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Punctual Hilbert Schemes of Small Length in Dimensions 2 and 3

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ABSTRACT. The biregular geometry of punctual Hilbert schemes in dimensions 2 and 3, i.e., of schemes parametrizing fixed-length zero-dimensional subschemes supported at a given point on a smooth surface or a smooth three-dimensional variety, is studied. A precise biregular description of these schemes has only been known for the trivial cases of lengths 3 and 4 in dimension 2. The next case of length 5 in dimension 2 and the two first nontrivial cases of lengths 3 and 4 in dimension 3 are considered. A detailed description of the biregular properties of punctual Hilbert schemes and of their natural desingularizations by varieties of complete punctual flags is given.

KEY WORDS: punctual Hilbert scheme, complete punctual flag, biregular description, desingularization, Ext-groups, Stein expansion, Briançon classification.

Introduction

The *punctual Hilbert scheme of length* d on a surface (in a space) is the Hilbert scheme

$$H_d(0) = \mathrm{Hilb}^d \operatorname{Spec} k[[x,y]]_{\mathrm{red}} \quad (H_d(0) = \mathrm{Hilb}^d \operatorname{Spec} k[[x,y,z]]_{\mathrm{red}}, \text{ respectively}),$$

which parametrizes the zero-dimensional subschemes of length d supported at a given point 0 on the surface (in the space, respectively); for brevity, we denote it also by $\mathrm{Hilb}^d k[[x,y]]$ (by $\mathrm{Hilb}^d k[[x,y,z]]$, respectively). The study of general properties of the schemes $H_d(0)$ was initiated by Briançon [1], Iarrobino [2], Granger [3], and others and continued by many authors (see, e.g., the surveys [4, 5]). But a precise biregular description of these schemes was only known in the trivial cases of $d=1$ and 2 and in the first nontrivial cases $d=3$ and 4 in dimension 2 (see [6]). In this paper, we consider the next case $d=5$ in dimension 2 and the two first nontrivial cases $d=3$ and 4 in dimension 3. We examine in detail the biregular geometry of the schemes $H_d(0)$ and their natural desingularizations by varieties of complete punctual flags in these cases. Our main method of study is to obtain schemes Z_d supported at the point 0 from the schemes Z_{d-1} by the operation of "adding the point 0"; in the language of schemes, this operation is expressed by the exact triple

$$0 \to k(0) \to \mathcal{O}_{Z_d} \to \mathcal{O}_{Z_{d-1}} \to 0.$$

All such extensions, which are classified according to the corresponding Ext-groups, give the description of the punctual Hilbert schemes $H_d(0)$. The base field k is assumed to be algebraically closed.

1. The punctual Hilbert scheme $\mathrm{Hilb}^5 k[[x,y]]$

1.1. Preliminaries. In this section, we consider the case of dimension 2. As the initial surface, for convenience we take the projective plane $\mathbf{P}^2$. First, we cite some known results on the punctual Hilbert schemes $H_5(0) = \mathrm{Hilb}^5 k[[x,y]]$ and varieties X_4 of complete punctual flags (their definition is given later on), which are used in what follows. Briançon [1] classified the zero-dimensional punctual schemes of length 5 in dimension 2 into the following five isomorphism classes, which are determined by the ideals $\mathcal{I}$ of the schemes in the ring $k[[x,y]]$:

(i) $\mathcal{I} = (y, x^5)$;
(ii) $\mathcal{I} = (y^2 + x^3, xy)$;
(iii) $\mathcal{I} = (y^2, xy, x^4)$;
(iv) $\mathcal{I} = (x^2 + y^2, x^2y, x^3)$;
(v) $\mathcal{I} = (y^2, x^2y, x^3)$.

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The set

$$H_5^c(0) = \{Z_5 \in H_5(0) \mid Z_5 \text{ is a scheme of type (i), i.e., a curvilinear scheme}\}$$

is dense and open in $H_5(0)$, and according to Granger [3],

$$\operatorname{Sing} H_5(0) = H_5(0) \setminus H_5^c(0) = \{Z_5 \in H_5(0) \mid Z_5 \text{ is a scheme of one of types (ii)–(v)}\}$$

is the closure in $H_5(0)$ of the set

$$\operatorname{Sing} H_5(0)^* = \{Z_5 \in H_5(0) \mid Z_5 \text{ is a scheme of type (ii)}\};$$

we have $\operatorname{codim}_{H_5(0)} \operatorname{Sing} H_5(0) = 1$, and the variety $H_5(0)$ is analytically isomorphic to $\operatorname{Sing} H_5(0) \times C$, where C is the curve given by the equation $\{x^2 + y^3 = 0\}$ in $\mathbf{A}^2$, in a neighborhood of a generic point from $\operatorname{Sing} H_5(0)$. In addition,

$$K := \{Z_5 \in H_5(0) \mid Z_5 \text{ is a scheme of type (v)}\}$$

is an irreducible curve isomorphic to $\mathbf{P}^1$.

Next, for any $d \geq 1$,

$$H_d^c(0) := \{Z_d \in H_d(0) \mid Z_d \text{ is a curvilinear scheme}\}$$

is a smooth irreducible variety which is a dense open subset of $H_d(0)$ [1, 3]. Thereby

$$X_d^c := \{(Z_2, Z_3, \dots, Z_d) \in H_2^c(0) \times H_3^c(0) \times \dots \times H_d^c(0) \mid Z_2 \subset Z_3 \subset \dots \subset Z_d\}$$

is also a smooth irreducible variety. Its closure X_d in $H_2(0) \times H_3(0) \times \dots \times H_d(0)$ is called the *variety of complete punctual flags of length* $\leq d$. Obviously, X_1 is the one-point set $\{0\}$. According to the main result of [6], we have the isomorphism of varieties $X_d \simeq \mathbb{P}(\mathcal{E}_{d-1}^\vee)$ for $d = 2$, 3, and 4; here

$$\mathcal{E}_{d-1} := \mathcal{E}xt^1_{p_2}(\mathcal{O}_{T_{d-1}}, k(0) \boxtimes \mathcal{O}_{X_{d-1}})$$

is a locally free sheaf of rank 2, $T_{d-1} \subset \mathbf{P}^2 \times X_{d-1}$ is the universal cycle of length $d-1$ over X_{d-1}, and $p_2 \colon \mathbf{P}^2 \times X_{d-1} \to X_{d-1}$ is the projection. Under this isomorphism, the natural projection (forgetful morphism)

$$X_d \to X_{d-1} \colon (Z_1, Z_2, \dots, Z_d) \mapsto (Z_1, Z_2, \dots, Z_{d-1})$$

coincides with the structural morphism $\pi_d \colon \mathbb{P}(\mathcal{E}_{d-1}^\vee) \to X_{d-1}$.

