Journal of Mathematical Sciences, Vol. 100, No. 1, 2000

AUTOMORPHISMS OF ONE-ROOTED TREES: GROWTH, CIRCUIT STRUCTURE, AND ACYCLICITY

Said Sidki

UDC 512.544.42

A natural interpretation of automorphisms of one-rooted trees as output automata permits the application of notions of growth and circuit structure in their study. New classes of groups are introduced corresponding to diverse growth functions and circuit structure. In the context of automorphisms of the binary tree, we discuss the structure of maximal 2-subgroups and the question of existence of free subgroups. Moreover, we construct Burnside 2-groups generated by automorphisms of the binary tree which are finite state, bounded, and acyclic.

1. Introduction

Automorphisms of one-rooted infinite regular trees have been the focus of increasing investigations in recent years for their connections with problems in different areas of mathematics ranging from Group Theory to Dynamical Systems. As it is not our intention to give a survey of these developments, we have listed in the reference section publications which reflect the range of work that has been done.

Our main purpose is to apply basic notions such as growth and circuit structure from finite automata and their graphs to the study of the structure of the group of automorphisms $\mathcal{A}(Y)$ of one-rooted regular trees, where Y stands for the set of first-level vertices of the tree. These notions are directly applicable in view of a natural interpretation of tree automorphisms as output automata having Y as both the input and output alphabet. The set of states of $\alpha \in \mathcal{A}(Y)$ is denoted by $Q(\alpha)$, and the initial state is α. A group H of tree automorphisms is*state closed* provided $Q(\alpha)$ is contained in H for all $\alpha \in H$, and is *layered* provided the set of functions $\mathcal{F}(Y, H)$ is a subgroup of H. Layered subgroups are state closed. When Y is finite, the set of automorphisms α in $\mathcal{A}(Y)$ for which $Q(\alpha)$ is finite form the enumerable subgroup $F(Y)$ of *finite state automorphisms*, which is a layered group.

The infinite Burnside p-groups of Aleshin, Sushchanskii, Grigorchuk, and those of Gupta–Sidki, all of which are necessarily not linear, afford faithful representations into the group of finite-state automorphisms of a p-adic tree. One property some of these p-groups enjoy is that their proper quotients are finite and therefore are also solvable. Periodicity of these groups is not essential for guaranteeing the latter extremal property. As a matter of fact, recently a 2-generator torsion-free nonsolvable group all of whose proper quotients are solvable was constructed in [6] as a subgroup of the finite-state automorphisms of the binary tree F. The group of finite-state automorphisms also covers the linear phenomenon, for it was shown in [5] that there exists a faithful representation of the linear group $GL(n, \mathbb{Z})$ into the group of $F(Y)$, where Y has 2^n elements.

Let D be a directed graph, W a property of vertices, and v a fixed vertex. The *W-growth* of D starting at v, denoted by $\theta(W; k+1, v)$, is the number of distinct directed paths of length $k \geq 0$ which start at v and end at a vertex with property W; the paths may be self-intersecting. In the case of a tree automorphism α, the graph of the automaton with vertex set $Q(\alpha)$ will be referred to as the graph of α and $\theta(W; k+1, \alpha)$ as its growth function. One of the properties we consider is whether or not a state of α is active; for this property we simply write $\theta(k+1, \alpha)$.

Given a vertex property and a measure of growth, we prove in Sec. 2.4 that the automorphisms of the tree whose automata have the given growth form a layered subgroup. It is a difficult problem in general to distinguish isomorphically among groups related to different choices of vertex properties and measures of growth.

A finite directed graph can be represented by an adjacency matrix with nonnegative integer entries. An elementary result in Sec. 2.5 on the growth of the entries of powers of such matrices implies

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 58, Algebra-12, 1998.

Theorem A. *Let $\alpha \in F(Y)$ with m states. Then $\theta(W; k+1, \alpha)$ seen as a function of k either grows exponentially or has an eventually polynomial growth of degree at most $m-1$.*

The set of finite-state automorphisms α for which $\theta(k+1, \alpha)$ is eventually zero form a layered subgroup which is the *base group* $G(Y)$, while those α for which $\theta(k+1, \alpha)$ has polynomial growth of degree at most m form a layered subgroup $F_m(Y)$. The elements of $F_0(Y)$ are called *bounded* automorphisms.

A *simple circuit* in a graph D is a circuit without self-intersections, and its length is the number of distinct vertices lying on this circuit. The graph D has *m-circuit type* provided $m = c(D)$ is the maximum length of its simple circuits, and it is *acyclic* provided $m \leq 1$.

Let D be the graph of a tree automorphism α and assume that the simple circuits of D have largest length $c(D) = m$. If D is a tree, or if the identity automorphism e is one of the states of α and the only circuit is from e to itself, then we say that α has *0-circuit type*. Otherwise, α has *m-circuit type* where $m = c(\alpha) \geq 1$. Furthermore, α is called *acyclic* if $c(\alpha) \leq 1$.

We observe that the tree automorphisms with $c(\alpha)$ bounded above by m do not form in general a subgroup. However, we prove in Sec. 2.5 that elements of $F(Y)$ which have circuit type at most 1 do form a subgroup $F_{*,1}(Y)$.

In Sec. 3, we prove that the cycle structures for finite-state bounded automorphisms behave quite well with respect to the group operation. As a matter of fact, the following holds:

Theorem B. *Let m be a natural number. Then the set*

$$F_{0,m}(Y) = \{\alpha \in F_0(Y) \mid c(\alpha) = 0, \text{ or } c(\alpha) \text{ divides } m\}$$

is a subgroup of $F_0(Y)$ and it is layered.

In Secs. 4–6, considerations are restricted to the automorphism group of the binary tree. As a test of the strength of the notions introduced so far, we undertake in Sec. 6 a detailed study of the group $F_{0,1}$ of automorphisms of the binary tree which are finite state, bounded, and acyclic.

Let $Y = \{0, 1\}$, and let σ denote both the transposition $(0, 1)$ and its so-called rigid extension to an automorphism of the binary tree. The group of automorphisms $\mathcal{A} = \mathcal{A}(Y)$ is a recursive wreath product $\mathcal{A} = \mathcal{A}\ wr\langle\sigma\rangle$. One special element of $\mathcal{A}$ is the *binary adding machine* which corresponds to adding 1 modulo 2. This machine is represented by $\tau = (e, \tau)\sigma$, and it is easy to see that $\tau \in F_{0,1}$. One of the characteristic properties of τ is that its centralizer subgroup in $\mathcal{A}$ is isomorphic to the ring of dyadic integers. The special properties of $\langle\tau\rangle$ are used in Sec. 4 to distinguish between $F_{0,1}$ and F_0.

Theorem C. *The group $F_{0,1}$ is not isomorphic to F_0.*

Moreover, it is possible to obtain the following nonfreeness criterion.

Theorem D. *Let H be a noncyclic group of finite-state binary tree automorphisms which contains $\langle\tau\rangle$. Then H is not a free group.*

In Sec. 5, maximal 2-subgroups of layered groups are considered. Define $\omega = (\sigma, \omega)$, an involutary element of $F_{0,1}$.

Theorem E. *Let $\mathcal{H}$ be a layered group of binary tree automorphisms. Assume that $\mathcal{H}$ contains the base group G and the element $\omega = (\sigma, \omega)$. Then there are infinitely many conjugacy classes of maximal 2-subgroups, and infinitely many conjugacy classes of maximal locally finite subgroups in $\mathcal{H}$.*

In Sec. 6, the group $F_{0,1}$ is shown to factor as a product of certain special subgroups. In the same section we prove

Theorem F. *The base subgroup G is a maximal locally finite subgroup of $F_{0,1}$.*

In Secs. 6.2 and 6.3, we discuss a number of nontorsion criteria for subgroups of $F_{0,1}$. These results are obtained in the context of constructing infinite Burnside 2-groups within $F_{0,1}$. The following group is the first of an infinite family of such examples.

Theorem G. *Let $b = ((\sigma, \sigma), e)$ be an element of the base group G, and let $\beta = (b, \beta) \in F_{0,1}$. Then the group B generated by the states of β is an infinite Burnside 2-group.*

We remark that Grigorchuk's Burnside 2-group [7] defined on the binary tree and generated by the states of $\alpha = (e, (\sigma, (\sigma, \alpha)))$ belongs properly to $F_{0,3}$, since $c(\alpha) = 3$.

It was shown in [10] that given a tree with alphabet $Y = \{0, 1, \dots, n-1\}, n \geq 3$, and given its two automorphisms, the rigid extension of the permutation $\sigma = (0, 1, \dots, n-1)$ and $\gamma = (\gamma, \sigma, e, \dots, e, \sigma^{n-1})$, the group $\langle \sigma, \gamma \rangle$ is an infinite Burnside n-subgroup of $F_{0,1}(Y)$. Thus, with this last construction, it is proven that $F_{0,1}(Y)$ contains infinite Burnside groups for all $|Y| \geq 2$.

Acknowledgment. Support from the CNPq and FAPDF of Brazil are acknowledged.

2. Tree Automorphisms

2.1. Preliminaries. A one-rooted regular tree $\mathcal{T}$ may be identified with the monoid $\mathcal{M}$ freely generated by a set Y and ordered by the relation $v \leq u$ if and only if u is a prefix of v; the identity element of $\mathcal{M}$ is the empty sequence ϕ.

Let $\mathcal{A} = \mathrm{Aut}(\mathcal{T})$ be the automorphism group of the tree $\mathcal{T}$. The permutations $P(Y)$ of the set Y can be extended "rigidly" to automorphisms of $\mathcal{A}$ by

$$(y.u)\sigma = (y)\sigma.u, \quad \forall y \in Y, \quad \forall u \in \mathcal{M},$$

and this gives us an embedding of $P(Y)$ into $\mathcal{A}$.

An automorphism $\alpha \in \mathcal{A}$ induces a permutation $\sigma_\phi(\alpha)$ on the set Y, which we identify with its rigid extension to the whole tree. Therefore, the automorphism affords the representation $\alpha = \alpha' \sigma_\phi(\alpha)$, where α' fixes Y pointwise. Furthermore, α' induces for each $y \in Y$ an automorphism $\alpha'(y)$ of the subtree whose vertices form the set $y.\mathcal{M}$. On using the canonical isomorphism $yu \to u$ between this subtree and the tree $\mathcal{T}$, we may consider (or renormalize) α' as a function from Y into $\mathcal{A}$; in notational form, $\alpha' \in \mathcal{F}(Y, \mathcal{A})$. Thus, the group $\mathcal{A}$ factors as

$$\mathcal{A} = \mathcal{F}(Y, \mathcal{A}).P(Y).$$

It is convenient to denote α by $\alpha(\phi)$ and $\alpha'(y)$ by $\alpha(y)$. In order to describe $\alpha(y)$, we use the same procedure as in the case of α. Successive applications produce the set

$$\Sigma(\alpha) = \{\sigma_u(\alpha) \mid u \in \mathcal{M}\}$$

of permutations of Y which describes faithfully the automorphism α. Another by-product of the procedure is the set *states* of α:

$$Q(\alpha) = \{\alpha_u \mid u \in \mathcal{M}\}.$$

States will be considered with respect to a number of properties such as the property of being active. A state α_u of a tree automorphism α is said to be *active* provided $\sigma_u(\alpha) \neq e$.

The definition of the product of automorphisms implies the following important properties of the Q function:

$$\begin{aligned} Q(\alpha^{-1}) &= Q(\alpha)^{-1}, \\ Q(\alpha\beta) &\subseteq Q(\alpha)Q(\beta), \forall \alpha, \beta \in \mathcal{A}. \end{aligned}$$

Given a group H and a set Y, define the groups

$$\begin{aligned} \mathcal{F}_0(Y, H) &= H,\ \mathcal{F}_1(Y, H) = \mathcal{F}(Y, H), \\ \mathcal{F}_i(Y, H) &= \mathcal{F}(Y, \mathcal{F}_{i-1}(Y, H)), \forall i \geq 1. \end{aligned}$$

Given a subgroup H of $P(Y)$, we may define $H^\#$ to be the subgroup of $\mathcal{A}$ generated by $\mathcal{F}_i(Y, H)$ for all $i \geq 0$. The factorization

$$H^\# = \mathcal{F}(Y, H^\#).H$$

can be verified directly.

