Siberian Mathematical Journal, Vol. 40, No. 6, 1999

TWO CLASSES OF CENTRAL SIMPLE n-LIE ALGEBRAS

A. P. Pozhidaev UDC 512.554

Introduction. It is well known that the classification problem for simple binary algebras in a variety is reduced to the classification of algebras which remain simple under every extension of the ground field. The notions of centroid and central simple algebra (see, for example, [1, Section 10, §1]) play the key role in the classification. These notions extend easily to Ω-algebras.

In the previous article [2], the author considered the two classes $A(H,t)$ and $E(H) = E(H,t,\mathcal{J})$ of n-Lie algebras (see (1) and (8)) and specified necessary and sufficient conditions for simplicity of the algebras in these classes. In the present article, we study central simplicity of these algebras; namely, we prove that these algebras are simple (Theorems 2.2 and 3.2).

Recall some definitions.

An *Ω-algebra* over a field Φ is a vector space over Φ furnished with a system $\Omega = \{\omega_i : |\omega_i| = n_i \in N,\ i \in I\}$ of multilinear algebraic operations, where $|\omega_i|$ stands for the arity of the operation ω_i. In what follows, an Ω-algebra is simply called an algebra.

An *n-Lie algebra* over Φ is an Ω-algebra L over Φ with one anticommutative n-ary operation $[x_1,\dots,x_n]$ satisfying the identity

$$[[x_1,\dots,x_n],y_2,\dots,y_n] = \sum_{i=1}^{n}[x_1,\dots,[x_i,y_2,\dots,y_n],\dots,x_n].$$

Henceforth we denote by $\langle\Upsilon\rangle_\Phi$ (or simply $\langle\Upsilon\rangle$ if the field is clear from the context) the vector space over Φ spanned by the family $\{\Upsilon\}$. Given $k \in N$, we put $N_k = \{1,\dots,k\} \subset N$.

The article is organized as follows. In § 1 we formulate the basic properties of central simple Ω-algebras; in § 2 and § 3 we prove central simplicity of the algebras $A(H,t)$ and $E(H)$; and in § 4 we study the Cartan subalgebras of $A(H,t)$ and $E(H)$, construct Cartan decompositions for these algebras over a field of prime characteristic, and prove an existence theorem for simple modular n-Lie algebras with Cartan subalgebras of various dimensions.

The research was carried out under the supervision of V. T. Filippov to whom the author expresses his sincere gratitude.

1. The centroid of an Ω-algebra. Let L be an Ω-algebra over a field Φ. Given an operation $\omega_i \in \Omega$ and arbitrary $x_1,\dots,\hat{x}_j,\dots,x_{n_i}$ in L, we define the operator $M_j(x_1,\dots,\hat{x}_j,\dots,x_{n_i})$ of right multiplication as the linear mapping $y \mapsto \omega_i(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_{n_i})$. The subalgebra $M = M(L)$ of the algebra $End_\Phi(L)$ of linear transformations of L which is generated by all left and right multiplications and the identity mapping is called the *multiplication algebra* of L.

By the *centroid* $\Gamma(L)$ of an Ω-algebra L we mean the following subalgebra of $End_\Phi(L)$:

$$\Gamma(L) = \left\{ \begin{array}{c} \phi \in End_\Phi(L) : \phi(\omega_i(a_1,\dots,a_{n_i})) = \omega_i(a_1,\dots,\phi(a_j),\dots,a_{n_i}) \\ \text{for all } \omega_i \in \Omega,\ a_1,\dots,a_{n_i} \in L,\ j \in N_{n_i} \end{array} \right\}.$$

Clearly, $\Gamma(L)$ is a unital associative algebra over Φ.

We now state without proof some theorems on the centroid of a simple Ω-algebra.[1)] Proofs repeat those in [1] almost verbatim.

Let $L^1 = \langle ML\rangle_\Phi$ be the square of L.

[1)] It is V. T. Filippov who pointed out to the author the possibility of translating the assertions of [1, Section 10, § 1] to the case of Ω-algebras.

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 40, No. 6, pp. 1313–1322, November–December, 1999. Original article submitted February 19, 1998.

 0037-4466/99/4006-1112 $22.00

Lemma 1.1. *If $L^1 = L$ then $\Gamma(L)$ is commutative.*

A subspace I of an Ω-algebra L which is invariant under the action of the multiplication algebra $M(L)$ is called an *ideal.* An Ω-algebra L is called *simple* if $L^1 \neq 0$ and L lacks ideals other than 0 and L. Therefore, an algebra L is simple if and only if $M(L)$ is an irreducible algebra of linear transformations. The following theorem is an immediate consequence of Lemma 1.1 and Schur's lemma.

Theorem 1.2. *The centroid of a simple Ω-algebra is a field.*

Recall that an Ω-algebra L is called central if $\Gamma(L) = \Phi$.