Now, consider the variety X_4 of complete punctual flags of length ≤ 4 in more detail. On X_4, we have the standard invertible sheaves $\mathcal{O}\tau_4 = \mathcal{O}_{X_4/X_3}(1)$, $\mathcal{O}\tau_3 = \pi_4^* \mathcal{O}_{X_3/X_2}(1)$, and $\mathcal{O}\tau_2 = (\pi_3 \cdot \pi_4)^* \mathcal{O}_{X_2/X_1}(1)$ and the universal flag

$$\{0\} \times X_4 = \mathbf{T}_1 \subset \mathbf{T}_2 \subset \mathbf{T}_3 \subset \mathbf{T}_4 = T_4,$$

where $\mathbf{T}_2$ and $\mathbf{T}_3$ are lifted from $\mathbf{P}^2 \times X_2$ and $\mathbf{P}^2 \times X_3$, respectively. In particular, $\mathbf{T}_3 = (1 \times \pi_4)^{-1}(T_3)$, and the projection is as in the diagram

$$\begin{array}{ccc} \mathbf{P}^2 \times X_4 & \xrightarrow{1 \times \pi_4} & \mathbf{P}^2 \times X_3 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\ X_4 & \xrightarrow{\pi_4} & X_3 \end{array} . \qquad (1)$$

According to [6, Sec. 1.2], the triple

$$0 \to k(0) \boxtimes \mathcal{O}_{X_4}(\tau_4) \to \mathcal{O}_{\mathbf{T}_4} \to \mathcal{O}_{\mathbf{T}_3} \to 0 \qquad (2)$$

is exact.

Finally, consider the closed subsets

$$W_i = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_i \text{ is not a locally complete intersection}\},$$

where $i = 3, 4$, of X_4. By the main theorem from [6], W_3 and W_4 are irreducible divisors on X_4. Note that, if $(Z_2, Z_3, Z_4) \in W_3$ is a generic point, then the zero-dimensional scheme Z_4 is determined by an ideal in $k[[x, y]]$ isomorphic to the ideal

$$\mathcal{I} = (x^2, y^2). \qquad (3)$$

1.2. Basic evaluation. Let us apply the functor $\mathcal{E}xt^{\bullet}_{p_2}(-, k(0) \boxtimes \mathcal{O}_{X_4})$ to the triple (2). Denote

$$\mathcal{E}_4 = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_4}, k(0) \boxtimes \mathcal{O}_{X_4}), \qquad \mathcal{F}_4 = \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_4}, k(0) \boxtimes \mathcal{O}_{X_4}),$$
$$\mathcal{E}_3 = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{T_3}, k(0) \boxtimes \mathcal{O}_{X_3}), \qquad \mathcal{F}_3 = \mathcal{E}xt^2_{p_2}(\mathcal{O}_{T_3}, k(0) \boxtimes \mathcal{O}_{X_3}).$$

The obvious isomorphisms

$$\mathcal{E}xt^0_{p_2}(k(0) \boxtimes \mathcal{O}_{X_4}(\tau_4), k(0) \boxtimes \mathcal{O}_{X_4}) \simeq \mathcal{E}xt^2_{p_2}(k(0) \boxtimes \mathcal{O}_{X_4}(\tau_4), k(0) \boxtimes \mathcal{O}_{X_4}) \simeq \mathcal{O}_{X_4}(-\tau_4),$$
$$\mathcal{E}xt^1_{p_2}(k(0) \boxtimes \mathcal{O}_{X_4}(\tau_4), k(0) \boxtimes \mathcal{O}_{X_4}) \simeq 2\mathcal{O}_{X_4}(-\tau_4),$$
$$\mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_3}, k(0) \boxtimes \mathcal{O}_{X_4}) \simeq \mathcal{O}_{X_4} \simeq \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_4}, k(0) \boxtimes \mathcal{O}_{X_4})$$

and the equalities

$$\mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3}, k(0) \boxtimes \mathcal{O}_{X_4}) = \pi_4^*\mathcal{E}_3, \qquad \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_3}, k(0) \boxtimes \mathcal{O}_{X_4}) = \pi_4^*\mathcal{F}_3,$$

which are obtained by change of base in diagram (1), give the exact sequence

$$0 \to \mathcal{O}_{X_4}(-\tau_4) \to \pi_4^*\mathcal{E}_3 \to \mathcal{E}_4 \xrightarrow{f} 2\mathcal{O}_{X_4}(-\tau_4) \xrightarrow{g} \pi_4^*\mathcal{F}_3 \xrightarrow{h} \mathcal{F}_4 \xrightarrow{\varepsilon} \mathcal{O}_{X_4}(-\tau_4) \to 0. \tag{4}$$

Consider the divisor

$$W = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_4 \text{ is not a locally complete intersection}\}$$

on X_4. By construction, W is a section of the projection $\pi_4 \colon X_4 \to X_3$; hence

$$\mathcal{O}_{X_4}(W) = \mathcal{O}_{X_4}(\tau_4 + \mathcal{L}), \qquad \mathcal{L} = \pi_4^*L, \quad L \in \operatorname{Pic} X_3. \tag{5}$$

Lemma 1. *We have*

$$\mathcal{F}_4 \otimes k(x) = \operatorname{Ext}^2(\mathcal{O}_{Z_4}, k(0)) = \begin{cases} k & \text{if } x = (Z_2, Z_3, Z_4) \notin W, \\ k^2 & \text{if } x = (Z_2, Z_3, Z_4) \in W. \end{cases}$$

