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UNBOUNDED SEMIDISTRIBUTIVE LATTICES

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In memory of Viktor A. Gorbunov

We consider two properties which are close to being lower bounded in the class of finite join semidistributive lattices. An example is constructed in which a finite join semidistributive lattice has both these two properties, but it is not lower bounded.

The purpose of this account is to illustrate a construction technique which has proved useful, and apply it to solve an interesting problem. Namely, we answer in the negative the question as to whether a finite lattice satisfying properties (P) and (Q), which are close to boundedness, is bounded. For a more complete discussion of the theory of bounded homomorphisms and lattices, see [1, Ch. II].

1. UNBOUNDED LATTICES

We begin with a well known criterion for meet semidistributivity. If $\mathbf{L}$ is a finite lattice and $a \in \mathrm{J}(\mathbf{L})$, let $\kappa(a)$ be the largest element above a_* but not above a, if such an element exists. We regard $\kappa : \mathrm{J}(\mathbf{L}) \to \mathrm{M}(\mathbf{L})$ as a partial map.

LEMMA 1. Let $\mathbf{L}$ be a finite lattice. Then $\mathbf{L}$ satisfies $\mathrm{SD}_\wedge$ if and only if $\kappa(a)$ exists for each $a \in \mathrm{J}(\mathbf{L})$.

Moreover, if a finite lattice $\mathbf{L}$ satisfies $\mathrm{SD}_\wedge$, then κ maps $\mathrm{J}(\mathbf{L})$ onto $\mathrm{M}(\mathbf{L})$. If $\mathbf{L}$ also satisfies $\mathrm{SD}_\vee$, then κ is one-to-one, and the dual map $\kappa^d : \mathrm{M}(\mathbf{L}) \to \mathrm{J}(\mathbf{L})$ is its inverse.

We define the standard dependency relations on $\mathrm{J}(\mathbf{L})$ as follows (assuming $\mathrm{SD}_\wedge$ for the first three):

$aAb \Leftrightarrow b < a < \kappa(b)^*$;

$aBb \Leftrightarrow a \neq b$, $b \not\leq \kappa(a)$, and $b_* \leq \kappa(a)$;

$aCb \Leftrightarrow aAb$ or aBb;

$aDb \Leftrightarrow$ there exists $x \in L$ such that $a \leq b \vee x$ but $a \not\leq b_* \vee x$.

Thus $A \cup B = C \subseteq D$.

The dual relations are defined on $\mathrm{M}(\mathbf{L})$. In semidistributive lattices, they behave particularly nicely, as is shown by the following result.

LEMMA 2 (see [2]). Let $\mathbf{L}$ be a finite semidistributive lattice and let $a, b \in \mathrm{J}(\mathbf{L})$. We have:

(1) aAb if and only if $\kappa(a)B^d\kappa(b)$.

(2) aBb if and only if $\kappa(a)A^d\kappa(b)$.

Recall the basic results on boundedness (in the sense of McKenzie) and D-cycles.

THEOREM 3. A finite lattice $\mathbf{L}$ is lower bounded if and only if $\mathbf{L}$ contains no D-cycle. Moreover, every lower bounded lattice satisfies $\mathrm{SD}_\vee$.

THEOREM 4. A finite semidistributive lattice $\mathbf{L}$ is bounded if and only if $\mathbf{L}$ contains no C-cycle.

