Siberian Mathematical Journal, Vol. 40, No. 6, 1999

TRANSITIVE ARRANGEMENTS OF ALGEBRAIC SYSTEMS†)

S. V. Sudoplatov

UDC 510.67

In this article we study the possibility of preserving various model-theoretic properties under embeddings of algebraic systems into algebraic systems of expanded signature with transitive automorphism group. We describe the spectrum functions of the corresponding elementary theories.

We use without specification the notions of [1] together with the standard model-theoretic terminology of [2, 3].

Henceforth we assume that $\mathcal{M}_i$ are countable models of some complete elementary theories T_i such that the Morley process applies to their predicate signatures; i.e., for each formula $\varphi(\bar{x})$ of T_i, there is a predicate symbol R_φ satisfying $T_i \vdash \varphi(\bar{x}) \leftrightarrow R_\varphi(\bar{x})$.

Consider an exact pseudoplane $\mathcal{P} = \langle P, L, \in \rangle$ without nontrivial polygons such that ω lines pass through each point, each line comprises ω points, and every two points are joined by a polygonal line. Such a pseudoplane $\mathcal{P}$ is called *free.* Define the model $\mathcal{M}_{fp} = \langle P, R^{(3)} \rangle$ corresponding to a pseudoplane $\mathcal{P}$ by the following condition: $\models R(a,b,c) \Leftrightarrow a, b$, and c lie on one line.

A *transitive arrangement* of systems $\mathcal{M}_i$, $i \in \omega$, in a system $\mathcal{M}_{fp}$ is a system $\mathcal{M}^* \rightleftharpoons \mathcal{M}_{fp}\langle \mathcal{M}_i \mid i \in \omega \rangle$ of signature $\langle \bigcup_{i\in\omega} \Sigma(\mathcal{M}_i), R^{(3)} \rangle$ with universe P which possesses the following properties:

(1) $\mathcal{M}^*|_R = \mathcal{M}_{fp}$;

(2) if $Q^{(n)} \in \Sigma(\mathcal{M}_i)$ and $(a_1, \dots, a_n) \in Q$ then $a_1, \dots, a_n \in l$ for some line $l \in L$;

(3) for every line $l \in L$, there is a system $\mathcal{M}_i$ such that

$$(\mathcal{M}^* \cap l)|_{\Sigma(\mathcal{M}_i)} \simeq \mathcal{M}_i,$$

and if $Q^{(n)} \in \Sigma(\mathcal{M}_j) \setminus \Sigma(\mathcal{M}_i)$ then $(Q \cap l^n) \setminus \mathrm{id}_l = \varnothing$;

(4) for every point $p \in P$, every system $\mathcal{M}_i$, and an element $a \in \mathcal{M}_i$, there are infinitely many lines $l \in L$ such that $p \in l$ and there is an isomorphism $f : \mathcal{M}_i \tilde{\to} (\mathcal{M}^* \cap l)|_{\Sigma(\mathcal{M}_i)}$ with $f(a) = p$.

Since the theory $T_{fp} = \mathrm{Th}(\mathcal{M}_{fp})$ is transitive (i.e., $|S^1(\varnothing)| = 1$), small (i.e., $|S(\mathrm{Th}(\mathcal{M}_{fp}))| = \omega$), ω-stable of Morley rank ω, not weakly normal [4], and has no minimal model; therefore, the following assertion is valid for $T^* = \mathrm{Th}(\mathcal{M}^*)$.

Proposition 1. *1. The theory T^* is transitive.*

2. The theory T^* is stable (ω-stable, superstable, small) if and only if the theories T_i are stable (ω-stable, superstable, small) for all $i \in \omega$.

3. The theory T^* has a simple (atomic, coutable universal, countable homogeneous, countable saturated, countable weakly saturated nonsaturated) model $\mathcal{M}^*$ if and only if the theories T_i have simple (atomic, countable universal, countable homogeneous, countable saturated, countable weakly saturated nonsaturated) models $\mathcal{M}_i$ for all $i \in \omega$. Moreover, the number of 1-types over $\mathcal{M}^*$ equals the maximal number of 1-types over $\mathcal{M}_i$, $i \in \omega$.