Proof. (i) If Z_4 is a curvilinear scheme, then, obviously, $\operatorname{Hom}(k(0), \mathcal{O}_{Z_4}) = k$, and by the Serre duality on S [7], $\operatorname{Ext}^2(\mathcal{O}_{Z_4}, k(0)) = k$. If Z_4 is not a curvilinear scheme but still is a locally complete intersection, then its ideal $\mathcal{I}_{Z_4}$ in the local ring $k[[x, y]]$ is isomorphic to (x^2, y^2); therefore, any nonzero morphism $k(0) \xrightarrow{i} \mathcal{O}_{Z_4}$ can be extended to the exact triple

$$0 \to k(0) \xrightarrow{i} \mathcal{O}_{Z_4} \xrightarrow{r} \mathcal{O}_{Z_3} \to 0,$$

where $\mathcal{O}_{Z_3} = 0^{(1)}$ is the first infinitesimal neighborhood of the point 0 and the morphism r is necessarily proportional to the restriction morphism $\cdot \otimes \mathcal{O}_{0^{(1)}}$. Thereby we again obtain the required equality $\operatorname{Hom}(k(0), \mathcal{O}_{Z_4}) = k$.

(ii) Now, suppose that the scheme Z_4 is not a locally complete intersection; then its ideal $\mathcal{I}_{Z_4}$ is isomorphic to (x^3, xy, y^2). In this case, the cokernel of any nonzero morphism from $\operatorname{Hom}(k(0), \mathcal{O}_{Z_4})$ is a sheaf $\mathcal{O}_{Z_3}$ with Z_3 tangent to the line $y = 0$ at 0; since all such Z_3 are parametrized by the projective line $\mathbf{P}^1$, which is isomorphic to $P(\operatorname{Hom}(k(0), \mathcal{O}_{Z_4}))$, we have $\operatorname{Hom}(k(0), \mathcal{O}_{Z_4}) = k^2$, and by Serre duality [7], $\operatorname{Ext}^2(\mathcal{O}_{Z_4}, k(0)) = k^2$.

Finally, the equality $\mathcal{F}_4 \otimes k(x) = \operatorname{Ext}^2(\mathcal{O}_{Z_4}, k(0))$, where $x = (Z_2, Z_3, Z_4) \in X_4$, is a base change isomorphism for the relative $\mathcal{E}xt_{p_2}$-sheaf $\mathcal{F}_4$. $\square$

Using Lemma 1 and the epimorphism ε in the long exact sequence (4), we obtain the following assertion.

Corollary 1. *The epimorphism* ε *in* (4) *coincides with the canonical map* $\mathcal{F}_4 \overset{\text{can}}{\to} \mathcal{F}_4^{\vee\vee}$, *and its kernel* $\mathcal{T}ors(\mathcal{F}_4) = \ker(\text{can})$ *is an invertible sheaf on some scheme* $\overline{W}$ *with support* W.

Proof. Indeed, since the sheaf $\mathcal{F}_4^{\vee\vee}$ is invertible, we have

$$\mathcal{T}or_1^{\mathcal{O}_{X_4}}(\mathcal{F}_4^{\vee\vee}, k(x)) = \mathcal{T}or_1^{\mathcal{O}_{X_4}}(\mathcal{O}_{X_4}(-\tau_4), k(x)) = 0, \qquad x \in X_4.$$

Therefore, considering the tensor products of the exact triple

$$0 \to \mathcal{T}ors(\mathcal{F}_4) \to \mathcal{F}_4 \overset{\text{can}}{\to} \mathcal{F}_4^{\vee\vee} \to 0$$

and $k(x)$ for $x \in X_4$ and applying Lemma 1, we obtain $\operatorname{Supp}(\mathcal{T}ors(\mathcal{F}_4)) = W$ and $\mathcal{T}ors(\mathcal{F}_4) \otimes k(x) = k$ for $x \in W$; thus the sheaf $\mathcal{T}ors(\mathcal{F}_4)$ is invertible on a suitable scheme $\overline{W}$ with support W. □

Suppose that T is a maximal subsheaf of dimension ≤ 1 in $\mathcal{T}ors(\mathcal{F}_4)$ and $\mathcal{M} := \mathcal{T}ors(\mathcal{F}_4)/T$, i.e., the triple

$$0 \to T \overset{i}{\to} \mathcal{T}ors(\mathcal{F}_4) \overset{e}{\to} \mathcal{M} \to 0 \tag{6}$$

is exact. Corollary 1 implies that $\mathcal{M}$ is an invertible sheaf on a divisor of W of multiplicity n for some $n \geq 1$, i.e., on the subscheme W_n in X_4 determined by the sheaf of ideals $\mathcal{I}_{W_n, X_4} = \mathcal{O}_{X_4}(-nW)$. This, in particular, implies $\mathcal{T}or_1^{\mathcal{O}_{X_4}}(\mathcal{M}, k(x)) = k$ for $x \in W$. Therefore, if $Y := \operatorname{Supp} T \neq \varnothing$, then the tensor multiplication of the exact triple (6) by $k(x)$, $x \in Y$, gives the exact sequence $k \to T \otimes k(x) \to k \to k \to 0$, whence $T \otimes k(x) \simeq k$; so T is an invertible sheaf on some scheme with support Y. Thus we obtain the following assertion.

Corollary 2. *There exist sheaves* $\mathcal{M}$ *and* T *such that* $\mathcal{M}$ *is an invertible sheaf on a divisor of* W *of multiplicity* n *for some* $n \geq 1$, *i.e., on the subscheme* W_n *in* X_4 *determined by the sheaf of ideals* $\mathcal{I}_{W_n, X_4} = \mathcal{O}_{X_4}(-nW)$; T *is either zero or invertible on some subscheme* Y *of dimension* ≤ 1 *with* $\operatorname{Supp} Y \subset W$; *and we have the exact triple* (6).

