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On generalized energy equality of the Navier-Stokes equations

Yasushi Taniuchi
Graduate School of Polymathematics
Nagoya University
Nagoya 464-01 JAPAN

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Abstract

We show that for every initial data $a \in L^2(\Omega)$ there exists a weak solution u of the Navier-Stokes equations satisfying the generalized energy inequality introduced by Caffarelli-Kohn-Nirenberg for $n = 3$. We also show that if a weak solution $u \in L^s(0,T; L^q(\Omega))$ with $2/q + 2/s \leq 1$ and $3/q + 1/s \leq 1$ for $n = 3$, or $2/q + 2/s \leq 1$ and $q \geq 4$ for $n \geq 4$, then u satisfies both the generalized and the usual energy equalities. Moreover we show that the generalized energy equality holds only under the local hypothesis that $u \in L^s(\varepsilon, T; L^q(K))$ for all compact sets $K \subset\subset \Omega$ and all $0 < \varepsilon < T$ with the same (q,s) as above when $3 \leq n \leq 10$.

1 Introduction

Let Ω be a domain in $\boldsymbol{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$. On the space-time cylinder $\Omega \times (0,\infty)$, we consider the nonstationary Navier-Stokes problem:

$$\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } x \in \Omega, t > 0, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega,\ t > 0, \quad u|_{t=0} = a, \tag{1.2}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and pressure, respectively; $a = a(x) = (a^1(x), a^2(x), \ldots, a^n(x))$ is the initial velocity field; $f = f(x,t) = (f^1(x,t), f^2(x,t), \ldots, f^n(x,t))$ is a given external force. Here we use the notation:

$$u \cdot \nabla v = \sum_{j=1}^{n} u^j \frac{\partial v}{\partial x^j}, \quad \nabla \cdot u = \sum_{j=1}^{n} \frac{\partial u^j}{\partial x^j}$$

for vector functions u and v. Up to now the fundamental problem of existence and uniqueness of global solutions to (1.1)-(1.2) is unsolved. Leray [11] and Hopf [10] showed the existence of a global weak solution to (1.1)-(1.2). However, uniqueness and regularity of weak solutions are still open problems. When Ω is $\boldsymbol{R}^3$, Leray [11] introduced a class of weak solutions with the energy inequality of the strong form which are called turbulent solutions (see Masuda [12]). He constructed turbulent solutions u for every $a \in L^2(\boldsymbol{R}^3)$ and proved that $u(t) \in C^\infty(\boldsymbol{R}^3)$ for almost all $t \in (0,\infty)$. After his pioneer work, the singular set $\Sigma \equiv \{t \in (0,\infty); u(t) \notin C^\infty(\Omega)\}$ *in time* was studied by Foias-Temam [4], Giga [6], and many authors and it is known that $\mathrm{H}^{1/2}(\Sigma) = 0$. Here H^s denotes the s-dimensional Hausdorff measure. On the other hand, the singular set $S \equiv \{(x,t) \in \Omega \times (0,\infty); u \notin L^\infty(B_r(x,t))$ for all $r > 0\}$ in *space-time* was investigated by the series of papers of Scheffer [16]-[20], where $B_r(x,t) = \{(y,s) \in |y-x|+|s-t| < r\}$. Then Caffarelli-Kohn-Nirenberg [3] improved Scheffer's result and introduced a new class of *suitable* weak solutions satisfying a local version of the energy inequality, i.e., the generalized energy inequality:

$$(1.3)\quad 2\int_0^\infty\int_\Omega |\nabla u|^2\phi\, dxdt \le \int_0^\infty\int_\Omega [|u|^2(\phi_t+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2(u\cdot f)\phi]dxdt$$

for all $\phi \in C_0^\infty(\Omega \times (0,\infty))$ with $\phi \ge 0$. Actually, Caffarelli-Kohn-Nirenberg [3] constructed the suitable weak solution and proved that $\mathrm{H}^1(S) = 0$ for all such solutions if $n = 3$. Concerning the initial data a, they imposed the condition that $a \in L^2(\boldsymbol{R}^3)$ in $\boldsymbol{R}^3$. In bounded domains, however, they assumed that $a \in L^2(\Omega) \cap W^{2/5,5/4}(\Omega)$.

The first purpose of the present paper is to remove the superfluous hypothesis $a \in W^{2/5,5/4}(\Omega)$ imposed by [3], and as in the case of the whole space $\boldsymbol{R}^3$, the condition $a \in L^2(\Omega)$ is sufficient for construction of suitable weak solutions. Our result covers not only the case when Ω is a bounded domain but also the case when Ω is an *exterior* domain and the *half-space* $\boldsymbol{R}^3_+$. For this purpose, we need to estimate an associated pressure only under the condition $a \in L^2(\Omega)$. Here we adopt the method introduced by Miyakawa-Sohr [14]. Investigating this procedure more precisely, we shall derive a larger class of weak solutions satisfying the generalized energy inequality.

In addition to the uniqueness result, Serrin [15] showed that if a weak solution $u \in L^s(0,T;L^q(\Omega))$ for

$$(1.4)\qquad n/q + 2/s \le 1,$$

then u satisfies the energy equality

$$(1.5)\quad \|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 dt = \|u(s)\|_2^2 + 2\int_s^t (f,u)dt, \quad 0 \le s \le t \le T.$$

Shinbrot [22] derived the same conclusion if $u \in L^s(0,T;L^q(\Omega))$ for

$$(1.6)\qquad 2/q + 2/s \le 1, \text{ and } q \ge 4.$$

Shinbrot's hypothesis is weaker than Serrin's if $n \ge 4$, but not the case for $n = 3$. Thus it occurs the question which class of solutions guarantees the generalized energy equality. Here we mean by the generalized energy equality

$$(1.7)\quad 2\int_0^T\int_\Omega |\nabla u|^2\phi\, dxdt = \int_0^T\int_\Omega [|u|^2(\phi_t+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2(u\cdot f)\phi]dxdt$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$. The second purpose of this paper is to give a partial answer to this question. We show that if $u \in L^s(0,T;L^q(\Omega))$ with

$$(1.8)\qquad 2/q + 2/s \le 1, \text{ and } 3/q + 1/s \le 1 \text{ for } n = 3,$$

$$(1.9)\qquad 2/q + 2/s \le 1, \text{ and } q \ge 4 \text{ for } n \ge 4,$$

then u satisfies both usual (1.5) and generalized (1.7) energy equalities. For $n = 3$ our class (1.8) is *larger* than both Serrin's and Shinbrot's results. For $n \geq 4$ our class (1.9) coincides with Shinbrot's. Sohr-von Wahl [24] showed that if $u^iu^j \in L^2(0,T; L^2(\Omega))$ for $i,j = 1,2,\cdots,n$, then u satisfies the energy equality. Although the hypotheses (1.8) and (1.9) yield the class of Sohr-von Wahl, we can characterize the class with the energy equality by means of solutions u itself directly, not by u^iu^j. In view of the structure of the generalized energy equality, it is reasonable to expect that one can get the generalized energy equality only by the information on *local* behavior of u. Compared with the usual parabolic equations, we need to obtain the information on the local behavior of the associated pressure p. However, such behavior of p cannot be easily derived from the local behavior of the velocity u, because p is determined by the global properties of the domain Ω, i.e., the Helmholtz decomposition. To overcome this difficulty, we shall make use of the cut-off technique to reduce the local problem to the global one. Bogovski's result on the operator *div* plays an important role for this reduction. In fact we shall show that if

$$u \in L^s(\varepsilon, T; L^q(K)) \text{ for all } 0 < \varepsilon < T, \text{ and all compact domain } K \subset\subset \Omega$$
$$\text{with } (1.8) - (1.9) \text{ for } 3 \leq n \leq 10,$$

then u satisfies the generalized energy equality. Our result may be regarded as a local version of Serrin [15] and Shinbrot [22]. Moreover, we can estimate the singular set of weak solution u as in Caffarelli-Kohn-Nirenberg [3] only by using local behavior of u.

In Section 2, we shall state the main results. Section 3 is devoted to preparing some fundamental lemmas for the approximate solutions. Then we shall prove the main results in Sections 4, 5 and 6.

2 Results

Throughout this paper we impose the following assumption on the domain.