REMARK. The notion of the Martindale centroid is soundly defined for Ω-algebras as well; it coincides with the notion of centroid in the case of simple Ω-algebra. Also, a theorem is known asserting that a simple finite-dimensional Ω-algebra over an algebraically closed field is central. Results relevant to these question can be found in [3].

Theorem 1.3. *Let L be a simple Ω-algebra over a field Φ and let Γ $(\supseteq \Phi)$ be the centroid of L. Consider L as an algebra over Γ, defining $\gamma \cdot a = \gamma(a)$, $a \in L$, $\gamma \in \Gamma$. Then L is a central simple algebra over Γ. The multiplication algebra of L over Γ has the same set of transformations as the multiplication algebra of L over Φ.*

Theorem 1.4. *If L is a central simple Ω-algebra over Φ and P is an extension of Φ then L_P is a central simple algebra over P. If L is an arbitrary Ω-algebra over Φ, Δ is a subfield (over Φ) of the centroid of L, and the Δ-algebra $\Gamma \otimes_\Delta L$ is simple, then L is simple over Φ and $\Delta = \Gamma$.*

Theorem 1.5. *If L is a finite-dimensional simple Ω-algebra with centroid Γ and multiplication algebra $M(L)$ then $M(L)$ is the set of all linear transformations of L regarded as a vector space over Γ.*

2. Central simplicity of the algebras $A(H,t)$. Let Φ be a field, let Φ^n be the Cartesian nth power of the additive group of Φ, let H be a subgroup in Φ^n, and let t be a fixed element of H. In what follows, we assume that there exists a set $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_n\} \subseteq H \setminus \{t\}$ such that $\varepsilon_1, \dots, \varepsilon_n$ are linearly independent elements of Φ^n regarded as a vector space over Φ.

Put $A_H = \langle \bar{e}_a : a \in H \rangle$; it is a vector space over Φ. Fix $t \in H$ and furnish A_H with the n-ary operation

$$[\bar{e}_{a_1}, \dots, \bar{e}_{a_n}] = |a_1, \dots, a_n| \cdot \bar{e}_{a_1+\dots+a_n+t}, \tag{1}$$

where $|a_1, \dots, a_n|$ is the determinant constructed from $a_1, \dots, a_n \in \Phi^n$.

By [4, Theorem 2.1], the so-constructed Ω-algebra $\bar{A}(H,t)$ is an n-Lie algebra. Put

$$\tilde{A}(H,t) = \begin{cases} \bar{A}(H,0) & \text{if } t = 0; \\ \langle \bar{e}_a : a \in H \setminus \{t\} \rangle & \text{if } t \neq 0. \end{cases}$$

Using (1), it is easy to verify that $\tilde{A}(H,t)$ is a subalgebra of $\bar{A}(H,t)$.

Let $A(H,t) = \tilde{A}(H,t)/\Phi\bar{e}_0$ be the quotient algebra of the n-Lie algebra $\tilde{A}(H,t)$ by the one-dimensional ideal $\Phi\bar{e}_0$. By definition,

$$A(H,t) = \langle e_a = \bar{e}_a + \Phi e_0 : a \in H' = H \setminus \{0,t\} \rangle.$$

Simplicity of $A(H,t)$ was proved in [2] for every $t \in H$. Before proving that this algebra is central, we prove the following

Lemma 2.1. *Assume that $H \trianglelefteq \Phi^n$, t is a fixed element of H, $H' = H \setminus \{0,t\}$, and a_1 and b are arbitrary elements of H'. Then we may choose elements $a_2, \dots, a_{n-1} \in H'$ so that $h = \sum_{i=1}^{n-1} a_i + b + t \in H'$ and $a_1, \dots, a_{n-1}$ are linearly independent elements of Φ^n.*

PROOF. Let $a_1, \dots, a_{n-1}$ be linearly independent elements of Φ^n and $h \notin H'$. The two cases are possible:

Case I. $h = \sum_{i=1}^{n-1} a_i + b + t = t$.

In this case, it suffices to take $a_2' = a_2 - a_1$ instead of a_2, provided that $a_2' \neq t$. If $a_2 - a_1 = t$ and $n > 3$, then it suffices to take a_3 instead of a_2. If $n = 3$ and char $\Phi \neq 2$ then put $a_2' = a_2 - 2a_1$, whereas if char $\Phi = 2$ then we have $h = a_1 + a_2 + b + t$, $a_2 - a_1 = t$; whence $h = b \in H'$.

Case II. $h = \sum_{i=1}^{n-1} a_i + b + t = 0$.

In this case, we take $a_2' = a_2 + a_1$ instead of a_2 and follow the arguments of Case I. The lemma is proved.

Given $a \in H$, denote by $\langle a \rangle$ the subgroup of the Abelian group H which is defined as follows: $\langle a \rangle = \{b \in H : b = \gamma a \text{ for some } \gamma \in \Phi\} \trianglelefteq H$.