A subgroup L of the tree automorphisms $\mathcal{A}$ is called a *layer subgroup,* or simply *layered,* provided $\mathcal{F}(Y, L)$ is a subgroup of L. Thus $H^\#$ is an example of a layer subgroup. If $H = P(Y)$, then $H^\#$ is called the *base group* and is denoted by $G(Y)$.

Now let $G_{0,k-1}(Y)$ be the subgroup of $G(Y)$ generated by $\mathcal{F}_{i-1}(Y, P(Y))$ for all $0 \leq i \leq k-1$. Then $G_{0,k-1}(Y)$ is the group of automorphisms of the subtree formed by the vertices u of length at most k, and $G(Y)$ is the union of $G_{0,k-1}(Y)$ for $k \geq 1$. A nontrivial element h of $G(Y)$ has *depth* $k-1$ provided k is the least integer such that $h \in G_{0,k-1}(Y)$; thus, nontrivial elements of $P(Y)$ have depth zero. Since

$$\mathcal{A} = \mathcal{F}_k(Y, \mathcal{A}).G_{0,k-1}(Y),$$

the decomposition of an automorphism α can be continued to the kth level of the tree in the form $\alpha = f_k(\alpha)\gamma_{k-1}(\alpha)$, where $f_k(\alpha) \in \mathcal{F}_k(Y, \mathcal{A})$ and $\gamma_{k-1}(\alpha) \in G_{0,k-1}(Y)$.

The product of two automorphisms α, β, is developed as follows:

$$\alpha\beta = f_k(\alpha\beta)\gamma_{k-1}(\alpha\beta),$$

where $f_k(\alpha\beta) = f_k(\alpha) \cdot \gamma_{k-1}(\alpha) f_k(\beta)\gamma_{k-1}(\alpha)^{-1}$, and $\gamma_{k-1}(\alpha\beta) = \gamma_{k-1}(\alpha)\gamma_{k-1}(\beta)$. Also, $\alpha^{-1} = f_k(\alpha^{-1}) \cdot \gamma_{k-1}(\alpha)^{-1}$, where

$$\begin{aligned} f_k(\alpha^{-1}) &= \gamma_{k-1}(\alpha)^{-1} f_k(\alpha)^{-1}\gamma_{k-1}(\alpha), \\ \gamma_{k-1}(\alpha^{-1}) &= \gamma_{k-1}(\alpha)^{-1}. \end{aligned}$$

Given α, we define $\alpha^{(0)} = \alpha$, $\alpha^{(1)} = f \in \mathcal{F}(Y, \mathcal{A})$, where $f(y) = \alpha$ for all $y \in Y$, and define inductively $\alpha^{(k)} = f \in \mathcal{F}_k(Y, \mathcal{A})$ such that $f(y) = \alpha^{(k-1)}$ for all $y \in Y$.

When $Y = \{0, 1\}$, the tree $\mathcal{T}$ is the binary tree. Here the symbol σ is reserved to indicate the transposition $(0, 1)$ and its rigid extension to a tree automorphism. For any binary tree automorphism α, define inductively the automorphisms $\alpha_0 = \alpha, \alpha_1 = (e, \alpha_0)$, and $\alpha_k = (e, \alpha_{k-1})$ for $k \geq 1$. We note that for $k \geq 0$, $G_{0,k}(Y)$ is generated by $\{\sigma_i \mid 0 \leq i \leq k\}$ and is isomorphic to the k-fold wreath product of the cyclic group of order two. We also note that the set $\{\sigma^{(i)} \mid i \geq 0\}$ generates freely an elementary abelian 2-group.

2.2. Automata representation. Automorphisms of the tree $\mathcal{T}$ can be interpreted in terms of output (Mealey) automata. Such an automaton is a Turing machine defined by a sextuple $(Q, L, \Gamma, f, l, q_0)$, where Q is the set of states, L is the input alphabet, Γ is the output alphabet, $f : Q \times L \to Q$ is the state transition function, $l : Q \times L \to \Gamma$ is the output function, and q_0 is the initial state. For a given automorphism α of the tree $\mathcal{T}$, the input and output alphabets are the same set Y, and its set of states is the set $Q(\alpha)$ defined above. The transition function is defined by

$$y : \alpha(u) \to \alpha(u.z)$$

and the output function by

$$\alpha(u) : y \to z,$$

where z is the image of y under $\sigma_u(\alpha)$.

Proposition 1. *Let Y be a finite set and $F(Y)$ be the set of finite-state automorphisms of the tree $\mathcal{T}(Y)$. Then $F(Y)$ is an enumerable layer group which factors as $F(Y) = \mathcal{F}(Y, F(Y))P(Y)$.*

Proof. This result is a direct consequence of the two properties of the Q state function above and of the enumerability of finite automata.

2.3. State-closed subgroups. It is evident that if L is a subgroup of the tree automorphisms $\mathcal{A}$, the states of its elements are not necessarily again elements of L. In this regard, we recall that the group L is said to be *state closed* provided $Q(\alpha) \subseteq L$ for all $\alpha \in L$.

Proposition 2. (i) *The set of state-closed subgroups of $\mathcal{A}$ form a lattice.*

(ii) *The layer subgroups of $\mathcal{A}$ are state closed. Furthermore, if Y is finite, then the layer subgroups form a lattice.*

(iii) *Any subgroup of the group of permutations $P(Y)$ is state closed.*

(iv) *Let $\alpha \in \mathcal{A}$. Then $\langle\alpha\rangle$ is state closed if and only if $\alpha = f.\sigma$, where f is a map such that $f(y) \in \langle\alpha\rangle$ for all $y \in Y$ and $\sigma \in P(Y)$.*

Proof. Let H, K be state-closed subgroups and $L = \langle H, K\rangle$. Also, let $\alpha = \beta\gamma\ldots\beta'\gamma' \in L$, where $\beta,\ldots,\beta' \in H$ and $\gamma,\ldots,\gamma' \in K$. Since the state function is sub-multiplicative, it follows that $Q(\alpha)$ is contained in $Q(\beta)Q(\gamma)\ldots Q(\beta')Q(\gamma')$, a subset of $\langle H, K\rangle$. The other items are easy to prove.

Proposition 3. *Let $Y = \{0,1\}, g \in G = G(Y), K = \langle Q(g)\rangle$. Then,*

(i) *K is an abelian group if and only if $g \in \langle \sigma^{(i)} \mid i \geq 0\rangle$;*

(ii) *K is a nonabelian dihedral group if and only if $g = (g_0, g_1)\sigma$, where $g_0 \in \langle \sigma^{(1)}, \sigma\rangle$ and $g_1 = g_0^{-1}\sigma$.*

Proof. The group K is state closed. Since $a \in K$ implies that $\langle Q(a)\rangle$ is a subgroup of K, we conclude that $\sigma \in Q(h)$ for all nontrivial $h \in G$.

(i) Assume that K is an abelian group. Then σ commutes with g, and therefore $g_0 = g_1$. As $g_u \in Q(g)$, we obtain that $g_{u0} = g_{u1}$. If the depth of g is s, then $g \in \langle\sigma, \sigma^{(1)}, \ldots, \sigma^{(s)}\rangle$. Since $\langle\sigma, \sigma^{(1)}, \ldots, \sigma^{(s)}\rangle$ is an abelian group, the converse statement follows.

(ii) Assume that K is a nonabelian dihedral group. It is clear that $(e,\sigma), (\sigma, e)$, or $\sigma^{(1)} = (\sigma,\sigma) \in K$. Since $\sigma \in Q(g)$, it follows that if $(e,\sigma) \in K$, then (σ, e), $\sigma^{(1)} \in K$. In all cases, $\sigma^{(1)} \in K$. Now a central involution is a square. Therefore, no active involution can be central; in particular, σ is a noncentral involution. The centralizer of σ in K is $\langle\sigma^{(1)}, \sigma\rangle$.

Let $a \in K$ such that $a^2 = (\sigma, \sigma)$. Then $a = (a_0, a_1)\sigma$. Therefore, $a^2 = (a_0a_1, a_1a_0) = (\sigma,\sigma), a_0a_1 = a_1a_0 = \sigma$. Also, from $a^\sigma = (a_1, a_0)\sigma = a^{-1} = (a_1^{-1}, a_0^{-1})\sigma$, we conclude that $a = (a_0, a_0\sigma)\sigma$, $a_0^2 = e$, and a_0 commutes with σ. As $a_0 \in K$, $a_0 \in \langle\sigma^{(1)}, \sigma\rangle$. Hence $o(a) = 4$.

Assume that g is inactive. Then $g = a^{2i} = \sigma^{(1)i}$ and $Q(g) = \{e, \sigma\}$; a contradiction. Therefore g is active. Now $g = a^i$, or $a^i\sigma$. In the first case, i is odd and a may be taken to be g. In the second case, i is even and K is abelian, which is a contradiction.

2.4. Growth functions and characters. Let D be the graph of the automata which represents a tree automorphism α. A vertex v of this graph is said to be *active* provided the state α_v associated to the vertex v is active. For the property W, the vertex v is active, and the notation for the growth function is simplified as

$$\theta(k+1,\alpha) = \#\{u : \sigma_u(\alpha) \neq e, \mid u \mid = k\} \quad \text{for all} \quad k \geq 0.$$

Thus $\theta(k+1,\alpha)$ is the number of entries of $f_k(\alpha)$ which, as automorphisms of the tree $\mathcal{T}$, are active on the 1st level; if $|Y| = n$, then $\theta(k+1,\alpha) \leq n^k$.

In the above definition the condition $\sigma_u(\alpha) \neq e$ is equivalent to $\alpha_u \notin \mathcal{F}(Y, \mathcal{A})$. More generally, let $\mathcal{H}$ be a subgroup of $\mathcal{A}$ and let W indicate the property that $\alpha_u \notin \mathcal{H}$. In view of this, we may define for the growth function

$$\theta_{\mathcal{H}}(k+1,\alpha) = \#\{u : \alpha_u \notin \mathcal{H}, \mid u \mid = k\} \quad \text{for all} \quad k \geq 0$$

which measures the *$\mathcal{H}$-inactivity* of α. When $\mathcal{H}$ is the identity subgroup, we define for $k \geq 0$

$$\xi(k+1,\alpha) = \#\{u : \alpha_u \neq e, |u| = k\},$$

the *frequency* of nontrivial states of α. It is clear that $\theta(k,\alpha) \leq \xi(k,\alpha)$.

The function $\theta_{\mathcal{H}}(k,-) : \mathcal{A} \to \mathbb{N}$ satisfies the following properties: for all $\alpha, \beta \in \mathcal{A}$

$$\begin{aligned} \theta_{\mathcal{H}}(k,\alpha\beta) &\leq \theta_{\mathcal{H}}(k,\alpha) + \theta_{\mathcal{H}}(k,\beta) \\ \theta_{\mathcal{H}}(k,\alpha^{-1}) &= \theta_{\mathcal{H}}(k,\alpha). \end{aligned}$$

To justify these assertions, first we observe that for $\alpha, \beta \in \mathcal{F}_k(Y, \mathcal{A})$, we have

$$\{u : (\alpha\beta)_u \notin \mathcal{H}\} \subseteq \{u : \alpha_u \notin \mathcal{H} \text{ or } \beta_u \notin \mathcal{H}\}.$$

Since $f_k'(\alpha) = \gamma_{k-1}(\alpha)f_k(\alpha)\gamma_{k-1}(\alpha)^{-1}$ is simply $f_k(\alpha)$ with its entries permuted by $\gamma_{k-1}(\alpha)^{-1}$, we have that $f_k(\alpha)$ and $f_k'(\alpha)$ have the same number of entries which do not belong to $\mathcal{H}$. Now as multiplication in $\mathcal{F}_k(Y, \mathcal{A})$ is effected coordinate-wise, the above properties follow.