Assumption 2.1 *$\Omega \subset \boldsymbol{R}^n (n \geq 3)$ is a bounded domain with smooth boundary, an exterior domain with smooth boundary, the half-space $\boldsymbol{R}^n_+$ or the whole space $\boldsymbol{R}^n$.*

Before stating our results, we introduce some notations and function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that div $\phi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\| \ \|_r$; $(\cdot,\cdot)$ denotes the L^2- inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued)L^r-space over Ω, $1 < r < \infty$. $H^{1,r}_{0,\sigma}$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j; i,j = 1,\cdots,n)$. When X is a Banach space, its norm is denoted by $\| \cdot \|_X$. Then $C^m([t_1,t_2); X)$ is the usual Banach space, where $m = 0,1,2,\cdots$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. In this paper, we denote by C various constants. In particular, $C = C(*,\cdots,*)$ denotes the constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G_r(\text{direct sum}), \quad 1 < r < \infty,$$

where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$. For the proof, see Fujiwara-Morimoto [5], Miyakawa [13], Simader-Sohr [21] and Borchers-Miyakawa [1]. P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in W^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. It is known that

$$(L^r_\sigma)^*(\text{the dual space of} \quad L^r_\sigma) = L^{r'}_\sigma, \quad A^*_r(\text{the adjoint operator of} \quad A_r) = A_{r'},$$

where $1/r + 1/r' = 1$. In particular, $A \equiv A_2$ is a non-negative self-adjoint operator in L^2_σ and there holds

$$D(A^{1/2}) = H^{1,2}_{0,\sigma} \quad \text{with} \quad \|A^{1/2}u\| = \|\nabla u\| \quad \text{for } u \in D(A^{1/2}).$$

Here and in what follows, for simplicity we abbreviate A_2 and the L^2-norm $\|\cdot\|$ as A and $\|\cdot\|$, respectively. It is shown by Giga [7], Giga-Sohr [9] and Borchers-Miyakawa [1] that for every $\frac{\pi}{2} < \omega < \pi$ and every $1 < r < \infty$, the resolvent set $\rho(-A_r)$ of $-A_r$ contains the sector $\Sigma_\omega \equiv \{\lambda \in C; |\arg\lambda| < \omega\}$ and there is a constant $M_{r,\omega}$ depending only on r and ω such that

$$\|(A_r + \lambda)^{-1}\|_{B(L^r_\sigma)} \leq M_{r,\omega}|\lambda|^{-1} \tag{2.1}$$

holds for all $\lambda \in \Sigma_\omega$. Therefore $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t \geq 0\}$ of class C_0 in L^r_σ. Moreover, there holds

$$\|u\|_{W^{2,r}} \leq C\|(1 + A_r)u\|_r \quad \text{for all } u \in D(A_r)$$

with a constant $C = C(r)$.

Let $J_k = (1 + k^{-1}A)^{-(1+[n/4])}$ for $k = 1, 2, \cdots$. By (2.1), we have

$$\|J_k w\| \leq \|w\| \quad k = 1, 2, \cdots, \quad J_k w \to w \quad \text{in } L^2_\sigma \text{ as } k \to \infty \tag{2.2}$$

for all $w \in L^2_\sigma$. Our definition of a weak solution to (1.1)-(1.2) is as follows.

Definition 2.1 *Let $a \in L^2_\sigma$ and let $f \in L^2(0,T; L^2_\sigma)$ for all $T > 0$. A measurable function u on $\Omega \times (0,\infty)$ is called a weak solution of (1.1)-(1.2) if for all $T > 0$*

(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; D(A^{1/2}))$;

(ii)

$$\int_0^T \{-(u(t), \frac{\partial}{\partial t}\phi(t)) + (\nabla u(t), \nabla\phi(t)) + (u \cdot \nabla u(t), \phi(t))\}dt = (a, \phi(0)) + \int_0^T (f(t), \phi(t))dt$$

for all $\phi \in \{C_0^\infty(\Omega \times [0,T))\}^n$ with $\nabla \cdot \phi = 0$.

Our results are stated as follows.

Theorem 2.1 *Let* $n = 3$ *and let* $a \in L^2_\sigma$ *and* $f \in L^2(0,T;L^2_\sigma)$ *for all* $T > 0$. *Then there exists at least one weak solution* u *of the problem (1.1)-(1.2) together with the associated pressure* p *in* $L^{5/4}_{loc}(\Omega\times(0,\infty))$ *satisfying the following generalized energy inequality;*

$$(2.3)\quad 2\int_0^\infty\int_\Omega |\nabla u|^2\phi dxdt \le \int_0^\infty\int_\Omega [|u|^2(\phi_t+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2(u\cdot f)\phi]dxdt$$

for all $\phi \in C_0^\infty(\Omega\times(0,\infty))$ *with* $\phi \ge 0$.

This generalized energy inequality was introduced by Caffarelli-Kohn-Nirenberg [3, pp. 779-780]. By their result, we can estimate the size of singular set of the above solutions.

Corollary 2.1 *In addition to the hypotheses of Theorem 2.1 , assume moreover that* $f \in L^q_{loc}(\Omega\times(0,\infty))$ *for some* $q > 5/2$. *Then every weak solution* u *as in Theorem 2.1 has the property that*

$$\mathrm{H}^1(S) = 0,$$

where $S \equiv \{(x,t) \in \Omega\times(0,\infty); u \notin L^\infty(B_r(x,t)) \text{ for all } r > 0\}$ *denotes the singular set of* u *in space-time.*

Remark 2.1. It will be shown that above u is actually a suitable weak solution defined by Caffarelli-Kohn-Nirenberg [3],which yields Corollary 2.1. In bounded domains Ω, however, they showed the existence of suitable weak solutions under the stronger assumption that the initial data $a \in L^2_\sigma \cap W^{2/5,5/4}(\Omega)$, while the Leray-Hopf weak solution can be constructed for arbitrary $a \in L^2_\sigma$. Hence our theorem states that suitable weak solutions are obtained for the same class of the initial data as Leray-Hopf's. Furthermore, we can include the case where Ω is an exterior domain and the half-space $\mathbf{R}^3_+$.

Next we investigate the class of weak solutions which fulfill the usual and generalized energy equalities.

Theorem 2.2 *Let* $a \in L^2_\sigma(\Omega), f \in L^2(0,T;L^2_\sigma(\Omega)), \varepsilon \in (0,T)$, K *be a domain having compact* $\overline{K} \subset \overline{\Omega}$ *and let* u *be a weak solution of the problem (1.1)-(1.2). If* $u \in L^s(\varepsilon,T;L^q(K))$, $1 < q,s < \infty$, *with*

$$2/q+2/s \le 1, \text{ and } 3/q+1/s \le 1 \text{ for } n = 3,$$
$$2/q+2/s \le 1, \text{ and } q \ge 4 \text{ for } 4 \le n \le 10,$$
$$2/q+2/s \le 1, q \ge 4, \text{ and } n/q+2/s \le 3 \text{ for } n \ge 11$$

then u *satisfies the generalized energy equality on* $(\varepsilon,T)\times K$.

Corollary 2.2 *Let* $n = 3$ *and* $a \in L^2_\sigma(\Omega), f \in L^2(0,T;L^2_\sigma(\Omega)) \cap L^\delta_{loc}(\Omega\times(0,T))$ *for some* $\delta > 5/2$, *and let* u *be a weak solution of the problem (1.1)-(1.2). If* $u \in L^s(\varepsilon,T;L^q(K))$ *for all* $\varepsilon \in (0,T)$ *and all compact* $K \subset\subset \Omega$ *with*

$$2/q+2/s \le 1, \text{ and } 3/q+1/s \le 1 (1 < q,s < \infty),$$

then u *has the property that* $\mathrm{H}^1(S) = 0$.

Remark 2.2. Note that we assume only local behavior of u in Theorem 2.2 and Corollary 2.2. To get the generalized energy equality, we need to show $\int\int up \cdot \nabla\phi dx < \infty$ for all $\phi \in C_0^\infty(\Omega \times (0,T))$. However, the information on p cannot be easily derived from the local behavior of u, because p is determined the global properties of the domain Ω, i.e., the Helmholtz decomposition. To overcome this difficulty, we make use of cut-off technique and Bogovski's lemma.

When $n \geq 11$, compared with Shinbrot's hypothesis, we need to assume $n/q + 2/s \leq 3$. However, under the global assumption $u \in L^s(0,T; L^q(\Omega))$ we can remove this hypothesis as follows.

Theorem 2.3 *Let $a \in L^2_\sigma(\Omega), f \in L^2(0,T; L^2_\sigma(\Omega))$, and let u be a weak solution of the problem (1.1)-(1.2). If $u \in L^s(0,T; L^q(\Omega))$, $1 < q, s < \infty$, with*

$$2/q + 2/s \leq 1, \text{ and } 3/q + 1/s \leq 1 \text{ for } n = 3,$$
$$2/q + 2/s \leq 1, \text{ and } q \geq 4 \text{ for } n \geq 4,$$

then u satisfies both usual and generalized energy equalities.

Remark 2.3. Serrin [15] and Shinbrot [22] found the class of weak solutions with the energy equality. For $n = 3$, there is no inclusion between their results. For $n \geq 4$, the class of Shinbrot is larger than that of Serrin. Our class covers both their results.

3 Preliminaries

Now we state the definition of a suitable weak solution introduced by Caffarelli-Kohn-Nirenberg as follows.