Theorem 2.2. *$A(H,t)$ is a central simple n-Lie algebra.*

PROOF. We claim that the centroid Γ of $A(H,t)$ coincides with the ground field Φ. Let ϕ be an element of Γ, $a \in H'$, and $\phi(e_a) = \sum_{h \in H'} \zeta(a,h)e_h$, where $\zeta : H \times H' \mapsto \Phi$ and $\zeta(a,h) = \zeta(h,a) = 0$ if $a = 0, t$. Then, for all $a_1, \dots, a_n \in H'$, $i \in N_n$, we have

$$\begin{aligned}\phi([e_{a_1}, \dots, e_{a_n}]) &= [e_{a_1}, \dots, e_{a_{i-1}}, \phi(e_{a_i}), e_{a_{i+1}}, \dots, e_{a_n}] \\ &= \sum_{h \in H'} \zeta(a_i, h)[e_{a_1}, \dots, e_{a_{i-1}}, e_h, e_{a_{i+1}}, \dots, e_{a_n}] \\ &= \sum_{h \in H'} \zeta(a_i, h)|a_1, \dots, a_{i-1}, h, a_{i+1}, \dots, a_n| e_{a_1 + \dots + a_n + h + t - a_i}.\end{aligned} \tag{2}$$

On the other hand,

$$\phi([e_{a_1}, \dots, e_{a_n}]) = \phi(|a_1, \dots, a_n| e_{a_1 + \dots + a_n + t}) = |a_1, \dots, a_n| \sum_{h \in H'} \zeta\Big(\sum_{i=1}^n a_i + t, h\Big) e_h. \tag{3}$$

Comparing (2) and (3) and using the unique decomposition of a vector in a basis, we conclude that

$$|a_1, \dots, a_{i-1}, h - t, a_{i+1}, \dots, a_n| \zeta(a_i, a_i + h - t - \sum_{i=1}^n a_i) = |a_1, \dots, a_n| \zeta\Big(\sum_{i=1}^n a_i + t, h\Big) \tag{4}$$

for all $h, a_1, \dots, a_n \in H'$, $i \in N_n$. Put in (4) $a_i = a_j$ with $i \neq j$. We have

$$|a_1, \dots, a_{i-1}, h - t, a_{i+1}, \dots, a_n| \zeta\Big(a_j, a_j + h - t - \sum_{i=1}^n a_i\Big) = 0. \tag{5}$$

Let b be an element of H'. Put $h = \sum_{i=1}^n a_i + b + t - a_j$. Without loss of generality we may assume that $i = n$ and that $h \in H'$ by Lemma 2.1. Then (5) is reduced to the equality

$$|a_1, \dots, a_{n-1}, b| \zeta(a_j, b) = 0$$

which, by the arbitrariness of $a_1, \dots, a_{n-1} \in H'$, implies that $\zeta(a, b) = 0$ whenever $a \notin \langle b \rangle$. Verify that $\zeta(a, b) \neq 0$ for $a \in \langle b \rangle$ only if $a = b$.

Indeed, let $a_1, \dots, a_n, h \in H'$ be such that $\zeta(\sum_{i=1}^n a_i + t, h) \neq 0$; i.e., the equality

$$\sum_{i=1}^n a_i + t = \alpha h \tag{6}$$

holds for some $\alpha \in \Phi^*$. By (4), for every $j \in N_n$ we have $\zeta(a_j, a_j + h - t - \sum_{i=1}^n a_i) \neq 0$; i.e., $a_j + h - t - \sum_{i=1}^n a_i = \beta_j a_j$ for some $\beta_j \in \Phi^*$. Summing the last equality and (6), we obtain $(\alpha - 1)h + (\beta_j - 1)a_j = 0$; whence $\alpha = \beta_j = 1$ by the arbitrariness of $j \in N_n$. Since $A(H,t)$ is simple; therefore, for every $b \in H'$, there exist linearly independent elements $a_1, \dots, a_n \in H'$ such that $\sum_{i=1}^{n} a_i + t = b$. We thus proved that $\zeta(a, b) \neq 0$ only if $a = b$.

Verify that $\zeta(a, a) = \zeta(b, b)$ for all $a, b \in H'$.

Indeed, by (4) we have

$$\zeta(a_i, a_i) = \zeta\Big(\sum_{i=1}^{n} a_i + t, \sum_{i=1}^{n} a_i + t\Big); \tag{7}$$

i.e., $\zeta(a_i, a_i) = \zeta(a_j, a_j)$ with a_i, $a_j \in H'$ such that $a_i \notin \langle a_j \rangle$. If $a_i \in H'$ and $\alpha a_i \in H'$ for some $\alpha \in \Phi^*$, then by (7) $\zeta(\alpha a_i, \alpha a_i) = \zeta(a_j, a_j)$, where a_i, $a_j \in H'$ are as above. Comparing the last equalities, we arrive at the required result. Thus, the mapping ϕ is the multiplication by an element of the field; i.e., $\Gamma = \Phi$. The theorem is proved.