Consider the triple

$$0 \to \mathcal{O}_D(-\tau_2) \overset{e_1}{\to} \pi_4^* \mathcal{F}_3 \overset{e_2}{\to} \mathcal{O}_{X_4}(-\tau_3) \to 0, \tag{7}$$

where D is a divisor on X_4 of the form

$$D = \pi_4^{-1}(l_0) = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_3 \text{ is not a locally complete intersection}\}$$

for an exceptional line l_0 on X_3. According to [6, Proposition 2.2], this triple is exact. In particular, by the main Theorem from [6], we have

$$\mathcal{O}_{X_4}(D) = \mathcal{O}_{X_4}(\tau_3 - 2\tau_2). \tag{8}$$

Next, consider the curve $C = D \cap W$. The divisors D and W are irreducible and intersect along C transversally; hence, by Corollary 1, the composition of morphisms

$$\mathcal{O}_D(-\tau_2) \overset{e_1}{\to} \pi_4^* \mathcal{F}_3 \overset{h}{\to} \mathcal{T}ors(\mathcal{F}_4) \overset{e}{\to} \mathcal{M}$$

(see (4), (7), and (6)) is zero, and we can define a morphism $h' \colon \mathcal{O}_D(-\tau_2) \to T$ such that $i \cdot h' = h \cdot e_1$ and, accordingly, an epimorphism $h'' \colon \mathcal{O}_{X_4}(-\tau_3) \twoheadrightarrow \mathcal{M}$ such that $h'' \cdot e_2 = e \cdot h$.

Thus we have $\mathcal{M} = \mathcal{O}_{W_n}(-\tau_3)$ by Corollary 2, and the triples (7) and (6) are included in the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{Y'',D}(-\tau_2) & \to & \mathcal{O}_D(-\tau_2) & \xrightarrow{h'} & T & \to & \operatorname{coker} h' \to 0 \\
 & & \downarrow & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle i} & & \\
0 & \to & \operatorname{im} g & \to & \pi_4^* \mathcal{F}_3 & \xrightarrow{h} & \mathcal{T}ors(\mathcal{F}_4) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e} & & \\
0 & \to & \mathcal{O}_{X_4}(-\tau_3 - nW) & \to & \mathcal{O}_{X_4}(-\tau_3) & \xrightarrow{h''} & \mathcal{O}_{W_n}(-\tau_3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} h' & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\quad, \qquad (9)
$$

where $\operatorname{coker} h' = T|Y'$ (by Corollary 2) and Y' is a subscheme (possibly empty) of dimension ≤ 1 in Y; accordingly, $\operatorname{im} h' = \mathcal{O}_{Y''}(-\tau_2)$ and Y'' is a subscheme (possibly empty) of dimension ≤ 1 in D, and $\operatorname{Supp} Y \supset \operatorname{Supp} Y''$. The left vertical sequence in diagram (9) gives the isomorphism $\operatorname{coker} h' = \mathcal{O}_{X_4}(-\tau_3 - nW)|Y'$ and the exact triple

$$0 \to \mathcal{I}_{Y'',D}(-\tau_2) \to \operatorname{im} g \xrightarrow{\theta} \mathcal{I}_{Y',X_4}(-\tau_3 - nW) \to 0,$$

which, together with the morphism g, gives the exact triples

$$0 \to \ker(\theta \cdot g) \to 2\mathcal{O}_{X_4}(-\tau_4) \xrightarrow{\theta \cdot g} \mathcal{I}_{Y',X_4}(-\tau_3 - nW) \to 0, \qquad (10)$$

$$0 \to \ker g \to \ker(\theta \cdot g) \to \mathcal{I}_{Y'',D}(-\tau_2) \to 0. \qquad (11)$$

Since $\dim Y' \leq 1$, (10) implies that $\ker(\theta \cdot g)$ is an invertible sheaf on X_4. Moreover, the condition $n \geq 1$ and equality (5) give $n = 1$; therefore, $\ker(\theta \cdot g) = \mathcal{O}_{X_4}(-\tau_4 + \tau_3 + \mathcal{L})$. This equality, the triple (11), and the condition $\dim Y'' \leq 1$ readily imply $\dim Y'' = 1$; thus Y'' is a divisor on D. Taking into account (8), we obtain the equalities

$$\mathcal{I}_{Y'',D}(-\tau_2) = \mathcal{O}_D(-\tau_2 - Y'') = \mathcal{O}_D(-\tau_4 + \tau_3 + \mathcal{L}), \qquad \ker g = \mathcal{O}_{X_4}(-\tau_4 + 2\tau_2 + \mathcal{L}). \qquad (12)$$

Next, according to Lemma 1.6 from [6], $\mathcal{E}_3$ is a reflexive and, hence, is a locally free sheaf of rank 2. A repetition of the proof of this lemma shows that $\mathcal{E}_4$ is also a reflexive sheaf of rank 2 (on X_4). According to (4), $\ker f$ is a sheaf of rank 1 on X_4; by virtue of the relation $\operatorname{im} f = \ker g$ and the second equality in (12), it is included in the exact triple

$$0 \to \ker f \to \mathcal{E}_4 \to \mathcal{O}_{X_4}(-\tau_4 + 2\tau_2 + \mathcal{L}) \to 0.$$

Since $\mathcal{E}_4$ is reflexive, this sheaf is locally free. This proves the following proposition.

Proposition 1. *$\mathcal{E}_4 = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_4}, k(0) \boxtimes \mathcal{O}_{X_4})$ is a locally free sheaf of rank 2 on X_4.*

1.3. A description of the forgetful morphism $\sigma\colon X_5 \to H_5(0)$. Consider the variety X_5 of complete punctual flags of length ≤ 5. Proposition 1 and [6, Secs. 1.2, 3] give the isomorphism $X_5 \simeq \mathbb{P}(\mathcal{E}_4^\vee)$ of smooth varieties; under this isomorphism, the natural projection (forgetful morphism)

$$\pi_5\colon X_5 \to X_4\colon\ (Z_2, Z_3, Z_4, Z_5) \mapsto (Z_2, Z_3, Z_4)$$

coincides with the structural morphism $\pi_5\colon \mathbb{P}(\mathcal{E}_4^\vee) \to X_4$.

To describe the forgetful morphism $\sigma\colon X_5 \to H_5(0)$: $(Z_2, Z_3, Z_4, Z_5) \mapsto Z_5$, consider the irreducible divisors

$$D_i = \pi_5^{-1}(W_i) = \{(Z_2, Z_3, Z_4, Z_5) \in X_5 \mid Z_i \text{ is not a locally complete intersection}\},$$

where $i = 3, 4$, on X_5. The Briançon classification of zero-dimensional schemes of length 5 (see 1.1) and formulas (3) directly imply that, if $(Z_2, Z_3, Z_4, Z_5) \in D_3$ is a generic point, then the scheme Z_5 is of type (v); hence $\sigma(D_3) = K \simeq \mathbf{P}^1$. Thus the forgetful morphism σ contracts the divisor D_3.