Definition 3.1 *The pair $\{u, p\}$ is a suitable weak solution of the Navier-Stokes system (1.1) on an open set $\boldsymbol{D} \in \boldsymbol{R}^3 \times \boldsymbol{R}$ with force f if the following conditions are satisfied:*

1. *u, p and f are measurable functions on $\boldsymbol{D}$ and*
   (a) *$f \in L^q(\boldsymbol{D})$ for some $q > 5/2$, and $\nabla \cdot f = 0$,*
   (b) *$p \in L^{5/4}(\boldsymbol{D})$,*
   (c) *for some constants $E_0, E_1 < \infty$,*

$$\int_{\boldsymbol{D}_t} |u|^2 dx \leq E_0 \quad , \quad \boldsymbol{D}_t = \boldsymbol{D} \cap (\boldsymbol{R}^3 \times \{t\}),$$

for almost every t such that $\boldsymbol{D}_t \neq \emptyset$ (empty set), and

$$\int\int_{\boldsymbol{D}} |\nabla u|^2 dx dt \leq E_1.$$

2. *u, p and f satisfy (1.1) in the sense of distribution on $\boldsymbol{D}$.*
3. *Generalized energy inequality (2.3) is valid for each real-valued $\phi \in C_0^\infty(\boldsymbol{D})$ with $\phi \geq 0$.*

These suitable weak solutions have the following property [3].

Lemma 3.1 (Caffarelli-Kohn-Nirenberg [3]) *When $n = 3$, for any suitable weak solution of the Navier-Stokes system (1.1) on an open set in space time, the associated singular set satisfies* $H^1(S) = 0$.

Before stating Sohr's approximate solutions to (1.1)-(1.2), we should state following lemma which plays an important role in constructing the approximate solutions.

Lemma 3.2 *(1) If $a \in D(A_2^{1/2}), T > 0$, and if $f \in L^2(0,T;L^2_\sigma)$, the function u*

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A} f(s)ds \quad , \tag{3.1}$$

belongs to $L^2(0,T;D(A_2)) \cap W^{1,2}(0,T;L^2_\sigma)$ and solves the problem

$$u' + A_2 u = f \quad , \quad a.e. \quad t \in (0,T) \quad ; \quad u(0) = a.$$

Furthermore u satisfies the estimate

$$\int_0^T (\|u'\|_2^2 + \|Au\|_2^2)dt \le C(\|A^{1/2}a\|_2^2 + \int_0^T \|f\|_2^2 dt)$$

with C independent of a, f, and T.

(2) If $a \in D(A_r)$ and $f \in L^\theta(0,T;L^r_\sigma)$, for some $1 < r, \theta < \infty$, then u defined by (3,1) is in $L^\theta(0,T;D(A_r)) \cap W^{1,\theta}(0,T;L^r_\sigma)$ and solves the problem

$$u' + A_r u = f \quad , \quad a.e. \quad t \in (0,T) \quad ; \quad u(0) = a.$$

Furthermore u satisfies the estimate

$$\int_0^T (\|u'\|_r^\theta + \|(1+A)u\|_r^\theta)dt \le C(\|a\|_{W^{2,r}} + \int_0^T \|f\|_r^\theta dt)$$

with C depending on T.

For the proof we refer the reader to Giga-Sohr [8] .

We now explain in Lemma 3.3 and 3.4 Sohr's approximate solutions to (1.1)-(1.2); see [14].

Lemma 3.3 (Miyakawa-Sohr) *(1) Let $n = 3, 4$, $a \in L^2_\sigma$ and $f \in L^2(0,T;L^2_\sigma)$. Then there is a unique u_k in $C([0,T];D(A_2^{1/2}))$ which solves integral equation*

$$u_k(t) = e^{-tA}a_k + \int_0^t e^{-(t-s)A}(F_k u_k + f_k)(s)ds. \tag{3.2}$$

Here, $a_k = J_k a$, $f_k = J_k f$, $F_k u = F_k(u,u)$, $F_k(u,v) = -P_2(J_k u \cdot \nabla)v$, and $J_k = (1 + k^{-1}A)^{-(1+[n/4])}$. ($[b]$ is the largest integer in the real number b.)

(2) The function u_k is in $L^2(0,T;D(A_2)) \cap W^{1,2}(0,T;L^2_\sigma)$ and satisfies

$$u_k' + A_2 u_k = f_k + F_k u_k \quad , \quad a.e. \quad t \in (0,T) \quad ; \quad u_k(0) = a_k, \tag{3.3}$$

$$\|u_k(t)\|_2^2 + 2\int_0^t \|A^{1/2}u_k\|_2^2 dt \le C(\|a\|_2^2 + \int_0^T \|f\|_2^2 dt), \tag{3.4}$$

with C independent of k. Therefore,

(3.5) $\{u_k\}$ *is bounded in* $L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;D(A_2^{1/2}))$.

(3) Let $u_k = v_k + w_k$, where

$$v_k(t) = e^{-tA}a_k + \int_0^t e^{-(t-s)A}f_k(s)ds, \quad w_k(t) = \int_0^t e^{-(t-s)A}F_k u_k(s)ds.$$

Then there are sequences $\{q_k\}$ and $\{\delta_k\}$ of scalar functions such that

$\{\nabla q_k\}$ *is bounded in* $L^2(\epsilon,T;G_2)$ *for each* $\epsilon > 0$,
$\{\nabla \delta_k\}$ *is bounded in* $L^{(n+2)/(n+1)}(0,T;G_{(n+2)/(n+1)})$, *and*
$\nabla q_k = \Delta v_k - v_k' + f_k, \quad \nabla\delta_k = \Delta w_k - w_k' - (J_k u_k)\cdot\nabla u_k$ *a.e.* $t\in(0,T)$,

where G_r is defined in Section.2. Let $p_k = q_k + \delta_k$. Then

(3.6) $\nabla p_k = \Delta u_k - u_k' - (J_k u_k)\cdot\nabla u_k + f_k$ *a.e.* $t\in(0,T)$.

Treating convergence of v_k and w_k separately, one can show strong convergence of u_k and $J_k u_k$ in $L^2(0,T;L^2(K))$ for each fixed compact set K in Ω, (see [14]). This lemma yields the following convergence of u_k.

Lemma 3.4 (Miyakawa-Sohr) *Let $n = 3,4$, $a\in L^2_\sigma$ and $f\in L^2(0,T;L^2_\sigma)$ for all $T>0$. Then there exist subsequences of u_k and $J_k u_k$ – again denoted by u_k and $J_k u_k$, for simplicity – with a function u such that*

(3.7) $u_k \to u$ *weak-star in* $L^\infty(0,T;L^2_\sigma)$ *and*
weakly in $L^2(0,T;D(A_2^{1/2}))$ *for all* $T>0$,

(3.8) $u_k\to u$ *strongly in* $L^2(K\times(0,T))$ *for all compact* $K\subset\subset\Omega, T>0$,

(3.9) $J_k u_k\to u$ *strongly in* $L^2(K\times(0,T))$ *for all compact* $K\subset\subset\Omega, T>0$.

Moreover u is a weak solution of the problem (1.1)-(1.2).

Miyakawa and Sohr showed the above two lemmas in [14], where they assumed that Ω was an exterior domain. However, in the cases a bounded domain, a half-space and the whole space the two lemmas can be shown in the same way.

The next lemma plays an important role in controlling behavior of the pressures p_k, and makes it possible to derive the generalized energy inequality (2.3) after passing to the limit.

Lemma 3.5 *Suppose that $\nabla f\in G_r$ for some $1<r<n$, and $f\in L^r_{loc}(\overline{\Omega})$. Then there is a unique function g in $L^{r^*}(\Omega)$, $1/r^* = 1/r - 1/n$, such that*
(1) $\nabla g = \nabla f$; *and* (2) $\|g\|_{r^*}\le\|\nabla g\|_r = \|\nabla f\|_r$,
where C is independent of ∇f.

Lemma 3.5 is shown e.g. in [8].

There are well-known lemmas stated as follows.

Lemma 3.6 *(1) If $1 \le q < r$, $u_k \to u$ strongly in L^q, and $\{u_k\}$ is bounded in L^r, then $u_k \to u$ strongly in L^s for $q < s < r$.*
(2) If B is a Banach space, $1 \le q < r$, $u_k \to u$ strongly in $L^q(0,T;B)$, and $\{u_k\}$ is bounded in $L^r(0,T;B)$, then $u_k \to u$ strongly in $L^s(0,T;B)$ for $q < s < r$.