3. Central simplicity of the algebras $E(H)$. In [2], there was distinguished some class $E(H, t, \mathcal{J})$ of subalgebras of the algebras $\bar{A}(H, t)$ and simplicity conditions were specified for the algebras in this class. It was shown that to study the algebras in $E(H, t, \mathcal{J})$ it suffices to study the algebras of the following form.

As above, let Φ be a field, H be a subgroup in Φ^n, and $t_1 = (1 - n, 0, \dots, 0) \in H$. Put $H_1 = \{h \in H : \pi_1(h) = 1\}$, where $\pi_i : \Phi^n \mapsto \Phi$ is the ith projection operator: $\pi_i(x_1, \dots, x_n) = x_i$. In what follows, we assume that there exists $\mathcal{E}' = \{\epsilon_1, \dots, \epsilon_n\} \subseteq H_1' = H_1 \setminus \{t_1\}$ such that $\epsilon_1, \dots, \epsilon_n$ are linearly independent in Φ^n. Put

$$\bar{E}_H = \langle e_a : a \in H_1 \rangle, \ E(H) = \begin{cases} \bar{E}_H & \text{if } n \not\equiv 0 \pmod{p}; \\ \bar{E}_H^1 & \text{if } n \equiv 0 \pmod{p}, \end{cases} \tag{8}$$

where $p = \operatorname{char} \Phi$ and $\bar{E}_H^1 = [\bar{E}_H, \dots, \bar{E}_H]$ is the square of $\bar{E}_H$.

It is easy to see that $\bar{E}_H$ is a subalgebra of $\bar{A}(H, t_1)$. By [2], $E(H) = \langle e_a : a \in H_1' \rangle$ is a simple n-Lie algebra.

Using these notations, we prove the following

Lemma 3.1. *Let a_1 and b be arbitrary elements of H_1'. Then we may choose $a_2, \dots, a_{n-1} \in H_1'$ so that $h = \sum_{i=1}^{n-1} a_i + b + t_1 \in H_1'$ and $a_1, \dots, a_{n-1}$ are linearly independent elements of Φ^n.*

PROOF. Let $a_1, \dots, a_{n-1}$ be linearly independent elements of Φ^n and $h \notin H_1'$; i.e., $h = \sum_{i=1}^{n-1} a_i + b + t_1 = t_1$ which is equivalent to $h' = \sum_{i=1}^{n-1} a_i + b = 0$. Note that $\pi_1(h') = n$ and hence $h \notin H_1'$ only if $n \equiv 0 \pmod{p}$. To prove the lemma, it suffices to take $a_2' = -\sum_{i=1}^{n-1} a_i$ instead of a_2. It is easy to see that $a_2' \in H_1'$. Otherwise $b = t_1$, which contradicts the assumption. Now, if $a_1 + a_2' + \sum_{i=3}^{n-1} a_i + b = 0$ then $b = a_2$ and, to prove the lemma, we have to repeat the argument with a_3 substituted for a_2. If $n = 3$, $a_1 + a_2 + b = 0$, and $b = a_2$ then $a_1 + 2a_2 = 0$ which amounts to $a_1 = a_2$, and we arrive at a contradiction with the choice of a_1 and a_2. The lemma is proved.

Theorem 3.2. *$E(H)$ is a central simple n-Lie algebra.*

PROOF. Demonstrate that the centroid Γ of $E(H)$ coincides with the ground field Φ. Let ϕ be an element of Γ and we have $\phi(e_a) = \sum_{h \in H_1'} \zeta(a, h) e_h$ for every $a \in H_1'$, where $\zeta : H_1 \times H_1 \mapsto \Phi$ and $\zeta(t_1, h) = \zeta(h, t_1) = 0$. Then, for arbitrary $a_1, \dots, a_n \in H_1'$, $i \in N_n$, we have

$$\begin{aligned} \phi([e_{a_1}, \dots, e_{a_n}]) &= [e_{a_1}, \dots, e_{a_{i-1}}, \phi(e_{a_i}), e_{a_{i+1}}, \dots, e_{a_n}] \\ &= \sum_{h \in H_1'} \zeta(a_i, h) [e_{a_1}, \dots, e_{a_{i-1}}, e_h, e_{a_{i+1}}, \dots, e_{a_n}] \\ &= \sum_{h \in H_1'} \zeta(a_i, h) |a_1, \dots, a_{i-1}, h, a_{i+1}, \dots, a_n| e_{a_1 + \dots + a_n + h + t_1 - a_i}. \end{aligned} \tag{9}$$