Next, consider the dense open set

$$X_4^* = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_4 \text{ is a curvilinear scheme}\} = X_4 \setminus (W_3 \cup W_4)$$

in X_4 and the closure D_2 of the set

$$D_2^* = \{(Z_2, Z_3, Z_4, Z_5) \in \pi_5^{-1}(X_4^*) \mid Z_5 \text{ is a noncurvilinear scheme}\}$$

in X_5. A simple local evaluation shows that, for an arbitrary point $w = (Z_2, Z_3, Z_4) \in X_4^*$, $D_2 \cap \pi_5^{-1}(w)$ is the point (Z_2, Z_3, Z_4, Z_5), where Z_5 is a scheme of type (iii). Therefore, D_2 is a divisor on X_5. According to Briançon [1], the set

$$S := \{Z_5 \in H_5(0) \mid Z_5 \text{ is a scheme of type (iii)}\}$$

is of dimension 2, and $\sigma(D_2^*) \subset S$; in addition, it is easy to see that $\sigma(D_2^*) = S$. Therefore, $\sigma(D_2) = S$, i.e., the morphism σ contracts the divisor D_2.

By the definitions of the divisors D_2, D_3, D_4, the set

$$X_5^c = \{(Z_2, Z_3, Z_4, Z_5) \in \pi_5^{-1}(X_4^*) \mid Z_5 \text{ is a curvilinear scheme}\}$$

coincides with $X_5 \setminus (D_2 \cup D_3 \cup D_4)$, and

$$\sigma|X_5^*\colon X_5^* \to H_5^c(0) = H_5(0) \setminus \operatorname{Sing} H_5(0)$$

is an isomorphism (recall that $\operatorname{codim}_{H_5(0)} \operatorname{Sing} H_5(0) = 1$). Since D_4 is an irreducible divisor on X_5 and the morphism σ contracts D_2 and D_3, we obtain $\operatorname{Sing} H_5(0) = \sigma(D_4)$. The description of X_5 in a neighborhood of $\operatorname{Sing} H_5(0)$ (see 1.1) readily implies that $\sigma|D_4\colon D_4 \to \operatorname{Sing} H_5(0)$ is a birational morphism; therefore, the morphism σ has a Stein expansion of the form $\sigma = \nu \cdot \sigma'$, where σ' is a contraction of the divisors D_2 and D_3 and $\nu\colon \sigma'(X_5) \to H_5(0)$ is the normalization morphism along the divisor $\sigma'(D_4)$.

Collecting the above assertions, we obtain the following result.

Theorem 1. (i) *The variety X_5 of complete punctual flags of length ≤ 5 in dimension 2 is a smooth irreducible variety isomorphic to $\mathbb{P}(\mathcal{E}_4^\vee)$, where $\mathcal{E}_4 = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathrm{T}_4}, k(0) \boxtimes \mathcal{O}_{X_4})$ is a locally free sheaf of rank 2 on X_4, and the forgetful morphism*

$$\pi_5\colon X_5 \to X_4\colon\ (Z_2, Z_3, Z_4, Z_5) \mapsto (Z_2, Z_3, Z_4)$$

coincides with the structural morphism $\mathbb{P}(\mathcal{E}_4^\vee) \to X_4$.

(ii) *The birational forgetful morphism*

$$\sigma\colon X_5 \to H_5(0)\colon\ (Z_2, Z_3, Z_4, Z_5) \mapsto Z_5$$

has a Stein expansion of the form $\sigma = \nu \cdot \sigma'$, where σ' is the contraction of the divisors D_3 and D_4 and $\nu\colon \sigma'(X_5) \to H_5(0)$ is the morphism of normalization along the divisor $\sigma'(D_2)$; here D_2 is the closure in X_5 of the set

$$\{(Z_2, Z_3, Z_4, Z_5) \in X_5 \mid Z_4 \textit{ is a curvilinear scheme and } Z_5 \textit{ is a scheme of type (iii)}\},$$

and $D_i = \{(Z_2, Z_3, Z_4, Z_5) \in X_5 \mid Z_i$ is not a locally complete intersection$\}$ for $i = 3, 4$.

2. The punctual Hilbert scheme $\mathrm{Hilb}^3 k[[x,y,z]]$

2.1. Preliminary evaluations. In this and the next sections, as the base three-dimensional variety, for convenience we take the projective space $\mathbf{P}^3$ in which a point 0 is fixed. By $G := G(1,3)$, we denote the Grassmannian of lines in $\mathbf{P}^3$; $\mathbf{P} = \{l \in G \mid l \ni 0\} \simeq P(T_0\mathbf{P}^3)$ is the α-plane on G;

$$\Sigma = \{(v,l,Y) \in \mathbf{P}^3 \times G \times \check{\mathbf{P}}^3 \mid 0 \in l \subset Y \ni v\}; \qquad \Pi = \{(l,Y) \in G \times \check{\mathbf{P}}^3 \mid 0 \in l \subset Y\};$$

$$\Gamma = \{(v,Y) \in \mathbf{P}^3 \times \check{\mathbf{P}}^3 \mid v \in Y \ni 0\}; \qquad \Pi \overset{\pi}{\leftarrow} \Sigma \overset{\widetilde{p}_2}{\to} \Gamma, \qquad \check{\mathbf{P}} \overset{\pi_0}{\leftarrow} \Gamma \overset{\rho}{\to} \mathbf{P}^3,$$

and $\mathrm{pr}_2\colon \mathbf{P}^3 \times \mathbf{P} \to \mathbf{P}$ are the natural projections; $\{0\} \times \mathbf{P} = T_1 \subset T_2$ is a universal flag of subschemes of length ≤ 2 in $\mathbf{P}^3 \times \mathbf{P}$; and $\mathbf{T}_1 \subset \mathbf{T}_2$ is the universal flag of subschemes of length ≤ 2 in Σ defined by $\mathbf{T}_i = \mathrm{pr}_{12}^{-1}(T_i)$ for $i = 1,2$, where

$$\mathrm{pr}_{12}\colon \Sigma \to \mathbf{P}^3 \times \mathbf{P}\colon\ (v,l,Y) \mapsto (v,l)$$

is the projection. We also use the notation

$$\mathcal{O}(l,m,n) = \mathcal{O}_{\mathbf{P}^3}(l) \boxtimes \mathcal{O}_{\mathbf{P}}(m) \boxtimes \mathcal{O}_{\check{\mathbf{P}}}(n)|_\Sigma, \quad \mathcal{O}(m,n) := \mathcal{O}_{\mathbf{P}}(m) \boxtimes \mathcal{O}_{\check{\mathbf{P}}}(n)|_\Pi, \qquad l,m,n \in \mathbb{Z}.$$

