manuscripta math. 94, 45–73 (1997)

manuscripta mathematica

Noncommutative spectral geometry of Riemannian foliations

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Received November 11, 1996;
in revised form April 11, 1997

Summary. We construct spectral triples in a sense of noncommutative differential geometry, associated with a Riemannian foliation on a compact manifold, and describe its dimension spectrum.

0. Introduction

According to [9, 8], the initial datum of noncommutative differential geometry is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (see Section 3.1 for the definition), which provides a description of the corresponding geometrical space in terms of spectral data of geometrical operators on this space.

The purpose of this paper is to construct spectral triples given by transversally elliptic operators with respect to a foliation on a compact manifold and describe its dimension. The first result of the paper is the following theorem:

Theorem 1. *Given a closed foliated manifold $(M, \mathcal{F})$, let a triple $(A, \mathcal{H}, D)$ be defined as follows:*

1. *$\mathcal{A}$ is the involutive algebra $C^\infty_c(G_\mathcal{F})$ of smooth, compactly supported functions on the holonomy groupoid $G_\mathcal{F}$ of the foliation $\mathcal{F}$;*
2. *$\mathcal{H}$ is the Hilbert space $L^2(M, E)$ of L^2-sections of a holonomy equivariant Hermitian vector bundle E equipped with the $*$-representation R_E of the algebra $\mathcal{A}$ (13);*
3. *D is a first order self-adjoint transversally elliptic operator in $L^2(M, E)$ with the holonomy invariant transversal principal symbol such that the operator D^2 is self-adjoint and has the scalar principal symbol.*

Then $(\mathcal{A}, \mathcal{H}, D)$ is a finite-dimensional spectral triple.

A geometrical example of spectral triples considered in Theorem 1 is given by the transverse signature operator on a Riemannian foliation.

Example 1. Let $(M, \mathcal{F})$ be a Riemannian foliation, equipped with a bundle-like metric g_M. Let $F = T\mathcal{F}$ be the integrable distribution in TM of tangent p-planes to the foliation, and $H = F^{\perp}$ be the orthogonal complement to F. So we have a decomposition of TM into a direct sum $TM = F \oplus H$ and the corresponding decomposition of the de Rham differential d in the form $d = d_F + d_H + \theta$, where the tangential de Rham differential d_F and the transversal de Rham differential d_H are first order differential operators, and θ is zeroth order.

To define a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we take the Hilbert space $\mathcal{H}$ to be the space $L^2(M, \Lambda^* H^*)$ of transversal differential forms, equipped with the natural action of the algebra $\mathcal{A} = C_c^\infty(G)$, and the operator D to be the transverse signature operator $d_H + \delta_H$ (see Section 3.2 for more details).

Transversally elliptic operators on manifolds, equipped with an action of a compact Lie group, were introduced by Atiyah and Singer in [2]. In the context of noncommutative differential geometry, these operators appeared in [6] to provide examples of Fredholm modules, associated with foliated manifolds. Namely, it was proved there that any zeroth order transversally elliptic operator with the holonomy invariant transversal principal symbol gives rise to a finite-dimensional Fredholm module over the foliation algebra $C_c^\infty(G_\mathcal{F})$ (see also [7, 17]). Theorem 1 provides an extension of the above mentioned result to the case of transversally elliptic operators of positive order.

The next problem is to describe dimension of the spectral triples in question. The usual notion of dimension for a general spectral triple $(\mathcal{A}, \mathcal{H}, D)$ ([7]) is given by the degree of summability d of the operator $(D-i)^{-1}$, that is, by the least p such that the operator $a(D-i)^{-1}, a \in \mathcal{A}$ is an operator of the Schatten ideal $\mathcal{L}^p(\mathcal{H})$. In the case under consideration, d is equal to the codimension q of the foliation $\mathcal{F}$ (see Proposition 17 for a proof). If we are looking at a geometrical space as a union of pieces of different dimensions, this notion of dimension of the corresponding spectral triple gives only an upper bound on dimensions of various pieces. To take into account lower dimensional pieces of the space under consideration, Connes and Moscovici [9] suggested that the correct notion of dimension is given not by a single real number d but by a subset $\mathrm{Sd} \subset \mathbb{C}$, which is called the dimension spectrum of the given triple (see Section 3.1 for the definition).

The second result of the paper is a description of the spectrum dimension of the spectral triples defined in Theorem 1.

Theorem 2. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as in Theorem 1 has discrete dimension spectrum* Sd, *which is contained in the set $\{v \in \mathbb{N} : v \leq q\}$ and simple.*

In [24], the author studied analytic properties of transversally elliptic operators with respect to noncompact Lie group actions. In particular, results of [24] allows us to define (finite-dimensional) Fredholm modules given by transversally elliptic operators with the invariant transversal principal symbol on a smooth manifold equipped with a Lie group action and claim that the corresponding spectral triples have discrete dimension spectrum (but the dimension spectrum might be not simple, if there are singular orbits). In this paper, we combine general methods of [24] with a further elaboration of pseudodifferential calculus on foliated manifolds [26, 25] to give a more precise description of the dimension spectrum for spectral triples associated with foliated manifolds.

This work concerns to the simplest examples of spectral triples associated with foliated manifolds. In the forthcoming paper [27], we will extend our considerations to foliated manifolds, equipped with a triangular transversal structure as in [9], using a transversal pseudodifferential calculus, modelled on the Beals-Greiner pseudodifferential calculus on Heisenberg manifolds (see [3, 9]).

Contents:

1. Transversal pseudodifferential calculus

1.1. Preliminaries

Throughout in the paper, we consider a closed, connected, oriented foliated manifold $(M,\mathcal{F})$, $\dim M = n$, $\dim \mathcal{F} = p$, $p+q=n$, and a complex vector bundle E on M of rank r. We fix a Riemannian metric on M with the corresponding distance ρ and an Hermitian structure on E.

We will denote by $G = G_{\mathcal{F}}$ the holonomy groupoid of $(M,\mathcal{F})$. G is equipped with the source and the target maps $s,r : G \to M$. We will make use of standard notation: $G^{(0)} = M$ is the set of objects, $G^x = \{\gamma \in G : r(\gamma) = x\}$, $G^x_x = \{\gamma \in G : s(\gamma) = r(\gamma) = x\}$, $x \in M$. For any $x \in M$, s defines a covering map from G^x to the leaf through the point x associated with the holonomy group G^x_x of the leaf. We will identify a point $x \in M$ with the identity element in G^x_x. Let dx be the Riemannian volume form on M, λ_L the Riemannian volume form on a leaf L of $\mathcal{F}$ and, for any $x \in M$, λ^x its lift to a density on the holonomy covering G^x. We will make use of notation $(x,y) \in I^p \times I^q$ ($I = (-1,1)$) for the local coordinates given by a foliated chart $\kappa : I^p \times I^q \to M$ and $(\xi,\eta) \in \mathbb{R}^p \times \mathbb{R}^q$ for the dual coordinates (in T^*M).

The holonomy groupoid G has the structure of a smooth manifold of dimension $2p+q$. Recall briefly the construction of an atlas on G [5]. Let $\kappa : I^p \times I^q \to M, \kappa' : I^p \times I^q \to M$, be two foliated charts, $\pi : \{0\} \times I^q \to \pi(\{0\} \times I^q) = D^q \subset M$, $\pi' : \{0\} \times I^q \to \pi'(\{0\} \times I^q) = D'^q \subset M$ be the corresponding transversals to the foliation. The foliation charts κ, κ' are called **compartible**, if, for any points $m \in U = \kappa(I^p \times I^q)$ and $m' \in U' = \kappa'(I^p \times I^q)$ such that $m = \kappa(x,y)$, $m' = \kappa'(x',y)$, there is a leafwise path γ from m to

m' such that the corresponding holonomy map h_γ maps the germ π_m of the transversal π at the point m to the germ $\pi'_{m'}$ of the transversal π' at the point m': $h_\gamma \pi_m = \pi'_{m'}$.

For any pair of compartible foliation charts, κ and κ', let $W(\kappa,\kappa')$ be a subset in $G_{\mathcal{F}}$:

$$W(\kappa,\kappa') = \{\gamma \in G : s(\gamma) = m \in U, r(\gamma) = m' \in U', h_\gamma \pi_m = \pi'_{m'}\},$$

equipped with a coordinate map

$$\Gamma : I^p \times I^p \times I^q \to W(\kappa,\kappa'), \tag{1}$$

which associates to any $(x,x',y) \in I^p \times I^p \times I^q$ the element $\gamma \in W(\kappa,\kappa')$ such that $s(\gamma) = m = \kappa(x,y)$, $r(\gamma) = m' = \kappa'(x',y)$ and $h_\gamma \pi_m = \pi'_{m'}$.

As shown in [5], the coordinate patches $W(\kappa,\kappa')$ form an atlas of a $2p+q$-dimensional manifold on G.

Denote by $C_c^\infty(G,\mathcal{L}(E))$ the space of smooth, compactly supported sections of the vector bundle $(s^*E)^* \otimes r^*E$ on G. Otherwise speaking, the value of $k \in C_c^\infty(G,\mathcal{L}(E))$ at any point $\gamma \in G$ is a linear map $k(\gamma) : E_{s(\gamma)} \to E_{r(\gamma)}$. Any element $k \in C_c^\infty(G,\mathcal{L}(E))$ defines an operator $R_E(k) : C^\infty(M,E) \to C^\infty(M,E)$ by the formula

$$R_E(k)u(x) = \int_{G^x} k(\gamma)u(s(\gamma))d\lambda^x(\gamma), u \in C^\infty(M,E), x \in M. \tag{2}$$

which is said to be a tangential operator on $(M,\mathcal{F})$ defined by the tangential kernel k.

1.2. Classes $\Psi^{m,-\infty}(M,\mathcal{F},E)$

In this section, we introduce the algebra $\Psi^{*,-\infty}(M,\mathcal{F},E)$ of transversal pseudodifferential operators on the foliated manifold $(M,\mathcal{F})$, which can be considered as an analogue of the algebra of pseudodifferential operators on a closed manifold. This algebra can be realized as a Guillemin-Sternberg algebra $\mathcal{R}_\Sigma$ [11], corresponding to a coisotropic conic submanifold Σ in the punctured cotangent bundle $\tilde{T}^*M$ (see below), therefore, many its properties can be deduced, just referring to the corresponding results for these general algebras. Here we prefer to give direct proofs (when it is possible), since this is simpler and allows us to extend these results to more general cases [27].

Recall that a function $k \in C^\infty(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ belongs to the class $S^m(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$, if, for any multiindices α and β, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$||\partial_\eta^\alpha \partial_{(x,x',y)}^\beta k(x,x',y,\eta)|| \le C_{\alpha\beta}(1+|\eta|)^{m-|\alpha|}, (x,x',y) \in I^p \times I^p \times I^q, \eta \in \mathbb{R}^q.$$

In what follows, we will consider only classical symbols. Recall that a function $k \in C^\infty(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ is a **classical symbol** of order $z \in \mathbb{C}$ ($k \in S^{z,-\infty}(I^p \times I^p \times I^q \times \mathbb{R}^n, \mathbb{R}^p, \mathcal{L}(\mathbb{C}^r))$), if k is represented as an asymptotic sum

$$k(x,x',y,\eta)\sim\sum_{j=0}^{\infty}\theta(\eta)k_{z-j}(x,x',y,\eta),$$

where $k_{z-j}\in C^\infty(I^p\times I^p\times I^q\times(\mathbb{R}^q\backslash\{0\}),\mathcal{L}(\mathbb{C}^r))$ is homogeneous in η of degree $z-j$, that is,

$$k_{z-j}(x,x',y,t\eta)=t^{z-j}k_{z-j}(x,x',y,\eta),t>0,$$

and θ is a smooth function on $\mathbb{R}^q$ such that $\theta(\eta)=0$ for $|\eta|\leq 1$, $\theta(\eta)=1$ for $|\eta|\geq 2$.

A symbol $k\in S^m(I^p\times I^p\times I^q\times\mathbb{R}^q,\mathcal{L}(\mathbb{C}^r))$ defines an operator $A: C_c^\infty(I^n,\mathbb{C}^r)\to C^\infty(I^n,\mathbb{C}^r)$ by the formula

$$Au(x,y)=(2\pi)^{-q}\int e^{i(y-y')\eta}k(x,x',y,\eta)u(x',y')\,dx'\,dy'\,d\eta, \tag{3}$$

where $u\in C_c^\infty(I^n,\mathbb{C}^r),x\in I^p,y\in I^q$. Denote by $\Psi^{m,-\infty}(I^n,I^p,\mathbb{C}^r)$ the class of operators of the form (3) with $k\in S^m(I^p\times I^p\times I^q\times\mathbb{R}^q,\mathcal{L}(\mathbb{C}^r))$ such that its Schwartz kernel is compactly supported in $I^n\times I^n$.