On the other hand,

$$\phi([e_{a_1}, \dots, e_{a_n}]) = \phi(|a_1, \dots, a_n| e_{a_1 + \dots + a_n + t_1}) = |a_1, \dots, a_n| \sum_{h \in H_1'} \zeta\Big(\sum_{i=1}^{n} a_i + t_1, h\Big) e_h. \tag{10}$$

Comparing (9) and (10), and using the unique decomposition of a vector in a basis, we conclude that the equality

$$|a_1, \dots, a_{i-1}, h - t_1, a_{i+1}, \dots, a_n|\zeta\Big(a_i, a_i + h - t_1 - \sum_{i=1}^{n} a_i\Big)$$
$$= |a_1, \dots, a_n|\zeta\Big(\sum_{i=1}^{n} a_i + t_1, h\Big) \tag{11}$$

holds for all $h, a_1, \dots, a_n \in H_1'$, $i \in N_n$. By setting $a_i = a_j$ with $i \neq j$ in (11), we obtain

$$|a_1, \dots, a_{i-1}, h - t_1, a_{i+1}, \dots, a_n|\zeta\Big(a_j, a_j + h - t_1 - \sum_{i=1}^{n} a_i\Big) = 0. \tag{12}$$

Let b be an element of H_1'. Put $h = \sum_{i=1}^{n} a_i + b + t_1 - a_j$. Without loss of generality we may assume that $i = n$ and that $h \in H_1'$ by Lemma 3.1. Then (12) is reduced to the equality

$$|a_1, \dots, a_{n-1}, b|\zeta(a_j, b) = 0$$

which, by the arbitrariness of $a_1, \dots, a_{n-1} \in H_1'$, implies that $\zeta(a, b) = 0$ whenever $a \notin \langle b \rangle$. This is equivalent to $a \neq b$, since $a, b \in H_1'$.

By (11), $\zeta(a_i, a_i) = \zeta(\sum_{i=1}^{n} a_i + t_1, \sum_{i=1}^{n} a_i + t_1)$; i.e., $\zeta(a_i, a_i) = \zeta(a_j, a_j)$ for all a_i, $a_j \in H_1'$. This implies that the mapping ϕ is multiplication by an element of the field and $\Gamma = \Phi$. The theorem is proved.

4. Cartan subalgebras of the algebras $A(H,t)$ and $E(H)$. In this section we define Cartan subalgebras of the algebras $A(H,t)$ and $E(H)$. By analogy to Block's theorem [5] concerning Lie algebras, we prove Theorem 4.4 on existence of simple n-Lie algebras of characteristic $p > 0$ with Cartan subalgebras of various dimensions. In what follows, we assume that the ground field Φ is algebraically closed. Recall some definitions.

An ideal I of an n-Lie algebra L is called *nilpotent* if $I^k = 0$ for some $k \in N$, where $I^1 = I$ and I^k with $k > 1$ is defined by induction: $I^k = [I^{k-1}, I, L, \dots, L]$.

Let $\mathcal{N}$ be a nilpotent subalgebra of an n-Lie algebra L. A function $\rho : \mathcal{N}^{n-1} \mapsto \Phi$ is called a *root* of L relative to $\mathcal{N}$ if there exists a nonzero $x \in L$ such that, for every right multiplication R_h with $h \in \mathcal{N}^{n-1}$, the element x is annihilated by some power of the operator $\tilde{R}_h = R_h - \rho(h) \cdot \mathrm{Id}$, where $\mathrm{Id} : L \mapsto L$ is the identity operator. The set of all such x's is called the *root subspace* L_ρ relative to the root ρ. Then $\rho = 0$ is a root and $L_0 = L_0(\mathcal{N})$ is a subalgebra of L containing $\mathcal{N}$. If $\dim L < \infty$ then L is decomposed as a vector space into the direct sum of the root subspaces L_ρ. If $\mathcal{N} = L_0$ then $\mathcal{N}$ is called a *Cartan subalgebra* of L, and the decomposition of L into the direct sum of root subspaces is called a *Cartan decomposition* of the n-Lie algebra L. The condition $\mathcal{N} = L_0$ is equivalent to the condition that the nilpotent subalgebra $\mathcal{N}$ coincides with its normalizer; i.e., the inclusion $[x, \mathcal{N}, \dots, \mathcal{N}] \subseteq \mathcal{N}$ implies $x \in \mathcal{N}$ [6]. If x is a regular element of L; i.e., if the subalgebra $L_0(\Phi x)$ has a minimal dimension, then $L_0(\Phi x)$ is a Cartan subalgebra of L [6]. Thus, every n-Lie algebra has a Cartan subalgebra.

In the case of characteristic zero, all Cartan subalgebras of an n-Lie algebra are conjugate [7]. However, this fails if the characteristic is prime, and the forthcoming Theorem 4.4 claims that, in the modular case, a Cartan subalgebra may fail to contain a regular element and that the Cartan decompositions relative to different Cartan subalgebras may differ.

