expressions are obtained from Eqs. (2.17) and (2.14) for a finite value of Ω:

$$r_1 \to \frac{a\,\Omega}{a_0} \to 0, \quad \bar{c} \to 0. \tag{2.20}$$

If the short-wavelength (high-frequency) approximation ($\Omega \to \infty$) is considered here, Eqs. (2.20) become invalid by virtue of Eqs. (2.17) and (2.14). Consequently, if the transition is made to an incompressible, viscous fluid by means of Eqs. (2.20), it is impossible to obtain results that also apply to the high-frequency approximation; the results obtained by means of Eqs. (2.20) are therefore applicable to an incompressible, viscous fluid for finite values of Ω. Note that conditions (2.19) are also satisfied for Eqs. (2.20).

3. Long-Wavelength (Low-Frequency) Approximation. In this case, it must be assumed that

$$\Omega \to 0, \tag{2.21}$$

where for finite values of $\nu^{(1)}$ the following expressions are obtained from Eqs. (2.17):

$$r_1 \to \frac{a\,\Omega}{a_0} \to 0, \quad r_2 \to \sqrt{i} \cdot 0\,. \tag{2.22}$$

If the transition is now made to an ideal fluid ($\nu^{(1)} \to 0$), the second expression in (2.22) becomes invalid by virtue of Eqs. (2.17). Consequently, if the transition is made to the long-wavelength (low-frequency) approximation for a compressible, viscous fluid by means of Eqs. (2.22), it is impossible to obtain results that also apply to a compressible, ideal fluid; the results obtained by the indicated procedure are therefore applicable to a compressible, viscous fluid for finite values of $\nu^{(1)}$. Note that conditions (2.19) are also satisfied for Eqs. (2.22).

4. Short-Wavelength (High-Frequency) Approximation. In this case it must be assumed that

$$\Omega \to \infty\,, \tag{2.23}$$

where for $\nu^{(1)} \neq 0$ the following expressions are obtained from Eqs. (2.17):

$$r_1 \to r_2 \left(\frac{\mu^{(1)}}{\lambda^{(1)} + 2\,\mu^{(1)}} \right)^{1/2}, \quad r_2 \to \sqrt{i} \cdot \infty\,. \tag{2.24}$$

If the transition is now made to an ideal fluid (in accordance with (1.41), $\nu^{(1)} \to 0$), the first expression in (2.24) becomes invalid by virtue of Eqs. (2.17) and (2.14). Consequently, if the transition is made to the short-wavelength (high-frequency) approximation for a compressible, viscous fluid by means of Eqs. (2.24), it is impossible to obtain results that also apply to a compressible, ideal fluid; the results obtained by the indicated procedure are therefore applicable to a compressible, viscous fluid for finite values of $\nu^{(1)}$.

5. Low Reynolds Number Fluid. In this case it must be assumed that

$$\bar{R} \to 0\,, \tag{2.25}$$

where for finite values of a_0 and $\Omega \nrightarrow 0$ the following expressions are obtained from Eqs. (2.17) and (2.14):

$$r_1 \to r_2 \left(\frac{\mu^{(1)}}{\lambda^{(1)} + 2\mu^{(1)}} \right)^{1/2}, \quad r_2 = \sqrt{i\bar{R}} \to \sqrt{i} \cdot 0 . \tag{2.26}$$

If the transition is now made to an incompressible, viscous fluid (in accordance with (1.38), $a_0 \to \infty$) or to the long-wavelength (low-frequency) approximation (in accordance with (2.21), $\Omega \to 0$), the first expression in (2.26) becomes invalid by virtue of Eqs. (2.17) and (2.14). Consequently, if the transition is made to a fluid with a low Reynolds number (in accordance with (2.25), $\bar{R} \to 0$) by means of Eqs. (2.26), it is impossible to obtain results that also apply to an incompressible, viscous fluid ($a_0 \to \infty$) or to the long-wavelength (low-frequency) approximation ($\Omega \to 0$); the results obtained by the indicated procedure are therefore applicable only to a compressible, viscous fluid for finite values of a_0 and $\Omega \nrightarrow 0$. It should be noted that when conditions (2.26) are used to go over to a compressible, viscous fluid with a low Reynolds number, conditions (2.19) are satisfied.