It is very useful to note that the algebra $\Psi^{*,-\infty}(I^n,I^p,\mathbb{C}^r)$ has the structure of a crossed product of an algebra of pseudodifferential operators on transversals to the foliation (in y variables) by the leafwise equivalence relation. More precisely, it can be formulated as follows. If we represent the space $L^2(I^n,\mathbb{C}^r)$ as the L^2 space of $L^2(I^q,\mathbb{C}^r)$)-valued functions on I^p, $L^2(I^n,\mathbb{C}^r)=L^2(I^p,L^2(I^q,\mathbb{C}^r))$, then an operator $A\in\Psi^{m,-\infty}(I^n,I^p,\mathbb{C}^r)$ can be written in the following form:

$$A\bar{u}(x)=\int A(x,x')\bar{u}(x')\,dx',x\in I^p, \tag{4}$$

where $\bar{u}\in C_c^\infty(I^p,L^2(I^q,\mathbb{C}^r))$ such that $\bar{u}(x)\in C_c^\infty(I^q,\mathbb{C}^r)$ for any $x\in I^p$, and, for any $x\in I^p$, $x'\in I^p$, the operator $A(x,x')$ is a pseudodifferential operator of order m on I^q:

$$A(x,x')v(y)=(2\pi)^{-q}\int e^{i(y-y')\eta}k(x,x',y,\eta)v(y')\,dy'\,d\eta,v\in C_c^\infty(I^p,\mathbb{C}^r). \tag{5}$$

The principal symbol of $A\in\Psi^{m,-\infty}(I^n,I^p,\mathbb{C}^r)$ is defined to be a matrix-valued function σ_A on $I^p\times I^p\times I^q\times(\mathbb{R}^q\backslash\{0\})$ given by the formula

$$\sigma_A(x,x',y,\eta)=k_m(x,x',y,\eta), \tag{6}$$

where k_m is the homogeneous (of degree m) component of the complete symbol k of the operator A. We have the following properties of the principal symbols of operators from $\Psi^{m,-\infty}(I^n,I^p,\mathbb{C}^r)$.

Lemma 1. *(1) Given $A\in\Psi^{m_1,-\infty}(I^n,I^p,\mathbb{C}^r)$ and $B\in\Psi^{m_2,-\infty}(I^n,I^p,\mathbb{C}^r)$, the composition $C=AB$ belongs to the class $\Psi^{m_1+m_2,-\infty}(I^n,I^p,\mathbb{C}^r)$, and its principal symbol σ_C is given by the formula*

$$\sigma_C(x,x',y,\eta)=\int\sigma_A(x,x'',y,\eta)\sigma_B(x'',x',y,\eta)dx''.$$

(2) The principal symbol σ_A of an operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ is transformed under a foliated coordinate change $x_1 = \phi(x,y), y_1 = \psi(y)$, via the following formula

$$\sigma_{A_1}(\phi(x,y), \phi(x',y), \psi(y), (d\psi(y)^*)^{-1}(\eta)) = \sigma_A(x,x',y,\eta),$$

where the operator A is assumed to be written in the coordinates (x,y) and A_1 denotes the operator A, written in the coordinates (x_1, y_1).

Proof. If $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ is written in the form (4), the principal symbol σ_A of A can be expressed in terms of the principal symbols $\sigma_{A(x,x')}$ of the operators $A(x,x')$ as follows:

$$\sigma_A(x,x',y,\eta) = \sigma_{A(x,x')}(y,\eta).$$

If operators $A \in \Psi^{m_1,-\infty}(I^n, I^p, \mathbb{C}^r)$ and $B \in \Psi^{m_2,-\infty}(I^n, I^p, \mathbb{C}^r)$ are written in the form (4) with the corresponding families $A(x,x') \in \Psi^{m_1}(I^q, \mathbb{C}^r)$ and $B(x,x') \in \Psi^{m_2}(I^q, \mathbb{C}^r)$ accordingly, then it is easy to see that the composition $C = AB$ is written in the form (4) with

$$C(x,x') = \int A(x,x'')B(x'',x')dx''.$$

Using these facts and standard pseudodifferential calculus, the lemma can be easily proved.

If $\kappa : I^p \times I^q \to U = \kappa(I^p \times I^q) \subset M, \kappa' : I^p \times I^q \to U' = \kappa'(I^p \times I^q) \subset M$, are two compartible foliated charts on M equipped with trivializations of the vector bundle E over them, we can transfer an operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ to an operator $A' : C_c^\infty(U,E) \to C_c^\infty(U',E)$, which extends in a trivial way to an operator in $C^\infty(M,E)$, denoted also by A'. The resulting operator A' is said to be **an elementary operator** of class $\Psi^{m,-\infty}(M, \mathcal{F}, E)$.

Definition 1. *The class $\Psi^{m,-\infty}(M,\mathcal{F},E)$ consists of operators A, acting from $C^\infty(M,E)$ to $C^\infty(M,E)$, such that A can be represented in the form $A = \sum_{i=1}^k A_i + K$, where A_i are elementary operators of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$, corresponding to some pairs κ_i, κ_i' of compartible foliated charts, $K \in \Psi^{-\infty}(M,E)$.*

To give an invariant definition of the principal symbol for operators of $\Psi^{m,-\infty}(M,\mathcal{F},E)$, let us show how these operators can be represented as Fourier integral operators, associated with some canonical relation on the punctured cotangent space $\tilde{T}^*M = T^*M \setminus \{0\}$.

It is well-known that the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}_N$ in the punctured conormal bundle $\tilde{N}^*\mathcal{F}$, which is transversally parallelizable and, therefore, has trivial holonomy (see [28]). In local coordinates (x,y,η) on $\tilde{N}^*\mathcal{F}$ given by a foliated chart on M, plaques of the foliation $\mathcal{F}_N$ are defined by $y = \text{const}, \eta = \text{const}$. It is easy to see that the leaf $\tilde{L}_\eta$ of the foliation $\mathcal{F}_N$ through a point $\eta \in \tilde{N}^*\mathcal{F}$ is diffeomorphic to the holonomy covering $G_{\mathcal{F}}^x$ of the leaf $L_x, x = \pi(\eta)$, of the foliation $\mathcal{F}$ through the point x, therefore, we can give the following description of the holonomy groupoid of $\mathcal{F}_N$.

Recall that, for any smooth leafwise path γ from $x \in M$ to $y \in M$, there is defined the map $dh_\gamma^* : N_y^*\mathcal{F} \to N_x^*\mathcal{F}$, being the codifferential of the holonomy map h_γ, corresponding to γ (cf., for instance, [7]). The holonomy groupoid $G_{\mathcal{F}_N}$ of the lifted foliation $\mathcal{F}_N$ consists of all $(\gamma,\eta)\in G_{\mathcal{F}}\times \tilde{N}^*\mathcal{F}$ such that $r(\gamma)=\pi(\eta)$ with the source map $s: G_{\mathcal{F}_N}\to \tilde{N}^*\mathcal{F}, s(\gamma,\eta)=dh_\gamma^*(\eta)$ and the target map $r: G_{\mathcal{F}_N}\to \tilde{N}^*\mathcal{F}, r(\gamma,\eta)=\eta$. The bundle map $\pi:\tilde{N}^*\mathcal{F}\to M$ induces a map $\pi_G: G_{\mathcal{F}_N}\to G_{\mathcal{F}}$ by

$$\pi_G(\gamma,\eta)=\gamma,(\gamma,\eta)\in G_{\mathcal{F}_N}.$$

There is also a symplectic description of the lifted foliation. Consider T^*M as a symplectic manifold, equipped with the canonical symplectic structure. Then $\tilde{N}^*\mathcal{F}$ is a coisotropic submanifold in $\tilde{T}^*M$, and the foliation $\mathcal{F}_N$ is the corresponding null-foliation. It is well-known that the mapping

$$(r,s): G_{\mathcal{F}_N}\to \tilde{T}^*M\times \tilde{T}^*M \tag{7}$$

defines an immersed canonical relation in $\tilde{T}^*M$, which is often called by the flowout of the coisotropic submanifold $\tilde{N}^*\mathcal{F}$.

It can be easily checked that the algebra of Fourier integral operators, associated with this canonical relation, is the algebra $\Psi^{*,-\infty}(M,\mathcal{F},E)$ introduced above. We only need to be more precise about the immersed canonical relation (7). Namely, let us define the space $I^m(M\times M,G'_{\mathcal{F}_N})$ of compactly supported Lagrangian distributions, taking finite sums of elementary Lagrangian distributions as we did above in the definition of classes $\Psi^{*,-\infty}(M,\mathcal{F},E)$. A precise statement is that the class $\Psi^{m,-\infty}(M,\mathcal{F},E)$ consists of all operators in $C^\infty(M,E)$ with Schwartz kernels from the space $I^{m-p/2}(M\times M,G'_{\mathcal{F}_N})$.

Now let us show how the notion of the principal symbol of operators of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$ as Fourier integral operators agrees with the local definition given by (6). According to [18, Section 25.1], the principal symbol of an operator of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$ as a Fourier integral operator is a half-density on $G_{\mathcal{F}_N}$ homogeneous of degree $m+q/2$ defined as follows. Let $\kappa: I^p\times I^q\to U=\kappa(I^p\times I^q)\subset M, \kappa': I^p\times I^q\to U'=\kappa'(I^p\times I^q)\subset M$, be two compartible foliated charts on M equipped with trivializations of the vector bundle E over them. Define a foliated coordinate map $\Gamma_N: I^p\times I^p\times I^q\times\mathbb{R}^q\to G_{\mathcal{F}_N}$ by $\Gamma_N(x,x',y,\eta)=(\Gamma(x,x',y),(d\kappa'^*)^{-1}(x',y,\eta))$, where $(x,x',y,\eta)\in I^p\times I^p\times I^q\times\mathbb{R}^q$, Γ is the coordinate map given by (1) and $(d\kappa'^*)^{-1}: I^p\times I^q\times\mathbb{R}^q\to \tilde{N}^*\mathcal{F}$ is the inverse to the codifferential of κ'. In this coordinate chart, the half-density principal symbol σ_A of an elementary operator $A\in\Psi^{m,-\infty}(M,\mathcal{F},E)$ (given by (3) in $W(\kappa,\kappa')$) is defined to be equal to

$$k_m(x,x',y,\eta)(dx\,dx'\,dy\,d\eta)^{1/2}, \tag{8}$$

where k_m is the homogeneous component of the complete symbol k of degree m. The half-density (8) can be identified with a leafwise half-density, using the canonical transversal symplectic form $dy\,d\eta$, and, moreover, with a smooth section from $C^\infty(G_{\mathcal{F}_N},\mathcal{L}(\pi^*E))$, using the fixed leafwise density $\lambda=\{\lambda_L: L\in M/\mathcal{F}\}$.

Let $S^m(G_{\mathcal{F}_N},\pi^*E)$ be the space of all sections $s\in C^\infty(G_{\mathcal{F}_N},\mathcal{L}(\pi^*E)), s=s(\gamma,\eta)$ homogeneous in η of degree m such that $\pi_G(\operatorname{supp} s)$ is compact in $G_{\mathcal{F}}$. Then the principal symbol σ_A of an operator $A\in\Psi^{m,-\infty}(M,\mathcal{F},E)$ is globally defined as an element of $S^m(G_{\mathcal{F}_N},\pi^*E)$. We can also consider the principal symbol of the operator A as the corresponding tangential operator $R_{\pi^*E}(\sigma_A)$ on $\tilde{N}^*\mathcal{F}$ with respect to $\mathcal{F}_N$.

The space

$$S^*(G_{\mathcal{F}_N}, \pi^*E) = \bigcup_m S^m(G_{\mathcal{F}_N}, \pi^*E)$$

carries the structure of an involutive algebra, defined by its embedding into the foliation algebra $C_c^\infty(G_{\mathcal{F}_N}, \pi^*E)$. By Lemma 1, we obtain symbolic properties of $\Psi^{m,-\infty}(M,\mathcal{F},E)$ (see also [11, Section 2] for the corresponding general result for Fourier integral operators).