Lemma 3.7 (Bogovski) *Let $K \subset \boldsymbol{R}^n (n \ge 2)$ be a bounded Lipshitz domain, $1 < r < \infty$, and m be a nonnegative integer. Then there exists a linear operator $R = R_K^{m,r}$ from $W_0^{m,r}(K)$ into $(W_0^{m+1,r}(K))^n$ with the following properties :*
(a) $divRf = f$ for all $f \in W_0^{m,r}(K)$ with $\int_K f dx = 0$,
(b) $\|\nabla^{m+1} Rf\|_r \le c\|\nabla^m f\|_r$ for all $f \in W_0^{m,r}(K)$
where $c = c(K,m,r) > 0$ is a constant,
(c) $Rf \in (C_0^\infty(K))^n$ if $f \in C_0^\infty(K)$, and $Rf = R_K^{m,r} f$ depends only on f and K if $f \in C_0^\infty(K)$.
(d) $R\frac{d}{dt} f = \frac{d}{dt} Rf$ if $f \in C_0^\infty(K \times (0,T))$.

Lemma 3.7 (a)-(c) are shown e.g. in [2]. When K is starlike with respect to some ball B such that $\overline{B} \subset K$, we can show the above (d) by Bogovski's formula, i.e.,

$$Rf(x,t) = \int_K G(x,y) f(y,t) dy, \quad G(x,y) = (x-y)\int_1^\infty h(y + s(x-y)) s^{n-1} ds,$$

where $h \in C_0^\infty(B)$ with $\int_B h dx = 1$. When K is a general bounded Lipshitz domain, the decomposition of K into finite starlike domains as in [2] yields the conclusion.

4 Proof of Theorem 2.1 and Corollary 2.1

In this Section we assume $n = 3$ and prove Theorem 2.1 and Corollary 2.1, showing that the function u obtained in Lemma 3.4 is actually a suitable weak solution. By Lemma 3.3, and Lemma 3.5, we may assume that

$$\{q_k\} \text{ is bounded in } L^2(\epsilon, T; L^6(\Omega)), \quad \{\delta_k\} \text{ is bounded in } L^{5/4}(0,T; L^{15/7}(\Omega)).$$

This implies $\{p_k\}$ is bounded in $L^{5/4}(\epsilon, T; L^{15/7}(K))$ for all ϵ, T with $0 < \epsilon < T$, and all compact $K \subset\subset \Omega$. Then, - an appropriate subsequence of $\{p_k\}$ being again denoted by $\{p_k\}$ -,there exists a function p such that

$$(4.1) \qquad p \in L^{5/4}(\epsilon, T; L^{15/7}(K)), p_k \to p \text{ weakly in } L^{5/4}(\epsilon, T; L^{15/7}(K)),$$

for all ϵ, T with $0 < \epsilon < T$, and all compact $K \subset\subset \Omega$.
By (3.6) , for any $\phi = (\phi^1, \phi^2, \phi^3) \in \{C_0^\infty(\Omega \times (0,\infty))\}^3$, we have

$$\iint [-p_k \nabla \cdot \phi] dx dt = \iint [-\nabla u_k^i \cdot \nabla \phi^i + u_k \cdot \frac{\partial \phi}{\partial t} + (J_k u_k) u_k^i \cdot \nabla \phi^i + f_k \cdot \phi] dx dt$$

Then, using (2.2), (3.7), (3.8), and (3.9), we obtain

$$\iint [-p \nabla \cdot \phi] dx dt = \iint [-\nabla u^i \cdot \nabla \phi^i + u \cdot \frac{\partial \phi}{\partial t} + u u^i \cdot \nabla \phi^i + f \cdot \phi] dx dt,$$

for any $\phi \in \{C_0^\infty(\Omega \times (0,\infty))\}^3$. Hence $\{u,p\}$ satisfies the condition 2 of Definition 3.1.

Next, we shall show $\{u,p\}$ satisfies the condition 3 of Definition 3.1, i.e., the generalized energy inequality. Let ϕ be a non-negative smooth function with compact support in $\Omega \times (0,\infty)$. Multiplying both sides of (3.6) by $2u_k\phi$ and integrating the result identity, we get

$$\iint [2u_k \cdot u_k'\phi + 2J_k u_k \cdot \nabla u_k^i u_k^i \phi - 2\Delta u_k^i u_k^i \phi + 2\nabla p_k \cdot u_k \phi - 2f \cdot u_k \phi] dxdt = 0.$$

By integration by parts,

$$\iint 2J_k u_k \cdot \nabla u_k^i u_k^i \phi dxdt = \iint J_k u_k \cdot (\nabla(|u_k|^2))\phi dxdt = -\iint J_k u_k |u_k|^2 \cdot \nabla\phi dxdt,$$

and

$$\iint 2\Delta u_k^i u_k^i \phi dxdt = -2\iint |\nabla u_k|^2 \phi dxdt + \iint |u_k|^2 \Delta\phi dxdt.$$

Therefore

$$(4.2)\quad 2\iint |\nabla u_k|^2 \phi dxdt = \iint |u_k|^2(\phi_t + \Delta\phi) + \iint 2p_k(u_k \cdot \nabla\phi)dxdt + \iint 2f_k \cdot u_k \phi dxdt + \iint |u_k|^2 J_k u_k \cdot \nabla\phi dxdt.$$

We discuss the convergence of each term of (4.2) as $k \to 0$. The Sobolev inequality yields

$$\|u_k\|_{L^{10/3}(\Omega)} \leq C\|u_k\|_{L^2(\Omega)}^{2/5} \|\nabla u_k\|_{L^2(\Omega)}^{3/5}, \text{ (where } C \text{ is independent of } u_k,)$$

$$\|u_k\|_{L^{10/3}(\Omega\times(0,T))} \leq C\|u_k\|_{L^\infty(0,T:L^2(\Omega))}^{2/5} \left\{\int_0^T \|\nabla u_k\|_{L^2(\Omega)}^2 dt\right\}^{3/10}.$$

Hence by (3.5), we see that

(4.3) $\{u_k\}$ is bounded in $L^{10/3}(\Omega \times (0,T))$ for all $T > 0$.

Since $\|J_k u_k\|_{L^{10/3}(\Omega)} \leq C\|u_k\|_{L^{10/3}(\Omega)}$ with C independent of u_k, we have

(4.4) $\{J_k u_k\}$ is bounded in $L^{10/3}(\Omega \times (0,T))$ for all $T > 0$.

Using (3.8), (3.9), (4.3), (4.4), and Lemma 3.6 (1), we obtain that

(4.5) $u_k \longrightarrow u$ strongly in $L^3(K \times (0,T))$ for all compact $K \subset\subset \Omega, T > 0$.
(4.6) $J_k u_k \longrightarrow u$ strongly in $L^3(K \times (0,T))$ for all compact $K \subset\subset \Omega, T > 0$.

By (3.5), u_k is bounded in $L^6(0,T; L^{15/8}(K))$ for all compact $K \subset\subset \Omega, T > 0$, and (3.8) implies also $u_k \longrightarrow u$ strongly in $L^2(0,T; L^{15/8}(K))$ for all compact $K \subset\subset \Omega$, $T > 0$. Hence from Lemma 3.6 (2), we see that

(4.7) $u_k \longrightarrow u$ strongly in $L^5(0,T; L^{15/8}(K))$ for all compact $K \subset\subset \Omega, T > 0$.

By (4.1) and (4.7) we obtain

$$(4.8)\quad \iint p_k(u_k \cdot \nabla\phi)dxdt \longrightarrow \iint p(u \cdot \nabla\phi)dxdt \text{ for all } \phi \in C_0^\infty(\Omega \times (0,\infty)),$$

and by (2.2) and (3.8),

$$(4.9) \qquad \iint f_k \cdot u_k \phi dxdt \longrightarrow \iint f \cdot u\phi dxdt \text{ for all } \phi \in C_0^\infty(\Omega \times (0,\infty)).$$

Using (3.7), we see that

$$(4.10) \qquad \iint |\nabla u|^2 \phi dxdt \le \liminf_{k\to\infty} \iint |\nabla u_k|^2 \phi dxdt$$

for all $\phi \in C_0^\infty(\Omega \times (0,\infty))$ with $\phi \ge 0$. We next handle the 4th term on the right-side of (4.2). From the Hölder inequality, we see

$$\begin{aligned} & \left|\iint [|u_k|^2 J_k u_k \cdot \nabla\phi - |u|^2 u \cdot \nabla \phi] dxdt\right| \\ \le \; & \left|\iint (|u_k|^2 - |u|^2) J_k u_k \cdot \nabla \phi dxdt\right| + \left|\iint |u|^2 (J_k u_k - u) \cdot \nabla \phi dxdt\right| \\ \le \; & \|\nabla\phi\|_{L^\infty} \{ \|u_k - u\|_{3;Q} (\|u_k\|_{3;Q} + \|u\|_{3;Q}) \| J_k u_k\|_{3;Q} + \|u\|_{3;Q}^2 \|J_k u_k - u\|_{3;Q}\}, \end{aligned}$$

where $\|\cdot\|_{3;Q} = \|\cdot\|_{L^3(Q)}$, $Q = \text{supp}\,\phi$. Then (4.5) and (4.6) yield

$$(4.11) \qquad \iint |u_k|^2 J_k u_k \cdot \nabla \phi dxdt \to \iint |u|^2 u \cdot \nabla \phi dxdt,$$

for all $\phi \in C_0^\infty(\Omega \times (0,\infty))$. (3.8), (4.8), (4.9), (4.10), (4.11), and (4.2) yield the generalized energy inequality. On the other hand (4.1) implies $p \in L^{5/4}_{loc}(\Omega \times (0,\infty))$. This completes the proof of Theorem 2.1.