It is important to mention that the established results for low Reynolds number fluids [74–76, 83–85] have been obtained for an incompressible, viscous fluid and therefore, in light of the foregoing analysis, *cannot be obtained* from the corresponding results for a compressible, viscous fluid with a low Reynolds number by passing to the limit $a_0 \to \infty$.

The preceding information as to limiting transitions in the exact solutions for a compressible, viscous fluid is necessary in analyzing the results and corresponding conclusions of a physical character obtained on the basis of the given compressible, viscous fluid model. The indicated results pertaining to the limiting transitions are briefly covered in the paper [39] and in greater detail in the book [44]; misprints occurring in [44] have been eliminated from the presentation of the stated results in this section.

2.3. Forced Harmonic Vibrations of Spherical and Circular Cylindrical Rigid Bodies in a Compressible, Viscous Fluid at Rest. Results pertaining to the exact solution of the stated problems in the formulation described in this section, based on the general solutions set forth in the preceding section, have been published in journals [31, 90] and in the book [44]. Below, for forced vibrations in the form (2.11) or (2.12) we shall give only the final results associated with calculating the resisting force (reaction) of the fluid in the form of the principal vector $\vec{F}$ (2.8) in the presence of the excitation (2.11) or in the form of the principal moment $\vec{M}$ (2.8) in the presence of the excitation (2.12). These quantities are written as components along the jth axis in the form

$$F_j = (F_{j1} + i F_{j2}) \exp(-i\Omega\tau), \quad M_j = (M_{j1} + i M_{j2}) \exp(-i\Omega\tau) . \tag{2.27}$$

In Eq. (2.27) and below, we have introduced the following notation: F_{j1} and M_{j1} are the values of the components of the principal vector and principal moment (about the center of gravity of the body) of forces along the jth axis at the beginning of each period; F_{j2} and M_{j2} are the values of the components of the principal vector and principal moment (about the center of gravity of the body) of forces along the jth axis after a quarter-period. Below we give specific results for two problems, whose solutions for an incompressible, viscous fluid on the basis of the Oseen [95] and Stokes [96] approximations are classical and are used to investigate various problems in physics and mechanics.

1. Transverse Vibrations of a Circular Cylinder in a Compressible, Viscous Fluid [31, 44]. We consider a perfectly rigid (infinite in the direction of the y_3 axis with unit vector $\vec{e_3}$) cylinder having a circular cross section of radius a. In the cross-sectional plane, the cylinder executes forced vibrations along the y_1 axis with unit vector $\vec{e_1}$; in this case, according to Eqs. (2.11), the velocity of the center of inertia is written in the form

$$\vec{v_S} = \vec{e_1} v_0 \exp(-i\Omega\tau), \quad v_0 = \text{const}, \tag{2.28}$$

and the boundary conditions for the fluid have the form (2.9) with Eqs. (2.28) taken into account. We give the final result [31, 44] only for low Reynolds numbers, taking Eqs. (2.25) and (2.26) into account, and we restrict the expansions in the parameter $\bar{R}$ of Eqs. (2.14) and (2.16) to a single term. Using the notation (2.27), we can write the result in the form

$$F_{11} \approx -4\pi\mu^{(1)} v_0 \left(-\frac{1}{2} \frac{\lambda^{(1)} + 3\mu^{(1)}}{\lambda^{(1)} + 2\mu^{(1)}} \ln \bar{R} \right)^{-1}, \quad F_{12} \approx 0 . \tag{2.29}$$