Proposition 1. *(1) The principal symbol mapping*

$$\sigma : \Psi^{*,-\infty}(M,\mathcal{F},E) \to S^*(G_{\mathcal{F}_N}, \pi^*E)$$

is an algebra homomorphism. Otherwise speaking, if $A \in \Psi^{m_1,-\infty}(M,\mathcal{F},E)$ and $B \in \Psi^{m_2,-\infty}(M,\mathcal{F},E)$, then $C = AB$ belongs to $\Psi^{m_1+m_2,-\infty}(M,\mathcal{F},E)$ and $\sigma_{AB} = \sigma_A\sigma_B$.

(2) If the density dx on M is holonomy invariant, then σ is a $*$-homomorphism of involutive algebras, i.e., if $A \in \Psi^{m,-\infty}(M,\mathcal{F},E)$, then $A^* \in \Psi^{m,-\infty}(M,\mathcal{F},E)$ and $\sigma_{A^*} = (\sigma_A)^*$.

The next problem is to state L^2-continuity of operators from $\Psi^{0,-\infty}(M,\mathcal{F},E)$. We refer the reader to [20, Theorem 25.3.8] for the corresponding general result on L^2-continuity of Fourier integral operators, but, indeed, our case is a model case for this general theorem.

Proposition 2. *Any operator $A \in \Psi^{0,-\infty}(M,\mathcal{F},E)$ defines a bounded operator in the Hilbert space $L^2(M,E)$.*

Proof. The proposition follows immediately, if we make use of a representation of the operator A in the form (4) and apply the theorem on L^2 boundedness of zero-order pseudodifferential operators.

1.3. Anisotropic Sobolev spaces and classes $\Psi^{m,\mu}(M,\mathcal{F},E)$

Our norm estimates will be given in terms of the scale of anisotropic Sobolev spaces $H^{s,k}(M,\mathcal{F},E)$, $s \in \mathbb{R}, k \in \mathbb{R}$ ([25, 26]). Let us briefly recall its definition.

Definition 2. *The space $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p, \mathbb{C}^r)$ consists of all $u \in S'(\mathbb{R}^n, \mathbb{C}^r)$ such that its Fourier transform $\tilde{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^r)$ and*

$$||u||^2_{s,k} = \int\int |\tilde{u}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^s(1+|\xi|^2)^k d\xi d\eta < \infty. \tag{9}$$

The identity (9) serves as a definition of a Hilbert norm in $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p, \mathbb{C}^r)$.

Definition 3. *The space $H^{s,k}(M,\mathcal{F},E)$ consists of all $u \in \mathcal{D}'(M,E)$ such that, for any foliated coordinate chart $\kappa : I^p \times I^q \to U = \kappa(I^p \times I^q) \subset M$, for any trivialization of the bundle E over it and for any $\phi \in C_c^\infty(U)$, the function $\kappa^*(\phi u)$ belongs to the space $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p, \mathbb{C}^r)$.*

Fix a finite covering $\{U_i : i = 1, \ldots, d\}$ of M by foliated coordinate patches with foliated coordinate charts $\kappa_i : I^p \times I^q \to U_i = \kappa_i(I^p \times I^q)$, and a partition of unity $\{\phi_i \in C^\infty(M) : i = 1, \ldots, d\}$ subordinate to this covering. A scalar product in $H^{s,k}(M, \mathcal{F}, E)$ can be equivalently defined by the formula

$$(u, v)_{s,k} = \sum_{i=1}^{d} (\kappa^*(\phi_i u), \kappa^*(\phi_i v))_{s,k}, u, v \in H^{s,k}(M, \mathcal{F}, E). \tag{10}$$

Using the anisotropic Sobolev spaces $H^{s,k}(M, \mathcal{F}, E), s \in \mathbb{R}, k \in \mathbb{R}$, we can give a sufficient condition for a bounded operator T in $L^2(M, E)$ to be an operator of trace class, adapted to the foliated structure of M [26].

Proposition 3. *Let T be a bounded operator in $L^2(M, E)$ such that T defines a bounded operator from $L^2(M, E)$ to $H^{s,k}(M, \mathcal{F}, E)$ with some $s > q$ and $k > p$. Then T is an operator of trace class with the following estimate of its trace norm:*

$$\|T\|_1 \leq C\|T : L^2(M, E) \to H^{s,k}(M, \mathcal{F}, E)\|.$$

Now we introduce operator classes $\Psi^{m,\mu}(M, \mathcal{F}, E)$ associated with the scale $H^{s,k}(M, \mathcal{F}, E)$. A local description of these classes (denoted by $\Psi^{m,\mu}(M, \mathcal{F}, \delta)$ there) was given in [26, 25] by means of Hörmander classes of pseudodifferential operators with tempered metrics. Here we define classes $\Psi^{m,\mu}(M, \mathcal{F}, E)$ globally, using the definition of Fourier integral operators, corresponding to a pair (Λ_0, Λ_1) of intersected Lagrangian submanifolds [15, 16]: Λ_0 is the diagonal in $\tilde{T}^*M \times \tilde{T}^*M$, Λ_1 is the holonomy groupoid $G_{\mathcal{F}_N}$. One of essential advantages of our approach is possibility to make use of the symbolic calculus for Fourier integral operators.

Definition 4. *We say that a function $a \in C^\infty(I^p \times I^p \times I^q \times \mathbb{R}^n \times \mathbb{R}^p, \mathcal{L}(\mathbb{C}^r)), a = a(s, x, y, \xi, \eta, \sigma)$ belongs to the class $S^{m,\mu}(I^n \times \mathbb{R}^n, \mathbb{R}^p, \mathcal{L}(\mathbb{C}^r))$, if, for any multiindices α, β and γ, there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that*

$$\|\partial^\alpha_{(\xi,\eta)} \partial^\beta_\sigma \partial^\gamma_{(s,x,y)} a(s, x, y, \xi, \eta, \sigma)\| \leq C_{\alpha\beta\gamma}(1 + |\xi| + |\eta|)^{m-|\alpha|}(1 + |\sigma|)^{\mu-|\gamma|},$$
$$(s, x, y) \in I^p \times I^n, (\xi, \eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}^p.$$

A symbol $a \in S^{m,\mu}(I^n \times \mathbb{R}^n, \mathbb{R}^p, \mathcal{L}(\mathbb{C}^r))$ defines an operator A from $C_c^\infty(I^n, \mathbb{C}^r)$ to $C^\infty(I^n, \mathbb{C}^r)$ by the formula

$$\begin{aligned} Au(x, y) &= (2\pi)^{-2p-q} \int e^{i[(x-x'-s)\xi + (y-y')\eta + s\sigma]} a(s, x, y, \xi, \eta, \sigma) \\ &\quad u(x', y')\, ds\, dx'\, dy'\, d\xi\, d\eta\, d\sigma, \end{aligned} \tag{11}$$

where $u \in C_c^\infty(I^n, \mathbb{C}^r), x \in I^p, y \in I^q$.

If $\kappa : I^p \times I^q \to U = \kappa(I^p \times I^q) \subset M, \kappa' : I^p \times I^q \to U' = \kappa'(I^p \times I^q) \subset M$, are two compartible foliated charts on M, then we can transfer an operator A of the form (11) to an operator $A' : C_c^\infty(U, E) \to C^\infty(U', E)$. If, in addition, the kernel of the operator A' is compactly supported in $U \times U'$, then the operator A' maps $C_c^\infty(U, E)$ to $C_c^\infty(U', E)$, and we can prolong it in a trivial way to an operator $A' : C^\infty(M, E) \to C^\infty(M, E)$. We say that the operator A' obtained in such a way is an **elementary operator** of class $\Psi^{m,\mu}(M, \mathcal{F}, E)$.

Definition 5. *The class $\Psi^{m,\mu}(M,\mathcal{F},E)$ consists of operators A, acting from $C^\infty(M,E)$ to $C^\infty(M,E)$, such that A can be represented in the form $A = \sum_{i=1}^k A_i + K$, where A_i are elementary operators of class $\Psi^{m,\mu}(M,\mathcal{F},E)$, corresponding to some pairs κ_i,κ_i' of compartible foliated charts, $K \in \Psi^{-\infty}(M,E)$.*

Remark 1. In notation of [15, 16], the Schwartz kernel K_A of an operator $A \in \Psi^{m,\mu}(M,\mathcal{F},E)$ belongs to the class $I^{m-p/2,\mu+p/2}(M\times M,\Delta,G_{\mathcal{F}_N})$.

Example 2. Any tangential pseudodifferential operator $B \in \Psi^\mu(\mathcal{F},E)$ [25] belongs to the class $\Psi^{0,\mu}(M,\mathcal{F},E)$. Moreover, if B is given by the complete symbol $b(x,y,\xi)$ in some foliated coordinate chart κ, then, in the foliated chart $W(\kappa,\kappa)$, the Schwarz kernel of B is represented in the form (11) with

$$a(s,x,y,\xi,\eta,\sigma) = b(x,y,\sigma)$$

(for a global description, see also Example 3).

Example 3. Any operator $C \in \Psi^m(M,E)$ belongs to $\Psi^{m,0}(M,\mathcal{F},E)$. Moreover, if C is given by the complete symbol $c(x,y,\xi,\eta)$ in some foliated coordinate chart κ, then, in the foliated chart $W(\kappa,\kappa)$, the Schwarz kernel of C is represented in the form (11) with

$$a(s,x,y,\xi,\eta,\sigma) = c(x,y,\xi,\eta).$$

Example 4. It can be easily seen that two definitions of classes $\Psi^{m,-\infty}(M,\mathcal{F},E)$ are equivalent, that is,

$$\Psi^{m,-\infty}(M,\mathcal{F};E) = \bigcap_\mu \Psi^{m,\mu}(M,\mathcal{F},E).$$

The operators of class $\Psi^{m,\mu}(M,\mathcal{F},E)$ can be considered as a usual pseudifferential operators of order $m+\mu$ with the complete symbol, singular on the punctured conormal bundle $\tilde{N}^*\mathcal{F}$. Let us briefly mention about the corresponding symbolic calculus referring to [15, 16] for details.

Let A be an elementary operator of class $\Psi^{m,\mu}(M,\mathcal{F},E)$, given by the formula (11). Then the principal symbol of A is a function $\sigma_0(A)$ on $T^*M\backslash N^*\mathcal{F}$, given locally by the formula

$$\sigma_0(A)(x,y,\xi,\eta) = a_{m,\mu}(0,x,y,\xi,\eta,\xi), \xi\neq 0,$$

where $a_{m,\mu}$ is the bihomogeneous component of the complete symbol a of degree m in (ξ,η) and of degree μ in σ.

We say that an operator $A \in \Psi^{m,\mu}(M,\mathcal{F},E)$ is elliptic, if $\sigma_0(A)$ is invertible on $T^*M\backslash N^*\mathcal{F}$. By [15, Proposition 6.4], any elliptic operator $A \in \Psi^{m,\mu}(M,\mathcal{F},E)$ has a parametrix, i.e. an operator $P \in \Psi^{-m,-\mu}(M,\mathcal{F},E)$ such that

$$AP = I - R_1, PA = I - R_2, \tag{12}$$

where $R_j \in \Psi^{-1,0}(M,\mathcal{F},E) + \Psi^{0,-\infty}(M,\mathcal{F},E)$, $j=1,2$.

Proposition 4. *(1) If $A \in \Psi^{m_1,\mu_1}(M,\mathcal{F},E)$ and $B \in \Psi^{m_2,\mu_2}(M,\mathcal{F},E)$, then $C = AB \in \Psi^{m_1+m_2,\mu_1+\mu_2}(M,\mathcal{F},E)$ and $\sigma_0(C) = \sigma_0(A)\sigma_0(B)$.*

(2) Any operator $A \in \Psi^{m,\mu}(M,\mathcal{F},E)$ defines a continuous mapping

$$A : H^{s,k}(M,\mathcal{F},E) \to H^{s-m,k-\mu}(M,\mathcal{F},E)$$

for any s and k.

Proof. 1) A proof is given in [1] (see also [16, Proposition 1.39]).

2) As usual, it suffices to consider the case when A is an elementary operator. Using the standard description of the Sobolev space $H^{s,k}(M,\mathcal{F},E)$ by means of elliptic operators of class $\Psi^{s,k}(M,\mathcal{F},E)$ (we may use the local description, using the operator $(1+D_x^2+D_y^2)^{s/2}(1+D_x^2)^{k/2})$ and a parametrix for elliptic operators (12), we can reduce the problem to the case $s=k=0$ and $\max(m,m+\mu)\le 0$. L^2 boundedness of operators from $\Psi^{m,\mu}(M,\mathcal{F},E)$ with $\max(m,m+\mu)\le 0$ can be stated by imitating of the Hörmander's proof of L^2 boundedness of zero-order pseudodifferential operators. For details see [16, Theorem 3.3].