On Σ, we have the exact triple $0 \to \mathcal{O}_{\mathbf{T}_1}(0,1,0) \to \mathcal{O}_{\mathbf{T}_2} \to \mathcal{O}_{\mathbf{T}_1} \to 0$. Applying the functor $\mathcal{E}xt^{\bullet}_\pi(-,\mathcal{O}_{\mathbf{T}_1})$ to this triple, we obtain

$$\begin{aligned} 0 &\to \mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1}(0,1,0),\mathcal{O}_{\mathbf{T}_1}) \\ &\xrightarrow{f} \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) \xrightarrow{g} \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \xrightarrow{h} \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_1}(0,1,0),\mathcal{O}_{\mathbf{T}_1}) \\ &\xrightarrow{j} \mathcal{E}xt^2_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^2_\pi(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \xrightarrow{e} \mathcal{E}xt^2_\pi(\mathcal{O}_{\mathbf{T}_1}(0,1,0),\mathcal{O}_{\mathbf{T}_1}) \to 0. \end{aligned} \tag{13}$$

Obviously, $\mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1}(0,1,0),\mathcal{O}_{\mathbf{T}_1}) = \mathcal{O}_\Pi(-1,0)$ and $\mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) = \mathcal{O}_\Pi$. Since $\mathbf{T}_1 = \widetilde{p}_2^{-1}(\mathbf{P}_0)$, where $\mathbf{P}_0 = \rho^{-1}(\{0\}) \overset{\pi_0}{\underset{\simeq}{\to}} \check{\mathbf{P}}$, it is easy to see that the triple

$$0 \to T_{\Gamma/\check{\mathbf{P}}}|\mathbf{P}_0 \to T_0\mathbf{P}^3 \otimes \mathcal{O}_{\check{\mathbf{P}}} \to \mathcal{O}_{\check{\mathbf{P}}}(1) \to 0,$$

which coincides with the exact Euler sequence on $\check{\mathbf{P}}$, is exact; hence $T_{\Gamma/\check{\mathbf{P}}}|\mathbf{P}_0 \simeq \Omega_{\check{\mathbf{P}}}(1)$. On the other hand, clearly, $T_{\Gamma/\check{\mathbf{P}}}|\mathbf{P}_0 \overset{\pi_{0*}}{\to} \mathcal{E}xt^1_{\pi_0}(\mathcal{O}_{\mathbf{P}_0},\mathcal{O}_{\mathbf{P}_0})$ is an isomorphism. This gives $\mathcal{E}xt^1_{\pi_0}(\mathcal{O}_{\mathbf{P}_0},\mathcal{O}_{\mathbf{P}_0}) \simeq \Omega_{\check{\mathbf{P}}}(1)$. Using the notation p_2 for the projection $\Pi \to \check{\mathbf{P}}$ and applying the projection formula and base change, we obtain

$$\mathcal{G} := \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) = p_2^*\mathcal{E}xt^1_{\pi_0}(\mathcal{O}_{\mathbf{P}_0},\mathcal{O}_{\mathbf{P}_0}) = \mathcal{O}_{\mathbf{P}} \boxtimes \Omega_{\check{\mathbf{P}}}(1)|\Pi.$$

Therefore, $\det\mathcal{G} = \mathcal{O}(0,-1)$. This implies that the morphism f in (13) is injective and

$$\operatorname{im} g = \mathcal{O}(1,-1), \qquad \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_1}(0,1,0),\mathcal{O}_{\mathbf{T}_1}) = \mathcal{G}(-1,0). \tag{14}$$

Now, $\omega_{\Gamma/\check{\mathbf{P}}} \simeq \mathcal{O}_{\mathbf{P}^3}(-3) \boxtimes \mathcal{O}_{\check{\mathbf{P}}}(1)|\Gamma$; hence $\omega_{\Gamma/\check{\mathbf{P}}}|\mathbf{P}_0 \simeq \mathcal{O}_{\check{\mathbf{P}}}(1)$, and thereby $\omega_{\Sigma/\Pi}|\mathbf{T}_1 = \widetilde{p}_2^*(\omega_{\Gamma/\check{\mathbf{P}}}|\mathbf{P}_0)$. The relative Serre duality for the flat smooth morphism π and the projection formula give

$$\mathcal{E}xt^2_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) \simeq \mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}(0,1))^\vee = ((0,1) \otimes \mathcal{E}xt^0_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}))^\vee = ((0,1)\otimes\mathcal{O}_\Pi)^\vee = \mathcal{O}(0,-1)$$

and show that the morphism e in (13) is an isomorphism. Therefore, (13) and (14) imply

$$\operatorname{im} h = \det\mathcal{G}(-1,0) \otimes (\mathcal{E}xt^2_\pi(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}))^{\otimes -1} = \mathcal{O}(-2,0),$$

i.e., the triple $0 \to \mathcal{O}(1,-1) \to \mathcal{E}_3 \to \mathcal{O}(-2,0) \to 0$, where $\mathcal{E}_3 := \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1})$, is exact. By virtue of the obvious equalities

$$\mathrm{Ext}^1(\mathcal{O}(-2,0),\mathcal{O}(1,-1)) = H^1(\mathcal{H}om_{\mathcal{O}_\Pi}(\mathcal{O}(-2,0),\mathcal{O}(1,-1))) = H^1(\mathcal{O}(3,-1)) = 0,$$

this triple splits, i.e.,

$$\mathcal{E}_3 = \mathcal{E}xt^1_\pi(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \simeq \mathcal{O}(1,-1) \oplus \mathcal{O}(-2,0). \tag{15}$$

Now, let us apply the canonical identification $H_2(0) \overset{\sim}{\to} \mathbf{P}\colon\ Z_2 \mapsto l = \operatorname{Span} Z_2$. Using the local freeness of the sheaf $\mathcal{E}_3$, the relative Serre duality for the projection π, and the results of [6, Sec. 1.2], we obtain the following proposition just as in the two-dimensional case (cf. 1.3).