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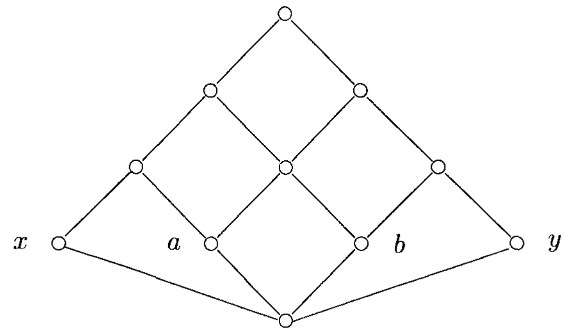

Fig. 1

We will be interested in semidistributive lattices which satisfy the condition

$$\text{(P)} \qquad \text{if } a, b \in \mathrm{J}(\mathbf{L}) \text{ and } b < a, \text{ then } aAb$$

and its dual

$$\text{(Q)} \qquad \text{if } p, q \in \mathrm{M}(\mathbf{L}) \text{ and } q > p, \text{ then } pA^d q.$$

Note that pA^dq is equivalent to $\kappa^d(p)B\kappa^d(q)$.

In [3], Caspard proved that the lattice $\mathbf{S}_n$ of all permutations of an n-element set is bounded. In [4], she proved that these lattices satisfy (P) and (Q), and used this to give a nice characterization of the linear orders on $\mathrm{J}(\mathbf{S}_n)$ which are consistent with the dependency relation D. Thus it seems natural to ask the following:

If a finite semidistributive lattice satisfies (P) and (Q), must it be bounded?

We will show that the answer is 'no.'

2. SEMIDISTRIBUTIVE LATTICES

LEMMA 5. If $\mathbf{L}$ is a finite lattice which fails $\mathrm{SD}_\vee$, then there exist distinct elements $a, b \in \mathrm{J}(\mathbf{L})$ and $c \in L$ such that $a \vee c = b \vee c \succ c$, $a_* \leq c$, and $b_* \leq c$.

Proof. Suppose $a_0 \vee c_0 = b_0 \vee c_0 > (a_0 \wedge b_0) \vee c_0$ in $\mathbf{L}$. Choose c such that $a_0 \vee b_0 \succ c \geq (a_0 \wedge b_0) \vee c_0$. Then choose a minimal such that $a \leq a_0$, but $a \not\leq c$, and choose b minimal such that $b \leq b_0$ but $b \not\leq c$.

As an immediate application, we have the following results, which characterize join semidistributivity as a sort of weak lower boundedness.

THEOREM 6. Let $\mathbf{L}$ be a finite lattice. Then $\mathbf{L}$ fails $\mathrm{SD}_\vee$ if and only if there exist distinct elements $a, b \in \mathrm{J}(\mathbf{L})$ and $x \in L$ such that $a \vee x = b \vee x$, $a \not\leq b_* \vee x$, and $b \not\leq a_* \vee x$.

COROLLARY 7. If $\mathbf{L}$ is a finite lattice which fails $\mathrm{SD}_\vee$, then $\mathbf{L}$ is not lower bounded. In fact, there exist $a, b \in \mathrm{J}(\mathbf{L})$ such that $aDbDa$ via the same element x.

Figure 1 gives a lattice which satisfies $\mathrm{SD}_\vee$ but which contains a short cycle $aDbDa$, via distinct elements x and y.

THEOREM 8. Let $\mathbf{L}$ be a finite lattice which satisfies $\mathrm{SD}_\wedge$. Then $\mathbf{L}$ fails $\mathrm{SD}_\vee$ if and only if there exist $a, b \in \mathrm{J}(\mathbf{L})$ such that $aBbBa$.

Proof. If $\mathbf{L}$ fails $\mathrm{SD}_\vee$, then we can obtain elements $a, b \in \mathrm{J}(\mathbf{L})$ and $c \in L$ as in Lemma 5. Now $a \leq b \vee c$ implies that at least one of b, c is not below $\kappa(a)$. Since $a_* \leq c$ and $a \not\leq c$, we have $c \leq \kappa(a)$, and so $b \not\leq \kappa(a)$. However, $b_* \leq c \leq \kappa(a)$, whence aBb. Similarly, bBa.

3. A COUNTEREXAMPLE

In this section we will construct an example of a lattice which is semidistributive, satisfies properties (P) and (Q), but which is not bounded.