4. The theory T^* is not weakly normal, has no minimal model, and $U(T^*) \geq \omega$.

Corollary 2. *There is a countable, transitive, small, stable, and nonsuperstable theory of binary predicates which has a simple model $\mathcal{M}$ such that $|S(\mathcal{M})| = \omega$.*

PROOF. In view of Proposition 1, it suffices to satisfy the conditions of smallness, stability, nonsuperstability, and existence of a simple model $\mathcal{M}$ with $|S(\mathcal{M})| = \omega$ in some signature of binary

†) The research was supported by the Russian Foundation for Basic Research (Grant 96–01–01675).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 40, No. 6, pp. 1347–1351, November–December, 1999. Original article submitted October 15, 1997.

0037-4466/99/4006-1142 $22.00 © 1999 Kluwer Academic/Plenum Publishers

predicate symbols E_i, $i \in \omega$. The sought properties are easy to verify for the model $\mathcal{M}$ of signature $\langle E_i \mid i \in \omega \rangle$ which is determined by the following conditions:

(a) for each i, the relation $E_i(x, y)$ is an equivalence on the set $\exists y E_i(\mathcal{M}, y)$;

(b) $\exists y E_0(\mathcal{M}, y) = \mathcal{M}$, $\exists y E_i(\mathcal{M}, y) \supset \exists y E_j(\mathcal{M}, y)$, $i < j < \omega$;

(c) the relation $E_i(x, y)$ splits the set $\exists y E_i(\mathcal{M}, y) \setminus \exists y E_{i+1}(\mathcal{M}, y)$ into infinitely many classes each of which is infinite;

(d) the relation $E_{i+1}(x, y)$ splits each equivalence class of $E_i(x, y)$ on the set $\exists y E_{i+1}(\mathcal{M}, y)$ into infinitely many classes;

(e) each element of the model $\mathcal{M}$ belongs to finitely many equivalence classes of E_i, $i \in \omega$.

This finishes the proof of the corollary.

The interest in the above assertion is explained by the fact that the existence of the above-indicated theory is necessary for the existence of a small stable trigonometric theory induced by a digraph without circuits.

DEFINITION. A theory T is called *locally closed* if, for every tuple $\bar{a}$, there is a tuple $\bar{b}$ containing all elements of $\bar{a}$ and such that the type $\mathrm{tp}(\bar{b})$ is determined by a collection of quantifier-free formulas.

Corollary 3. *There is a countable, transitive, small, and not locally closed theory.*

PROOF. Define the sequence of graphs Γ_n, $n \in \omega \setminus \{0\}$ by the following conditions: $\Gamma_1 = \langle \{a_0, a_1\}, \{(a_0, a_1)\} \rangle$, $\Gamma_{n+1} = \langle M_n \cup \{a_{n+1}\}, R_n \cup \{(a_n, a_{n+1})\} \rangle$, where $\Gamma_n = \langle M_n, R_n \rangle$. Take as Γ the disjoint union of the graphs Γ_n, $n \geq 1$, and take as $\mathcal{M}_i$ the copies of Γ. Obviously, the theory $\mathrm{Th}(\mathcal{M}_{fp}\langle \mathcal{M}_i \mid i \in \omega \rangle)$ is small, while the fact that it is not locally closed follows from the existence of a model of the theory $\mathrm{Th}(\Gamma)$ with an infinite connected component such that all graphs Γ_n, $n \geq 1$, embed in this component. This finishes the proof of the corollary.

We recall that a theory T is called Δ-*based* if each formula of T is equivalent in T to some Boolean combination of formulas in Δ. A theory T is said to be an n-*theory* if T is Δ-based with Δ the set of formulas in T having at most n free variables.

Lemma 4. *The theory T_{fp} is a 2-theory.*

PROOF is analogous to the proof of Theorem 2.1 in [5]. It suffices to show that the following choice condition holds for every tuple $\bar{a}$ of pairwise distinct elements connected by polygonal lines: there is an element $a_i \in \bar{a}$ such that the type $\mathrm{tp}(a_i, \bar{a} \setminus \{a_i\})$ is determined by the set of formulas $\varphi(x, a_j)$, $j \neq i$.

Fix the tuple $\bar{a} = (a_0, \dots, a_n)$. It is easy to see that there is an element a_i for which we can find a line l such that every polygonal line, joining a_i to some element $a_j \in \bar{a} \setminus \{a_i\}$), includes l. Since every shortest polygonal line joining two distinct points in the pseudoplane $\mathcal{P}$ is unique, the type $\mathrm{tp}(a_i, \bar{a} \setminus \{a_i\})$ is determined by the set of formulas $\varphi(x, a_j)$, $j \neq i$, that describe the shortest polygonal lines between a_j and a_i. Thereby the choice condition is satisfied, and the lemma is proved.