1.4. Symbolic properties of $\Psi^{m,-\infty}(M,\mathcal{F},E)$

Here we turn to more elaborate symbolic properties of classes $\Psi^{m,-\infty}(M,\mathcal{F},E)$. While the algebraic symbolic properties of the section follow directly from the corresponding properties of the algebras $\mathcal{R}_\Sigma$ of [11] (see [11, Section 3]), the presence of the Sobolev space scale in our geometric case provides norm estimates in addition to the algebraic symbolic results of [11].

Recall that the principal symbol p_m of an operator $P\in\Psi^m(M,E)$ is a smooth section of the vector bundle $\mathcal{L}(\pi^*E)$ on $\tilde{T}^*M$, where $\pi:T^*M\to M$ is the natural projection.

Definition 6. The transversal principal symbol σ_P *of an operator* $P\in\Psi^m(M,E)$ *is the restriction of its principal symbol* p_m *on* $\tilde{N}^*\mathcal{F}$.

Proposition 5. *If* $A\in\Psi^{m_1}(M,E)$ *and* $B\in\Psi^{m_2,-\infty}(M,\mathcal{F},E)$, *then* AB *and* BA *in* $\Psi^{m_1+m_2,-\infty}(M,\mathcal{F},E)$ *and*

$$\begin{aligned}\sigma_{AB}(\gamma,\eta) &= \sigma_A(\eta)\sigma_B(\gamma,\eta),(\gamma,\eta)\in G_{\mathcal{F}_N},\\ \sigma_{BA}(\gamma,\eta) &= \sigma_B(\gamma,\eta)\sigma_A(dh_\gamma^*(\eta)),(\gamma,\eta)\in G_{\mathcal{F}_N}.\end{aligned}$$

Proof. This Proposition follows from the composition theorem of Fourier integral operators (see, for instance, [20]).

From now on, we will assume that E is **holonomy equivariant**, that is, there is an isometrical action

$$T(\gamma):E_x\to E_y,\gamma\in G,\gamma:x\to y$$

of the holonomy groupoid G in fibres of E. We have an inclusion $C_c^\infty(G)\subset C_c^\infty(G,\mathcal{L}(E))$, given by $k(\gamma)\mapsto k(\gamma)T(\gamma)$, and, by (2), a representation R_E of the algebra $C_c^\infty(G)$ in $L^2(M,E)$, which is a $*$-representation, if the density dx is holonomy invariant. For any $k\in C_c^\infty(G)$, the operator $R_E(k)$ is given by the formula

$$R_E(k)u(x)=\int_{G^x}k(\gamma)T(\gamma)u(s(\gamma))d\lambda^x(\gamma),x\in M,u\in C^\infty(M,E). \tag{13}$$

Our norm estimates will be given in terms of the following seminorms on $C_c^\infty(G)$:

$$||k||_{s,t,t-l}=||R_E(k):H^{s,t}(M,\mathcal{F},E)\to H^{s,t-l}(M,\mathcal{F},E)||, \tag{14}$$

where $k\in C_c^\infty(G)$ and $s\in\mathbb{R},t\in\mathbb{R},l\in\mathbb{R}$. There are defined the corresponding functional classes:

Definition 7. *The class* $\mathrm{OP}^l(G)$ *consists of all distributions* $k \in \mathcal{D}'(G)$ *such that the operator* $R_E(k)$ *defines a continuous mapping*

$$R_E(k) : H^{s,t}(M, \mathcal{F}, E) \to H^{s,t-l}(M, \mathcal{F}, E)$$

for any real s *and* t.

Recall that an open subset U in T^*M is **a conic neighborhood** of $N^*\mathcal{F}$, if U is a neighborhood of $N^*\mathcal{F}$, which is invariant under the action of $\mathbb{R}_+$ by multiplication. It is clear that a basis of conic neighborhoods of $N^*\mathcal{F}$ is formed by sets $\kappa(U_\varepsilon), \varepsilon > 0$, where U_ε is given by

$$U_\varepsilon = \{(x, y, \xi, \eta) \in I^n \times \mathbb{R}^n : |\xi| < \varepsilon|\eta|\},$$

and $\kappa : I^p \times I^q \to M$ is a foliated coordinate chart.

Definition 8. *We say that an operator* $P \in \Psi^l(M, E)$ *has* **transversal order** $m \leq l$ *(*$P \in \Psi^m(N^*\mathcal{F}, E)$*), if* P *has order* m *in some conic neighborhood of* $N^*\mathcal{F}$*, that is, its complete symbol* p *in any foliated coordinate system satisfies the following condition: there is a* $\varepsilon > 0$ *such that, for any multiindices* α, β, *there is a constant* $C_{\alpha,\beta} > 0$ *such that*

$$|\partial^\alpha_{(\xi,\eta)}\partial^\beta_{(x,y)}p(x, y, \xi, \eta)| \leq C_{\alpha\beta}(1 + |\xi| + |\eta|)^{m-|\alpha|}, (x, y, \xi, \eta) \in U_\varepsilon. \qquad (15)$$

The main fact, which relates the notion of transversal order with the classes $\Psi^{m,\mu}(M, \mathcal{F}, E)$, consists in the following inclusion:

$$\Psi^l(M, E)\bigcap\Psi^m(N^*\mathcal{F}, E) \subset \Psi^{m,l-m}(M, \mathcal{F}, E). \qquad (16)$$

Indeed, it can be easily checked by a straightforward calculation, that, if $p \in S^l(I^n \times \mathbb{R}^n)$ satisfies (15), then $p(x, y, \xi, \eta)(1 + |\xi|^2)^{m-l}$ belongs to $S^m(I^n \times \mathbb{R}^n)$, from where (16) follows immediately (see also [19, Theorem 18.1.35]).

By Proposition 4 and (16), any operator $P \in \Psi^m(N^*\mathcal{F}, E)\bigcap\Psi^l(M, E)$ defines a continuous mapping

$$P : H^{s,k}(M, \mathcal{F}, E) \to H^{s-m,k-l+m}(M, \mathcal{F}, E), \qquad (17)$$

that, in its turn, gives immediately the following proposition.

Proposition 6. *Suppose that an operator* $P \in \Psi^l(M, E)$ *has transversal order* $m \leq l$. *Then, for any* $k \in C^\infty_c(G, \mathcal{L}(E))$, *the operators* $R_E(k)P$ *and* $PR_E(k)$ *belong to* $\Psi^{m,-\infty}(M, \mathcal{F}, E)$ *with the following norm estimates*

$$\begin{aligned}
||R_E(k)P : H^{s,t}(M, \mathcal{F}, E) \to H^{s-m,r}(M, \mathcal{F}, E)|| &\leq C||k||_{s-m,t-l+m,r}, \\
||PR_E(k) : H^{s,t}(M, \mathcal{F}, E) \to H^{s-m,r}(M, \mathcal{F}, E)|| &\leq C||k||_{s,t,r+l-m}.
\end{aligned}$$

We will denote by $\operatorname{ad} T(\gamma)$ the action of G in fibres of the bundle $\mathcal{L}(\pi^*E)$, induced by $T(\gamma)$.

Definition 9. *We say that the transversal principal symbol of an operator* $P \in \Psi^m(M, E)$ *is* **holonomy invariant**, *if, for any smooth leafwise path* γ *from* x *to* y, *the following equality holds:*

$$\operatorname{ad} T(\gamma)[\sigma_P(dh^*_\gamma(\xi))] = \sigma_P(\xi), \xi \in N^*_y\mathcal{F}.$$

Remark 2. The assumption of existence of positive order pseudodifferential operators with the holonomy invariant transversal principal symbol implies rather strong restrictions on the foliated manifold under consideration. There are examples of such operators on every Riemannian foliation given by the transverse signature operator, and, in fact, this assumption is equivalent to a slightly more general assumption on the foliation to be transversally Finsler (as introduced in [25]). For general foliations, one can use a generalized notion of holonomy invariance, based on more sophisticated transversal pseudodifferential calculus. As an example, we point out the treatment of triangular Riemannian manifolds, based on hypoelliptic operators and Beals-Greiner pseudodifferential calculus [9]. We will discuss this subject in more details in [27].

By Proposition 5, if P is a pseudodifferential operator from $\Psi^m(M,E)$ with the holonomy invariant transversal principal symbol, and $k \in C_c^\infty(G)$, the operator $[P, R_E(k)]$ belongs to the class $\Psi^{m-1,-\infty}(M,\mathcal{F},E)$. Moreover, we have the following norm estimate:

Proposition 7. *Let P be a pseudodifferential operator from $\Psi^m(M,E)$ with the holonomy invariant transversal principal symbol. Then, for any $k \in \mathrm{OP}^l(G)$, the operator $[P, R_E(k)]$ defines a continuous map*

$$[P, R_E(k)] : H^{s,t}(M,\mathcal{F},E) \to H^{s-m+1,t-l-1}(M,\mathcal{F},E)$$

with the following norm estimate

$$\begin{aligned}\|[P, R_E(k)] \quad &: \quad H^{s,t}(M,\mathcal{F},E) \to H^{s-m+1,t-l-1}(M,\mathcal{F},E)\| \\ &\leq C \max(\|k\|_{s,t,t-l}, \|k\|_{s-m+1,t-1,t-l-1}).\end{aligned}$$

Proof. Let $p \in S^m(I^n, \mathcal{L}(\mathbb{C}^r))$ be the complete symbol of the operator P in some foliated chart with a complete asymptotic expansion $p \sim \sum_{j=0}^\infty p_{m-j}$, $p_{m-j}(x,y,\xi,\eta)$ is homogeneous in (ξ,η) of degree $m-j$. By the holonomy invariance assumption, we can choose a trivialization of the bundle E so that the transversal principal symbol in this coordinate system will be a matrix-valued function, independent of x:

$$p_m(x,y,0,\eta) = p_m(y,\eta).$$

Using the Taylor formula, we represent p_m in the form

$$p_m(x,y,\xi,\eta) = p_m(y,\eta) + \sum_{i=1}^p p_{m,i}(x,y,\xi,\eta)\xi_i,$$

where $p_{m,i}$ are homogeneous in (ξ,η) of degree $m-1$.

Let P_1 be an operator with the complete symbol $p_m(y,\eta)$, R_1 be an operator with the complete symbol $\sum_{i=1}^p p_{m,i}(x,y,\xi,\eta)\xi_i$. Gluing together these local operators into global ones in a standard way, we get a representation $P = P_1 + R_1$, where $P_1 \in \Psi^m(M,E)$, $[P_1, R_E(k)] \in \Psi^{-\infty}(M,E)$, $R_1 \in \Psi^{m-1,1}(M,\mathcal{F},E)$. So we have $[P, R_E(k)] = [R_1, R_E(k)] \bmod \Psi^{-\infty}(M,E)$, from where Proposition 7 is immediate.

1.5. Residue trace

Now we turn to a trace extension and a Wodzicki type residue for operators of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$. These results are particular cases of results [12, 14] on Fourier integral operators, but we make use of the local structure of operators in question given by (4) to derive directly all necessary facts

For any $\sigma \in C^\infty(\mathbb{R}^q\setminus\{0\})$, homogeneous of order q, i.e. $\sigma(\lambda\eta) = \lambda^d\sigma(\eta)$ for any $\eta \neq 0$ and $\lambda \in \mathbb{R}^*_+$, let

$$S(\sigma) = \int_{|\eta|=1} \mathrm{Tr}\ \sigma(\eta)d\eta.$$

For any function ϕ on $\mathbb{R}^q\setminus\{0\}$, let

$$\phi_\lambda(\eta) = \lambda^q\phi(\lambda\eta), \lambda > 0, \eta \in \mathbb{R}^q\setminus\{0\}.$$

Recall the following fact on continuation of a homogeneous smooth function on $\mathbb{R}^q\setminus\{0\}$ to a homogeneous distribution in $\mathbb{R}^q$, see [18], Theorems 3.2.3 and 3.2.4.

Lemma 2. *Let $\sigma \in C^\infty(\mathbb{R}^q\setminus\{0\})$ be homogeneous of order d in $\eta \in \mathbb{R}^q$.*

(1) If $d \notin \{-q-k : k \in \mathbb{N}\}$, σ extends to a homogeneous distribution τ on $\mathbb{R}^q$.