We first find the Cartan subalgebras of $A = A(H,t)$ and $E = E(H)$. Let us agree that the set $\mathcal{E}$ is chosen so that $t \in \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$.

Proposition 4.1. *Let $K = \{h \in H : h \in \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle\}$ and $A_K = \langle e_a : a \in K' = K \cap H' \rangle$. Then A_K is a Cartan subalgebra of $A(H,t)$.*

PROOF. Firstly, A_K is an Abelian subalgebra. Next, suppose that $x = \sum_{i\in I} \gamma_i e_{v_i} \in A_0(A_K)$. Then for some $m = m(x, h) \in N$ we have $xR_h^m = 0$, where $h = (e_{\varepsilon_1}, \dots, e_{\varepsilon_{n-1}}) \in A_K^{n-1}$. Applying (1) to the last equality, we obtain:

$$\sum_{i\in I} \gamma_i [e_{v_i}, e_{\varepsilon_1}, \dots, e_{\varepsilon_{n-1}}] = \sum_{i\in I} \gamma_i \tau_i e_{u_i} = 0, \tag{13}$$

where $\tau_i = \prod_{k=0}^{m-1} |v_i + kt, \varepsilon_1, \dots, \varepsilon_{n-1}|$, $u_i = v_i + m\varepsilon_1 + \ldots + m\varepsilon_{n-1} + mt$.

Note that $u_i \neq u_j$ with $i \neq j$, and we may assume that $u_j \neq 0$ for any fixed $j \in I$, since otherwise we may take $\varepsilon_1' = \varepsilon_1 + \varepsilon_2$ instead of ε_1 or $\varepsilon_2' = \varepsilon_2 + \varepsilon_1$ instead of ε_2, and follow the arguments of Lemma 1.1.

By (13), $\tau_j = 0$; i.e., $|v_j + kt, \varepsilon_1, \dots, \varepsilon_{n-1}| = 0$ for some $k \in \{0, \dots, m-1\}$, which implies the membership $v_j \in K'$ by the choice of $\mathcal{J}$. Since $j \in I$ is arbitrary, we conclude that $x \in A_K$, and the proposition is proved.

If char $\Phi = p > 0$ then for arbitrary $a \in H'$ and $h = (e_{a_1}, \dots, e_{a_{n-1}}) \in A_K^{n-1}$ we have $e_a R_h^p = g(a, h) e_a$, where $g(a, h) = \prod_{k=0}^{p-1} |a + kt, a_1, \dots, a_{n-1}| \in \Phi$. Note that in the case of $e_a \notin A_K$ we have $g_a \not\equiv 0$, where $g_a : A_K^{n-1} \mapsto \Phi$, $g_a(h) = g(a, h)$. From the construction of the Cartan subalgebra it follows that, if $a_1, \dots, a_{n-1}$ are independent, then there exist $\alpha_1, \dots, \alpha_{n-1} \in \Phi$ such that $\alpha_1 a_1 + \ldots + \alpha_{n-1} a_{n-1} = t$ which implies $g(a, h) = \rho(a, h)^p$ for every $h = (e_{a_1}, \dots, e_{a_{n-1}}) \in A_K^{n-1}$, where $\rho(a, h) = |a, a_1, \dots, a_{n-1}|$. We then have $e_a(R_h - \rho(a, h) \cdot \mathrm{Id})^p = 0$ and so $\rho_a = \rho(a, h)$ is a root of A relative to A_K. Furthermore, we have $g(b, h) = g(a, h)$ if and only if $b \in a + K$, which implies the Cartan decomposition of $A(H, t)$ relative to $A_K : A(H, t) = \oplus \sum_{\rho\in\Delta} A_\rho$, where $\Delta = \{\rho_a : A_K^{n-1} \mapsto \Phi : a \in H/K,\ \rho_a(h) = \rho(a', h),\ a = a' + K,\ h = (a_1, \dots, a_{n-1})\}$, and $A_0 = A_K$.

If we choose $\varepsilon_1, \dots, \varepsilon_k \in \Phi^n$, with $k \geq n$, linearly independent over F_p and consider H as a vector space over F_p with the basis $\varepsilon_1, \dots, \varepsilon_k$, then we obtain an example of a central simple n-Lie algebra $A(H, t)$ of dimension $p^k - 2$ ($p^k - 1$ if $t = 0$) with a Cartan subalgebra of dimension $p^{n-1} - 2$ ($p^{n-1} - 1$).

Before constructing a Cartan subalgebra of $E(H)$, let us agree that the set $\epsilon_1, \dots, \epsilon_{n-1} \in \mathcal{E}'$ is chosen so that $t_1 \in M_0 = \langle \epsilon_1, \dots, \epsilon_{n-1} \rangle$.