The function $\theta_{\mathcal{H}}(k,\alpha)$ is not conjugate invariant in general. For instance, consider σ and $\lambda=(\lambda,\lambda)\sigma$, automorphisms of the binary tree. Then $\lambda=\sigma^{(\lambda,e)}$ and we have $\theta(k,\sigma)=0$ for all $k\geq 2$, while $\theta(k,\lambda)=2^{k-1}$ for all $k\geq 1$. A conjugate-invariant function is obtained upon replacing θ by

$$\widetilde{\theta_{\mathcal{H}}}(k,\alpha)=\max\{\theta_{\mathcal{H}}(k,\gamma)\mid \gamma \text{ is conjugate to } \alpha\}.$$

Norms and metrics may be defined on $\mathcal{A}$ using the function $\theta_{\mathcal{H}}(k,\alpha)$. For example, for $n\geq 1$, let $s_k=(n-1)/n^{2k-1}$ and define

$$\mid\alpha\mid=\sum\{\theta_{\mathcal{H}}(k,\alpha)s_k : k\geq 1\},\ d(\alpha,\beta)=\mid\alpha\beta^{-1}\mid.$$

Remark. Bhatacharjee, in proving in [2] the ubiquity of free subgroups within $\mathcal{A}$, introduces the function $m(\alpha,\beta)=\min\{|u|\mid (u)\alpha\neq(u)\beta\}$ for $\alpha\neq\beta$, which she uses to define the invariant metric

$$d(\alpha,\beta)=\begin{cases}0 & \text{if}\quad \alpha=\beta,\\ 2^{-m(\alpha,\beta)} & \text{if}\quad \alpha\neq\beta.\end{cases}$$

Let $\mathbf{B}$ be a set of functions defined on the natural numbers, contained in $\mathcal{F}(\mathbb{N},\mathbb{N})$, which satisfy the following two properties:

(i) the constant function $0\in\mathbf{B}$,

(ii) $(\forall a,b\in\mathbf{B})(\exists q\in\mathbf{B})(a+b\leq q)$.

A first example of such a class is the set $\mathbf{Z}$ of eventually zero functions. A second example is the set of functions with growth k^m, where for given $m\geq 0$,

$$\mathbf{P}_m=\{f: f(k)\leq ck^m \quad\text{for some}\quad c>0\},$$

and a third is $\mathbf{P}=\bigcup\{\mathbf{P}_m\mid m\geq 0\}$, which consists of functions of *polynomial growth.* In the case $\mathcal{H}=\mathcal{F}(Y,\mathcal{A})$, we use the same symbol $\mathbf{B}$ for the class of functions as well as for the corresponding group.

Proposition 4. *Let $\mathcal{H}$ be a subgroup of $\mathcal{A}$ and let $\mathbf{B}$ be a set of functions as above. Then* $\mathbf{B}(\mathcal{H})=\{\alpha\in\mathcal{A}\mid (\exists b\in\mathbf{B})(\forall k\geq 0)(\theta_{\mathcal{H}}(k+1,\alpha)\leq b)\}$ *is a layer subgroup of $\mathcal{A}$.*

Let Y be a finite set, $F_m(Y)$ the set of finite-state automorphisms α whose activity functions $\theta(k,\alpha)$ belong to $\mathbf{P}_m$, and let $F_{\Pi}(Y)=\bigcup\{F_m(Y)\mid m\geq 0\}$. A special instance of the previous proposition is

Corollary 5. *The set $F_m(Y)$ is a layered subgroup of $F(Y)$ for all $m\geq 0$.*

Problem 1. Let the class of functions $\mathbf{B}$ be fixed and let $\mathcal{H},\mathcal{K}$ be subgroups of $\mathcal{A}$. When are $\mathbf{B}(\mathcal{H})$ and $\mathbf{B}(\mathcal{K})$ equal? isomorphic?

Let Y be finite, $H\leq P(Y)$, U an abelian group, and φ a homomorphism from H into U. Define a sequence of *characters* φ_k from $\mathbf{B}(H)$ into U as follows:

$$\varphi_1(\alpha)=\varphi(\sigma_\phi(\alpha)),\ \varphi_{k+1}(\alpha)=\prod\{\varphi(\sigma_u(\alpha))\mid |u|=k\}.$$

Associated to these characters are the normal subgroups

$$N(\varphi)=\{\alpha\mid(\exists m)(\forall k\geq m)(\varphi_k(\alpha)=e)\},$$
$$\tilde{N}(\varphi)=\{\alpha\mid(\exists m)(\forall k\geq m)(\varphi_k(\alpha)=\varphi_m(\alpha))\},$$

which are layer but not state-closed subgroups of $\mathcal{A}$. Moreover, $N(\varphi)$ admits a *total character* defined by $\hat{\varphi}(\alpha)=\prod\{\varphi_k(\alpha)\mid k\geq 1\}$.

For the binary tree, let $U=\{\bar{0},\bar{1}\}$ be the integers modulo 2 and φ the isomorphism from $\langle\sigma\rangle$ onto U. Then $\varphi_k(\alpha)=\theta(k,\alpha)$ modulo 2, which we denote by $\bar{\theta}(k,\alpha)$, and denote the corresponding total character by $\hat{\theta}(\alpha)=\sum\{\bar{\theta}(k,\alpha)\mid k\geq 1\}$.

2.5. Adjacency matrices and circuits. Given a finite directed graph D having k vertices and an initial vertex v, enumerate its vertices as $V = \{v_1(=v), \dots, v_k\}$ and let s_{ij} denote the number of directed edges which connect v_i to v_j. This can be expressed in terms of m linear equations $v_i = \sum\{s_{ij}v_j : 1 \leq j \leq k\}$, where s_{ij} is the number of directed edges from v_i to v_j. The *adjacency matrix* of D is the $m \times m$ matrix $[D] = (s_{ij})$. If some of the vertices of D are distinguished by some property W, then the *W-frequency* can be recorded in a column vector $\vec{\theta}_1 = (\theta(W;1,v_j))$. For $k \geq 0$, let D_k denote the graph having the same vertices as D and having a directed edge connecting v_i to v_j for every directed path of length k connecting v_i to v_j. We note that $[D]^k = [D_k]$, the vector $\vec{\theta}_{k+1} = [D]^k\vec{\theta}_1$ represents the W-frequency of D_k, and the first entry of $\vec{\theta}_{k+1}$ is the W-growth function $\theta(W;k+1,v)$.

Theorem 6. *Let A be an $m \times m$ matrix with nonnegative integer entries, and define the functions $f_{ij}(k) = (A^k)_{ij}$. Then for each pair (i,j), the function f_{ij} either grows exponentially or is a polynomial function of degree at most $m-1$.*

Proof. We will proceed by induction on the dimension m.

For $m = 1$, the assertion is obvious. We will assume $1 < m$, and that the assertion is true for all $m' < m$.

Case 1. Let $i = j$. Clearly, $f_{ii}(k) = (A^k)_{ii} \geq A_{ii}^k$ and therefore f_{ii} grows exponentially unless $A_{ii} = 0, 1$. Assume that f_{ii} does not grow exponentially, so $f_{ii}(k) = 0, 1$, for all k, and if $f_{ii}(h) = 1$ for some h, then $f_{ii}(k) = 1$ for all $k \geq h$. Thus $f_{ii} = 0$, or f_{ii} is zero at first and then eventually becomes constant with value 1.

Case 2. Let $i \neq j$.

(2.1) Assume that f_{ii} grows exponentially. Then if for some h we have $f_{ij}(h) \neq 0$, then $f_{ij}(k+h) \geq f_{ii}(k)f_{ij}(h)$ and thus grows exponentially. Thus either f_{ij} is eventually 0 or it grows exponentially.

(2.2) Assume that $f_{ii} = 0$, or f_{ii} is eventually 1. Assume, furthermore, that f_{ij} does not grow exponentially. To simplify the notation, we will let $i = 1$, $j = 2$. Then, $A = \begin{bmatrix} a & v \\ w & B \end{bmatrix}$, where $a \in \{0,1\}$, $v = (v_1, v_2, \dots, v_{m-1})$, and B is an $(m-1) \times (m-1)$ matrix with nonnegative integer entries. Define the functions $g_{ij}(k) = (B^k)_{ij}$.

(2.2.1) Assume that $f_{11} = 0$. Then $A^k = \begin{bmatrix} 0 & vB^{k-1} \\ * & * \end{bmatrix}$ and $f_{12}(k) = (vB^{k-1})_1 = \sum\{v_i g_{i1}(k-1) \mid 1 \leq i \leq m-1\}$. Since f_{12} is not exponential, by induction g_{i1} is eventually polynomial of degree at most $m-2$ whenever $v_i \neq 0$. Clearly, if $v_i = 0$ then f_{12} is not affected by g_{i1}.

(2.2.2) Assume that there exists an $s \geq 1$ such that $f_{11}(k) = 1$ for all $k \geq s$. Then $A^s = \begin{bmatrix} 1 & vB^{s-1} \\ * & * \end{bmatrix}$,

$$A^{s+1} = A^s A = \begin{bmatrix} 1 & vB^{s-1} \\ * & * \end{bmatrix} \begin{bmatrix} a & v \\ w & B \end{bmatrix} = \begin{bmatrix} 1 & v(1+B^s) \\ * & * \end{bmatrix},$$

$$A^{s+k} = \begin{bmatrix} 1 & v.(q_k(B) + B^{s+k-1}) \\ * & * \end{bmatrix},$$

where $q_k(B) = 1 + B + \ldots + B^{k-1}$. Therefore,

$$f_{12}(k) = (vq_k(B) + vB^{s+k-1})_1 = \sum\{v_i g_{i1}(j) \mid 1 \leq i \leq m-1, 0 \leq j \leq k-1\} + \sum\{v_i g_{i1}(s+k-1) \mid 1 \leq i \leq m-1\}.$$

By induction, whenever $v_i \neq 0$, g_{i1} is a polynomial of degree at most $m-2$, and so $\sum\{g_{i1}(j) : 0 \leq j \leq k-1\}$ is a polynomial in k of degree at most $m-1$. Hence f_{12} is a polynomial of degree at most $m-1$.

Corollary 7. *Let D be a finite graph with m vertices, W a vertex property, and v a fixed vertex of D. Then the W-growth function $\theta(W;k+1,v)$ either grows exponentially or is eventually a polynomial function of degree at most $m-1$.*

Let α be an automorphism of a one-rooted tree, and let D be the graph of the automata associated to α. When α is a finite-state automorphism we enumerate the states of α as $Q(\alpha) = \{\alpha_1(=\alpha), \alpha_2, \ldots, \alpha_m\}$, setting $\alpha_m = e$ whenever this element is a state of α; we hope that this notation α_i does not get confused with α_y, where $y \in Y$. Here the $m \times m$ adjacency matrix of α is $[\alpha] = [D] = (s_{ij})$; clearly, $\sum\{s_{ij} \mid 1 \leq j \leq m\} = n$. The column vector $\vec{\theta_1} = (\theta(W, 1, \alpha_j))$ registers whether or not α_j satisfies the W property.

Given the adjacency matrix of a finite-state automorphism, we conclude:

Theorem 8. *Let $\alpha \in F(Y)$ with m states. Then the function $\theta(W; k+1, \alpha)$ of k either grows exponentially or has an eventually polynomial growth of degree at most $m-1$.*

We can now prove that the groups $F_m(Y)$ of tree automorphisms with polynomial growth of degree m are distinct.

Proposition 9. *Let Y be a finite set with at least two elements. Then groups of finite-state automorphisms $F_m(Y)$ are distinct for all integers $m \geq 0$.*

Proof. We will prove the assertion for the binary tree, the general case being similar.

Let $m \geq 0$. Consider the $(m+2)$-dimensional upper triangular matrix $\mathrm{S} = \begin{bmatrix} \mathrm{T} & u \\ 0 & 2 \end{bmatrix}$, where $\mathrm{T} = \mathrm{I} + \sum \mathrm{E}_{i,i+1}$ is $(m+1)$-dimensional, E_{ij} is the elementary transformation with 1 in the ijth position, and $u = \begin{bmatrix} 0_m \\ 1 \end{bmatrix}$. Also let $\vec{\theta_1} = \begin{bmatrix} 0_m \\ 1 \\ 0 \end{bmatrix}$. Then the pair $(\mathrm{S}, \vec{\theta_1})$ represents an automorphism of the binary tree having $m+2$ states. Now, $\mathrm{S}^k = \begin{bmatrix} \mathrm{T}^k & * \\ 0 & * \end{bmatrix}$ and $\mathrm{T}^k = \mathrm{I} + \sum \binom{k}{j} \mathrm{E}_{i,i+j}$. The first term of the vector $\mathrm{S}^k\vec{\theta_1}$ is the $(m+1)$-th term of $\mathrm{T}^k \begin{bmatrix} 0_m \\ 1 \end{bmatrix}$ and so, $\theta(1+k, \alpha) = \binom{k}{m}$, which is a polynomial in k having degree m when $k \geq m$.