We note that the corresponding results for an incompressible, viscous fluid (a simpler model) have been obtained in the Oseen approximation [95], which is a refinement of the Stokes approximation [96]; this result is represented by Eq. (26.27) on p. 534 of [74] or by Eq. (20.19) in [76], which is still known [74] as the Lamb equation. It should also be mentioned that the above-stated result for an incompressible, viscous fluid corresponds to steady flow about a cylinder; consequently, it can only be applied approximately to vibrations of a cylinder at comparatively low frequencies (long-wavelength approximation) to estimate the maximum reaction of the fluid. If the first term of the expansion for low Reynolds numbers ($R << 1$) is calculated in the above-cited result [74, 76], we obtain the following expression (taking into account the direction of motion assumed in Eq. (2.28)):

$$F_{11} \approx -4\pi\mu^{(1)} v_0 (-\ln R)^{-1}. \tag{2.30}$$

It follows from a comparison of Eqs. (2.29) and (2.30) that these results (2.29) (compressible, viscous fluid, forced vibrations of a cylinder, first term of the expansion of (2.14) and (2.16) in the parameter $\bar{R}$, consistent with low Reynolds numbers) and (2.30) (incompressible, viscous fluid, steady flow around a body based on the Oseen approximation [95], first term of the expansion for low Reynolds numbers) are, to a certain extent, mutually consistent. We also note that in analyzing the physical significance of Eq. (2.29), it is necessary to take into account the results described in the preceding subsection regarding limiting transitions in the exact solutions for a compressible, viscous fluid.

2. *Longitudinal Vibrations of a Sphere* [31, 44]. We consider a perfectly rigid sphere of radius a executing forced vibrations (motions) along the y_3 axis with unit vector $\vec{e_3}$; the investigations are carried out within the framework of the axisymmetric problem in spherical coordinates r, θ, and φ, where the angle θ is measured from the y_3 axis. According to Eqs. (2.11), the velocity of the center of inertia is written in the form

$$\vec{v_S} = \vec{e_3} v_0 \exp(-i\Omega\tau), \quad v_0 = \text{const}, \tag{2.31}$$

and the boundary conditions for the fluid have the form (2.9) with Eq. (2.31) taken into account. It is important to note that in this situation a final result of the type given by the first expression (2.27) can be written in a very compact form by means of the representations of cylindrical Hankel functions of {integer + 1/2} order; therefore, taking Eq. (2.27) into account, we obtain [31, 44]

$$F_{31} + iF_{32} = 8\pi a^3 \Omega \rho_0 v_0 i \left[1 - i(r_1 + r_2) - r_1 r_2 - \frac{1}{6}(r_1^2 + r_2^2) + \frac{1}{6} i r_1 r_2 (r_1 + r_2)\right]$$
$$\times \left[-(r_1^2 + 2r_2^2) + i r_1 r_2 (r_1 + 2r_2) + r_1^2 r_2^2\right]^{-1}. \tag{2.32}$$

The notation in (2.13) and (2.17) is used for r_1 and r_2 in Eq. (2.32). The result (2.32) using the notation in (2.13) and (2.17) is obtained from the exact solution without any assumptions as to limiting cases, and it subsumes several special cases. As one such special case, we give the results of [31, 34] for low Reynolds numbers, taking Eqs. (2.25) and (2.26) into account and restricting the discussion to the first terms of the expansions of (2.14) and (2.16) in the parameter $\bar{R}$. Using the notation (2.27), we write this result in the form

$$F_{31} \approx -6\pi a\mu^{(1)} v_0 \left(\frac{4}{3}\frac{\lambda^{(1)} + 2\mu^{(1)}}{2\lambda^{(1)} + 5\mu^{(1)}}\right), \quad F_{32} \approx 0. \tag{2.33}$$

We note that the corresponding results for an incompressible, viscous fluid (a simpler model) have been obtained in the Stokes approximation [96], and the corresponding equation is known as the Stokes equation. It should also be mentioned that the indicated Stokes equation for an incompressible, viscous fluid corresponds to the case of steady flow around a sphere; hence, it can be applied only approximately to the case of vibrations of a sphere at comparatively low frequencies (long-wavelength approximation) to estimate the maximum reaction of the fluid. Taking into consideration the assumed direction of motion in Eq. (2.31), we can write the Stokes equation in the form

$$F_{31} \approx -6\pi a\mu^{(1)} v_0. \tag{2.34}$$

It follows from a comparison of Eqs. (2.33) and (2.34) that the results (2.33) (compressible, viscous fluid, forced vibrations of a sphere, first term of the expansion of Eqs. (2.14) and (2.16) in the parameter $\overline{R}$, corresponding to a low Reynolds number) and (2.34) (incompressible, viscous fluid, steady flow around a body in the Stokes approximation [96], first term of the expansion in low Reynolds numbers) are mutually consistent in a certain sense. We also note that the results given in the preceding subsection on limiting transitions in the exact solutions for a compressible, viscous fluid must be taken into account in analyzing the physical significance of Eq. (2.33).