Proposition 2. *The smooth variety* $\mathbb{P}(\check{\mathcal{E}}_3) = \mathbb{P}(\mathcal{O}(2,0) \oplus \mathcal{O}(-1,1))$ *coincides with the variety*

$$\widetilde{X}_3 = \{(Z_2, Z_3, Y) \in H_2(0) \times H_3(0) \times \check{\mathbf{P}} \mid Z_2 \subset Z_3 \subset Y\},$$

and the structural morphism $\mu\colon \mathbb{P}(\check{\mathcal{E}}_3) \to \Sigma$ *coincides with the forgetful morphism* $\widetilde{X}_3 \to \Pi\colon (Z_2, Z_3, Y) \mapsto (l = \operatorname{Span} Z_2, Y)$.

2.2. Some Ext-groups. In what follows, we need to know the dimensions of certain Ext-groups on $\mathbf{P}^3$. They are given by the following lemma.

Lemma 2. *The following formulas hold:*

$$\operatorname{Ext}^i(k(0), k(0)) = \begin{cases} k, & i = 0, 3, \\ k^3, & i = 1, 2, \end{cases} \qquad \operatorname{Ext}^i(\mathcal{O}_{Z_2}, k(0)) = \begin{cases} k, & i = 0, 3, \\ k^3, & i = 1, 2, \end{cases}$$

$$\operatorname{Ext}^i(\mathcal{O}_{Z_3}, k(0)) = \begin{cases} k, & i = 0, 3, \\ k^3, & i = 1, 2 \end{cases} \qquad \text{if } Z_3 \notin \operatorname{Sing} H_3(0),$$

$$\operatorname{Ext}^i(\mathcal{O}_{Z_3}, k(0)) = \begin{cases} k, & i = 0, \\ k^4, & i = 1, \\ k^5, & i = 2, \\ k^2, & i = 3 \end{cases} \qquad \text{if } Z_3 \in \operatorname{Sing} H_3(0).$$

Proof. Consider two cases:

(i) Z is one of the schemes $k(0)$, Z_2, and Z_3, where $Z_3 \notin \operatorname{Sing} H_3(0)$;

(ii) $Z = Z_3 \in \operatorname{Sing} H_3(0)$.

(i) Obviously; in this case, Z has the free resolvent

$$K^\bullet : 0 \to \mathcal{O}_U \xrightarrow{\partial_4} 3\mathcal{O}_U \xrightarrow{\partial_3} 3\mathcal{O}_U \xrightarrow{\partial_2} \mathcal{O}_U \xrightarrow{\partial_1} \mathcal{O}_Z \to 0$$

in a suitable neighborhood $U \subset \mathbf{P}^3$ of the point 0. Applying the functor $\mathcal{H}om^\bullet_{\mathcal{O}_U}(-, k(0))$ to the complex $K^\bullet$, i.e., taking the complex $\check{K}^\bullet$ dual to $K^\bullet$ and multiplying it by $k(0)$, we obtain the complex

$$0 \to k(0) \xrightarrow{\check{\partial}_1 \otimes k(0)} k(0)^3 \xrightarrow{\check{\partial}_2 \otimes k(0)} k(0)^3 \xrightarrow{\check{\partial}_3 \otimes k(0)} k(0) \xrightarrow{\check{\partial}_4 \otimes k(0)} 0,$$

in which all the differentials $\check{\partial}_i \otimes k(0)$ are obviously zero. Since the cohomology of this complex is formed by the sheaves $\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{O}_Z, k(0))$, we have

$$\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{O}_Z, k(0)) = \begin{cases} k(0), & i = 0, 3, \\ k(0)^3, & i = 1, 2. \end{cases}$$

This and the spectral sequence of local and global $\mathcal{E}xt$'s (which, obviously, degenerates, because the sheaves $\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{O}_Z, k(0))$ have zero-dimensional supports) give the required formulas for the Ext-groups.

(ii) In this case, Z coincides with the transversal intersection $Y \cap l^{(1)}$, where $Y = \operatorname{Span} Z$ is the plane containing the scheme Z and $l^{(1)}$ is the first infinitesimal neighborhood of a line l in $\mathbf{P}^3$ that intersects the plane Y at 0. It is easy to see that, in a suitable neighborhood $U \subset \mathbf{P}^3$ of 0, we have the free resolvents

$$K_1^\bullet : 0 \to \mathcal{O}_U \to \mathcal{O}_U \to \mathcal{O}_{Y\cap U} \to 0, \qquad K_2^\bullet : 0 \to 2\mathcal{O}_U \to 3\mathcal{O}_U \to \mathcal{O}_U \to \mathcal{O}_{l^{(1)}\cap U} \to 0$$

of the sheaves $\mathcal{O}_{Y\cap U}$ and $\mathcal{O}_{l^{(1)}\cap U}$. Since the intersection $Z = Y \cap l^{(1)}$ is transversal, the resolvent of the sheaf $\mathcal{O}_{Z\cap U}$ is the total complex

$$K^\bullet = \operatorname{tot}(K_1^\bullet \otimes K_2^\bullet) : 0 \to 2\mathcal{O}_U \xrightarrow{\partial_4} 5\mathcal{O}_U \xrightarrow{\partial_3} 4\mathcal{O}_U \xrightarrow{\partial_2} \mathcal{O}_U \xrightarrow{\partial_1} \mathcal{O}_{Z\cap U} \to 0.$$

A repetition of the argument from (i) for the complex $K^\bullet$ give the required formulas for the Ext-groups. This completes the proof of the lemma. □