(2) If $d = -q-k$, there is an extension τ of σ, satisfying the condition

$$(\tau,\phi) = \lambda^{-q-k}(\tau,\phi_\lambda) + \log\lambda \sum_{|\alpha|=k} S(\eta^\alpha\sigma)\partial^\alpha_\eta\phi(0)/\alpha!, \alpha > 0.$$

In particular, the obstruction to an extension of $\sigma \in \mathcal{D}'(\mathbb{R}^q)$, homogeneous in η, is given by $S(\eta^\alpha\sigma)$, $|\alpha| = k$.

Let a functional L be given by the formula

$$L(\sigma) = (2\pi)^{-q}\int \mathrm{Tr}\ \sigma(\eta)d\eta,$$

which is well-defined on symbols $\sigma \in S^m_{cl}(\mathbb{R}^q)$ of order $m < -q$.

Lemma 3 ([22, 9]). *The functional L has an unique holomorphic extension $\tilde{L}$ to the space of classical symbols $S^z_{cl}(\mathbb{R}^q)$ of non-integral order z. The value of $\tilde{L}$ on a symbol $\sigma \sim \sum \sigma_{z-j}$ is given by*

$$\tilde{L}(\sigma) = (2\pi)^{-q}\int \mathrm{Tr}\ (\sigma - \sum_0^N \tau_{z-j})d\eta,$$

where τ_{z-j} is the unique homogeneous extension of σ_{z-j}, given by Lemma 2, $N \geq \mathrm{Re}\, z + q$.

The trace of a pseudodifferential operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ given by (3) with $m < -q$ is given by the formula

$$\mathrm{tr}\,(A) = (2\pi)^{-q} \int \mathrm{Tr}\, k(x,x,y,\eta)\,dx\,dy\,d\eta.$$

The following formula provides an extension of the trace to a pseudodifferential operator $A \in \Psi^{z,-\infty}(I^n, I^p, \mathbb{C}^r)$ of arbitrary non-integral order $z \in \mathbb{C}\backslash\mathbb{Z}$:

$$\mathrm{TR}\,(A) = \int \tilde{L}(k(x,x,y,\eta))\,dx\,dy.$$

This definition can be extended to elementary operators, and, by linearity, to all operators $P \in \Psi^{z,-\infty}(M, \mathcal{F}, E), z \in \mathbb{C}\backslash\mathbb{Z}$.

If an operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ is written in the form (4), then TR (A) is given by

$$\mathrm{TR}\,(A) = \int \mathrm{TR}\,(A(x,x))\,dx, \tag{18}$$

where TR $(A(x,x))$ denotes the extension of the usual trace of the pseudodifferential operator $A(x,x)$ on I^q, defined in [22, 23]. Using (18) and [22, 23], we immediately obtain the following proposition.

Proposition 8. *The linear functional* TR *on the class* $\Psi^{\alpha,-\infty}(M,\mathcal{F},E)$ *of classical pseudodifferential operators of orders* $\alpha \in m + \mathbb{Z}, m \in \mathbb{C}\backslash\mathbb{Z}$, *has the following properties:*

(1) It coincides with the usual trace tr *for* $\mathrm{Re}\,\alpha < -q$.

(2) It is a trace functional, i.e. TR $([A,B]) = 0$ *for any* $A \in \Psi^{\alpha_1,-\infty}_{cl}(M,\mathcal{F},E)$ *and* $B \in \Psi^{\alpha_2,-\infty}_{cl}(M,\mathcal{F},E)$, $\alpha_1 + \alpha_2 \in m + \mathbb{Z}$.

Now let us turn to the residue trace of operators from $\Psi^{m,-\infty}(M,\mathcal{F},E)$. As above, it suffices to define the residue trace for an elementary operator. Given an operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$, we define its residue form ρ_A as

$$\rho_A = \mathrm{Tr}\, k_{-q}(x,x,y,\eta)\,dx\,dy\,d\eta,$$

and the residue trace $\tau(A)$ as

$$\tau(A) = \int_{|\eta|=1} \mathrm{Tr}\, k_{-q}(x,x,y,\eta)\,dx\,dy\,d\eta.$$

If an operator $A \in \Psi^{m,-\infty}(I^n, I^p, \mathbb{C}^r)$ is written in the form (4), then its residue trace $\tau(A)$ is given by

$$\tau(A) = \int \tau(A(x,x))\,dx, \tag{19}$$

where $\tau(A(x,x))$ denotes the residue trace of the pseudodifferential operator $A(x,x)$ on I^q due to [13, 30].

Using (19), it can be easily checked that, for any $A \in \Psi^{m,-\infty}(M,\mathcal{F},E)$, its residue form ρ_A is an invariantly defined form on $\tilde{N}^*\mathcal{F}$, and the residue trace $\tau(A)$ is given by integration of the residue form ρ_A over the spherical conormal bundle $SN^*\mathcal{F} = \{\nu \in N^*\mathcal{F} : |\nu| = 1\}$:

$$\tau(A) = \int_{SN^*\mathcal{F}} \rho_A.$$

Remark 3. Let $A \in \Psi^{m,-\infty}(M,\mathcal{F},E)$, and $K_A \in I^{m-p/2}(M\times M, G'_{\mathcal{F}_N})$ be its Schwarz kernel. In terms of [12], the residue trace $\tau(A)$ of A is defined as the residue pairing of K_A with the delta function δ_Δ of the diagonal Δ in T^*M. The holonomy groupoid $G_{\mathcal{F}_N}$ and Δ are intersected cleanly in the $p+2q$-submanifold $G^{(0)}_{\mathcal{F}_N} = N^*\mathcal{F}$, and, therefore, this residue pairing is defined by [12, Theorem 4.1]. The main properties, concerning to the trace extension and the residue form for general Fourier integral operators, are due to [12, Theorem 2.1].

Now we relate the trace extension and the residue trace for operators of $\Psi^{m,-\infty}(M,\mathcal{F},E)$. At first, let us give a definition of a holomorphic family of pseudodifferential operators of class $\Psi^{*,-\infty}(M,\mathcal{F},E)$. As usual, it is sufficient to do this for elementary operators.

Definition 10. *We say that a family $A(z) \in \Psi^{f(z),-\infty}_{cl}(I^n, I^p, \mathbb{C}^r)$ is holomorphic (in a domain $D\subset\mathbb{C}$), if:*

(1) the order $f(z)$ is a holomorphic function;

(2) $A(z)$ is given by a classical symbol $k(z) \in S^{f(z),-\infty}(I^p\times I^p\times I^q\times\mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$, represented as an asymptotic sum

$$k(z,x,x',y,\eta) \sim \sum_{j=0}^{\infty} \theta(\eta) k_{f(z)-j}(z,x,x',y,\eta),$$

which is uniform in z, and the homogeneous components $k_{z-j}(z,x,x',y,\eta)$ are holomorphic in z.

Proposition 9. *For any holomorphic family $A(z) \in \Psi^{m+\operatorname{Re} z,-\infty}_{cl}(M,\mathcal{F},E)$, $z\in D\subset\mathbb{C}$, the function $z\mapsto \mathrm{TR}\,(A(z))$ is meromorphic with no more than simple poles at $z_k = -m-q+k \in D\bigcap\mathbb{Z}, k\geq 0$ and with*

$$\operatorname{res}_{z=z_k}\mathrm{TR}\,(A(z)) = \tau(A(z_k)).$$

Proof. The proposition is an immediate consequence of (19) and of the similar fact for usual pseudodifferential operators [22, 23].

2. Transversally elliptic operators

2.1. Definition and basic properties

As above, we assume that M is a closed foliated manifold, and E is a holonomy equivariant Hermitian vector bundle E on M.

Definition 11. *We say that an operator $P\in\Psi^m(M,E)$ is* **transversally elliptic,** *if the transversal principal symbol of P is invertible for any $\xi\in\tilde{N}^*\mathcal{F}$.*

Remark 4. The condition of transversal ellipticity implies that the transversal principal symbol of P is invertible for any ξ in some conic neighborhood of $N^*\mathcal{F}$. It also implies that, in any foliated coordinate system, there are $\varepsilon>0$ and $c>0$ such that

$$|(p_m(x,y,\xi,\eta)v,v)| \geq c(1+|\xi|+|\eta|)^m\|v\|, (x,y,\xi,\eta)\in U_\varepsilon, v\in\mathbb{C}^r,$$

where p_m is the homogeneous component of degree m in (ξ,η) of the complete symbol of P.

Performing the standard parametrix construction in the conic neighborhood U_ε of $N^*\mathcal{F}$ given by Remark 4, it is easy to get the following proposition (see also [24]).

Proposition 10. *For a transversally elliptic operator $P \in \Psi^m(M,E)$, there exists a parametrix, that is, an operator $Q \in \Psi^{-m}(M,E)$ such that*

$$PQ = I - R_1, QP = I - R_2, \tag{20}$$

where $R_j \in \Psi^0(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E), j=1,2$.

Existence of a parametrix implies a transverse elliptic regularity theorem in a usual manner:

Proposition 11. *Given a transversally elliptic operator $P \in \Psi^m(M,E)$, and a section u such that $u \in H^{s+m-N,k+N}(M,\mathcal{F},E)$ for some $N > 0$ and $Pu \in H^{s,k}(M,\mathcal{F},E)$, we have $u \in H^{s+m,k}(M,\mathcal{F},E)$ and*

$$||u||_{s+m,k} \leq C(||Pu||_{s,k} + ||u||_{s+m-N,k+N}).$$

2.2. Complex powers

Throughout in this section, we assume that an operator $A \in \Psi^m(M,E)$ satisfies the following conditions:

(T1) A is a transversally elliptic pseudodifferential operator with the positive transversal principal symbol;

(T2) A is essentially self-adjoint on the initial domain $C^\infty(M,E)$, and its closure is invertible and positive definite as an unbounded operator in the Hilbert space $L^2(M,E)$.

Remark 5. The assumption (T2) may be considered as an equivariance type condition, which is, usually, assumed for transversally elliptic operators.

Proposition 12. *Let $A \in \Psi^m(M,E)$ be as above. Then, for any $\lambda \notin \mathbb{R}_+$, the resolvent operator $(A-\lambda)^{-1}$ is represented as*

$$(A-\lambda)^{-1} = P(\lambda) + R_1(\lambda)(A-\lambda)^{-1}, (A-\lambda)^{-1} = P(\lambda) + (A-\lambda)^{-1}R_2(\lambda), \tag{21}$$

where:

(1) $P(\lambda) \in \Psi^{-m}(M,E)$ is an operator, which complete symbol in any foliated coordinate system is supported in some conical neighborhood U_ε of $N^*\mathcal{F}$ and satisfies the estimates

$$\begin{aligned} |D^\beta_{(x,y)}D^\alpha_{(\xi,\eta)}p(x,y,\xi,\eta,\lambda)| \quad &\leq \quad C_{\alpha\beta}(1+|\xi|+|\eta|+|\lambda|^{1/m})^{-m} \\ &(1+|\xi|+|\eta|)^{-|\alpha|}, (x,y,\xi,\eta) \in U_\varepsilon, \lambda \in \Lambda_\delta \end{aligned}$$

for any $\delta > 0$ and for any multi-indices α and β;

(2) $R_j(\lambda) \in \Psi^0(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E), j=1,2$, with the complete symbol $r_j(\lambda)$, satisfying the following estimates:

$$|D^{\beta}_{(x,y)}D^{\alpha}_{(\xi,\eta)}r_j(x,y,\xi,\eta,\lambda)| \leq C_{\alpha\beta N}(1+|\xi|+|\eta|+|\lambda|^{1/m})^{-m}$$
$$(1+|\xi|+|\eta|)^{-|\alpha|+m-N},(x,y,\xi,\eta)\in U_{\varepsilon_1};$$
$$|D^{\beta}_{(x,y)}D^{\alpha}_{(\xi,\eta)}r_j(x,y,\xi,\eta,\lambda)| \leq C_{\alpha\beta}(1+|\xi|+|\eta|)^{-|\alpha|},$$
$$(x,y,\xi,\eta)\in I^p\times I^q\times\mathbb{R}^p\times\mathbb{R}^q,\lambda\in\Lambda_\delta \quad (22)$$

for any $\delta>0$, for any natural N and for any multi-indices α and β.

Moreover, the principal symbol $p_{-m}(\lambda)$ of $P(\lambda)$ is equal to $(a_m-\lambda)^{-1}$ in some conic neighborhood of $N^*\mathcal{F}$ with a_m, being the principal symbol of A.