Now, we assume $f \in L^q_{loc}(\Omega \times (0,\infty))$ for some $q > 5/2$. Then $\{u,p\}$ is a suitable weak solution on every bounded open set $D \subset\subset \Omega \times (0,\infty)$. From Lemma 3.1 we conclude $H^1(S \cap D) = 0$ for every bounded open set $D \subset\subset \Omega \times (0,\infty)$. One can take a sequence of bounded open sets $\{A_k\}_{k=1}^\infty$ such that

$$A_k \subset\subset \Omega \times (0,\infty), k = 1,2,\cdots, \text{ and } \Omega \times (0,\infty) = \bigcup_{k=1}^{\infty} A_k.$$

From the property of Hausdorff measure, we get

$$H^1(S) = H^1(\bigcup_{k=1}^{\infty} (A_k \cap S)) \le \sum_{k=1}^{\infty} H^1(A_k \cap S) = 0$$

This completes the proof of Corollary 2.1.

Remark 4.1. Miyakawa-Sohr [14] showed the existence of a weak solution satisfying the energy inequality for $n = 3,4$. Compared with the energy inequality, however, it seems to be difficult to prove the generalized energy inequality for $n = 4$. Because in the case of $n = 4$, we have a lack of information corresponding to (4.5), (4.6) and (4.11). There are other difficulties for $n = 4$.

5 Key lemmas for the proof of Theorem 2.2

For the proof of Theorem 2.2 and proof of 2.3, we need more precise information on $\{u,p\}$.

Lemma 5.1 *Let $a \in L^2_\sigma(\Omega)$, $f \in L^2(0,T;L^2_\sigma(\Omega))$, and let u be a weak solution of (1.1)-(1.2). If $u\cdot\nabla u \in L^\theta(0,T;L^r(\Omega))$, $1 < r, \theta < \infty$, then there are two pair $\{u_1,p_1\}$ and $\{u_2,p_2\}$ of functions such that*

$$
\begin{aligned}
&u_1 + u_2 = u \ \textit{a.e}\ \Omega\times(0,T);\\
&u_1 \ \in L^2(\varepsilon,T;D(A_2))\cap H^{1,2}(\varepsilon,T;L^2_\sigma), \nabla p_1 \in L^2(\varepsilon,T;G_2) \textit{ for all } 0<\varepsilon<T;\\
&u_2 \ \in L^\theta(0,T;D(A_r))\cap H^{1,\theta}(0,T;L^r_\sigma), \nabla p_2 \in L^\theta(0,T;G_r);\\
&\tfrac{\partial}{\partial t}u_1 - \Delta u_1 + \nabla p_1 = f \ \textit{in}\ L^2(\varepsilon,T;L^2(\Omega));\\
&\tfrac{\partial}{\partial t}u_2 - \Delta u_2 + \nabla p_2 = -u\cdot\nabla u \ \textit{in}\ L^\theta(0,T;L^r(\Omega)),\ \textit{and}\ u_2(0)=0.
\end{aligned}
$$

Lemma 5.1 implies that u always has an associated pressure $p(=p_1+p_2)$ with $\nabla p \in \{L^2(\varepsilon,T;G_2)+L^{\theta_0}(\varepsilon,T;G_{r_0})\}$ for all $(r_0,\theta_0)\in(1,n/(n-1))\times(1,2)$ with $n/r_0+2/\theta_0 \le n+1$, since $u\cdot\nabla u \in L^{\theta_0}(0,T;L^{r_0}(\Omega))$. Applying Lemma 3.5, we see that $p \in L^{\frac{n+2}{n}}_{loc}(\Omega\times(0,T))$, and that in particular $p\in L^{\frac{5}{3}}_{loc}(\Omega\times(0,T))$ for $n=3$. Using Lemma 5.1, we have:

Lemma 5.2 *Let $a\in L^2_\sigma(\Omega)$, $f\in L^2(0,T;L^2_\sigma(\Omega))$, $\varepsilon\in(0,T)$, K be a domain having compact $\overline{K}\subset\overline{\Omega}$, and let u be a weak solution of (1.1)-(1.2). If $u\cdot\nabla u \in L^\alpha(\varepsilon,T;L^\beta(K))$, $1<\alpha,\beta<\infty$, then u has the associated pressure p such that*

$$\nabla p \in \{L^2(\varepsilon',T;G_2(K')) + L^\theta(\varepsilon',T;G_{r^*}(K')) + L^\alpha(\varepsilon',T;G_\beta(K'))\},$$

for all $\varepsilon<\varepsilon'<T$, all domain K' with $\overline{K'}\subset K$, all $r\in(1,n/(n-1))$, and all $\theta\in(1,2)$ with $n/r+2/\theta\le n+1$. Here $1/r^*=1/r-1/n$.

Proof of Lemma 5.1.
Let $\Lambda_p = -P_p\Delta_p + I$, $\Im_k = (I+\Lambda_2/k)^{-\delta}$, where $\delta = n/4$ and define $u_k=\Im_k u$. Then it follows from Sohr-von Wahl [25, p.435] that u_k satisfies the following equation

$$(5.1)\qquad u_k' - P_2\Delta u_k + \Im_k P_r(u\cdot\nabla u) = \Im_k f \quad \text{in } L^2(0,T;L^2(\Omega)).$$

By assumption and Sohr-von Wahl [25, p.435] we have $\Im_k P_r(u\cdot\nabla u)\in L^2(0,T;L^2(\Omega))\cap L^\theta(0,T;L^r(\Omega))$. This implies that u_k is a strong solution of the linear Stokes equation

$$(5.2)\ u_k' + A_2 u_k = F,\ \ F = \Im_k f - \Im_k P_r(u\cdot\nabla u) \text{ a.e. } t\in(0,T),\ u_k(0)=\Im_k u(0).$$

On the other hand, there exist two solutions $v_k, w_k \in L^2(0,T;D(A_2))\cap W^{1,2}(0,T;L^2_\sigma)$ of the equations

$$(5.3)\qquad \frac{\partial v_k}{\partial t} + A_2 v_k = \Im_k f,\quad v_k(0)=u_k(0)=\Im_k a,$$

$$(5.4)\qquad \frac{\partial w_k}{\partial t} + A_2 w_k = -\Im_k P_r(u\cdot\nabla u),\quad w_k(0)=0,$$

respectively. From the uniqueness of strong solutions of the linear Stokes equation, we have

$$(5.5)\qquad u_k(t) = v_k(t)+w_k(t)\quad a.e.\quad t\in(0,T)$$

Let us estimate v_k and w_k in the similar manner to Miyakawa-Sohr [14]. By Lemma 3.2 we see that

(5.6) $\{w_k\}$ is bounded in $L^\theta(0,T;D(A_r))\cap W^{1,r}(0,T;L^r_\sigma)$,

(5.7) $\int_\varepsilon^T(\|v_k'\|_2^2+\|Av_k\|_2^2)dt \le C(\|A^{1/2}v_k(\varepsilon)\|_2^2+\int_\varepsilon^T\|f\|_2^2dt)$ for $0<\varepsilon<T$.

with C independent of ε, since $\|\Im_k f\|_2\le C\|f\|_2, \|\Im_k P_r(u\cdot\nabla u)\|_r\le C\|u\cdot\nabla u\|_r$. Integrating (5.7) in ε on $(0,T)$, we have

$$(5.8)\qquad \int_0^T t(\|v_k'\|_2^2+\|Av_k\|_2^2)dt \le C(\int_0^T\|A^{1/2}v_k(\varepsilon)\|_2^2dt+T\int_0^T\|f\|_2^2dt).$$

Multiplying (5.3) by $2v_k$, we obtain

$$\frac{\partial\|v_k\|_2^2}{\partial t}+2\|A_2^{1/2}v_k\|_2^2=2(\Im_k f,v_k)\le\|\Im_k f\|_2^2+\|v_k\|_2^2\le\|f\|_2^2+\|v_k\|_2^2.$$

Application of Gronwall's lemma yields

$$(5.9)\qquad \begin{aligned}\|v_k(t)\|_2^2+2\int_0^T\|A_2^{1/2}v_k\|_2^2dt &\le C(\|v_k(0)\|_2^2+\int_0^T\|f\|_2^2dt)\\ &= C(\|\Im_k a\|_2^2+\int_0^T\|f\|_2^2dt)\\ &\le C(\|a\|_2^2+\int_0^T\|f\|_2^2dt).\end{aligned}$$

By (5.8) and (5.9), there holds

(5.10) v_k is bounded in $L^2(\varepsilon,T;D(A_2))\cap W^{1,2}(\varepsilon,T;L^2_\sigma)$ for each fixd $\varepsilon>0$.