Theorem 5. *The theory T^* is an n-theory if and only if all theories T_i, $i \in \omega$, are n-theories.*

PROOF. Necessity of the requirement that all theories T_i be n-theories for T^* to be an n-theory is obvious. Prove sufficiency. Clearly, each type of elements lying on one line l is determined by formulas in two free variables, which "assert" that the elements lie on one line, together with formulas in at most n free variables which determine the type of the corresponding elements of the model $\mathcal{M}$ of the theory T_i such that $\mathcal{M} \simeq l|_{\Sigma(\mathcal{M}_i)}$. However, if the elements of the tuple $\bar{a}$ do not belong to one line then by Lemma 4 the type $\mathrm{tp}(\bar{a})$ is determined by the set of formulas in two free variables, which describe the shortest polygonal lines between the elements of the tuple, together with the set of formulas in at most n free variables which determine the types of elements in $\bar{a}$ lying on one line.

Thus, each type $\mathrm{tp}(\bar{a})$ is determined by formulas in at most n free variables. Therefore, T^* is an n-theory. The proof of the theorem is over.

Henceforth we denote by $I(T, \lambda)$ the number of pairwise nonisomorphic models of a theory T which are of cardinality λ.

Lemma 6. *The spectrum function of the theory T_{fp} is as follows:*

$$I(T_{fp},\omega_\alpha)=\begin{cases}\omega_0 & \text{if } \alpha=0,\\ |\alpha+1|^\omega & \text{if } \alpha\geq 1.\end{cases}$$

PROOF. We first suppose that $\alpha = 0$. In this case, ω lines pass through each point and each line comprises ω points. Therefore, every two countable connected models of the theory T_{fp} are isomorphic. Hence, each countable model is determined by the number of connected components; i.e., $I(T_{fp},\omega_0)=\omega_0$.

Now, suppose that $\alpha\geq 1$ and consider an arbitrary connected model $\mathcal{M}_\alpha$ of cardinality ω_α. It is easy to see that the model $\mathcal{M}_\alpha$ can be represented as the union of an increasing chain of subsystems $\mathcal{M}_{\alpha,n}$, $n\in\omega$, that appear as a result of the following induction process.

1. Take as $\mathcal{M}_{\alpha,0}$ any one-element subsystem and take as $\mathcal{M}_{\alpha,1}$ the subsystem containing the element p of $\mathcal{M}_{\alpha,0}$ and all points that belong to the lines passing through p.

2. If a subsystem $\mathcal{M}_{\alpha,n}$ has been already constructed then take as $\mathcal{M}_{\alpha,n+1}$ the subsystem in which, for every point p of $\mathcal{M}_{\alpha,n}$, we add to the universe of $\mathcal{M}_{\alpha,n}$ all points that belong to the lines passing through p.

Since the cardinality of the set of lines passing through each given point is between ω and ω_α and since, on each line, there exist ω to ω_α points; there are $|\alpha+\omega|$ pairwise nonisomorphic subsystems of the form $\mathcal{M}_{\alpha,1}$. Similarly, for each $n\geq 2$ there are $|\alpha+\omega|$ pairwise nonisomorphic subsystems of the form $\mathcal{M}_{\alpha,n}$.

Thus, there are $|\alpha+\omega|^\omega$ pairwise nonisomorphic models of cardinality ω_α. Since, from $|\alpha+\omega|^\omega$ connected components we can arrange $|\alpha+\omega|^\omega\cdot|\alpha+\omega|$ models of cardinality ω_α, we obtain $I(T_{fp},\omega_\alpha)=|\alpha+\omega|^\omega=|\alpha+1|^\omega$, which finishes the proof of the lemma.

We say that theories T_i, $i\in\omega$, *induce a new system* if there is an algebraic system $\mathcal{A}$ of signature $\bigcup_{i\in\omega}\Sigma(T_i)$ such that every proposition in $\mathrm{Th}(\mathcal{A})$ is consistent with some theory T_i but $\mathcal{A}|_{\Sigma(T_i)}$ is not a model of T_i for any i.

Denote by λ_α the cardinal that equals the maximum of $|\alpha+\omega|$, the number of models of the theories T_i of cardinality $\leq\alpha$, and the number of new models of the theories T_i of cardinality $\leq\alpha$.