Proof. We will prove the proposition, performing the standard construction due to Seeley of a parametrix $P(\lambda)$ for the operator $A-\lambda$ as an operator with a parameter in some conic neighborhood of $N^*\mathcal{F}$.

Denote by Λ_δ the angle in the complex plane:

$$\Lambda_\delta=\{\lambda\in\mathbb{C}: |\arg\lambda|>\delta\}.$$

Fix some foliated coordinate system. Let $a\sim\sum_{j=0}^{\infty}a_j$ be an asymptotic expansion of the complete symbol of the operator A in this system. By (T1), there is a $\varepsilon>0$ such that

$$a_m(x,y,\xi,\eta)\geq C(1+|\xi|+|\eta|)^m,(x,y,\xi,\eta)\in U_\varepsilon.$$

For any $\delta>0$, define functions $p_{-m-l}(\lambda),\lambda\in\Lambda_\delta, l=0,1,\ldots,$ in U_ε by the following system

$$(a_m-\lambda)p_{-m}=1,$$
$$(a_m-\lambda)p_{-m-l}+\sum_{j<l,j+k+|\alpha|=l}\partial^{\alpha}_{\xi}b_{-m-j}D^{\alpha}_{x}a_{m-k}/\alpha!=0,\quad l>0.$$

It can be easily checked that the functions $p_{-m-l}(\lambda)$ satisfy the following estimates

$$|D^{\beta}_{(x,y)}D^{\alpha}_{(\xi,\eta)}p_{-m-l}(x,y,\xi,\eta,\lambda)| \leq C_{\alpha\beta}(1+|\xi|+|\eta|+|\lambda|^{1/m})^{-m}$$
$$(1+|\xi|+|\eta|)^{-|\alpha|},(x,y,\xi,\eta)\in U_\varepsilon,\lambda\in\Lambda_\delta,$$

where α and β are any multi-indices. Take p as an asymptotic sum of symbols with a parameter: $p\sim\sum_{j=0}^{+\infty}p_{-m-j}$. Then p satisfies the estimates

$$|D^{\beta}_{(x,y)}D^{\alpha}_{(\xi,\eta)}p(x,y,\xi,\eta,\lambda)| \leq C(1+|\xi|+|\eta|+|\lambda|^{1/m})^{-m}(1+|\xi|+|\eta|)^{-|\alpha|},$$
$$(x,y,\xi,\eta)\in U_\varepsilon,\lambda\in\Lambda_\delta. \quad (23)$$

Let $\theta_0\in C^\infty_c(\mathbb{R})$ such that $\operatorname{supp}\theta_0\subset(-\varepsilon,\varepsilon)$, $\theta_0(\tau)=1$ for any $\tau\in(-\varepsilon_1,\varepsilon_1)$ with some $\varepsilon_1<\varepsilon$, and $\theta\in C^\infty(\mathbb{R}^p\times\mathbb{R}^q)$ be given by $\theta(\xi,\eta)=\theta_0(\eta/\xi)$, if $|\eta|<\varepsilon|\xi|$, and $\theta(\xi,\eta)=0$ in the opposite case. Let us take a covering of M by foliation charts, construct in any foliation patch of this covering an operator with the complete symbol $\theta p(\lambda)$, and glue these local operators in a global operator $P(\lambda)\in\Psi^{-m}(M,E),\lambda\in\Lambda_\delta$ by means of a partition of unity. It can be easily seen that

$$P(\lambda)(A-\lambda) = I - R_1(\lambda), (A-\lambda)P(\lambda) = I - R_2(\lambda), \lambda \in \Lambda_\delta, \quad (24)$$

where $R_j(\lambda) \in \Psi^0(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E), j = 1,2$, has the complete symbol $r_j(\lambda)$, satisfying the estimates (22). By (T2), the operator $A-\lambda$ is invertible as an unbounded operator in $L^2(M,E)$ for all $\lambda \notin \mathbb{R}_+$ with the following estimate for the norm of its inverse:

$$\|(A-\lambda)^{-1}\| \leq C/|\lambda|.$$

Using (24), we get the representation (21) for the resolvent, that completes the proof.

Proposition 13. *Let $A \in \Psi^m(M,E)$ be as above. Then, for any $\lambda \notin \mathbb{R}_+$, the resolvent operator $(A-\lambda)^{-1}$ can be represented as*

$$(A-\lambda)^{-1} = P(\lambda) + T(\lambda), \quad (25)$$

where:

(1) $P(\lambda) \in \Psi^{-m}(M,E)$ satisfies the following norm estimates:

$$\|P(\lambda): H^{s,k}(M,\mathcal{F},E) \to H^{s,k}(M,\mathcal{F},E)\| \leq C_{s,k,\delta}(1+|\lambda|)^{-1}, \lambda \in \Lambda_\delta,$$

$$\|P(\lambda): H^{s,k}(M,\mathcal{F},E) \to H^{s+m,k}(M,\mathcal{F},E)\| \leq C_{s,k,\delta}, \lambda \in \Lambda_\delta,$$

for any $s \in \mathbb{R}$, $k \in \mathbb{R}$ and $\delta > 0$;

(2) $T(\lambda)$ satisfies the following norm estimates:

$$\|T(\lambda): H^{t,-t}(M,\mathcal{F},E) \to H^{s,-s}(M,\mathcal{F},E)\| \leq C_{s,t,\delta}(1+|\lambda|)^{-1}, \lambda \in \Lambda_\delta,$$

for any s,t and $\delta > 0$.

Proof. By (21), $(A-\lambda)^{-1} = P(\lambda) + R_1(\lambda)P(\lambda) + R_1(\lambda)(A-\lambda)^{-1}R_2(\lambda)$, so we get (25) with

$$T(\lambda) = R_1(\lambda)P(\lambda) + R_1(\lambda)(A-\lambda)^{-1}R_2(\lambda). \quad (26)$$

Since $R_j(\lambda) \in \Psi^0(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E), j = 1,2$, by (17), R_j defines a continuous map from $H^{t,-t}(M,\mathcal{F},E)$ to $H^{s,-s}(M,\mathcal{F},E)$ for any s and t, that implies the same is true for $T(\lambda)$. The desired norm estimates for operators $P(\lambda)$ and $T(\lambda)$ follow immediately from the symbol estimates (23) and (22).

Now we turn to a construction of complex powers A^z for a transversally elliptic operator $A \in \Psi^m(M,E)$, satisfying to the conditions (T1) and (T2).

Let Γ be a contour in the complex plane of the form $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $\lambda = re^{i\alpha}, +\infty > r > \rho$, on Γ_1, $\lambda = \rho e^{i\phi}, \pi > \phi > -\pi$, on Γ_2, $\lambda = re^{-i\alpha}, \rho < r < +\infty$, on Γ_3, where $\alpha \in (0,\pi)$ is arbitrary, and the constant $\rho > 0$ is chosen in such a way that the disk of the radius ρ, centered at the origin, isn't contained in $\sigma(D)$.

A bounded operator A^z, $\operatorname{Re} z < 0$, in $L^2(M,E)$ is defined by the formula

$$A^z = \frac{i}{2\pi}\int_\Gamma \lambda^z (A-\lambda)^{-1} d\lambda,$$

where a branch of the analytic function λ^z is chosen so that $\lambda^z = e^{z\ln\lambda}$ for $\lambda > 0$. This definition is extended to all z by

$$A^z = A^{z-k}A^k \quad (27)$$

for any z, $\operatorname{Re} z < k$, where k is natural and A^k is the usual power of the operator A. The following proposition provides a descripiton of the complex powers A^z.

Proposition 14. *Under the current hypotheses on the operator A, the operator A^z has the form*

$$A^z = P(z) + T(z), \tag{28}$$

where $P(z)$ is a holomorphic family of pseudodifferential operators of class $\Psi^{mz}(M,E)$ with the principal symbol $p_{m\mathrm{Re}\, z}(z)$, being equal to $(a_m)^z$ in some conic neighborhood of $N^*\mathcal{F}$ (a_m is the principal symbol of A), and, for any s,l and z, $\mathrm{Re}\ z \le k$, $T(z)$ defines a continuous mapping

$$\begin{aligned} T(z) : H^{s,-s}(M,\mathcal{F},E) &\to H^{l,-l-k}(M,\mathcal{F},E), k > 0, \\ T(z) : H^{s,-s}(M,\mathcal{F},E) &\to H^{l,-l}(M,\mathcal{F},E), k \le 0. \end{aligned}$$

Proof. A proof of the proposition can be obtained by a straighforward repetition of the proof of [24, Proposition 7.3]. Namely, let us write A^z for $\mathrm{Re}\ z < 0$ as $A^z = P(z) + T(z)$, where

$$P(z) = \frac{i}{2\pi}\int_\Gamma \lambda^z P(\lambda) d\lambda, T(z) = \frac{i}{2\pi}\int_\Gamma \lambda^z T(\lambda) d\lambda,$$

$P(\lambda)$ and $T(\lambda)$ are given by Proposition 13. All the statements about $P(z)$ and $T(z)$ can be easily checked in a standard way (see [24] for more details).

Remark 6. We can get $P(z)$ to be an elliptic operator of class $\Psi^{m\mathrm{Re}\, z}(M,E)$ with the positive principal symbol, adding to $P(z)$ an appropriate operator of class $\Psi^{m\mathrm{Re}\, z}(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E)$,.

2.3. G-trace

Before going to the distributional zeta-function of transversally elliptic operators, we introduce a general scheme of defining distributional spectral invariants for transversally elliptic operators based on the notion of the G-trace and prove some existence results for such invariants, that may have its own interest.

Definition 12. *We say that a bounded operator T in $L^2(M,E)$ is* **an operator of G-trace class**, *if, for any $k \in C_c^\infty(G)$, the operator $R_E(k)T$ is a trace class operator, and a functional $\mathrm{tr}_G\ (T)$ on $C_c^\infty(G)$, defined by the formula*

$$(\mathrm{tr}_G\ (T), k) = \mathrm{tr}\ R_E(k)T, k \in C_c^\infty(G),$$

is a distribution on G. In this case, the distribution $\mathrm{tr}_G\ (T) \in \mathcal{D}'(G)$ is called **the G-trace** *of the operator T.*

For any integral operator T on $C^\infty(M,E)$ with the smooth kernel K_T, its G-trace $\mathrm{tr}_G\ (T)$ is a smooth function on the holonomy groupoid G, given by the formula

$$\mathrm{tr}_G\ (T)(\gamma) = K_T(r(\gamma), s(\gamma)), \gamma \in G.$$

Otherwise speaking, the G-trace $\mathrm{tr}_G\ (T)$ is obtained by pulling back of the integral kernel K_T via the map $(r,s) : G \to M \times M$.

By Propositions 3 and 4, we immediately obtain that any $P \in \Psi^{m,\mu}(M,\mathcal{F},E)$ with $m < -q$ is an operator of G-trace class. Indeed, the following, more general proposition is valid.

Proposition 15. *Let T be a bounded operator in $C^\infty(M,E)$, which extends to a bounded operator from $L^2(M,E)$ to $H^{-m,-\mu}(M,\mathcal{F},E)$ with some $m<-q$ and μ. Then T is an operator of G-trace class with the following estimate of its G-trace:*

$$|(\mathrm{tr}_G\ (T),k)|\le C_\varepsilon \|k\|_{q+\varepsilon,-m-\mu-q-\varepsilon,p+\varepsilon_1},$$

where $\varepsilon>0$ and $\varepsilon_1>0$ are arbitrary constants such that $s>q+\varepsilon$.

Proof. Let us fix some $\varepsilon>0$ and $\varepsilon_1>0$ such that $m<-q-\varepsilon$. By Proposition 3, we have

$$|\mathrm{tr}\ R_E(k)P|\le C\|R_E(k)P:L^2(M,E)\to H^{q+\varepsilon,p+\varepsilon_1}(M,\mathcal{F},E)\|.$$

Using the embedding $H^{-m,-\mu}(M,\mathcal{F},E)\subset H^{q+\varepsilon,-m-\mu-q-\varepsilon}(M,\mathcal{F},E)$, we get

$$\begin{aligned}\|R_E(k)P\ &:\ L^2(M,E)\to H^{q+\varepsilon,p+\varepsilon_1}(M,\mathcal{F},E)\|\\ &\le\ \|R_E(k):H^{q+\varepsilon,-m-\mu-q-\varepsilon}(M,\mathcal{F},E)\to H^{q+\varepsilon,p+\varepsilon_1}(M,\mathcal{F},E)\|\\ &\quad\ \|P:L^2(M,E)\to H^{-m,-\mu}(M,\mathcal{F},E)\|,\end{aligned}$$

that completes the proof.