Then there exist subsequences of $\{v_k\},\{w_k\}$, which we denote by $\{v_k\},\{w_k\}$ themselves for simplicity, and functions v,w such that

$$\begin{aligned}&v\in L^2(\varepsilon,T;(A_2))\cap W^{1,2}(\varepsilon,T;L^2_\sigma(\Omega));\\ &w\in L^\theta(0,T;(A_r))\cap W^{1,\theta}(0,T;L^r_\sigma(\Omega)));\\ &v_k\longrightarrow v \text{ weakly in } L^2(\varepsilon,T;(A_2));\\ &\frac{\partial}{\partial t}v_k\longrightarrow\frac{\partial}{\partial t}v \text{ weakly in } L^2(\varepsilon,T;L^2_\sigma(\Omega));\\ &w_k\longrightarrow w \text{ weakly in } L^\theta(0,T;(A_r));\\ &\frac{\partial}{\partial t}w_k\longrightarrow\frac{\partial}{\partial t}w \text{ weakly in } L^\theta(0,T;L^r_\sigma(\Omega)),\end{aligned}$$

for all $0<\varepsilon<T$. Moreover, from (5.3),(5.4), and (5.5) we have

(5.11) $u=v+w$ a.e.$\Omega\times(0,T)$,

(5.12) $v_t+A_2v=f$ in $L^2(\varepsilon,T;L^2_\sigma(\Omega))$ for all $0<\varepsilon<T$,

(5.13) $w_t+A_2w=-P_r(u\cdot\nabla u)$ in $L^\theta(0,T;L^r_\sigma(\Omega))$, $w(0)=0$.

Clearly, $f-v_t+\Delta v\in L^2(\varepsilon,T;G_2))$, $-u\cdot\nabla u-w_t+\Delta u\in L^\theta(0,T;G_r)$. Then there exist $\nabla p_1\in L^2(\varepsilon,T;G_2))$, $\nabla p_2\in L^\theta(0,T;G_r)$ such that

$$\begin{aligned}&\nabla p_1=f-v_t+\Delta v \text{ in } L^2(\varepsilon,T;L^2_\sigma(\Omega)) \text{ for all } 0<\varepsilon<T,\\ &\nabla p_2=-u\cdot\nabla u-w_t+\Delta w \text{ in } L^\theta(0,T;L^r_\sigma(\Omega)).\end{aligned}$$

This completes the proof Lemma 5.1.

Next we prove Lemma 5.2, using Lemma 3.7

Proof of Lemma 5.2.
Let us fix $r, \theta \in (1, \infty)$ so that $n/r + 2/\theta \geq n + 1$. Since $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^{1,2}_{0,\sigma})$ implies $u \cdot \nabla u \in L^\theta(0,T; L^r(\Omega))$, by Lemma 5.1 there exist two pairs $\{u_1, p_1\}, \{u_2, p_2\}$ such that

$$u = u_1 + u_2, \quad p = p_1 + p_2 \text{ a.e. } \Omega \times (0,T); \tag{5.14}$$

$$u_1 \in L^2(\varepsilon, T; D(A_2)) \cap H^{1,2}(\varepsilon, T; L^2_\sigma), \nabla p_1 \in L^2(\varepsilon, T; G_2); \tag{5.15}$$

$$u_2 \in L^\theta(0,T; D(A_r)) \cap H^{1,\theta}(0,T; L^r_\sigma), \nabla p_2 \in L^\theta(0,T; G_r); \tag{5.16}$$

$$\tfrac{\partial}{\partial t} u_1 - \Delta u_1 + \nabla p_1 = f \text{ in } L^2(\varepsilon, T; L^2(\Omega)); \tag{5.17}$$

$$\tfrac{\partial}{\partial t} u_2 - \Delta u_2 + \nabla p_2 = -u \cdot \nabla u \text{ in } L^\theta(0,T; L^r(\Omega)), \text{ and } u_2(0) = 0. \tag{5.18}$$

for all $0 < \varepsilon < T$. Let K and K' be compact domains in Ω satisfying $\overline{K'} \subset K$ with smooth boundaries ∂K and $\partial K'$. Let ϕ be a smooth scalar function satisfying $\phi \in C_0^\infty(K)$ and $\phi = 1$ in K'. From Lemma 3.5, we have

$$p_1 \in L^2(\varepsilon, T; L^{2^*}(\Omega)), p_2 \in L^\theta(0,T; L^{r^*}(\Omega)),$$

where $1/2^* = 1/2 - 1/n$, and $1/r^* = 1/r - 1/n$. Since $r, \theta < 2$ and since K is compact, this yields

$$p \in L^\theta(\varepsilon, T; L^{r^*}(K)), \nabla p \in L^\theta(\varepsilon, T; L^r(K)), \tag{5.19}$$

(5.14)-(5.16) imply

$$\begin{aligned} u \in\, & L^\infty(0,T; L^2(K)) \cap L^2(0,T; W^{1,2}(K)) \\ & \cap L^\theta(\varepsilon, T; W^{2,r}(K)) \cap H^{1,\theta}(\varepsilon, T; L^r(K)); \end{aligned} \tag{5.20}$$

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = -u \cdot \nabla u + f \in L^\theta(\varepsilon, T; L^r(K)). \tag{5.21}$$

Now, we recall Lemma 3.7 (Bogovski's lemma). We easily prove that if $1 < \gamma < \infty, 1 < \gamma_1 \leq \gamma_2 < \infty$, then

$$\begin{aligned} & R_K^{2,\gamma} g = R_K^{0,\gamma} g \text{ for all } g \in W_0^{2,\gamma}(K); \\ & R_K^{m,\gamma_1} g = R_K^{m,\gamma_2} g \text{ for all } g \in W_0^{m,\gamma_2}(K); \\ & \frac{d}{dt} R_K^{0,\gamma} g = R_K^{0,\gamma} \frac{d}{dt} g \text{ for all } g \in H^{1,\theta}(0,T; L^\gamma(K)). \end{aligned}$$

Set $\tilde{u} = \phi u, v = R_K^{1,2}(\nabla\phi \cdot u), w = \tilde{u} - v$, and $\tilde{p} = \phi p$. We find that

$$\begin{aligned} \tilde{u}, \nabla\tilde{u} (= \nabla\phi \cdot u) \in\, & L^\infty(0,T; L^2(K)) \cap L^2(0,T; W_0^{1,2}(K)) \\ & \cap L^\theta(\varepsilon, T; W_0^{2,r}(K)) \cap H^{1,\theta}(\varepsilon, T; L^r(K)), \end{aligned} \tag{5.22}$$

$$\nabla\tilde{p} = \phi \nabla p + p \nabla\phi \in L^\theta(\varepsilon, T; L^r(K)), \tag{5.23}$$

$$v = R_K(\nabla\phi \cdot u) = R_K^{2,r}(\nabla\phi \cdot u) = R_K^{1,2}(\nabla\phi \cdot u), \tag{5.24}$$

$$\begin{aligned} v \in\, & L^\infty(0,T; W_0^{1,2}(K)) \cap L^2(0,T; W_0^{2,2}(K)) \\ & \cap L^\theta(\varepsilon, T; W_0^{3,r}(K)) \cap H^{1,\theta}(\varepsilon, T; W_0^{1,r}(K)), \text{ and} \end{aligned} \tag{5.25}$$

$$\operatorname{div} v = \nabla\tilde{u} = \nabla\phi \cdot u. \tag{5.26}$$

In (5.26) note that $\int \nabla\phi \cdot u dx = 0$. By (5.21), we have

$$\frac{\partial \tilde{u}}{\partial t} - \Delta\tilde{u} + \nabla\tilde{p} = \phi(f - u\cdot\nabla u) - 2\nabla\phi\cdot\nabla u - u\Delta\phi + p\nabla\phi \text{ in } L^\theta(\varepsilon, T; L^r(K)).$$

By (5.22),(5.25),we can take $\varepsilon'(\varepsilon' > \varepsilon)$ such that $\tilde{u}(\varepsilon'), v(\varepsilon') \in W_0^{1,2}(K) \cap W_0^{2,r}(K)$. Let $A_{r,K}$ be the Stokes operator on $L^r_\sigma(K)$ and $P_{r,K}$ be the projection operator from $L^r(K)$ onto $L^r_\sigma(K)$. Then we see that $w(= \tilde{u} - v)$ is a unique strong solution of the following Stokes equation,