Proposition 4.2. *Let $M = H_1' \cap M_0$ and $E_M = \langle e_a : a \in M \rangle$. Then E_M is a Cartan subalgebra of $E = E(H)$.*

PROOF. Firstly, E_M is an Abelian subalgebra. Next, suppose that $x = \sum_{i\in I} \gamma_i e_{v_i} \in E_0(E_M)$. Then we have $xR_h^m = 0$ with $h = (e_{\epsilon_1}, \dots, e_{\epsilon_{n-1}}) \in E_M^{n-1}$ and some $m = m(x, h) \in N$. Applying (1) to the last equality, we obtain:

$$\sum_{i\in I} \gamma_i [e_{v_i}, e_{\epsilon_1}, \dots, e_{\epsilon_{n-1}}] = \sum_{i\in I} \gamma_i \tau_i e_{u_i} = 0, \tag{14}$$

where $\tau_i = \prod_{k=0}^{m-1} |v_i + kt_1, \epsilon_1, \dots, \epsilon_{n-1}|$, $u_i = v_i + m\epsilon_1 + \ldots + m\epsilon_{n-1} + mt_1$.

Note that $u_i \neq u_j$ with $i \neq j$. Then, by (14), $\tau_j = 0$; i.e., $|v_j + kt_1, \epsilon_1, \dots, \epsilon_{n-1}| = 0$ for some $k \in \{0, \dots, m-1\}$; whence $v_j \in M$ by the choice of $\epsilon_1, \dots, \epsilon_{n-1}$. Since $j \in I$ is arbitrary, we conclude that $x \in E_M$, and the proposition is proved.

If char $\Phi = p > 0$ then, for arbitrary $a \in H_1'$ and $h = (e_{a_1}, \dots, e_{a_{n-1}}) \in E_M^{n-1}$, we have $e_a R_h^p = g(a, h) e_a$, where $g(a, h) = \prod_{k=0}^{p-1} |a + kt_1, a_1, \dots, a_{n-1}| \in \Phi$. Note that in the case of $e_a \notin E_M$ we have $g_a \not\equiv 0$, where $g_a : E_M^{n-1} \mapsto \Phi$, $g_a(h) = g(a, h)$. From the construction of the Cartan subalgebra it follows that, if $a_1, \dots, a_{n-1}$ are independent, then there exist $\alpha_1, \dots, \alpha_{n-1} \in \Phi$ such that $\alpha_1 a_1 + \ldots + \alpha_{n-1} a_{n-1} = t_1$, which implies $g(a, h) = \rho(a, h)^p$ for every $h = (e_{a_1}, \dots, e_{a_{n-1}}) \in E_M^{n-1}$, where $\rho(a, h) = |a, a_1, \dots, a_{n-1}|$. We then have $e_a(R_h - \rho(a, h) \cdot \mathrm{Id})^p = 0$ and so $\rho_a = \rho(a, h)$ is a root of E relative to E_M. Furthermore, we have $g(b, h) = g(a, h)$ if and only if $b - a \in M_0$, which implies the Cartan decomposition of $E(H)$ relative to $E_M : E(H) = \oplus \sum_{\rho\in\Delta'} E_\rho$, where $\Delta' = \{\rho_a : E_M^{n-1} \mapsto \Phi : a \in H \bullet M = \{m + M_0 : m \in H_1'\},\ \rho_a(h) = \rho(a', h),\ a = a' + M_0,\ h = (a_1, \dots, a_{n-1})\}$, and $E_0 = E_M$.

If, as above, we choose $\epsilon_1, \dots, \epsilon_k \in \Phi^n$, with $k \geq n$, linearly independent over F_p and consider H as a vector space over F_p with the basis $\epsilon_1, \dots, \epsilon_k$, then we obtain an example of a central simple n-Lie algebra $E(H)$ of dimension p^{k-1} ($p^{k-1} - 1$ if $n \equiv 0 \pmod p$) with a Cartan subalgebra of dimension p^{n-1} ($p^{n-1} - 1$).

Before constructing an example of an n-Lie algebra with Cartan subalgebras of various dimensions, we introduce the following definition.

Fix $r \in N$, and put $N^*_{r,n} = \{(i_1, \dots, i_{n-1}) \in N^{n-1}_{r-1} : i_k \neq i_s \text{ with } k \neq s\}$. We call the set $\Upsilon = \{v_1, \dots, v_r\} \subset \Phi^n$ n-independent over Φ if $v_{i_1}, \dots, v_{i_n}$ are linearly independent over Φ for every $(i_1, \dots, i_n) \in N^*_{r+1,n+1}$.