Let $A\in\Psi^m(M,E)$ be a transversally elliptic operator. We assume that A considered as an unbounded operator in $L^2(M,E)$ with the domain $C^\infty(M,E)$ is essentially self-adjoint. For any mesurable, bounded function f on $\mathbb{R}$, the bounded operator $f(A)$ in $L^2(M,E)$ is defined via the spectral theorem.

Proposition 16. *For any mesurable, bounded function f on $\mathbb{R}$ such that*

$$|f(\lambda)|\le C(1+|\lambda|)^{-l},\lambda\in\mathbb{R},$$

with some $C>0$ and $l>q/m$, the operator $f(A)$ is an operator of G-trace class with the following estimate for its G-trace functional:

$$|(\mathrm{tr}_G\ (f(A)),k)|\le C_\varepsilon\|k\|_{ml,-ml,n+\varepsilon-ml},k\in C^\infty_c(G).$$

for any $\varepsilon>0$.

Proof. By (28) and Remark 6, we have

$$A^l=P(l)+T(l),\qquad(29)$$

where $P(l)$ is an elliptic operator of class $\Psi^{ml}(M,E)$ with the positive principal symbol, and, for any real s, $T(l)$ defines a continuous mapping

$$T(l):L^2(M,E)\to H^{s,-s-ml}(M,\mathcal{F},E).$$

Let $Q\in\Psi^{-ml}(M,E)$ be a parametrix for P, that is,

$$QP(l)=I-K,K\in\Psi^{-\infty}(M,E).\qquad(30)$$

Let $B\in\Psi^{0,-\infty}(M,\mathcal{F},E)$ and $u\in C^\infty(M,E)$. By (30), we have the estimate

$$\|Bf(A)u\|_{ml,t}\le\|BQP(l)f(A)u\|_{ml,t}+\|BKf(A)u\|_{ml,t}$$

The second term in the right-hand side of the last estimate can be estimated as follows:

$$||BKf(A)u||_{ml,t} \leq C||B||_{ml,t,t} \sup |f(\lambda)| \, ||u||.$$

Let us turn to the first term. By (29), we have

$$||BQP(l)f(A)u||_{ml,t} \leq ||BQA^l f(A)u||_{ml,t} + ||BQT(l)f(A)u||_{ml,t}.$$

Finally, the terms in the right-hand side of the last estimate can be estimated as follows:

$$\begin{aligned} ||BQA^l f(A)u||_{ml,t} &\leq C||B||_{ml,0,t} \sup |(1+|\lambda|)^l f(\lambda)| \, ||u||, \\ ||BQT(l)f(A)u||_{ml,t} &\leq C||B||_{ml,-ml,t} \sup |f(\lambda)| \, ||u||. \end{aligned}$$

Taking $t = n + \varepsilon - ml$ and applying Proposition 3, we complete the proof.

Remark 7. Using Proposition 16, we can define distributional spectral invariants of transversally elliptic operators like as a spectrum distribution function, a zeta function etc. We refer the reader to [24] for analogous results for transversally elliptic operators on manifolds equipped with a smooth action of a (noncompact) Lie group.

2.4. Zeta-function

As in Section 2.2, we assume that $A \in \Psi^m(M, E)$ is a transversally elliptic classical pseudodifferential operator with the positive transversal principal symbol, which is essentially self-adjoint, invertible and positive definite in the Hilbert space $L^2(M, E)$ (see (T1) and (T2) above). By Proposition 16, for any Re $z > q/m$, the operator A^{-z} is an operator of G-trace, and **the distributional zeta-function** of the operator A is defined as follows:

$$\zeta_A(z) = \mathrm{tr}_G \, (A^{-z}), \ \mathrm{Re}\, z > q/m. \tag{31}$$

Moreover, Proposition 16 provides the following estimate for the distributional zeta-function with any $\varepsilon > 0$:

$$|(\zeta_A(z), k)| \leq C_\epsilon ||k||_{q,-q,p+\epsilon}, k \in C_c^\infty(G), \mathrm{Re}\, z > q/m. \tag{32}$$

Now we turn to the problem of meromorphic continuation of $\zeta_A(z)$. Actually, we consider a little bit more general situation.

Theorem 3. *Let A as above and $Q \in \Psi^{l,-\infty}(M, \mathcal{F}, E)$, $l \in \mathbb{Z}$. Then the function $z \mapsto$ Tr (QA^{-z}) is holomorphic for* Re $z > l + q/m$ *and admits a (unique) meromorphic continuation to $\mathbb{C}$ with at most simple poles at points $z_k = k/m$ with integer $k \leq l + q$. Its residue at $z = z_k$ is given by*

$$\mathrm{res}_{\,z=z_k} \, \mathrm{tr}\, (QA^{-z}) = q\tau(QA^{-k/m}).$$

Proof. Using (28), we construct a meromorphic continuation of tr (QA^{-z}) as follows:

$$\mathrm{tr}\, (QA^{-z}) = \mathrm{TR}\, (QP(z)) + \mathrm{tr}\, (QT(z)).$$

Here $QP(z)$ is a holomorphic family of operators of class $\Psi^{mz+l,-\infty}(M, \mathcal{F}, E)$ and the meromorphic extension of its trace is given by TR $(QP(z))$ due to Proposition 9. Further, $QT(z)$ defines a continuous mapping $QT(z) : L^2(M, E) \to H^s(M, E)$ for any s. Therefore, by Proposition 3, the operator $QT(z)$ is of trace class for any $z \in \mathbb{C}$ with tr $(QT(z))$, being an entire function of z.

As a corollary, we have the following result on a meromorphic continuation of the zeta-function, considered as a distribution on G.

Theorem 4. *For any $k \in C_c^\infty(G)$, the zeta function $(\zeta_A(z), k)$ of the operator A extends to a meromorphic function on the complex plane with simple poles at points $z = q/m, (q-1)/m, \dots$.*

For future references, we state the following theorem, which can be proved in the same manner (see [23, Proposition 4.2]).

Theorem 5. *Let A as above and $Q_j \in \Psi^{l_j,-\infty}(M,\mathcal{F},E)$, $l_j \in \mathbb{Z}$, $j = 1,\dots,N$. Then the function*

$$\zeta(z_1,\dots,z_N) = \mathrm{tr}\,(Q_1 A^{-z_1} \dots Q_N A^{-z_N}), (z_1,\dots,z_N) \in \mathbb{C}^N$$

admits a (unique) meromorphic continuation to $\mathbb{C}^N$ with at most simple poles on the hyperplanes $\sum_{j=1}^N l_j - m\sum_{j=1}^N z_j = k \in \mathbb{Z}, k \geq -q$. Its residue at this hyperplane is given by

$$\mathrm{res}\,\zeta(z_1,\dots,z_N) = q\tau(Q_1 A^{-z_1} \dots Q_N A^{-z_N}).$$

3. Spectral triples of Riemannian foliations

3.1. Proof of main theorems

In this section, we complete proofs of our main theorems. Recall [9, 8] that a spectral triple is a triple $(\mathcal{A},\mathcal{H},D)$, where:

1. $\mathcal{A}$ is an involutive algebra;
2. $\mathcal{H}$ is a Hilbert space equipped with a $*$-representation of the algebra $\mathcal{A}$;
3. D is a (unbounded) selfadjoint operator in $\mathcal{H}$ such that

   1. for any $a \in \mathcal{A}$, the operator $a(D-i)^{-1}$ is a compact operator in $\mathcal{H}$;

   2. D almost commutes with any $a \in \mathcal{A}$ in a sense that $[D,a]$ is bounded for any $a \in \mathcal{A}$.

One of the basic geometrical examples of spectral triples is given by a triple $(A,\mathcal{H},D)$, associated with a compact Riemannian manifold M:

1. The involutive algebra $\mathcal{A}$ is the algebra $C^\infty(M)$ of smooth functions on M;
2. The Hilbert space $\mathcal{H}$ is the L^2 space $L^2(M,\Lambda^*M)$ of differential forms on M, on which the algebra $\mathcal{A}$ acts by multiplication;
3. The operator D is the signature operator $d + d^*$.

In this section, we will consider spectral triples $(\mathcal{A},\mathcal{H},D)$ associated with a compact foliated manifold $(M,\mathcal{F})$:

1. The involutive algebra $\mathcal{A}$ is the algebra $C_c^\infty(G)$;
2. The Hilbert space $\mathcal{H}$ is the space $L^2(M,E)$ of L^2-sections of a holonomy equivariant Hermitian vector bundle E, on which an element k of the algebra $\mathcal{A}$ is represented via the $*$-representation R_E;
3. The operator D is a first order self-adjoint transversally elliptic operator with the holonomy invariant transversal principal symbol such that the operator D^2 is self-adjoint and has the scalar principal symbol.

The following theorem is Theorem 1 of Introduction.

Theorem 6. *Let $(M,\mathcal{F})$ be a closed foliated manifold. Then a spectral triple $(A,\mathcal{H},D)$ as above is a finite-dimensional spectral triple, that is:*

1. for any $k\in C_c^{-\infty}(G)$, the operator $R_E(k)(D-i)^{-1}$ is a compact operator in $L^2(M,E)$;

2. for any $k\in C_c^{-\infty}(G)$, $[D,R_E(k)]$ is bounded in $L^2(M,E)$.

Proof. The following proposition applied to $A=D-i$ implies, in particular, the first part of this theorem.

Proposition 17. *Let $A\in\Psi^1(M,E)$ be a transversally elliptic operator, invertible in the Hilbert space $L^2(M,E)$. Then, for any $k\in \mathrm{OP}^{-n/q}(\mathcal{F})$, the operator $R_E(k)A^{-1}$ defines a continuous map from $L^2(M,E)$ to $H^{1,p/q}(M,\mathcal{F},E)$. In particular, the operator $R_E(k)A^{-1}$ is a compact operator in $L^2(M,E)$.*

Proof. By (20), we have the following representation:

$$R_E(k)A^{-1}=R_E(k)P+R_E(k)R_2A^{-1}, \tag{33}$$

where $P\in\Psi^{-1}(M,E)$ and $R_2\in\Psi^0(M,E)\bigcap\Psi^{-\infty}(N^*\mathcal{F},E)$. Since $k\in \mathrm{OP}^{-n/q}(\mathcal{F})$, by Proposition 4, the first term, $R_E(k)P$, in the right-hand side of (33) defines a continuous mapping from $H^{s,t}(M,\mathcal{F},E)$ to $H^{s+1,t+n/q}(M,\mathcal{F},E)$ for any s and t. By (17), the operator $R_E(k)R_2A^{-1}$ defines a continuous mapping from $L^2(M,E)$ to $H^{N,n/q-N}(M,\mathcal{F},E)$ for any N. So we get that the operator $R_E(k)A^{-1}$ defines a continuous map from $L^2(M,E)$ to $H^{1,p/q}(M,\mathcal{F},E)$.

The second part of this theorem, concerning to boundedness of commutators $[D,R_E(k)]$, follows from Proposition 7.

Remark 8. By Proposition 17 and Proposition 3, it is easy to see that, for any $k\in C_c^{\infty}(G)$, the operator $R_E(k)(D-i)^{-1}$ belongs to the Schatten ideal $\mathcal{L}^{q+\varepsilon}(L^2(M,E))$ for any $\varepsilon>0$, therefore, the spectral triple in question has the finite dimension d, which is equal to q.

Now we turn to a description of the dimension spectrum for the spectral triples under consideration. First, recall briefly the definition of the dimension spectrum [9, 8]. Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple. Denote by δ an (unbounded) derivation on the algebra $\mathcal{L}(\mathcal{H})$ of all linear operators in $\mathcal{H}$, given by the formula

$$\delta(T)=[|D|,T], T\in\mathcal{L}(\mathcal{H}). \tag{34}$$

Assume that, for any $a\in\mathcal{A}$,

$$a\in\bigcap_{n>0}\mathrm{Dom}\,\delta^n, [D,a]\in\bigcap_{n>0}\mathrm{Dom}\,\delta^n, \tag{35}$$

and denote by $\mathcal{B}$ the algebra generated by the elements $\delta^n(a), a\in\mathcal{A}, n\in\mathbb{N}$. Then the operator $b|D|^{-z}$ is of trace class for $Re\ z>d$, where d is the top spectrum dimension and $b\in\mathcal{B}$, and we can define the distributional zeta function $\zeta_b(z)$ of the operator $|D|$ by the formula

$$\zeta_b(z)=\mathrm{tr}\,(b|D|^{-z}), b\in\mathcal{B}, \mathrm{Re}\, z>d.$$

Definition 13. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has* **discrete dimension spectrum** $\mathrm{Sd} \subset \mathbb{C}$, *if* Sd *is a discrete subset in* $\mathbb{C}$, *the triple satisfies the assumptions (35), and, for any $b \in \mathcal{B}$, the distributional zeta function $\zeta_b(z)$ extends holomorphically to* $\mathbb{C} \backslash \mathrm{Sd}$ *such that $\Gamma(z)\zeta_b(z)$ is of rapid decay on vertical lines $z = s + it$, for any s with* $\mathrm{Re}\, s > 0$.