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + \nabla\tilde{p} = F \text{ in } L^\theta(\varepsilon, T; L^r(K)), \\ \nabla\cdot w = 0, \\ w|_{\partial K} = 0, \quad w(\varepsilon') = \tilde{u}(\varepsilon') - v(\varepsilon') \in W_0^{2,r}(K) \cap L^r_\sigma(K) = D(A_{r,K}), \end{cases}$$

where $F = \phi(f - u\cdot\nabla u) - 2\nabla\phi\cdot\nabla u - u\Delta\phi + p\nabla\phi - \dfrac{\partial v}{\partial t} + \Delta v$. As in the proof of Lemma 5.1, we decompose this Stokes equation as

$$\begin{cases} \dfrac{\partial w_1}{\partial t} - \Delta w_1 + \nabla p_1 = F_1 \text{ in } L^\theta(\varepsilon', T; L^r(K)), \\ \operatorname{div} w_1 = 0, \quad w_1|_{\partial K} = 0, \\ w_1(\varepsilon') = \tilde{u}(\varepsilon') - v(\varepsilon') \in D(A_{2,K}^{1/2}) \cap D(A_{r,K}), \end{cases}$$

$$\begin{cases} \dfrac{\partial w_2}{\partial t} - \Delta w_2 + \nabla p_2 = F_2 \text{ in } L^\theta(\varepsilon', T; L^r(K)), \\ \operatorname{div} w_2 = 0, \quad w_2|_{\partial K} = 0, \\ w_2(\varepsilon') = 0, \end{cases}$$

$$\begin{cases} \dfrac{\partial w_3}{\partial t} - \Delta w_3 + \nabla p_3 = F_3 \text{ in } L^\theta(\varepsilon', T; L^r(K)), \\ \operatorname{div} w_3 = 0, \quad w_3|_{\partial K} = 0, \\ w_3(\varepsilon') = 0, \end{cases}$$

where

$$F = F_1 + F_2 + F_3 \quad \text{with} \quad F_1 = \phi f - 2\nabla\phi\cdot\nabla u - u\Delta\phi + \Delta v,$$
$$F_2 = p\nabla\phi - \frac{\partial v}{\partial t}, \quad F_3 = \phi(-u\cdot\nabla u).$$

These Stokes equations have unique strong solutions w_1, w_2, w_3 in $L^\theta(\varepsilon', T; D(A_{r,K})) \cap H^{1,\theta}(\varepsilon', T; L^r_\sigma(K))$, respectively. Clearly it follows that $w = w_1 + w_2 + w_3$, $\tilde{p} = p_1 + p_2 + p_3$ a.e.$(x,t) \in K \times (\varepsilon', T)$. By (5.22),(5.25) and (5.19) we obtain

$$F_1 \in L^2(\varepsilon', T; L^2(K)), \quad F_2 \in L^\theta(\varepsilon', T; L^{r^*}(K)),$$
$$F_3 \in L^\alpha(\varepsilon', T; L^\beta(K)).$$

Hence by Lemma 3.2 we have

$$\nabla p_1 \in L^2(\varepsilon', T; G_2(K)), \quad \nabla p_2 \in L^\theta(\varepsilon', T; G_{r^*}(K)),$$
$$\nabla p_3 \in L^\alpha(\varepsilon', T; G_\beta(K)),$$

which yields

$$\nabla p \in \{L^2(\varepsilon', T; G_2(K')) + L^\theta(\varepsilon', T; G_{r^*}(K')) + L^\alpha(\varepsilon', T; G_\beta(K'))\},$$

since $\tilde{p} = p$ in K'. This is valid for a.e.$\varepsilon' \in (\varepsilon, T)$, and this completes the proof of Lemma 5.2.

6 Proof of Theorems 2.2 and 2.3

For the proof of Theorems 2.2 and 2.3 we state the following lemma.

Lemma 6.1 *If* $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \cap L^s(0,T;L^q(\Omega))$, *where*

$$1/q+1/s \le 1/2, \quad q \ge 4 \quad \textit{for} \quad n \ge 4,$$
$$1/q+1/s \le 1/2, \quad 3/q+1/s \le 1 \quad \textit{for} \quad n=3,$$

then $v\cdot\nabla v \in L^{s'}(0,T;L^{q'}(\Omega))$ *where* $1/s+1/s'=1,\ 1/q+1/q'=1$. *Moreover* $v \in L^4(\Omega\times(0,T))$.

Proof of Lemma 6.1.

Let us first find the exponents r and θ guaranteeing $v \in L^\theta(0,T;L^r(\Omega))$ if $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \cap L^s(0,T;L^q(\Omega)), 2<q<\infty, 1<s<\infty$.

We have

$$\|v\cdot\nabla v\|_r \le \|v\|_2^{1-\lambda}\|v\|_q^\lambda\|\nabla v\|_2, \quad \lambda=\frac{2q}{q-2}(1-\frac{1}{r}) \text{ for } 1\le r\le \frac{2q}{q+2},$$
$$\|v\cdot\nabla v\|_r \le \|v\|_q^\kappa\|v\|_{W^{1,2}}^{2-\kappa}, \quad \kappa(\frac{1}{q}+\frac{1}{n}-\frac{1}{2})=\frac{1}{r}+\frac{1}{n}-1 \text{ for } \frac{2q}{q+2}<r\le\frac{n}{n-1}.$$

Letting $d>1$, we obtain

$$\int_0^T\|v\cdot\nabla v\|_r^\theta dt \le \sup_{0\le t\le T}\|v(t)\|_2^{(1-\lambda)\theta}\Big\{\int\|v\|_q^{\lambda\theta d}dt\Big\}^{\frac{1}{d}}\Big\{\int\|\nabla v\|_2^{\frac{\theta d}{d-1}}dt\Big\}^{\frac{d-1}{d}}, \text{ for } 1\le r\le\frac{2q}{q+2},$$
$$\int_0^T\|v\cdot\nabla v\|_r^\theta dt \le C\Big\{\int\|v\|_q^{\kappa\theta d}dt\Big\}^{\frac{1}{d}}\Big\{\int\|v\|_{W^{1,2}}^{\frac{(2-\kappa)\theta d}{d-1}}dt\Big\}^{\frac{d-1}{d}}, \text{ for } \frac{2q}{q+2}<r\le\frac{n}{n-1}.$$

Then $v\cdot\nabla v \in L^\theta(0,T;L^r(\Omega))$ if there is $d>1$ such that

$$\lambda\theta d\le s, \quad \frac{\theta d}{d-1}\le 2, \text{ for } 1\le r\le 2q/(q+2), \theta\ge 1, \text{ or}$$
$$\kappa\theta d\le s, \quad (2-\kappa)\frac{\theta d}{d-1}\le 2 \text{ for } 2q/(q+2)<r\le n/(n-1), \theta\ge 1.$$

Hence we see that $v\cdot\nabla v\in L^\theta(0,T;L^r(\Omega))$, where

$$\text{(6.1)} \quad 4q+s(q-2)\le\frac{2(q-2)s}{\theta}+\frac{4q}{r}, \quad \theta\ge 1, \quad 1\le r\le\frac{2q}{2+q}, \text{ or}$$

$$\text{(6.2)} \quad (s-2)(n-1)q+s(2n+2q-nq)\le\frac{n(s-2)q}{r}+\frac{(2n+2q-nq)s}{\theta},$$
$$\frac{2q}{2+q}<r\le\frac{n}{n-1}, \quad \theta\ge 1.$$

We can get a sufficient condition in order to $v\cdot\nabla v\in L^{s'}(0,T;L^{q'}(\Omega))$. This is guaranteed by (6.1) with (r,θ) replaced by (q',s'), i.e.,

$$2/q+2/s\le 1, \quad s>1, \quad q\ge 4.$$

Another sufficient condition is represented by (6.2) with (r,θ) replaced by (q',s'), i.e.,

$$n/q+(4-n)/s\le 1, \quad s>1, \quad n\le q<4.$$

Therefore under the hypothesis of Lemma 6.1, $v \cdot \nabla v \in L^{s'}(0,T;L^{q'}(\Omega))$.

We easily show that $v \in L^4(\Omega \times (0,T))$ under the hypothesis of Lemma 6.1. This completes the proof of Lemma 6.1.

Proof of Theorem 2.2.
Let K be a compact domain in $\overline{\Omega}$ with smooth boundary ∂K. By Lemma 6.1 and Lemma 5.2, $\{u,p\}$ satisfy that

(6.3) $\quad u \cdot \nabla u \in L^{s'}(\varepsilon, T : L^{q'}(K)),$

(6.4) $\quad \nabla p \in \{L^2(\varepsilon',T;G_2(K')) + L^{\theta}(\varepsilon',T;G_{r^*}(K')) + L^{s'}(\varepsilon',T;G_{q'}(K'))\},$

for all $\varepsilon' \in (\varepsilon,T)$, all $\overline{K'} \subset K$, and all $(r,\theta) \in (1,n/(n-1)) \times (1,2)$ with $n/r + 2/\theta \geq n+1$,

(6.5) $$\int_0^T \{-(u,\psi_t) + (\nabla u, \nabla\psi) + (u \cdot \nabla u, \psi) + (\nabla p, \psi) - (f,\psi)\}dt = 0,$$

for all $\psi \in \{C_0^\infty(\Omega \times (0,T))\}^n$.