We infer from these considerations that in the compressible, viscous fluid model, Eq. (2.29) corresponds, in a certain sense, to the Lamb equation obtained on the basis of the incompressible, viscous fluid model in the Oseen approximation [95] (first term of the expansion in the Reynolds number), and Eq. (2.33) corresponds to the Stokes equation obtained on the basis of the incompressible, viscous fluid model in the Stokes approximation [96] (first term of the expansion in the Reynolds number). We should also mention that other problems concerning forced vibrations of cylindrical and spherical bodies in a compressible, viscous fluid have been investigated in [31, 44] as well as in several other publications, complete with the derivation of corresponding exact solutions.

2.4. Forced Harmonic Vibrations of Spherical and Circular Cylindrical Rigid Bodies in a Moving Compressible, Viscous Fluid. Results pertaining to the exact solution of the stated problems in the formulation set forth in this section on the basis of the general solutions given in the preceding section have been published in several papers [30, 32, 34, 38, 40] and in the book [44]. A significant feature of the cited papers is the formulation of general problems involving vibrations of a circular cylinder and a sphere in a compressible, viscous fluid flow and the separation of the general problems into simpler special problems with the basic equations written in appropriate coordinate systems for determining the scalar potentials; these results are equally applicable to harmonic vibrations and to transient problems. We give the final results only in application to calculations of the resistance (reaction) of the fluid in the form of the principal vector $\vec{F}$ (2.8) and the principal moment $\vec{M}$ (2.8) about the center of gravity of the cross section; the corresponding expressions for the indicated principal force vector and principal moment along the jth axis are written in the form (2.27) in this case.

1. Dynamics of a Circular Cylinder in a Transverse Flow. We consider a perfectly rigid (infinite in the direction of the y_3 axis with unit vector $\vec{e_3}$) circular cylinder of radius a. A compressible, viscous fluid flows around the cylinder in the direction perpendicular to its axis (along the y_1 axis with unit vector $\vec{e_1}$) with a constant freestream velocity U "at infinity" (far from the cylinder). Bearing in mind the preceding discussion, in the vicinity of the rigid body in the fluid we can assume that the components of the vector are given by the following equations instead of (1.11) in the linearized formulation:

$$U\,\delta_n^1 + v_n\,(x_m, \tau\,), \quad n, m = 1, 2, 3, \tag{2.35}$$

where $v_n\,(x_m, \tau\,)$ or $v_n\,(\,y_m, \tau\,)$ denotes the perturbations of the components of the velocity vector in Cartesian coordinates. We assume that in addition to the velocity $\vec{V}$ of the center of inertia of the cross section we can also consider rotational motion about the y_3 axis (unit vector $\vec{e_3}$) with angular velocity ω. In this case, the kinematic conditions (2.10) on the surface of the cylinder for a compressible, viscous fluid can be written in the form

$$(\vec{e_1}\,U + \vec{v}\,)\,\big|_{r=a} = \vec{V} + \omega\, a\, \vec{e_\varphi}\,, \qquad \vec{V} = \vec{e_1}\,V_1 + \vec{e_2}\,V_2 + \vec{e_3}\,V_3\,. \tag{2.36}$$

In this formulation, the conditions for the fluid in the form (2.36) on the surface of the cylinder must be augmented with the basic equations (1.12)–(1.14), in which the x_3 axis must now be replaced by the x_1 axis in accordance with Eq. (2.35); the given formulation is therefore governed by Eqs. (1.12)–(1.14) with the indicated substitution, the boundary conditions (2.36), and corresponding extinction conditions at infinity.