Let $\mathbf{K}$ be the join semilattice with 0 generated by the set

$$G = \{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, x, y, z\}$$

subject to the following relations:

$$\begin{array}{lll} a_0 < a_1 < a_2, & b_0 < b_1 < b_2, & c_0 < c_1 < c_2, \\ a_2 \leq a_1 \vee x, & a_1 \leq a_0 \vee b_2, & a_1 \leq a_0 \vee y, \\ b_2 \leq b_1 \vee y, & b_1 \leq b_0 \vee c_2, & b_1 \leq b_0 \vee z, \\ c_2 \leq c_1 \vee z, & c_1 \leq c_0 \vee a_2, & c_1 \leq c_0 \vee x. \end{array}$$

We have implemented this construction using LISP, and it turns out that $\mathbf{K}$ has 198 elements. We claim that $\mathbf{K}$ has the desired properties.

By construction, $\mathrm{J}(\mathbf{K}) = G$, and each element of $\mathbf{K}$ can be represented as a join of these elements using at most one a_i, one b_i, and one c_i. Moreover, $a_2 \succ a_1 \succ a_0 \succ 0$, and similarly for the b_i's and c_i's, while x, y and z are atoms. In order to check that $\mathbf{K}$ satisfies $\mathrm{SD}_\wedge$, we must find $\kappa(a)$ for each $a \in G$. It is not hard to verify that the following list is correct:

$$\begin{array}{rclrcl} \kappa(a_2) & = & a_1 \vee b_2 \vee c_2 \vee y \vee z, & \kappa(c_2) & = & a_2 \vee b_2 \vee c_1 \vee x \vee y, \\ \kappa(a_1) & = & a_0 \vee b_1 \vee c_2 \vee x \vee z, & \kappa(c_1) & = & a_1 \vee b_2 \vee c_0 \vee y \vee z, \\ \kappa(a_0) & = & b_2 \vee c_2 \vee x \vee y \vee z, & \kappa(c_0) & = & a_2 \vee b_2 \vee x \vee y \vee z, \\ \kappa(b_2) & = & a_2 \vee b_1 \vee c_2 \vee x \vee z, & \kappa(x) & = & a_2 \vee b_2 \vee c_2 \vee y \vee z, \\ \kappa(b_1) & = & a_2 \vee b_0 \vee c_1 \vee x \vee y, & \kappa(y) & = & a_2 \vee b_2 \vee c_2 \vee x \vee z, \\ \kappa(b_0) & = & a_2 \vee c_2 \vee x \vee y \vee z, & \kappa(z) & = & a_2 \vee b_2 \vee c_2 \vee x \vee y. \end{array}$$

Using this, we can check that all the A and B relations holding in $\mathbf{K}$ are contained in the following list:

$$\begin{array}{ccc} a_2\,A\,a_1, & a_1\,A\,a_0, & a_2\,A\,a_0, \\ a_2\,B\,x, & a_1\,B\,b_2, & a_1\,B\,y, \\ b_2\,A\,b_1, & b_1\,A\,b_0, & b_2\,A\,b_0 \\ b_2\,B\,y, & b_1\,B\,c_2, & b_1\,B\,z, \\ c_2\,A\,c_1, & c_1\,A\,c_0, & c_2\,A\,c_0, \\ c_2\,B\,z, & c_1\,B\,a_2, & c_1\,B\,x. \end{array}$$

(These are just the relations you would expect from the defining relations. There are three more nontrivial join covers implied by them, viz., $a_1 \leq a_0 \vee b_1 \vee y$ and symmetrically, but these do not introduce any new dependency relation.)

Now we see that $\mathbf{K}$ satisfies $\mathrm{SD}_\vee$ because there is no cycle of the form $dBeBd$. However, $\mathbf{K}$ is unbounded because of the cycle

$$a_2 A a_1 B b_2 A b_1 B c_2 A c_1 B a_2.$$

It is easy to check property (P), and we verify (Q) in the form $\kappa(e) > \kappa(d)$ implies dBe.

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