The dimension spectrum is said to be **simple**, *if the singularities of the function $\zeta_b(z)$ at $z \in \mathrm{Sd}$ are at most simple poles.*

From now on, let D be a first order transversally elliptic operator with the holonomy invariant transversal principal symbol such that the operator D^2 has the scalar principal symbol, and D and D^2 are self-adjoint. We start with a description of the domain of the derivative $\delta : C_c^\infty(G) \to \mathcal{L}(L^2(M, E))$, given by (34). To do this, we slightly refine the description of the operator $|D|$, given by Proposition 14.

Lemma 4. *Under the current assumptions, the operator $|D|$ has the form*

$$|D| = P + T, \tag{36}$$

where P is a pseudodifferential operator of class $\Psi^1(M, E)$ with the scalar principal symbol and the holonomy invariant transversal principal symbol, and T defines a continuous mapping

$$T : H^{s,k}(M, \mathcal{F}, E) \to H^{s+t,k-t-1}(M, \mathcal{F}, E) \tag{37}$$

for any s, k and t.

Proof. By Proposition 14, the operator $|D| = (D^2)^{1/2}$ can be represented in the form (36) with P and T, satisfying almost all the conditions stated in the lemma. It only remains to prove (37) for any s, k and t, because, by Proposition 14, we know it is true for any s, k with $s + k = 0$. Recall that T is given by

$$T = \frac{i}{2\pi} \int_\Gamma \lambda^{-1/2} D^2 T(\lambda) d\lambda,$$

and $T(\lambda)$ is given by (26), therefore, it is easy to see that, in order to prove (37), it suffices to state the standard resolvent estimate in any Sobolev space:

$$\|(D^2 - \lambda)^{-1} : H^s(M, E) \to H^s(M, E)\| \leq C/|\lambda| \tag{38}$$

for any real s and $\lambda \in \Lambda_\delta$ with $|\lambda|$ large enough ($\delta \in (0, \pi)$ is arbitrary).

Recall that the estimate (38) for $s = 0$ is a direct consequence of self-adjointness of D^2. Let $\Lambda_s = (I + \Delta_M)^{s/2}$. Then $\Lambda_s D \Lambda_{-s} = D + B_s$, where B_s is a bounded operator in $L^2(M, E)$, and $\Lambda_s D^2 \Lambda_{-s} = D^2 + DB_s + B_s D + B_s^2$. It can be easily checked that, for any $\varepsilon > 0$, we have the estimate

$$((DB_s + B_s D + B_s^2)u, u) \leq \varepsilon \|Du\|^2 + C_\varepsilon \|u\|^2, u \in C^\infty(M, E),$$

which implies the estimate (38) for any s due to well-known facts of the perturbation theory of linear operators (see, for instance, [21]).

Proposition 18. *Under the current assumptions, the operator $\delta(K) = [|D|, K]$ is an operator of class $\Psi^{0,-\infty}(M, \mathcal{F}, E)$ for any $K \in \Psi^{0,-\infty}(M, \mathcal{F}, E)$.*

Proof. By (36), $\delta(K) = [P, K] + [T, K]$, where P and T are as in Lemma 4. Since P has the scalar principal symbol and the holonomy invariant transversal principal symbol, Proposition 5 implies that the operator $[P, K]$ belongs to the class $\Psi^{0,-\infty}(M, \mathcal{F}, E)$. By (37), the operator $[T, K]$ is a smoothing operator in the scale $H^{s,k}(M, \mathcal{F}, E)$, and, therefore, belongs to $\Psi^{-\infty}(M, E)$.

By Proposition 18, we get immediately the following characterization of the domain of the derivative δ.

Proposition 19. *Given an operator D as above, the class $\Psi^{0,-\infty}(M, \mathcal{F}, E) + \mathrm{OP}^{-1}(\mathcal{F})$ is contained in the domain of the derivative δ.*

For further references, we note the following, slightly more general assertion of the domain of the derivative δ, which follows from the estimates of Proposition 7.

Proposition 20. *Given a first order transversally elliptic operator D with the holonomy invariant transversal principal symbol, the class $\mathrm{OP}^{-1}(\mathcal{F})$ is contained in the domain of δ.*

Now we are ready to prove the second main result of the paper, Theorem 2 of Introduction.

Theorem 7. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as in Theorem 6 has discrete spectrum dimension* Sd, *which is contained in the set $\{v \in \mathbb{N} : v \leq q\}$ and simple.*

Proof. First of all, we have to verify (35). Given $\mathcal{A}$, being the algebra $C^\infty_c(G)$, by Propositions 18 and 19, all the assumptions on $\mathcal{A}$ formulated in (35) are satisfied, that is, for any $k \in C^\infty_c(G)$, $R_E(k) \in \bigcap_{n>0} \mathrm{Dom}\, \delta^n, [D, R_E(k)] \in \Psi^{0,-\infty}(M, \mathcal{F}, E) \subset \bigcap_{n>0} \mathrm{Dom}\, \delta^n$. Moreover, the algebra $\mathcal{B}$, generated by the elements $\delta^n(R_E(k)), k \in \mathcal{A}, n \in \mathbb{N}$, is contained in $\Psi^{0,-\infty}(M, \mathcal{F}, E)$, The rest of the proof follows immediately from Theorem 3.

3.2. Geometric example

In this section, we discuss an example of a spectral triple given by the transverse signature operator on a Riemannian foliation.

Let $(M, \mathcal{F})$ be a Riemannian foliation equipped with a bundle-like metric g_M. Let $F = T\mathcal{F}$ be the tangent bundle to $\mathcal{F}$, and $H = F^\perp$ be the orthogonal complement to F. So we have a decomposition of TM into a direct sum

$$TM = F \oplus H. \tag{39}$$

The de Rham differential d inherits the decomposition (39) in the form

$$d = d_F + d_H + \theta.$$

Here the tangential de Rham differential d_F and the transversal de Rham differential d_H are first order differential operators, and θ is zeroth order. Moreover, the operator d_F doesn't depend on a choice of g_M (see, for instance, [29]).

The conormal bundle $N^*\mathcal{F}$ has a leafwise flat connection (the Bott connection) defined by the lifted foliation $\mathcal{F}_N$. The parallel transport along leafwise paths with respect to this connection defines a representation of the holonomy groupoid G in fibres of $N^*\mathcal{F}$. Since $H^* \cong N^*\mathcal{F}$, the bundles H^* and $\Lambda^* H^*$ are holonomy equivariant.

Define a triple $(\mathcal{A}, \mathcal{H}, D)$ to be given by the space $\mathcal{H} = L^2(M, \Lambda^* H^*)$ of the transversal differential forms equipped with the action of the algebra $\mathcal{A} = C_c^\infty(G)$ and by the transverse signature operator $D = d_H + d_H^*$.

Let us check that this spectral triple satisfy all the assumptions of Section 3.1. Under the isomorphism $H^* \cong N^*\mathcal{F}$, the transversal principal symbol, $\sigma_D(\eta) \in \mathcal{L}(\Lambda^* N^* \mathcal{F}$, of D is given by the formula

$$\sigma_D(\eta) = e_\eta + i_\eta, \ \eta \in \tilde{N}^*\mathcal{F}$$

(e_η and i_η are the exterior and the interior multiplications by η accordingly), from where one can easily see holonomy invariance of σ_D. We also have

$$\sigma_{D^2}(\eta) = |\eta|^2 I_{\pi(\eta)}, \ \eta \in \tilde{N}^*\mathcal{F}.$$

Finally, essential self-adjointness of D and D^2 follows from the finite propagation speed arguments of [4].

3.3. Concluding remarks

1. Using the well-known relationship between the zeta-function and the heat trace, we can derive from Theorem 4 the following fact on heat trace asymptotics for transversally elliptic operators, which was used in [26].

Proposition 21. *Let $(M, \mathcal{F})$ be a compact foliated manifold and E be an Hermitian vector bundle on M. Given a differential operator $P \in \Psi^m(M, E), m > 0$, with the positive transversal principal symbol, which is self-adjoint and positive in $L^2(M, E)$, we have an asymptotic expansion of the G-trace of the parabolic semigroup operator e^{-tP}:*

$$(\mathrm{tr}_G\,(e^{-tP}), k) = \mathrm{tr}\ R_E(k) e^{-tP}, k \in C_c^\infty(G),$$

given by the formula

$$\mathrm{tr}_G\,(e^{-tP}) \sim \sum_{l=0}^{\infty} a_l t^{-q/m + l/m}, t \to 0+,$$

where $a_l, l = 0, 1, \ldots$ are distributions on G with a_0, given by the formula:

$$(a_0, k) = \int_M \Big(\int_{N_x^*\mathcal{F}} \mathrm{Tr}\, e^{-\sigma_P(\eta)} k(x) d\nu(\eta) \Big) dx, k \in C_c^\infty(G). \tag{40}$$

2. There are two observations, based on the explicit formula (40) for the leading coefficient in the heat trace expansion, or, equivalently, for the residue of the distributional zeta-function at the point $z = -q/m$, which have an interesting interpretation in terms of noncommutative spectral geometry.

Assume that $(M, \mathcal{F})$ is a compact Riemannian foliated manifold with a bundle-like metric g_M, and a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ be given by the transverse signature operator (as in Section 3.2).

Denote by $C_E^*(G)$ the closure of the set $R_E(C_c^\infty(G))$ in the uniform operator topology of $\mathcal{L}(L^2(M,E))$. It is easy to see that the $*$-homomorphism $R_E : C_c^\infty(G) \to C_E^*(G)$ extends to a $*$-homomorphism of the full C^*-algebra $C^*(G)$, $R_E : C^*(G) \to C_E^*(G)$. By [10], we have the natural projection $\pi_E : C_E^*(G) \to C_r^*(G)$. If $\mathcal{I}$ is an involutive ideal in $C_E^*(G)$, we may think of the spectral triple $(\mathcal{I},\mathcal{H},D)$ as a subset of our spectrally defined geometrical space and look for its dimension spectrum. Then Theorem 2 implies the following fact about the dimension spectra of "subsets" in the singular space $M/\mathcal{F}$.

Proposition 22. *Let $\mathcal{I}$ be an involutive ideal in $C_E^*(G)$. Then $q \in \mathrm{Sd}(\mathcal{I})$ iff $\pi_E(\mathcal{I}) \neq 0$. In particular, if $\pi_E(\mathcal{I}) = 0$, then the top spectral dimension of the spectral triple $(\mathcal{I},\mathcal{H},D)$ is less than q.*

This fact can be also interpreted as a fact about a noncommutative analogue of the integral in the case under consideration. Let I be a functional on $C_c^\infty(G)$, given by the formula

$$I(k) = \tau(R_E(k)|D|^{-q}), k \in C_c^\infty(G).$$

Proposition 23. *Under the current assumptions, the functional I is given by the following formula*

$$I(k) = \frac{q}{\Gamma(\frac{q}{2}+1)}\mathrm{tr}_{\mathcal{F}}\ \pi_E(k), k \in C_c^\infty(G),$$

and can be extended by continuity to a functional on C^*-algebra $C_E^*(G)$. In particular, $I(k) = 0$ for any $k \in C_E^*(G), \pi_E(k) = 0$.

Otherwise speaking, the functional I coincides on $C_c^\infty(G)$ (up to some multiple) with the von Neumann trace $\mathrm{tr}_{\mathcal{F}}$, given by the Riemannian transversal volume due to the noncommutative integration theory [5], and the support of I is a "regular" part of our geometrical space.

Acknowledgement. The work was done during visits to the Max Planck Institut für Mathematik at Bonn and to Institut des Hautes Etudes Scientifiques. I wish to express my gratitude to these institutes for their hospitality and support.

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