According to a *Proposition* proved in [34] and reconstructed in the book [44], in the general case the above-formulated problem of small forced motions (vibrations) of a circular cylinder in uniform transverse flow (perpendicular to the axis of the infinite cylinder) of a compressible, viscous fluid can be separated into the following dissociated problems: the steady-state problem of flow around a stationary cylinder and four dynamical (transient or involving harmonic vibrations) problems. The corresponding parts of the reaction of the fluid for each of the stated problems are determined independently. As mentioned, an essential feature of the above-cited results [34, 44] is the principle that for each of the stated problems the basic equations are written in terms of scalar potentials, each determined from separate equations.

It is also important to note that one or more (not four) of the stated problems can arise, depending on the type of kinematic excitation (2.36), in application to the four dynamical problems indicated above (transient or involving harmonic vibrations).

One of the indicated four dynamical problems comprises *forced harmonic vibrations of a circular cylinder along its axis in transverse flow*; specific results have been obtained in [34, 44] on the basis of the exact solution of this problem. The basic relations for the given problem in this case have been formulated [34, 44] within the framework of the antiplane problem in the x_1x_2 plane (in the transverse cross-sectional plane); according to these relations, the only nonzero component of the velocity vector is the one along the axis of the cylinder (v_3). Accordingly, the kinematic excitation (boundary conditions for the fluid) (2.36) is now written in the form

$$v_3 \mid_{r=a} = V_3^0 \exp(-i\,\Omega\,\tau), \quad V_3^0 = \text{const}. \tag{2.37}$$

The given problem is characterized by the two parameters [34, 44]

$$R_\infty = \frac{U a}{\nu^{(1)}}, \quad \overline{R} = \frac{\Omega a^2}{\nu^{(1)}}. \tag{2.38}$$

In Eqs. (2.38) and below, R_∞ denotes the Reynolds number associated with the freestream flow velocity (the velocity at infinity), and $\overline{R}$ is a parameter of the Reynolds number type (2.14)–(2.16) associated with vibrations of the cylinder and expressed in terms of this Reynolds number by Eqs. (2.16). For the given situation, the exact solution [34, 44] is found in terms of cylinder functions with the following parameter as their argument:

$$r_2^2 = i\,\overline{R} - \frac{1}{4} R_\infty^2. \tag{2.39}$$

From the foregoing discussion, we infer the existence of three distinct regimes of motion, which are defined by the inequalities

$$1)\ \frac{1}{4} R_\infty^2 > \overline{R}, \quad 2)\ \frac{1}{4} R_\infty^2 < \overline{R}, \quad 3)\ \frac{1}{4} R_\infty^2 = \overline{R}. \tag{2.40}$$

Based on considerations of a physical nature, the first regime in (2.40) is the most preferable and consistent with the formulation of linearized problems. We therefore consider [34, 44] the limiting transition to the case of low Reynolds numbers, bearing in mind the reasoning set forth in the second subsection of this section. Consequently, in accordance with Eqs. (2.25), (2.38), and (2.40) we investigate the case

$$\overline{R} << 1, \quad \overline{R} \to 0, \quad R_\infty << 1, \quad R_\infty \to 0, \quad \frac{1}{4} R_\infty^2 > \overline{R}. \tag{2.41}$$

In [34, 44], the expansion of (2.38) in the parameters R_∞ and $\overline{R}$ is restricted to the first term for the case (2.41), and the following result is obtained in accordance with the notation (2.27):

$$F_{31} + i F_{32} \approx 4\pi\mu^{(1)} V_3^0 \left[\ln\left(\overline{R} + \frac{i}{4} R_\infty^2\right)\right]^{-1}. \tag{2.42}$$

If we set $U = 0$ (i.e., assume zero freestream velocity) in Eq. (2.42) and use the notation (2.38), we obtain the following result from Eq. (2.43):

$$F_{31} + i F_{32} \approx 4\pi\mu^{(1)} V_3^0 (\ln \overline{R})^{-1} \tag{2.43}$$

which has also been obtained independently in [44] using the model of a compressible, viscous fluid at rest. It must be understood by virtue of the previously discussed *Proposition* [34, 44] that Eqs. (2.42) and (2.43) correspond only to the reaction (resisting force) in application to the investigated dynamical problem; to derive the complete equation for the resisting force, these expressions must be augmented with the fluid reaction corresponding to the steady-state problem. Moreover, it should be noted that Eqs. (2.42) and (2.43) can also be obtained on the basis of the incompressible, viscous fluid model; this situation is attributable to the invariance of the fluid volume in the antiplane problem.