A W-circuit is a circuit in the graph which passes through a W-vertex. Geometrically, the graphs with polynomial growth are described by

Proposition 10. *Let D be a finite graph, W a vertex property, and v a fixed vertex of D. Then the function $\theta(W; k+1, v)$ has polynomial growth if and only if no W-circuit in the graph D contains two distinct W-circuits.*

Remark. (i) Let Y be finite. Then the set of finite-state automorphisms of 0-circuit type is the base group $G(Y)$.

(ii) If Y is infinite, then the automorphisms of 0-circuit type of the binary tree do not form a group. An example is the following: let $\alpha = (e, (\alpha, e))\sigma$, $\beta = (\beta^2, \beta^3)\sigma$, and $\delta = (\beta^{-1}, \beta\alpha)\sigma$. Then α is finite state and has 2-circuit type, while the graphs of β and δ are infinite trees without loops; that is, β and δ are 0-circuit types. Now, $\delta^2 = (\alpha, \beta\alpha\beta^{-1})$, which has α as one of its states, and therefore δ^2 is not of 0-circuit type.

Proposition 11. *Let Y be a finite set. Then set of automorphisms in $F(Y)$ having circuit type at most* 1 *forms a layer group $F_{*,1}(Y)$.*

Proof. (i) Let α be an automorphism of the tree. We recall the decomposition $\alpha = f\sigma$, where f is the first-level pointwise stabilizer and σ is a permutation of Y. Then α is acyclic if and only if $f(y)$ is acyclic for all $y \in Y$.

(ii) Assume by contradiction that there exist α, β acyclic automorphisms of the tree such that $c(\alpha\beta) \geq 2$. Choose such elements α, β for which $|Q(\alpha)| + |Q(\beta)|$ is minimum. Decompose $\alpha = f\sigma, \beta = g\sigma'$. Then $\alpha\beta = h\sigma''$, where $h = fg^{\sigma^{-1}}, \sigma'' = \sigma\sigma'$, and $h(y) = f(y)g^{\sigma^{-1}}(y)$.

Let $y \in Y$ such that $h(y)$ is not acyclic. Since

$$Q(f(y)) \subseteq Q(\alpha),\ Q(g^{\sigma^{-1}}(y)) \subseteq Q(\beta),$$

by the minimality of the counterexample it follows that

$$|Q(f(y))| + \left|Q(g^{\sigma^{-1}}(y))\right| = |Q(\alpha)| + |Q(\beta)|$$

and so

$$|Q(f(y))| = |Q(\alpha)|, \left|Q(g^{\sigma^{-1}}(y))\right| = |Q(\beta)|.$$

Now since α, β are acyclic, we have $f(y) = \alpha, g^{\sigma^{-1}}(y) = \beta$, and $h(y) = \alpha\beta$. Thus for all $y \in Y$, the entry $h(y)$ is either acyclic or $h(y) = \alpha\beta$. But clearly $\alpha\beta$ is acyclic; a contradiction.

(iii) The proof that the inverse of an acyclic automorphism is acyclic proceeds in a similar manner to the previous item.

Remark. The product of two finite-state automorphisms having 2-circuit type can have larger circuit type: let $\alpha = (\alpha, \alpha\sigma)$, $\beta = (\sigma\beta, \beta\sigma)$ be automorphisms of the binary tree. Then both α and β have finite number of states and have 2-circuit type, while the product $\alpha\beta$ is of 4-circuit type.

3. Bounded Automorphisms

A tree automorphism α is called *bounded* provided its activity growth function $\theta(k, \alpha)$ is bounded ($\theta(k, \alpha) \in \mathbf{P}_0$). Recall that $\theta(k, \alpha) \leq \xi(k, \alpha)$ for all k. We will show that for a bounded finite-state automorphism, the state frequency $\xi(k, \alpha)$ is also bounded ($\xi(k, \alpha) \in \mathbf{P}\{e\}$).

Proposition 12. $F_0(Y) = F(Y) \cap \mathbf{P}_0 = F(Y) \cap \mathbf{P}_0\{e\}$.

Proof. Let the activity function $\theta(k, \alpha)$ be bounded above by c, and assume that α has unlimited state frequency $\xi(k, \alpha)$. Then since α has a finite number of states, there exists a level k such that some nontrivial state $\beta \in Q(\alpha)$ occurs more than c number of times on that level. But the fact $\beta \neq e$ implies that there exists an index u of length h for which $\sigma_u(\beta)$ is nontrivial. Thus at level $k + h$, the activity of α is greater than c; a contradiction.

Proposition 13. *Let $\alpha \in F(Y)$ with $e \in Q(\alpha)$, and let S be its adjacency matrix. Then*
(i) $\alpha \in F_0(Y)$ *if and only if there exist positive integers $p < q$ such that $\mathrm{S}_1^p = \mathrm{S}_1^q$;*
(ii) $\alpha \in G(Y)$ *if and only if $\mathrm{S}_1^p = 0$ for some p.*

Proof. Let $|Q(\alpha)| = m$. Since $e \in Q(\alpha)$, the adjacency matrix of α has the block form $\mathrm{S} = \begin{bmatrix} \mathrm{S}_1 & v \\ 0_{m-1} & n \end{bmatrix}$ and $\xi_1 = \begin{bmatrix} 1_{m-1} \\ 0 \end{bmatrix}$. Then $\xi_{1+k} = \mathrm{S}^k \xi_1$ and for $1 \leq i \leq m-1$, $\xi(1+k, \alpha_i)$ is the ith entry of $\mathrm{S}_1^k [1_{m-1}]$.

(i) Assume that $\alpha \in F_0(Y)$. Then there exists a constant c such that the entries of S_1^k are bounded by c for all k. Therefore S_1^k may be seen as an $(m-1)^2$ vector with entries from $\{0, 1, \ldots, c\}$. As k varies over all the natural numbers, it follows that there exist natural numbers $p < q$ such that $\mathrm{S}_1^p = \mathrm{S}_1^q$. The other direction is clear.

(ii) By definition, $\alpha \in G(Y)$ if and only if there exists a level k where $\alpha_u = e$ for all u of length k, which is equivalent to saying that $\mathrm{S}_1^k = 0$.

Bounded automorphisms can be described geometrically as follows.

Corollary 14. *Let α be a bounded finite-state automorphism. Then no two different circuits in the graph of the automata of α pass through the same vertex or are connected by a directed path.*

One feature which distinguishes internally the group of bounded finite-state automorphisms F_0 from the group of finite-state automorphisms F of the binary tree is:

Proposition 15. *The base group G of automorphisms of the binary tree is self-normalizing in F_0.*

Proof. Let $\mu \in F$ such that $\mu^{-1}G\mu \leq G$. Then from Proposition 7.3 of [4], $\mu = \mu^{(m)}g$ for some natural number m and some $g \in G$. The states of μ are of the form $\mu^{(j)}g'$, where $j \leq m, g' \in G$. If some such state is the identity, then $\mu^{(j)} \in G$, and therefore $\mu \in G$.

3.1. Circuitous automorphisms. A nontrivial finite-state automorphism α is called *circuitous* provided it is *bounded* and α lies on a circuit in its own graph. An example of a circuitous automorphism α of the binary tree is defined as follows: let $g_0, g_{11} \in G$, and let

$$\alpha = (g_0, \alpha_1), \quad \alpha_1 = (\alpha_{10}, g_{11})\sigma, \quad \alpha_{10} = \alpha.$$

We note that in this example, α factors as $\alpha = \alpha' g$, where $\alpha' = (e, (\alpha, e))$ and $g = (g_0, (e, g_{11})\sigma) \in G$. Furthermore, we note that $\alpha'_{10} = \alpha, g_{10} = e$, and that conjugation of α' by g simply permutes the entries of α'.

To a circuitous automorphism α there is associated a complete chain of indices $C(\alpha) = \{v \in \mathcal{M} \mid \alpha_v$ is circuitous$\}$. There exists an index u of smallest length k such that $\alpha_u = \alpha$, and this index is unique. Note that $u^i \in C(\alpha)$, for all $i \geq 0$, and that for any index $v = us$, $\alpha_v = \alpha_s$ holds. Now, in the same manner as in the above example, α factors as $\alpha = \alpha' g$, where $\alpha' = (e, \ldots, e, \alpha_u, e, ., e)$ is an element of the kth stabilizer, $\alpha_u = \alpha$, and $g \in G$, where $g_u = e$. This is a normal form for the circuitous automorphism α. Since $\alpha' = \alpha . g^{-1}$, it follows that for any tree level k, if γ is an element of the kth stabilizer subgroup having as a unique nonbase entry a circuitous automorphism α, then γ can be written as $\gamma = h\alpha h'$, for some $h, h' \in G$.

Proposition 16. *The group $F_0(Y)$ is generated by the base group $G(Y)$ together with the set of circuitous automorphisms.*

Proof. Given $\delta \in F_0(Y)$; then δ determines a unique set $S = S(\delta)$ of indices u of minimal length where a state of δ is either an element of $G(Y)$ or is circuitous. This set S is a minimal connecting set for the free monoid $\mathcal{M}$ of indices. Let S' be the subset of S formed by indices u for which δ_u is circuitous. Define δ' as follows: for $u \in S$, let $\delta'_u = \delta_u$ if $u \in S'$, and $\delta'_u = e$ if $u \in S - S'$. Then there exists $h \in G(Y)$ such that $\delta = \delta' h$. Assume that $u \in S$ is of length k and that δ_u is circuitous. Then δ_u appears as an entry of $(e, \ldots, e, \delta_u, e, \ldots, e) \in \mathcal{F}_k(Y, \mathcal{A})$, the kth level stabilizer of the tree, and so it can be written as a product of a circuitous automorphism and an element of $G(Y)$. Thus δ' is a product of circuitous automorphisms and base elements, and therefore the same holds for δ.

Remark. Let $\delta \in F_0(Y)$, and let $s' = \mid S' \mid$, as in the previous proposition. Then δ can be expressed as a word $\delta = h\alpha h'\beta \ldots h''\gamma h'''$ where $h, h', h'', h''' \in G$, and $\alpha, \beta, \ldots, \gamma$ are s' circuitous automorphisms. Indeed, δ cannot be written using less than s' circuitous automorphisms, which justifies calling s' the *syllable length* of δ.

Lemma 17. *Let $\delta \in F_0(Y)$. There exists a tree level l such that $\delta = \delta' q$, where δ' is an element of the l-th stabilizer subgroup such that $\delta'(v)$ is either circuitous or a base element, and $q \in G_{0,l-1}(Y)$.*

Proof. First we observe that if δ is a base element or is circuitous, then for any level l, δ factors as asserted. Furthermore, if δ is circuitous, then since δ is bounded, there is a unique index u of length l such that $\delta'(u)$ is circuitous, whereas $\delta'(v)$ is a base element for all indices v of length l and $v \neq u$.

Now we proceed by induction on the number of circuitous states of δ and assume that $\delta \notin G(Y)$. There exists a tree level k and an index u of length k such $\delta = \delta' q$, where δ' is an element of the kth stabilizer subgroup such that $\delta'(u)$ is circuitous and $q \in G_{0,k-1}(Y)$. Since δ is bounded, the number of circuitous states of $\delta'(v)$ is less than that of δ for all $v \neq u$. Thus, by induction, for each index v of length k, there exists a level l_v such that $\delta'(v)$ has the required factorization. It follows from the first observation that there exists a common level l such that for all indices v of length l, $\delta'(v) = \delta''(v)q'(v)$, where $\delta''(v)$ is an element of the lth stabilizer subgroup and $q'(v) \in G_{0,l-1}(Y)$. Let $t = k + l$. Then $\delta = \delta'' q' . q = \delta'' . (q' q)$, where δ'' is and element of the tth stablizer subgroup and $q'q \in G_{0,t-1}$.

Whereas for $m \geq 2$, the finite-state automorphisms of circuit m-type do not form a group, the situation for the bounded automorphisms is nicer. Indeed, on defining for every integer $m \geq 0$,

$$F_{0,m}(Y) = \{\alpha \in F_0(Y) \mid c(\alpha) = 0, \text{ or } c(\alpha) \text{ divides } m\}.$$

we will show that

Theorem 18. *The set $F_{0,m}(Y)$ is a subgroup of $F_0(Y)$ and it is layered.*

Proof. (i) First we prove that if $g \in G$ and α is a circuitous automorphism with $c(\alpha) > 0$, then $c(\alpha g) = c(\alpha)$.