Now we consider the mollification of u given by

$$u_m(x,t) = \int_0^T \int_\Omega \rho_m(x-x',t-t')u(x',t')dx'dt'.$$

Here

$$\rho_m(x,t) = \begin{cases} 0, & \text{for } |x|^2 + |t|^2 \geq 1/m^2, \\ Cm^{n+1}\exp\{(m^2|x|^2 + m^2|t|^2 - 1)^{-1}\}, & \text{for } |x|^2 + |t|^2 < 1/m^2, \end{cases}$$

where C are chosen so that $\int_{-\infty}^{\infty}\int_{R^n} \rho_1 dx = 1$. Since $u \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^s(\varepsilon,T;L^q(K))$, it follows that

(6.6) $\quad \|u_m - u\|_{L^2(0,T;W^{1,2}(\Omega))} \to 0$, as $m \to \infty$,

(6.7) $\quad \|u_m - u\|_{L^s(\varepsilon',T;L^q(K'))} \to 0$ as $m \to \infty$, for all $\varepsilon' \in (\varepsilon,T)$, all $\overline{K'} \subset K$.

Let $\phi \in C_0^\infty(K \times (\varepsilon,T))$. From (6.5) we obtain, for large m,

(6.8) $$\frac{\partial u_m}{\partial t} - \Delta u_m + (u \cdot \nabla u)_m + \nabla p_m = f_m, \quad \nabla \cdot u_m = 0$$

in a neighborhood of supp ϕ. Here $(u \cdot \nabla u)_m$ and p_m are the mollifications of $u \cdot \nabla u$ and p, respectively. Multiplying (6.8) by $2u_m\phi$ and integrating the result identity, we have

(6.9) $$2\int_0^T \int_\Omega |\nabla u_m|^2 \phi dxdt = \int_0^T \int_\Omega [|u_m|^2(\phi_t + \Delta\phi) - 2(u \cdot \nabla u)_m u_m \phi + 2p_m u_m \cdot \nabla\phi + 2(u_m \cdot f_m)\phi]dxdt.$$

Letting $\lambda = (r^*)^*$, *i.e.*, $1/(r^*)^* = 1/r - 2/n$, and applying Lemma 3.5, we obtain

$$p \in \{L^2(\varepsilon',T;L^2(K')) + L^{\theta}(\varepsilon',T;L^{\lambda}(K')) + L^{s'}(\varepsilon',T;L^{q'}(K'))\},$$

for all $\varepsilon' \in (\varepsilon,T)$, all $\overline{K} \subset K$, and all $(\lambda,\theta) \in (1,\infty) \times (1,2)$ with $n/\lambda + 2/\theta \geq n-1$. Since the hypotheses of Theorem 2.2 imply $n/q' + 2/s' \geq n-1$, we see that

(6.10) $$p \in \{L^2(\varepsilon',T;L^2(K')) + L^{s'}(\varepsilon',T;L^{q'}(K'))\}.$$

Letting $m\to\infty$ and using (6.9) and (6.10) we get

$$2\int_0^T\int_\Omega|\nabla u|^2\phi dxdt=\int_0^T\int_\Omega[|u|^2(\phi_t+\Delta\phi)-2(u\cdot\nabla u)u\phi+2pu\cdot\nabla\phi+2(u\cdot f)\phi]dxdt.$$

Since $u\in L^3_{loc}(\Omega\times(0,T))$, we have

$$\int_0^T(u\cdot\nabla u,\phi u)dt=-\frac{1}{2}\int_0^T\int_\Omega|u|^2u\cdot\nabla\phi dxdt.$$

Hence we get the generalized energy equality on $(\varepsilon,T)\times K$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3.
Let u satisfy the hypothesis of Theorem 2.3. Then Lemma 6.1 and [24, Theorem 2.1] yield the usual energy equality (1.5). As we have seen in the proof of Theorem 2.2, to get the generalized energy equality, it is essential to derive the class of the associated pressure p. In fact, by Lemma 5.1, we see

$$\nabla p\in L^2(\varepsilon,T;G_2(\Omega))+L^{s'}(\varepsilon,T;G_{q'}(\Omega))$$

for all $\varepsilon\in(0,T)$. Then using this instead of (6.10), we can argue similarly to the proof of Theorem 2.2 and get the conclusion.

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References

[1] Borchers, W., Miyakawa, T.: L^2 decay for Navier-Stokes flow in halfspaces. Math. Ann. 282 139-155 (1988).

[2] Borchers, W., Sohr, H.: On the equations rot $v=g$ and div $u=f$ with zero boundary conditions. Hokkaido. Math. J. 19 67-87 (1990).

[3] Caffarelli,L., Kohn,R., Nirenberg, L.: *Partial regularity of suitable weak solutions of the Navier-Stokes equations.* Comm. Pure Appl. Math. 35, 771-831 (1982).

[4] Foias, C., Temam, R.: *Some analytic and geometric properties on the solutions of the evolution Navier-Stokes equations.* J. Math. Pures et Appl. 58, 339-368 (1979).

[5] Fujiwara, D., Morimoto, H.: *An L^r-theorem of the Helmholtz decomposition of vector fields.* J.Fac. Sci. Univ. Tokyo, Sec IA. Math. 24, 685-700 (1977).

[6] Giga, Y.: *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system.* J. Differ. equations 62, 186-212 (1986).

[7] Giga, Y.: *Analyticity of the semigroup generated by the Stokes operator in L^r spaces.* Math. Z. 178, 297-329 (1981).

[8] Giga, Y., Sohr, H.: *Abstract L^p estimate for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains.* J. Func. Anal. 102, 72-94 (1991).

[9] Giga, Y., Sohr, H.: *On the Stokes operator in exterior domains.* J. Fac. Sci Univ. Tokyo, Sec IA. 36, 103-130 (1989).

[10] Hopf, E.: *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.* Math. Nachr. 4, 213-231 (1951).

[11] Leray, J.: *Sur le mouvement d'un liquide visqueux emplissant l'espace.* Acta Math. 63, 193-248 (1934).

[12] Masuda, K.: *On the stability of incompressible viscous fluid motions past objects.* J. Math. Soc. Japan 27,294-327 (1975).

[13] Miyakawa, T.: *On nonstationary solutions of the Navier-Stokes equations in an exterior domain.* Hiroshima Math. J. 12. 115-140 (1982).

[14] Miyakawa, T., Sohr. H.: *On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains.* Math. Z. 199. 455-478 (1988).

[15] Serrin, J.: *Initial value problem for the Navier-Stokes equations.* Nonlinear Problems, Rudolph E. Langer, University of Wisconsin Press, Madison 69-98 (1963).

[16] Scheffer, V.: *Turbulence and Hausdorff dimension, in Turbulence and the Navier-Stokes Equations* . Lecture Notes in Math. No.565, Springer-Verlag. 94-112 (1976).

[17] Scheffer, V.: *Partial regularity of solutions to the Navier-Stokes equations* . Pacific J. Math. 66. 535-552 (1976).

[18] Scheffer, V.: *Hausdorff measure and the Navier-Stokes equations.* Comm. Math. Phys. 55. 97-112 (1977).

[19] Scheffer, V.: *The Navier-Stokes equations in space dimension four.* Comm. Math. Phys. 61. 41-68 (1978).

[20] Scheffer, V.: *The Navier-Stokes equations on a bounded domain.* Comm. Math. Phys. 73. 1-42 (1980).

[21] Simader, C.G., Sohr, H.: *A new approach to the the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains.* Mathematical Problems Relating to the Navier-Stokes Equations, series on Advanced in Mathematics for Applied Sciences, G.P.Galdi,ed.(1992), World Scientific, Singapore-New Jersey-London-Hong Kong, 1-35.

[22] Shinbrot, M.: *The energy equation for the Navier-Stokes system.* SIAM J. Math. Anal. Vol. 5, No. 6. 948-954 (1974).

[23] Sohr, H.: *Optimale lokale Existenzsätze für die Gleichungen von Navier-Stokes.* Math. Ann. 267. 107-123 (1984).

[24] Sohr, H.,von Wahl, W.: *A New Proof of Leray's Structure Theorem and the Smoothness of Weak Solutions of Navier-Stokes Equations for Large* $|x|$. Bayreuther Math. Schr. 20 153-204 (1985).

[25] Sohr, H.,von Wahl, W.: *On the regularity of the pressure of weak solutions of Navier-Stokes equations.* Arch. Math. 46. 428-439 (1986).