2. Dynamics of a Sphere in a Flow. We consider a perfectly rigid sphere of radius a immersed in a compressible, viscous fluid flow (along the y_3 axis with unit vector $\vec{e_3}$) with a constant freestream ("at infinity," i.e., far from the sphere) velocity U; in accordance with the preceding discussion, the components of the velocity vector obey Eqs. (1.11). We assume that together with the velocity $\vec{V}$ of the center of the sphere we can also specify rotational motion of the sphere with an instantaneous angular velocity $\vec{\omega}$. In this case, the kinematic conditions (2.10) on the surface of the sphere for a compressible, viscous fluid can be written in the form

$$(\vec{e_3}\,U + \vec{v})\,\big|_{r=a} = \vec{V} + a\,\vec{\omega} \times \vec{e_r}, \tag{2.44}$$

where $\vec{v}$ is the velocity vector of the compressible, viscous fluid. In the given formulation, the boundary conditions for the fluid in the form (2.44) must be augmented with the basic equations (1.12)–(1.14), which are analyzed in spherical coordinates (the angle θ is measured from the unit vector $\vec{e_3}$, i.e., from the direction of the freestream velocity) or in circular cylindrical coordinates (whose axis is aligned with the unit vector $\vec{e_3}$).

The stated problem is therefore governed by Eqs. (1.12)–(1.14), by boundary conditions for the fluid in the form (2.44), whose right-hand sides characterize the kinematic excitation, and by appropriate extinction conditions at infinity; results obtained in this formulation have been published briefly in several papers [30, 32, 38, 40] and more completely in the book [44].

According to a *Proposition* proved in [40] and given in [44], in the general case the above-stated problem of small forced motions (vibrations) of a sphere in a uniform flow (with a constant freestream — far from the sphere — velocity U in the direction of the unit vector $\vec{e_3}$) can be separated into the following dissociated problems: the steady-state problem of flow around a stationary sphere and six dynamical (transient or involving harmonic vibrations) problems. The corresponding parts of the fluid reaction for each of these seven problems are determined independently. As mentioned, a significant result [40, 44] is the principle that, for each of the seven problems, the basic equations are written in terms of scalar potentials, each determined from separate equations. It is also important to note that one or more (not six) of the stated problems can arise, depending on the type of kinematic excitation (2.44), in application to the six dynamical problems indicated above (transient or involving harmonic vibrations).

One of the indicated six dynamical problems comprises *forced harmonic rotational vibrations of a sphere in a flow*; specific results have been obtained in [30, 32, 44] on the basis of the exact solution of this problem. The kinematic excitation of rotational vibrations of the sphere is investigated according to Eq. (2.44) with the angular velocity vector directed along the unit vector e_3 — see Eqs. (1.11) and (2.4) — i.e., along the freestream velocity

$$\vec{\omega} = \omega_0\,\vec{e_3}\exp(-i\,\Omega\,\tau), \qquad \omega_0 = \text{const}, \qquad \omega_3 = \omega_0\exp(-i\,\Omega\,\tau)\,. \tag{2.45}$$

This problem has been investigated [30, 32, 44] using the basic equations of the theory of rotational vibrations (Eqs. (1.35) and (1.36)), which imply that the only nonzero component of the velocity vector is the component v_φ. For this case, in accordance with Eqs. (2.44) and (2.45), we obtain boundary conditions for the fluid in the form

$$v_\varphi\,\big|_{r=a} = (\sin\theta)\,a\,\omega_0\exp(-i\,\Omega\,\tau). \tag{2.46}$$