Assume that $g \in G$ has depth k, and let $\beta = \alpha g$. Let $v \in C(\alpha)$ be an index of length greater than k. Then $\beta_v = (\alpha g)_v = \alpha_v$. Therefore, $c(\beta) \geq c(\alpha_v) = c(\alpha)$. As $c(\alpha) = c(\beta g^{-1}) \geq c(\beta_v) = c(\beta)$, we conclude that $c(\beta) = c(\alpha)$.

(ii) Next we prove that for any circuitous automorphisms α, β, we have that

$$c(\alpha\beta) \mid \text{l.c.m.}(c(\alpha), c(\beta)).$$

Let $\gamma = \alpha\beta$. Then, $\gamma_v = \alpha_v \beta_{v'}$, where $(v')^\alpha = v$. The circuit type of γ_v is either $c(\gamma_v) = 0$, $c(\alpha_v)$, or $c(\beta_{v'})$, unless possibly when $v \in C(\alpha)$ and $v' \in C(\beta)$. Assume that $v' \in C(\beta)$ whenever $v \in C(\alpha)$. Let $m = c(\alpha)$, $n = c(\beta)$, $l = \text{l.c.m.}(m, n)$, $m' = l/m$, $n' = l/n$. Furthermore let $u \in C(\alpha)$ be of length m, $w \in C(\beta)$ of length n, and let $v = u^{m'}$. Then, $\gamma_v = \alpha_v \beta_{v'}$, where $v' = w^{n'}$. Thus, $\gamma_v = \alpha\beta = \gamma$, from which it follows that $c(\gamma)$ divides l.

Let α be a circuitous automorphism, and let $y \in Y$ be the unique index for which the state α_y is circuitous. Then α is said to be *strongly active* provided y is not fixed by $\sigma = \sigma_\phi(\alpha)$. We note that in the case of the binary tree, the two notions active and strongly active coincide.

Proposition 19. *A circuitous automorphism $\alpha \in F_0(Y)$ has infinite order if and only if at least one of its nonbase states α_v is strongly active.*

Proof. We will prove the statement for the binary tree. The argument in the general case is similar.

Assume that all nonbase states of α are inactive. Then, $\alpha = (g_0, \alpha_1)$ or (α_0, g_1), where $g_0, g_1 \in G$; we will assume the first form. Thus, $\alpha^{o(g_0)} = (e, \alpha_1^{o(g_0)})$. Since $\alpha_v = \alpha$ for some index v, a repetition of the previous step shows that α has finite order.

Assume now that some nonbase state α_v of α is active and choose v of minimum length. We may assume that $\alpha = (g_0, \alpha_1)\sigma$. Let $K(\alpha)$ be the subgroup generated by the base states of α; this subgroup is clearly finite. Now, $\alpha^2 = (g_0\alpha_1, \alpha_1 g_0)$. Thus, α^2 projects onto $\alpha_1 g_0$, and if the latter element is inactive, then it projects onto $\alpha_{10}h$ or $\alpha_{11}h$, where $h \in K(\alpha)$ has smaller depth than g_0, and either α_{10} or α_{11} is a nonbase state. If we continue this sequence of projections we will reach an active automorphism $\beta = \alpha_u h'$, with α_u a nonbase state of α and $h' \in K(\alpha)$; this β could be α itself. We repeat the same operation on β as was done previously for α. So $\beta^2 (= \alpha^4)$ projects eventually onto $\delta = \alpha_v h''$, which is an active automorphism where α_v is a nonbase state of α and $h'' \in K(\alpha)$. As the number of states of α is finite and $K(\alpha)$ is a finite group, successive repetitions will produce at some stage an active automorphism $\gamma = \alpha_w h'''$ that had appeared previously. Therefore, for some $s \geq 1$, γ^{2^s} projects eventually onto γ. This proves that γ has infinite order, and consequently, α itself has infinite order.

Remark. The above proof provides an algorithm for deciding the order of an element of $F_0(Y)$, once it is given as a word in elements of the base group and in circuitous automorphisms.

4. The Binary Addition Machine

The automorphism of the binary tree $\tau = (e, \tau)\sigma$ corresponds to adding 1 modulo 2, which explains its denomination as the binary adding machine. This automorphism has a number of special properties such as τ permutes transitively the kth-level vertices for all $k \geq 0$, and its character values $\bar{\theta}_k(\tau) = 1$ for all $k \geq 1$. Indeed, each of these properties characterizes conjugates of τ.

Proposition 20. *Let α be an automorphism of the binary tree. Then α is conjugate to τ if and only if α permutes transitively the k-th-level vertices for all $k \geq 0$.*

Proof. Assume that α permutes transitively the kth-level vertices for all $k \geq 0$. Then α is active, $\alpha = (\alpha_0, \alpha_1)\sigma$. When α is conjugated by $\beta = (\alpha_0, e)$ it transforms into $\alpha^\beta = (e, \alpha_1\alpha_0)\sigma$. The automorphism $\alpha_1' = \alpha_1\alpha_0$ also permutes transitively the kth-level vertices for all $k \geq 0$, and thus in particular it is active,

$\alpha'_1 = (\alpha'_{10}, \alpha'_{11})\sigma$. Now we conjugate α^β by $\beta' = (\delta, \delta), \delta = (\alpha'_{10}, e)$, thus reducing α'_{10} to the identity. In this manner we produce an infinite sequence of conjugators $\beta, \beta', \ldots$, whose product γ is a well-defined element of $\mathcal{A}$ which conjugates α to τ.

Corollary 21. *Let α be an automorphism of the binary tree. Then α is conjugate to τ if and only if $\bar{\theta}_k(\alpha) = 1$ for all $k \geq 1$.*

Proof. Assume that $\bar{\theta}_k(\alpha) = 1$ for all $k \geq 1$. In particular, α is active. We observe in the above proof that $\bar{\theta}_2(\alpha^\beta) = \bar{\theta}_1(\alpha_1\alpha_0) = \bar{\theta}_1(\alpha_1) + \bar{\theta}_1(\alpha_0) = \bar{\theta}_2(\alpha)$, and, more generally, $\bar{\theta}_k(\alpha^\beta) = \bar{\theta}_k(\alpha)$ for all $k \geq 1$. Indeed, the successive conjugations in the above proof preserve the characters $\bar{\theta}_k$.

The structure of the centralizer of the cyclic group $\langle\tau\rangle$ allows us to distinguish isomorphically between the group of finite-state automorphisms F and its subgroup F_0 of bounded automorphisms.

Theorem 22. *The group F_0 is not isomorphic to F.*

We recall the following facts from [5].

Proposition 23. *Let $\mathbb{Z}_2$ be the ring of dyadic integers, and $\mathbb{Z}_{(2)}$ the localization of the rationals at the prime $p = 2$. Then $C_A(\tau) = \langle\tau\rangle^* = \{\tau^\xi : \xi \in \mathbb{Z}_2\}$. Furthermore, if $\alpha \in \mathcal{A}$ is such that $C_\mathcal{A}(\alpha) = \langle\alpha\rangle^*$, then α is conjugate to τ. In addition, $C_F(\tau) = \{\tau^\xi : \xi \in \mathbb{Z}_{(2)}\}$.*

Next we prove

Proposition 24. *The cyclic group $\langle\tau\rangle$ is self-normalizing in F_0.*

Proof. We will show that $\langle\tau\rangle$ is self-centralizing in F_0. Let $\xi \in \mathbb{Z}_2$ and assume that $\tau^\xi \in F_0$. We know that $\tau^\xi \in F$ if and only if $\xi \in \mathbb{Z}_{(2)}$, a rational number with odd denominator. Then $\xi = s + t\varsigma$ where s, t are nonnegative integers, and $\varsigma = 1 + 2^k + 2^{2k} + \ldots$. Assume that $t \neq 0$. Then $\tau^\xi \in F_0$ if and only if $\gamma = \tau^\varsigma \in F_0$. Since γ satisfies $\gamma = \gamma^{(k)}\tau$, the set of states of γ is $Q(\gamma) = \{\gamma^{(k-i)}, \gamma^{(k-i)}\tau \mid 1 \leq i \leq k\}$. Since the identity element e is in $Q(\gamma)$ it follows that $\gamma\tau = e$.

Now let s be an odd integer, $s = 2t + 1$, and let $\alpha = (\alpha, \alpha\tau^t)$. Then α conjugates τ into τ^s. It can be checked directly that $Q(\alpha)$ is a subset of $\{\alpha\tau^i : 0 \leq i \leq t\}$. Now if $\alpha \in F_0$, then $\alpha\tau^i = e$ for some i; therefore α centralizes τ.

Proof of Theorem 22. Assume by contradiction that $F_0 \cong F$ and let $\alpha \in F$ be the image of τ under such an isomorphism. Then $\langle\alpha\rangle$ is self-centralizing in F. However, $C_F(\alpha)$ contains $\{\alpha^\xi : \xi \in \mathbb{Z}_{(2)}\}$, a free abelian group of infinite rank; a contradiction.

Proposition 25. *Let $\Upsilon =< \tau, G >$. Then Υ is a layer group, and any finitely generated subgroup of Υ is an extension of a torsion-free abelian group of finite rank by a finite 2-group.*

Proof. Note that $\mathcal{F}(Y, \langle\tau\rangle)$, which is a direct product of two copies of $\langle\tau\rangle$, is a subgroup of Υ. Let $f \in \mathcal{F}(Y, \Upsilon)$. Then $f(y)$ is a product of elements from G and $\mathcal{F}(Y, \langle\tau\rangle)$. Therefore $\mathcal{F}(Y, \Upsilon) \subseteq \Upsilon$. Given a finitely generated subgroup of Υ, there exists some level k such that Υ is contained in $\mathcal{F}_k(Y, \langle\tau\rangle)G_{0,k-1}$, which is clearly torsion-free by a finite group.

Since $\langle\tau\rangle$ has a restricted centralizer, one might hope to find an overgroup of it which is a free group of rank 2. However, this does not happen within the group of finite-state automorphisms F.

Theorem 26. *Let $H = \langle\tau, \alpha\rangle$ be a free group of automorphisms of the binary tree, having rank 2. Then $\langle\alpha_u, \tau \mid |u| = k\rangle$ is a free group of rank $2^k + 1$, for all $k \geq 1$.*

Proof. Let H_1 be the first-level stabilizer subgroup. Since τ is active, H_1 has index 2 in H, and therefore H_1 is a free group of rank 3. To exhibit the generators of H_1 we have to consider two cases according to whether or not α is active. (i) Assume that $\alpha = (\alpha_0, \alpha_1)$. Then $\alpha^\tau = (\alpha_1^\tau, \alpha_0)$, and $H_1 = \langle\tau^2, \alpha, \alpha^\tau\rangle$. As $\tau^2 = (\tau, \tau)$, H_1 is a sub-direct product of $L \times L$, where $L = \langle\alpha_0, \alpha_1, \tau\rangle$. (ii) Assume that $\alpha = (\alpha_0, \alpha_1)\sigma$. Then $\alpha\tau = (\alpha_0\tau, \alpha_1)$, $\tau\alpha = (\alpha_1, \tau\alpha_0)$, and $H_1 = \langle\tau^2, \alpha\tau, \tau\alpha\rangle$, which again is a sub-direct product of $L = \langle\alpha_0, \alpha_1, \tau\rangle$.

It is easy to see that $L = \langle \alpha_0, \alpha_1, \tau \rangle$ is a free group of rank 3. On replacing H by L in the above argument, we arrive at the fact that $M = \langle \alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}, \tau \rangle$ is a free group of rank 5. Iterating the argument k times produces $\langle \alpha_u, \tau \mid |u| = k \rangle$ is a free group of rank $2^k + 1$.

A direct consequence of the above is

Corollary 27. *Let $H = \langle \tau, \alpha \rangle$ be a noncyclic group of finite-state automorphisms of the binary tree. Then H is not a free group.*

A faithful representation of $GL(2, \mathbb{Z})$ into the finite-state automorphisms of the regular 4-tree was produced in [5]. Since the free group of rank 2 is a subgroup of $GL(2, \mathbb{Z})$, a faithful representation of this free group as a finite-state automorphism group is consequently obtained.