Lemma 4.3. *Let Φ be an infinite field of characteristic p. For any $n, k \in N$ there exists an n-independent set $\Upsilon = \{v_1, \dots, v_k\} \subset \Phi^n$ over Φ with linearly independent elements over F_p.*

PROOF is carried out by induction on k. For $k = n$, we take $\{v_1, \dots, v_n\}$ equal to the elements of the basis. Now, assume that the lemma is proved for $k = r - 1 > n$ and that $\{v_1, \dots, v_{r-1}\}$ satisfy the conditions of the lemma. Put $v_r = \alpha_1 v_1 + \dots + \alpha_n v_n \in \Phi^n$. Then the set $\{v_1, \dots, v_r\}$ is n-independent if and only if $|v_r, v_{i_1}, \dots, v_{i_{n-1}}| \neq 0$ for any $\iota = (i_1, \dots, i_{n-1}) \in I = N^*_{r,n}$. Expand the determinant in the first row. We have $\alpha_1 \gamma^1_\iota + \dots + \alpha_n \gamma^n_\iota \neq 0$, where not all $\gamma^j_\iota = 0$ by the induction hypothesis. Thus, it suffices to choose $\alpha_1, \dots, \alpha_n \in \Phi$ so that $\alpha_1 \gamma^1_\iota + \dots + \alpha_n \gamma^n_\iota \neq 0$ for any $\iota \in I$.

For every $\iota \in I$, consider the function $f_\iota : \Phi^n \mapsto \Phi$ such that $f_\iota(x) = f_\iota(x_1, \dots, x_n) = x_1 \gamma^1_\iota + \dots + x_n \gamma^n_\iota$. Put $f = \prod_{\iota \in I} f_\iota : \Phi^n \mapsto \Phi$. Since Φ is infinite, there exist infinitely many $\alpha = (\alpha_1, \dots, \alpha_n) \in \Phi^n$ such that $f(\alpha) \neq 0$ (see, for example, [8, Proposition IV.2.5.8]). Since the set $\{v_1, \dots, v_{r-1}\}$ is finite, its linear span over F_p is finite. So we may also choose α so that $\{v_1, \dots, v_r\}$ are linearly independent over F_p. The lemma is proved.

Theorem 4.4. *For every prime p and every natural $k \in N$, there exists a simple n-Lie algebra of characteristic p with Cartan subalgebras of k different dimensions.*

PROOF. Take as a sought example the algebra $A = A(H, 0)$ over a field Φ with the group H constructed below. Take $r \in N$ so that $k \leq C^r_{n-1}$. Using Lemma 4.3, choose $\{\xi^0_1, \dots, \xi^0_r\} \subset \Phi^n$ n-independent and linearly independent over F_p. Now, choose $k_1, \dots, k_r \in N$ so that $\operatorname{card}\{k_{i_1} + \dots + k_{i_{n-1}} : (i_1, \dots, i_{n-1}) \in N^*_{r+1,n}\} = C^r_{n-1}$ (the sums of $n-1$ summands are all distinct). Consider the sets $\Xi_i = \{\xi^0_i, \xi^1_i, \dots, \xi^{k_i}_i\}$, where $i \in N_r$, $\xi^j_i = \beta_j \xi^0_i$, and $\beta_j \in \Phi$ are chosen so that the elements of Ξ_i are linearly independent over F_p. Let H be the vector space over F_p of dimension $\sum_{i=1}^r k_i + r$ with the basis $\Xi = \cup_{i=1}^r \Xi_i$. Put $A = A(H, 0)$. It is easy to see that, for every $\iota = (i_1, \dots, i_{n-1}) \in N^*_{r+1,n}$, $A_\iota = \langle e_a : a \in \langle \cup_{j=1}^{n-1} \Xi_{i_j} \rangle_{F_p} \rangle_\Phi$ is a Cartan subalgebra of A of dimension $p^{n-1} p^{k(\iota)} - 1$, where $k(\iota) = \sum_{j=1}^{n-1} k_{i_j}$. The claim follows by the choice of k_i. The theorem is proved.

References

1. N. Jacobson, Lie Algebras [Russian translation], Mir, Moscow (1964).
2. A. P. Pozhidaev, "Simple n-Lie algebras," Algebra and Logic, **38**, No. 3, 334–353 (1998).
3. Yu. P. Razmyslov, Identities of Algebras and Their Representations [Russian translation], Nauka, Moscow (1989).
4. A. P. Pozhidaev, "Monomial n-Lie algebras," Algebra and Logic, **37**, No. 5, 307–322 (1998).
5. R. E. Block, "New simple Lie algebras of prime characteristic," Trans. Amer. Math. Soc., **89**, 421-449 (1958).
6. Sh. M. Kasymov, "To the theory of n-Lie algebras," Algebra and Logic, **26**, No. 3, 277–297 (1987).
7. Sh. M. Kasymov, "Conjugation of Cartan subalgebras in n-Lie algebras," Algebra and Logic, **34**, No. 4, 405–420 (1995).
8. N. Bourbaki, Algebra (Polynomials and Fields. Ordered Groups) [Russian translation], Nauka, Moscow (1965).