Equations (2.38) and (2.40) hold for the given problem; therefore, as in the preceding problem for a cylinder (Eqs. (2.37)–(2.43)), we consider the limiting case of low Reynolds numbers, bearing in mind the reasoning set forth in the second subsection of this section. Here also we limit the discussion to the case (2.41), which, based on physical considerations, is the most preferable and consistent with the formulation of linearized problems. Initially, following [30, 32, 44] and to facilitate comparison with earlier results [74], we introduce the following notation in place of (2.27) for the component of the principal moment (about the center of the sphere) of reaction forces of the fluid (resisting forces):

$$M = -\vec{M}\cdot\vec{e_3} = (M_1 + i\,M_2)\exp(-i\,\Omega\,\tau)\,. \tag{2.47}$$

In (2.47), the following additional notation is introduced in accordance with Eqs. (2.27):

$$M = -M_3, \quad M_1 = -M_{31}, \quad M_2 = -M_{32}, \tag{2.48}$$

where the physical significance of the quantities M_{31} and M_{32} is explained in the text following Eqs. (2.27). We confine the ensuing discussion to the linear approximation in the parameter R_∞ (2.38) and take inequalities (2.41) into account; this case corresponds to the investigation of slow vibrations of the sphere, so that the fluid velocity induced by these rotational vibrations is an order of magnitude smaller than the freestream velocity of the fluid at infinity. The following expression has been obtained in [30, 32, 44] using the above-described approach:

$$M_1 + i M_2 \approx 8 \pi \mu^{(1)} a^3 \omega_0 \left[1 - i \frac{1}{3} \frac{\Omega a^2}{\nu^{(1)}} \left(1 + \frac{5 a}{2 \nu^{(1)}} | U | \right) \right]. \tag{2.49}$$

It follows from Eq. (2.49) that the presence of a flow with the velocity U ostensibly makes the fluid more "rigid" in keeping with physical considerations. If we set $U = 0$ in Eq. (2.49) (i.e., go over to the case of a fluid at rest), we obtain the following expression from Eq. (2.49):

$$M_1 + i M_2 \approx 8 \pi \mu^{(1)} a^3 \omega_0 \left[1 - i \frac{1}{3} \frac{\Omega a^2}{\nu^{(1)}} \right]. \tag{2.50}$$

It must be understood by virtue of the previously discussed *Proposition* [40, 44] that Eqs. (2.49) and (2.50) correspond only to the reaction (resisting force) for the investigated dynamical problem; to derive the complete equation for the resisting force, these expressions must be augmented with the fluid reaction corresponding to the steady-state problem. Moreover, Eqs. (2.49) and (2.50) can also be obtained on the basis of the incompressible, viscous fluid model; this situation is attributable to the invariance of the fluid volume in the rotational vibration problem.

We note that the quantity M_1 (2.50) exactly coincides with Eq. (22.7) on p. 503 in the book [74], which is obtained in the Stokes approximation [96] in the problem of slow rotation of a sphere of radius a with a constant angular velocity. For rotational vibrations of a sphere, the quantity M_1 in Eqs. (2.49) and (2.50) corresponds to the reaction of the fluid at the beginning of each period, and the quantity M_2 in Eqs. (2.49) and (2.50) corresponds to the reaction of the fluid after a quarter-period. Although M_2 in Eq. (2.50) is much smaller than M_1 in Eq. (2.50), the quantity M_2 (2.50) still cannot be calculated in the Stokes approximation [96].

In summary, the foregoing example reveals that even for slow vibrations (low-frequency approximation) the Stokes theory or Stokes approximation [96] cannot predict or determine the reaction of the fluid at an arbitrary time. This situation underscores the potential benefits of research on the theory of forced vibrations of rigid bodies in a compressible (and, as a special case, incompressible), viscous fluid.

Conclusion. In closing, we have only analyzed characteristic problems in somewhat abridged form. Publications devoted to investigations of other problems and questions pertinent to the scientific subject discussed in the article are listed in the bibliographic references.

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