Aleshin asserted in [1] that the group generated by

$$\alpha = (\alpha, (\alpha^\sigma)^{(1)})\sigma, \ \beta = (\beta, (\beta^\sigma)^{(3)})\sigma$$

is free of rank 2. Unfortunately, we have not been able to understand his proof. At any rate, his generators are conjugates of τ. This is so because $\theta_1(\alpha) = 1, \theta_2(\alpha) = 1$, and

$$\theta_k(\alpha) = \theta_{k-1}(\alpha) + \theta_{k-1}((\alpha^\sigma)^{(1)}) = \theta_{k-1}(\alpha) + 2\theta_{k-2}(\alpha),$$

which is an odd number for $k \geq 3$. Similarly, $\theta_k(\beta)$ is an odd number for $k \geq 1$.

Andrew Brunner and I propose considering the simpler looking group generated by the three states of $\alpha = (\alpha^\sigma, \alpha^{(1)})\sigma$ as a candidate for a free group of rank 3.

Problem 2. Prove that a representation of the free group of rank 2 into the group $F_{\mathbb{N}}(Y)$ of finite-state automorphisms having polynomial growth cannot be faithful for any finite set Y.

In surveying the overgroups of $\langle \tau \rangle$, we find in $F_{0,2}$ a torsion-free group generated by τ and one of its conjugates, having a rich recursive structure (see [6]).

Theorem 28. *Let H be the group of automorphisms of the binary tree generated by $\tau = (e, \tau)\sigma$ and $\mu = (e, \mu^{-1})\sigma$. Then H is a subgroup of $F_{0,2}$, is just-nonsolvable, and is residually a "torsion-free solvable group."*

4.1. Maximal 2-subgroups of binary tree automorphisms. The group of automorphisms $\mathcal{A}$ of the binary tree was shown in [15] to contain nondenumerably many nonconjugate maximal 2-subgroups. The argument was built upon the observation that the product of $\omega = (\sigma, \omega)$ with any conjugate of σ has infinite order.

Theorem 29. *Let $\mathcal{H}$ be a layered group of binary tree automorphisms. Assume that $\mathcal{H}$ contains the base group G and the element $\omega = (\sigma, \omega)$. Then there are infinitely many conjugacy classes of maximal 2-subgroups and infinitely many conjugacy classes of maximal locally finite subgroups in $\mathcal{H}$.*

First we reprove

Proposition 30. *A 2-subgroup of binary tree automorphisms that contains $\omega = (\sigma, \omega)$ is not conjugate to any 2-subgroup of binary tree automorphisms which contains σ.*

Proof. We claim that $o(\omega\sigma^\alpha)$ is infinite for any $\alpha \in \mathcal{A}$. As we may consider α modulo the centralizer of σ, it can be assumed that $\alpha = (e, \alpha_1)$. Thus, $\omega\sigma^\alpha = (\omega\alpha_1, \sigma\alpha_1^{-1})\sigma$ and

$$(\omega\sigma^\alpha)^2 = (\omega\sigma^\beta, \ \sigma\alpha_1^{-1}\omega\alpha_1),$$

where $\beta = \alpha_1^{-1}$. Therefore, $o(\omega\sigma^\alpha) \geq 2o(\omega\sigma^\beta)$. We note that the conjugator α of σ has changed to β inside the bracket, and now the argument may be repeated to obtain $o(\omega\sigma^\alpha)$ is infinite.

Proposition 31. *Let $\mathcal{H}$ be a layered group containing the base group G. Let N be a 2-subgroup of $\mathcal{H}$. Then N is a maximal 2-subgroup (maximal locally finite 2-subgroup) of $\mathcal{H}$ if and only if*

(i) *$N = H \times K$, where H and K are nonconjugate maximal 2-subgroups (maximal locally finite 2-subgroups) of $\mathcal{H}$ or*

(ii) *N is conjugate to $L = (K \times K).\langle \sigma \rangle$, where K is a maximal 2-subgroup (maximal locally finite 2-subgroup) of $\mathcal{H}$.*

Proof. Let K be the pointwise stabilizer of $0, 1$, within N, and let K_0, K_1, be the projections of K on the first and second coordinates, respectively.

(i) If $N = K$, then $N \leq K_0 \times K_1$, a 2-subgroup of $\mathcal{H}$. If K_1 is a conjugate of K_0, then N is conjugate to a subgroup of the larger 2-group $L = (K_0 \times K_0).\langle\sigma\rangle$.

(ii) If $N \neq K$, then N contains an element $a = (a_0, a_1)\sigma$. On conjugating N by (a_0, e), we may assume that $a_0 = e$. Since $a = (e, a_1)\sigma$, it follows that $a^2 = (a_1, a_1)$ and $a_1 \in K_0 \cap K_1$. Now since a normalizes K, we have that $K_0 = K_1$. Therefore, N is embeddable in the 2-subgroup $L = (K_0 \times K_0)\langle\sigma\rangle$.

Proof of Theorem 29. Let N be a maximal 2-subgroup of $\mathcal{H}$ which contains ω. Define the following sequence of subgroups of $\mathcal{H}$, $N_{(0)} = N$, $N_{(i)} = (N_{(i-1)} \times N_{(i-1)})\langle\sigma\rangle$ for all $i \geq 1$. Then by the above result, these are maximal 2-subgroups of $\mathcal{H}$. Clearly, since $N_{(1)}$ contains σ, it is not conjugate to $N_{(0)}$. By a direct argument it can be shown that no two subgroups in the set $\{N_{(i)} \mid i \geq 0\}$ are conjugate.

The proof remains essentially the same when the condition "maximal 2-subgroup" is replaced by "maximal locally finite 2-subgroup."

5. The Group of Acyclic Finite-State Binary Automorphisms

Let Y be finite. Then by Proposition 11, the finite-state acyclic automorphisms form a group $F_{*,1}(Y)$. In the case of the binary tree we denote it simply by $F_{*,1}$. Recall that $F_{\mathbb{N},1}$ stands for the group of automorphisms which have polynomial growth and are acyclic. The automorphism $\lambda = \lambda^{(1)}\sigma$ is an element of $F_{*,1}$ and has exponential growth. Define by induction $\lambda_0 = \lambda$ and $\lambda_k = (e, \lambda_{k-1})$ for all $k \geq 1$. Also, let Λ be the group generated by λ_k for $k \geq 0$. We observe that Λ contains G properly and is locally a finite 2-group.

Proposition 32. *The group $F_{*,1}$ factors as $F_{*,1} = \Lambda F_{\mathbb{N},1}$. Furthermore, $\Lambda \cap F_{\mathbb{N},1} = G$.*

Proof. Let $\alpha \in F_{*,1}$ have exponential growth. Then it has some nontrivial state which is a vertex of a multiple loop. This state is the unique $\lambda = \lambda^{(1)}\sigma$. Now gathering to the left the occurrences of λ in the development of α, we obtain the factorization $\alpha = \alpha'\alpha''$, where $\alpha' \in \Lambda$ and $\alpha'' \in F_{\mathbb{N},1}$. The next assertion is easy to prove.

5.1. The group $F_{0,1}$ of bounded acyclic automorphisms.

In the rest of the paper, we limit our considerations to the group $F_{0,1}$ of finite-state bounded acyclic automorphisms of the binary tree.

Given a subgroup H of the base group G, we define the corresponding circuitous subgroups $H_c = \{\alpha \mid \alpha = (g, \alpha), g \in H\}$, $H_{\bar{c}} = \{\alpha \mid \alpha = (\alpha, g), g \in H\}$, and $\Upsilon = \langle\tau, G\rangle$. It is clear that H_c and $H_{\bar{c}}$ are both isomorphic to H.

Theorem 33. *Let $\hat{G} = \langle G, G_c\rangle, \check{G} = \langle G, G_{\bar{c}}\rangle, \Upsilon = \langle\tau, G\rangle$. Then the group $F_{0,1}$ admits the factorizations $F_{0,1} = \Upsilon\hat{G}\Upsilon = \Upsilon\check{G}\Upsilon$. Furthermore, $\Upsilon \cap \hat{G} = G = \hat{G} \cap \check{G}$.*

First we prove some lemmas.

Lemma 34. *The automorphism $\tau = (e, \tau)\sigma$ conjugates the subgroup H_c into $H_{\bar{c}}$. The groups $\hat{G}, \check{G}$ are layer groups.*

Proof. Let $\alpha = (g, \alpha) \in G_c$. Then $\alpha^\tau = (g, \alpha)^\tau = (g, \alpha^\tau)^\sigma = (\alpha^\tau, g) \in G_{\bar{c}}$.

The proof of the second assertion is a routine matter.

Lemma 35. *Let $\alpha \in F_{0,1}$ be a circuitous automorphism. Then α is one of four types: $\alpha = (g, \alpha) \in G_c$, $\alpha = (\alpha, g) \in G_{\bar{c}}$, $\alpha = (g, \alpha)\sigma \in G_c\tau$, $\alpha = (\alpha, g)\sigma \in \tau^{-1}G_c$.*

Proof. By definition, $\alpha = (g, \alpha) \in G_c$. For $\alpha = (\alpha, g)$, we have $\beta = \tau\alpha\tau^{-1} = (g, \tau\alpha\tau^{-1}) \in G_c$ and so $\alpha \in \tau^{-1}G_c\tau$. For $\alpha = (g, \alpha)\sigma$, we have $\alpha\tau^{-1} = (g, \alpha\tau^{-1}) \in G_c$ and so $\alpha \in G_c\tau$. For $\alpha = (\alpha, g)\sigma$, we have $\tau\alpha = (g, \tau\alpha) \in G_c$ and $\alpha \in \tau^{-1}G_c$.

Lemma 36. *Let $\rho \in F_{0,1}$ and let $\rho = h\alpha h'\beta \dots \delta h''\gamma$ be a word of minimum syllable length s, where $\alpha, \beta, \dots, \delta, \gamma$ are circuitous, $h, h', \dots, h'' \in G$. Then $\alpha, \beta, \dots, \delta, \gamma \in Q(\rho)$. Furthermore, there exists a level l such that $\rho = \rho' q$, where ρ' is an element of the lth stabilizer subgroup, $q \in G_{0,l-1}$, and where the vector of nonbase entries of ρ' is a permuted form of $(\alpha, \beta, \dots, \delta, \gamma)$.*

Proof. The states ρ_0, ρ_1 are words in $\alpha, \beta, \dots, \delta, \gamma$ and certain base elements such that the sum of the syllable lengths of ρ_0, ρ_1 is at most s. By Lemma 17, there exists a minimum level l where ρ factors as $\rho = \rho' q$, where ρ' is an element of the lth stabilizer subgroup, $q \in G_{0,l-1}$, and where for each index v, $\rho'(v)$ is a circuitous automorphism or a base element. Let the number of circuitous $\rho'(v)$'s be s'. The $s' \le s$. Now as we had observed in Sec. 3.1, ρ' is a word of syllable length at most s' in the circuitous $\rho'(v)$'s and certain base elements. Therefore $s = s'$ and the subvector of ρ' formed by the circuitous $\rho'(v)$'s is a permuted form of $(\alpha, \beta, \dots, \delta, \gamma)$.

Proof of Theorem 33. Given $\alpha \in F_{0,1}$ there exists a level l such that $\alpha = h(\alpha_u, \dots, \alpha_v, \dots)g$, where $u, \dots, v, \dots$ are indices of length l, $\alpha_u, \dots, \alpha_v$ are circuitous automorphisms of the four types mentioned in Lemma 35, and $h, g \in G$. Thus $\alpha_v \in \langle\tau\rangle G_c \langle\tau\rangle$ and $(\alpha_u, \dots, \alpha_v) \in \Upsilon(\alpha'_u, \dots, \alpha'_v)\Upsilon$, where $\alpha'_u, \dots, \alpha'_v \in G_c$. Now, $(\alpha'_u, \dots, \alpha'_v) \in \hat{G}$ and therefore we conclude that $\alpha \in \Upsilon\hat{G}\Upsilon$.

Let $\gamma \in \Upsilon \cap \hat{G}$ and assume that $\gamma \notin G$. Then for some tree level l we have a first form for γ, $\gamma = \gamma' p$, where $\gamma' \in \mathcal{F}_l(Y, \langle\tau\rangle)$, $p \in G_{0,l-1}$. Since $\gamma \in \hat{G}$, by Lemma 36, there exists a tree level l' such that $\gamma = \gamma'' p'$, where γ'' is an element of the l'th stabilizer subgroup, $\gamma''(v) \in G_c \cup G$ for all indices v of length l', and $p \in G_{0,l-1}$; this is the second form for γ. Both forms can be developed to a common level l''. Thus we may assume that $l = l' = l''$, and

$$\gamma = (\tau^i, \dots, \tau^j)p = (\alpha_u, \dots, \alpha_v)q.$$

Therefore $p = q$, and since $\langle\tau\rangle$ intersects G_c trivially, it follows that $\gamma \in G$.