Theorem 7. *The spectrum function of the theory T^* is as follows:*

$I(T^*,\omega_0)=\omega_0$ if T_i, $i\in\omega$ are countable categorical theories not inducing new countable systems, and $I(T^*,\omega_0)=2^{\omega_0}$ otherwise;

$I(T^*,\omega_\alpha)=(\lambda_\alpha)^\omega$ for $\alpha\geq 1$.

PROOF of the equality $I(T^*,\omega_0)=\omega_0$ in the case when T_i are countable categorical theories not inducing new countable systems is analogous to the proof of Lemma 6. Otherwise there are λ_0 pairwise nonisomorphic systems consisting of lines, passing through some common point, with countable models of the theories T_i put on them. Then there are $\lambda_0^\omega=2^\omega$ pairwise nonisomorphic countable models of T^* in which arbitrary two elements are joined by polygonal lines; i.e., $I(T^*,\omega_0)=2^{\omega_0}$.

In the case of $\alpha\geq 1$, by analogy to the proof of Lemma 6 we can easily notice that in the models of cardinality ω_α there may exist λ_α subsystems consisting of lines passing through a common point. Therefore, there are $(\lambda_\alpha)^\omega$ models of T^* of cardinality ω_α; i.e., $I(T^*,\omega_\alpha)=(\lambda_\alpha)^\omega$, which finishes the proof of the theorem.

In conclusion we define some kind of transitive arrangement of algebraic systems which is close to that above and uses a *free directed pseudoplane*.

Consider a connected directed graph $\mathcal{M}_{fdp}=\langle P,R^{(2)}\rangle$ satisfying the following conditions:

(1) the graph Γ has neither edges nor loops;

(2) the undirected graph, obtained by replacing the arcs of Γ with edges, forms a tree;

(3) each point in the graph Γ has countably many images and inverse images under the relation R.

The free directed pseudoplane $\langle P,L,\in\rangle$ is defined by the condition $L=\{R(P,a)\mid a\in P\}$.

A *transitive arrangement* of systems $\mathcal{M}_i$, $i\in\omega$, in a system $\mathcal{M}_{fdp}$ is a system $\mathcal{M}^\vee\rightleftharpoons\mathcal{M}_{fdp}\langle\mathcal{M}_i\mid i\in\omega\rangle$ of signature $\langle\bigcup_{i\in\omega}\Sigma(\mathcal{M}_i),R^{(2)}\rangle$ with universe P possessing the following properties:

(1) $\mathcal{M}^\vee|_R=\mathcal{M}_{fdp}$;

(2) if $Q^{(n)} \in \Sigma(\mathcal{M}_i)$ and $(a_1, \dots, a_n) \in Q$ then $a_1, \dots, a_n \in l$ for some line $l \in L$;

(3) for every line $l \in L$, there is a system $\mathcal{M}_i$ such that $(\mathcal{M}^\vee \cap l)|_{\Sigma(\mathcal{M}_i)} \simeq \mathcal{M}_i$ and if $Q^{(n)} \in \Sigma(\mathcal{M}_j) \setminus \Sigma(\mathcal{M}_i)$ then $(Q \cap l^n) \setminus \mathrm{id}_l = \varnothing$;

(4) for every point $p \in P$, every system $\mathcal{M}_i$, and an element $a \in \mathcal{M}_i$, there are infinitely many lines $l \in L$ such that $p \in l$ and there exists an isomorphism $f : \mathcal{M}_i \tilde{\to} (\mathcal{M}^\vee \cap l)|_{\Sigma(\mathcal{M}_i)}$ with $f(a) = p$.

It is easy to verify that all assertions proven above for the theory T^* remain valid for the theory $\mathrm{Th}(\mathcal{M}^\vee)$.

References

1. S. V. Sudoplatov, "On trigonometries of groups on a projective plane," Sibirsk. Mat. Zh., **36**, No. 2, 419–431 (1995).

2. Handbook of Mathematical Logic. Vol. 1: Model Theory [Russian translation], Nauka, Moscow (1982).

3. A. Pillay, An Introduction to Stability Theory, Oxford Univ. Press, Oxford (1983).

4. A. Pillay, "Stable theories, pseudoplanes and the number of countable models," Ann. Pure Appl. Logic, **43**, No. 2, 147–160 (1989).

5. S. V. Sudoplatov, "On a certain complexity estimate in graph theories," Sibirsk. Mat. Zh., **37**, No. 3, 700–703 (1996).