The proof of the last statement $\hat{G} \cap \check{G} = G$ proceeds similarly.

5.2. Locally finite subgroups of $F_{0,1}$.

Theorem 37. *The base group G is a maximal locally finite subgroup of $F_{0,1}$.*

First, some criteria for finitely generated subgroups to be infinite will be developed. For the case of the binary tree, Proposition 19 reads:

Lemma 38. *Let $\alpha \in F_{0,1}$ be an active circuitous automorphism. Then α has infinite order.*

Proposition 39. *Let $\alpha \in F_{0,1}$ be a nonbase element and let $H = \langle Q(\alpha)\rangle$. Then H is an infinite group. Furthermore, if H is a 2-group, then $\alpha \in \langle G, G_c, G_{\bar{c}}\rangle$.*

Proof. Assume that H is a finite group. There exists a level k such that $\alpha = \gamma q$, where $\gamma = (\alpha_u, \dots, \alpha_w)$ is an element of the kth stabilizer subgroup with entries which are base or circuitous automorphisms and where $q \in G_{0,k-1}$ permutes the nontrivial circuitous entries of γ. As H is a finite group, $\alpha_u, \dots, \alpha_w \in G \cup G_c \cup G_{\bar{c}}$. Now since $Q(\alpha_u) \cup \dots \cup Q(\alpha_w) \subseteq Q(\alpha)$, we may assume α circuitous, $\alpha = (g, \alpha)$. Therefore, $Q(\alpha) = \{\alpha\} \cup Q(g)$, and since g is nontrivial, $\sigma \in Q(g)$. The 1st level stabilizer subgroup K_1 is a proper subgroup of K and is generated by $\{\alpha, \alpha^\sigma\} \cup \{g_u, g_u^\sigma$, for all indices $u\}$ if g_u is inactive, and by $\{\alpha, \alpha^\sigma\} \cup \{g_u\sigma, \sigma g_u$, for all indices $u\}$ if g_u is active. These generating sets project in their second coordinates onto $\{\alpha, g\} \cup \{g_{u1}, g_{u0} \mid |u| \ge 0\}$, which is none other than the generating set of K; a contradiction is reached.

Proof of Theorem 37. Assume that G is not maximal locally finite and let $\rho = \alpha h\beta h' \dots h''\gamma h''' \in F_{0,1} \backslash G$ be of minimum syllable length such that $H = \langle G, \rho\rangle$ is locally finite. There exists a depth k such that $\rho = (\rho_u, \dots, \rho_w)q$, where the set of circuitous entries of $(\rho_u, \dots, \rho_w)$ coincides with $\{\alpha, \beta, \dots, \gamma\}$ and $q \in G_{0,k-1}$. Then $\langle G, \rho_u\rangle, \dots, \langle G, \rho_w\rangle$ are locally finite, and in particular $\langle G, \alpha\rangle, \dots, \langle G, \gamma\rangle$ are locally finite. However, by Proposition 39, $\langle Q(\alpha)\rangle$ is an infinite subgroup of $\langle G, \alpha\rangle$; a contradiction is reached.

Another criterion for the infiniteness of a finitely generated subgroup is:

Lemma 40. *Let $g, h \in G$ be active elements and let $\alpha = (g, \alpha)$. Then $K = \langle h, \alpha\rangle$ is an infinite group.*

Proof. Write $K(h) = K = \langle h, \alpha\rangle$. From $h = (h_0, h_1)\sigma$, we calculate $\alpha^h = (\alpha^{h_1}, g^{h_0})$. Let $L(h) = \langle \alpha, \alpha^h\rangle$. Then $L(h)$ is a proper subgroup of $K(h)$, and $L(h)$ projects in the second coordinate onto $K(g^{h_0}) = \langle g^{h_0}, \alpha\rangle$. Successive applications of this process produce g^x, where $x \in \langle Q(h) \cup Q(g)\rangle$, which is a finite group. Therefore at some stage we have a repetition from which fact it follows that K is an infinite group.

Problem 3. Let $\alpha \in F_{0,1}$ be a nonbase element. By Proposition 39, $H = \langle Q(\alpha)\rangle$ is a finitely generated infinite group. Define necessary and sufficient conditions which α has to satisfy for H to be a 2-group.

5.3. Nontorsion subgroups. We consider in this subsection some criteria for subgroups of $F_{0,1}$ to be nontorsion. The following notation will be useful.

(i) Let $c = (a, b)\sigma^i \in \mathcal{A}$, $i = 0, 1$. If $i = 0$, then we indicate the projection of c on its first coordinate a by $c \to^1 a$, and on its second coordinate b by $c \to^2 b$. On the other hand, if $i = 1$, then we indicate the projection of c^2 on its first coordinate by $c \to^{s1} ab$ and on its second coordinate by $c \to^{s2} ba$.

(ii) Let $\alpha = (g, \alpha)$. Let also $h = (h_0, h_1)\sigma^i \in G$, $i = 0, 1$. If $i = 0$, then $\alpha h = (gh_0, \alpha h_1) \to^2 \alpha h_1$, while if $i = 1$, then $\alpha h = (gh_0, \alpha h_1)\sigma \to^{s2} \alpha h_1 g h_0$.When g is fixed we simplify the notation by writing $h \to h_1$ in the first case and $h \to h_1 g h_0$ in the second.

Lemma 41. *Let $g \in G$ such that g_u is inactive for all indices $u = \phi, 1, \ldots, 1^k$, and assume that $g_{1^{k+1}} = \sigma$. Let $\alpha = (g, \alpha) \in G_c$. Then $o(\alpha g)$ is infinite.*

Proof. We easily see that $\alpha g \to^2 \alpha g_1 \to^2 \ldots \to^2 \alpha\sigma \to^{s2} \alpha g$, and this repetition proves our claim.

Lemma 42. *Let $g \in G$ having total character $\hat{\theta}(g) = \sum\{\bar{\theta}_i(g) : i \geq 1\} = 1$ and let $\alpha = (g, \alpha) \in G_c$. Furthermore, let $\hat{g} = \Pi\{g_v \mid g_v \in Q(g)\}$, where the product is taken in some order. Then $o(\alpha\hat{g})$ is infinite.*

Proof. Since $\hat{\theta}(g) \neq 0$, it follows that $\hat{g}$ is active. We have $\hat{g} \to h = \hat{g}_1 g \hat{g}_0$, which is an active element since it is $\hat{g}$ in a permuted form. Successive projections of $\hat{g}$ produce active elements from the finite group $\langle Q(g)\rangle$, and so at some point a repetition occurs. Therefore $o(\alpha\hat{g})$ is infinite.

Proposition 43. *Let $g \in G, g \neq e, \alpha = (g, \alpha) \in G_c$. Also, let $K = \langle Q(g)\rangle$, $H = \langle Q(\alpha)\rangle$. If K is an abelian or dihedral group, then H is not a 2-group.*

Proof. We use the description of K in Proposition 3.

(i) Assume that $K = \langle Q(g)\rangle$ is abelian. Then $\langle Q(g)\rangle$ is a subgroup of the abelian group $\langle \sigma^{(i)} \mid i \geq 0\rangle$. If g is active, then $g_0 = g_1, g_0^2 = e$. Now αg has infinite order since $\alpha g \to^{s2} \alpha g_1 g g_0 = \alpha g_1 g_0 g = \alpha g$. On the other hand, if g is inactive, then there exists a first $u = 11\ldots1$ such that g_u is active and we apply Lemma 41 to conclude that αg has infinite order.

(ii) Let $K = \langle Q(g)\rangle$ be nonabelian. Then g is active. Now $g^2 = (\sigma, \sigma)$, $\alpha^2 = (g^2, \alpha^2)$, and $\alpha^2 g^2$ has infinite order since $\alpha^2 g^2 \to \alpha^2\sigma \to^{s2} \alpha^2 g^2$.

Remark. We make some observations about some elements $g \in (C_2 wr C_2) wr C_2$ with total character value $\hat{\theta}(g) = \sum\{\bar{\theta}_i(g) : i \geq 1\} = 0$. (i) Let $g = (\sigma, (\sigma, e)\sigma)\sigma$, $\alpha = (g, \alpha)$. Then, $o(g) = 4$. The following calculations show that $o(\alpha g^i)$ is finite for $i = 1, 2, 3$, yet $o(\alpha g_1)$ is infinite:

$$g \to g_1 g g_0 = ((e, \sigma), \sigma)\sigma \to \sigma g(e, \sigma) = (g_1\sigma, e) \to e,$$

$$g^2 = ((e, \sigma), (\sigma, e)) \to (\sigma, e) \to e,$$

$$g^{-1} = ((e, \sigma)\sigma, \sigma)\sigma \to \sigma g(e, \sigma)\sigma = ((\sigma, e)\sigma, e)\sigma \to eg(\sigma, e)\sigma = (\sigma, (\sigma, e)) \to (\sigma, e) \to e,$$

$$g_1 = (\sigma, e)\sigma \to e.g.\sigma = (\sigma, (\sigma, e)\sigma) \to g_1.$$

Is $B = \langle g, \alpha\rangle$ a 2-group?

(ii) Let $g = ((\sigma, \sigma), \sigma)\sigma$ and $\alpha = (g, \alpha)$. Then, $g^2 = ((\sigma, \sigma)\sigma, (\sigma, \sigma)\sigma)$. The following calculation shows that $o(\alpha^2 g^2)$ is infinite:

$$g^2 \to (\sigma, \sigma)\sigma \to \sigma g^2 \sigma = g^2.$$

(iii) Let $b = ((\sigma, e), e)\sigma$ and $\alpha = (b, \alpha) \in G_c$. Then $K = \langle Q(g) \rangle = (C_2 wr C_2) wr C_2$. Consider the group $B = \langle K, \alpha \rangle$. We have checked that the elements in the set $\langle \alpha \rangle K$ have finite order. Is B is a 2-group?

5.4. Word-length stability. Considerations concerning elements of infinite order lead to the following notion of word-length stability.

Definition 44. Let $a, b, \dots, g \in G$ such that $\hat{\theta}(ab \dots g) = 0$. Let $\alpha = (a, \alpha)$, $\beta = (b, \beta), \dots, \gamma = (g, \gamma) \in G_c$, and $h, h', \dots, h'', h''' \in G$. The element $w = \alpha h \beta h' \dots h'' \gamma h'''$ of syllable length s is said to be (length) *stable* provided that:

(i) if $w = (w_0, w_1)$, then its second coordinate $w_1 = \alpha h_0 \dots$ has syllable length s and is stable,

(ii) if $w = (w_0, w_1)\sigma$, then $w^2 = (w_0 w_1, w_1 w_0)$ and its second coordinate $w_1 w_0 = \alpha h_0 \dots$ has syllable length s and is stable.

Any element produced from a stable w by iterated application of projection π_2 on the second coordinate, or from squaring composed with π_2, has the same syllable length as w. Denote the set of elements produced from w using this process by $J(w)$.

Lemma 45. *Let $w = \alpha h \beta h' \dots h'' \gamma h'''$ be a word of syllable length s and assume that it is stable. If w is inactive, then $h, h', \dots, h'''$ are all inactive.*

Proof. If h is active, then $\alpha h \beta = (a h_0 \beta, \alpha h_1 b)\sigma$, and therefore the second coordinate of w has syllable length less than s, in contradiction with the stability of w. In this manner $h, h', \dots h''$ are inactive, and so it follows that h''' is inactive as well.

Proposition 46. *Let $\alpha, \beta, \dots, \gamma, h, h', \dots, h'''$, and w be as above. Assume that w is stable and assume that $\hat{\theta}(ab \dots g) = 0$. Then there exists an element in $J(w)$ which is inactive.*

Proof. Assume that all elements of $J(w)$ are active. In particular w is active; here we have $\bar{\theta}_1(w) = \bar{\theta}_1(hh' \dots h''') = \bar{\theta}_1(h) + \dots + \bar{\theta}_1(h''') = 1$.

On expanding w, we get $w = (w_0, w_1)\sigma$, where

$$\begin{aligned} w_0 &\in a\{h_0, h_1\}\{b, \beta\}\{h'_0, h'_1\} \dots \{g, \gamma\}\{h'''_0, h'''_1\}, \\ w_1 &\in \alpha\{h_1, h_0\}\{\beta, b\}\{h'_1, h'_0\} \dots \{\gamma, g\}\{h'''_1, h'''_0\}. \end{aligned}$$

That is, if h_0 appears after a in w_0, then h_1 appears after α in w_1, and so on. The exact form of w_0 is determined by the activity of each of the elements $h, h', \dots, h''$. Also, given the form of w_0, the form of w_1 is determined as is suggested by the displayed expressions. Observe that the sum of the syllable lengths of the words w_0, w_1 is at most s. Also, the G-terms are words in $h_0, h_1, h'_0, h'_1, \dots, h'''_0, h'''_1, a, b, \dots, g$; thus the depth of the h-terms in w_0, w_1 is smaller than those in the original word w.

Now $w^2 = (w_0 w_1, w_1 w_0)$, where $w_0 w_1$ is conjugate to $w_1 w_0$ and has syllable length s. We note from the formulas for w_0, w_1 that, irrespective of the different possibilities for their forms, the activity of $w_1 w_0$ is

$$\bar{\theta}_1(w_1 w_0) = \bar{\theta}_1(w_1) + \bar{\theta}_1(w_0) = \left(\bar{\theta}_1(a) + \bar{\theta}_1(b) + \dots + \bar{\theta}_1(g)\right) + \left(\bar{\theta}_2(h) + \dots + \bar{\theta}_2(h''')\right) = 1.$$

Applying the process of squaring and projection m times leads us to a word of the same syllable length as the original w with activity

$$\begin{aligned} &\sum\{\bar{\theta}_l(a) \mid 1 \le l \le m\} + \sum\{\bar{\theta}_l(b) \mid 1 \le l \le m\} + \dots \\ &+ \sum\{\bar{\theta}_l(g) \mid 1 \le l \le m\} + \left(\bar{\theta}_{m+1}(h) + \dots + \bar{\theta}_{m+1}(h''')\right) = 1. \end{aligned}$$

Now let t be the maximum depth of the elements $h, \dots, h'''$. Then for all $m \ge t$, we have

$$\sum\{\bar{\theta}_l(a) \mid 1 \le l \le m\} + \sum\{\bar{\theta}_l(b) \mid 1 \le l \le m\} + \dots + \sum\{\bar{\theta}_l(g) \mid 1 \le l \le m\} = 1,$$

which becomes $\hat{\theta}(ab \dots g)$ after passing the maximum depth of the elements $a, b, \dots, g$. However, this contradicts the hypothesis $\hat{\theta}(ab \dots g) = 0$.

The above proof implies

Corollary 47. *Let $\alpha \in F_{0,1}$ and $H = \langle Q(\alpha)\rangle$. Then H is a 2-group if and only if $\langle G, \alpha\rangle$ is a 2-group.*

5.5. Burnside 2-subgroups.

Proposition 48. *Let $b \in G, b \neq e, \beta = (b, \beta) \in G_c$, and $B = \langle Q(\beta)\rangle$. Furthermore, let $K = \langle Q(b)\rangle$ and K_1 be the first-level stabilizer subgroup of K. Then B is a Burnside 2-group if and only if $\hat{\theta}(b) = 0$ and $W = \langle \beta, K_1\rangle$ is a 2-group.*

Proof. We already know that B is a finitely generated infinite group and that $\hat{\theta}(b) = 0$ is a necessary condition for B to be a 2-group.

Assume that $\hat{\theta}(b) = 0$ and $W = \langle \beta, K_1\rangle$ is a 2-group, yet B is not a 2-group. Let $w \in B$ have infinite order of shortest syllable length, $w = \beta k \beta k' \dots \beta k''$. Therefore w is stable.

Now this w is active; otherwise by Lemma 45, $k, k', \dots, k'' \in K_1$ and therefore $w \in W$, which has finite order. Since $\hat{\theta}(b) = 0$, by Proposition 46, at some stage in the squaring-projection process we arrive at an inactive $w' \in J(w)$, and so w' has finite order. Hence w has finite order as well, which is a contradiction.

The following is the first Burnside 2-group that appears in $F_{0,1}$.

Theorem 49. *Let $b = (\sigma^{(1)}, e) \in G$, $\beta = (b, \beta) \in G_c$, and $B = \langle Q(\beta)\rangle$. Then B is a Burnside 2-group.*

Proof. Firstly, the set of states of β is $Q(\beta) = \{\beta, b, \sigma^{(1)}, \sigma\}$, and the first-level stabilizer in B is $B_1 = \langle \beta, \beta^\sigma, b, \sigma^{(1)}\rangle$. It is easy to see that B_1 projects onto B and therefore B has infinite order. Secondly, $Q(b)$ generates $K = \langle b, \sigma\rangle \oplus \langle \sigma^{(1)}\rangle$, where $\langle b, \sigma\rangle$ is dihedral of order 8 and $(b\sigma)^2 = \sigma^{(2)}$. Also, the first-level stabilizer in K is $K_1 = \langle \sigma^{(2)}, \sigma^{(1)}, b\rangle$.

Note that $\beta b = b\beta$ and that $o(\beta\sigma) = o(\beta\sigma^{(1)}) = 4$.

We verify that $W = \langle \beta, K_1\rangle$ is a finite group:

(i) $W = \langle \beta, K_1\rangle = \langle \beta, \sigma^{(2)}, \sigma^{(1)}, b\rangle \leq K \times V$, where $V = \langle \beta, \sigma^{(1)}, \sigma\rangle$;

(ii) $V = \langle \beta, \sigma^{(1)}, \sigma\rangle \leq (U \times U)\langle\sigma\rangle$, where $U = \langle \beta, b, \sigma\rangle$;

(iii) $U = \langle \beta, b, \sigma\rangle \leq (T \times T)\langle\sigma\rangle$, where $T = \langle \beta, b, \sigma^{(1)}\rangle = \langle \beta, \sigma^{(1)}\rangle \oplus \langle b\rangle$.

The proof is concluded by applying Proposition 48.

Corollary 50. *With the above notation, $\langle \beta, b\sigma^{(1)}, \sigma\rangle$ and $\langle \beta\sigma^{(1)}, b\sigma\rangle$ are Burnside subgroups of B.*

Proof. The proof is a routine exercise in calculating the first-level stabilizer and in projecting on the coordinates.

The Burnside group in the above theorem generalizes as follows.

Theorem 51. *Let $n \geq 1, b = (\sigma^{(n)}, e) \in G, \beta = (b, \beta) \in G_c$, and $B = \langle Q(\beta)\rangle$. Then B is a Burnside 2-group.*

Proof. We imitate the proof of the previous theorem. The set of states of β is $Q(\beta) = \{\beta, b, \sigma^{(n)}, \dots, \sigma^{(1)}, \sigma\}$, and the first-level stabilizer in B is

$$B_1 = \langle \beta, \beta^\sigma, b, b^\sigma, \sigma^{(n)}, \dots, \sigma^{(1)}\rangle,$$

which projects onto B. Secondly, $Q(b)$ generates

$$K = \langle b, \sigma\rangle \oplus \langle \sigma^{(n)}, \dots, \sigma^{(1)}\rangle,$$

where $\langle b, \sigma\rangle$ is dihedral of order 8 and $(b\sigma)^2 = \sigma^{(n+1)}$. Also,

$$K_1 = \langle \sigma^{(n+1)}, \sigma^{(n)}, \dots, \sigma^{(1)}, b\rangle.$$

We check that $\beta b = b\beta$, $o(\beta\sigma) = o(\beta\sigma^{(i)}) = 4$.

Now it is sufficient to prove that the subgroup $W = \langle \beta, K_1\rangle$ is finite. We observe that:

(i) $W = \langle \beta, \sigma^{(n+1)}, \dots, \sigma^{(1)}, b\rangle \leq K \times V$, where $V = \langle \beta, \sigma^{(n)}, \dots, \sigma\rangle$;

(ii) $V = \langle \beta, \sigma^{(n)}, \dots, \sigma\rangle \leq (U \times U)\langle\sigma\rangle$, where $U = \langle \beta, b, \sigma^{(n-1)}, \dots, \sigma\rangle$;

(iii) $U = \langle \beta, b, \sigma^{(n-1)}, \dots, \sigma\rangle \leq (T \times T)\langle\sigma\rangle$, where $T = \langle \beta, b, \sigma^{(n)}, \sigma^{(n-2)}, \dots, \sigma\rangle$;

(iv) the embeddings proceed until we reach the factor $\langle \beta, b, \sigma^{(n)}, \sigma^{(n-1)}, \dots, \sigma^{(1)} \rangle$ which is embedded in $\langle b, \sigma^{(n)}, \dots, \sigma \rangle \times \langle \beta, \sigma^{(n-1)}, \dots, \sigma \rangle$;

(v) the second factor in the previous step may replace V in step (ii). These steps can be iterated until we reach the factor $\langle \beta, \sigma \rangle$, which is finite.

The proof concludes as in the previous theorem.

REFERENCES

1. S. V. Aleshin, "A free group of finite automata," *Vestn. Mosk. Univ., Ser. Mat.,* **38**, 10–13 (1983).
2. M. Bhattacharjee, "The ubiquity of free subgroups in certain inverse limits of groups," *J. Algebra,* **172**, 134–146 (1995).
3. H. Bass, M. Otero-Espinar, D. Rockmore, and C. P. L. Tresser, "Cyclic renormalization and automorphism groups of rooted trees," In: *Lect. Notes Math.,* Vol. 1621, Springer, Berlin (1995).
4. A. M. Brunner and S. Sidki, "On the automorphism group of the one-rooted binary tree," *J. Algebra,* **195**, 465–486 (1997).
5. A. M. Brunner and S. Sidki, "The generation of $GL(n, \mathbb{Z})$ by finite state automata," *Int. J. Algebra Comput.,* **8**, 127–139 (1998).
6. A. M. Brunner, S. Sidki, and A. C. Vieira, "A just-nonsolvable torsion-free group defined on the binary tree," to appear in *J. Algebra.*
7. R. I. Grigorchuk, "On the Burnside problem for periodic groups," *Funkts. Anal. Prilozhen.,* **14**, 41–43 (1980).
8. R. I. Grigorchuk, "On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata," In: *Proc. Groups St. Andrews/Bath 1997.*
9. R. I. Grigorchuk and P. F. Kurchanov, "Some questions of group theory related to geometry," In: A. N. Parshin and I. R. Shafarevich, eds., *Algebra VII. Combinatorial Group Theory. Applications to Geometry.* Springer-Verlag (1991), pp. 167–240.
10. N. Gupta and S. Sidki, "Extensions of groups by tree automorphisms," In: *Contributions to Group Theory. Contemp. Math.,* Vol. 33, American Mathematical Society, Providence, Rhode Island (1984), pp. 232–246.
11. D. S. Passman and W. V. Temple, "Representations of the Gupta–Sidki group," *Proc. Amer. Math. Soc.,* **124**, 1403–1410 (1996).
12. A. V. Rozhkov, "The lower central series of one group of automorphisms of a tree," *Mat. Zametki,* **60**, 223–237 (1996).
13. M. Rubin, "The reconstruction of trees from their automorphism groups," In: *Contemp. Math.*, Vol. 151, American Mathematical Society, Providence, Rhode Island (1993).
14. S. Sidki, "A primitive ring associated to a Burnside 3-group," *J. London Math. Soc. (2),* **55**, 55–64 (1997).
15. S. Sidki, "Regular trees and their automorphisms," In: *Notes of a Course, XIV Escola de Álgebra, Rio de Janeiro, July 1996. Monografias de Matemática*, Vol. 56, IMPA, Rio de Janeiro (1998).
16. V. I. Sushchanskii, "Wreath products and periodic factorable groups," *Mat. Sb.,* **67**, No. 2, 535–553 (1990).
17. A. C. Vieira, "On the lower central series and the derived series of the Gupta–Sidki 3-group," *Commun. Algebra,* **26**, 1319–1333 (1998).
18. J. S. Wilson and P. A. Zalesskii, "Conjugacy separability of certain torsion groups," *Arch. Math.,* **68**, 441–449 (1997).