2.3. A description of the scheme $H_3(0)$ and the variety X_3 of complete punctual flags. Let $\mathbf{P} \overset{p_1}{\leftarrow} \Pi \overset{p_2}{\rightarrow} \check{\mathbf{P}}$ be the natural projections. For an arbitrary line $l \in \mathbf{P}$ through 0, we have

$$\check{l} := p_1^{-1}(l) \simeq p_2 p_1^{-1}(l) \simeq \mathbf{P}^1.$$

Put $\Sigma_l = \pi^{-1}(\check{l})$ and $\pi_l = \pi|\Sigma_l$. Let $Z_2 = Z_2(l) \in H_2(0)$ be the zero-dimensional scheme of length 2 corresponding to the line l under the canonical isomorphism $H_2(0) \tilde{\rightarrow} \mathbf{P}$; then

$$\mathbf{T}_2 \times_\Sigma \Sigma_l = \mathbf{T}_2 \cap \Sigma_l \simeq Z_2 \times \mathbf{P}^1.$$

We denote

$$\mathcal{E}_3(l) = \mathcal{E}xt^{\,i}_{\pi_l}(\mathcal{O}_{Z_2(l)\times\mathbf{P}^1}, \mathcal{O}_{\{0\}\times\mathbf{P}^1}).$$

By Lemma 2, for any plane $Y \in \check{\mathbf{P}}$, the dimensions of the spaces $\mathrm{Ext}^i_Y(k(0), k(0))$ with $0 \le i \le 2$ do not depend on the point Y; taking into account Proposition 2, we obtain the base change isomorphisms

$$\mathcal{E}_3(l) \simeq \mathcal{E}_3|\check{l} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}, \tag{16}$$

$$\mathcal{E}_3 \otimes k(\{Y\}) = \mathcal{E}_3(l) \otimes k(\{Y\}) \tilde{\rightarrow} \mathrm{Ext}^1_Y(\mathcal{O}_{Z_2(l)}, k(0)), \qquad Y \in \check{l} \simeq \mathbf{P}^1. \tag{17}$$

Consider the surface $S_l := \mathbb{P}(\check{\mathcal{E}}_3(l)) = \mathbb{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1})$, the structural morphism $\mu_l \colon S_l \to \check{l} \simeq \mathbf{P}^1$, and a Grothendieck sheaf $\mathcal{O}_{S_l}(1)$ such that $\mu_{l*}(\mathcal{O}_{S_l}(1)) = \check{\mathcal{E}}_3(l)$. Note that, for any point $Y \in \check{\mathbf{P}}$, the natural map

$$\mathrm{Ext}^1_Y(\mathcal{O}_{Z_2(l)}, k(0)) \to \mathrm{Ext}^1_{\mathbf{P}^3}(\mathcal{O}_{Z_2(l)}, k(0))$$

is a monomorphism; hence we have the embedding

$$P(\mathrm{Ext}^1_Y(\mathcal{O}_{Z_2(l)}, k(0))) \hookrightarrow P(\mathrm{Ext}^1_{\mathbf{P}^3}(\mathcal{O}_{Z_2(l)}, k(0))),$$

which, together with the isomorphism $P(\mathcal{E}_3(l)|\{Y\}) \simeq P(\mathrm{Ext}^1_Y(\mathcal{O}_{Z_2(l)}, k(0)))$, determines the following embedding f_l:

$$\mu_l^{-1}(\{Y\}) = P(\mathcal{E}_3(l)|\{Y\}) \hookrightarrow \mathbf{P}^2_l := P(\mathrm{Ext}^1_{\mathbf{P}^3}(\mathcal{O}_{Z_2(l)}, k(0))).$$

Here by Lemma 2, $\mathbf{P}^2_l$ is the projective plane, which is naturally embedded in the variety

$$X_3 = \{(Z_2, Z_3) \in H_2(0) \times H_3(0) \mid Z_2 \subset Z_3\}$$

(indeed, to each point $k\xi \in P(\mathrm{Ext}_{\mathbf{P}^3}(\mathcal{O}_{Z_2(l)}, k(0)))$ considered as an extension $\xi : 0 \to k(0) \to \mathcal{O}_{Z_3} \to \mathcal{O}_{Z_2(l)} \to 0$ corresponds the point $(Z_2(l), Z_3) \in X_3$). The map f_l can be globalized to the morphism $f_l \colon S_l \to \mathbf{P}^2_l$, which by construction coincides with the forgetful morphism

$$f_l \colon S_l \to \mathbf{P}^2_l \subset X_3 : \quad (Z_2(l), Z_3, Y) \mapsto (Z_2(l), Z_3). \tag{18}$$

Let $Z_3(l) \in H_3(0)$ be a zero-dimensional subscheme of length 3 on the line l (we also refer to this subscheme as *three collinear points*). Since S_l is a surface of type $\mathbb{F}_1$, description (18) directly implies that f_l is a blow-up of the projective plane $\mathbf{P}^2_l$ at the point $(Z_2(l), Z_3(l))$, i.e., the map determined by the complete linear series of the sheaf $|\mathcal{O}_{S_l}(1)|$. Thus we obtain the canonical isomorphism

$$\mathbf{P}^2_l \simeq \mathbb{P}(H^0(\mathcal{O}_{S_l}(1))) = \mathbb{P}((p_1\mu_l)_*\mathcal{O}_{S_l}(1))). \tag{19}$$

Since $X_3 = \bigcup_{l\in\mathbf{P}} \mathbf{P}^2_l$ and $\mathcal{O}_{S_l}(1) = \mathcal{O}_{\mathbb{P}(\check{\mathcal{E}}_3)}(1)|S_l$, obviously, the isomorphism (19) can be globalized to the isomorphism

$$X_3 \simeq \mathbb{P}((p_1\mu_l)_*\mathcal{O}_{\mathbb{P}(\check{\mathcal{E}}_3)}(1)) = \mathbb{P}(\check{\mathcal{E}}_3) = \mathbb{P}(p_{1*}(\mathcal{O}(2,0) \oplus \mathcal{O}(-1,1)) = \mathbb{P}(\mathcal{F}_3),$$

where $\mathcal{F}_3 = \mathcal{O}_{\mathbf{P}}(2) \oplus \Omega_{\mathbf{P}}(1)$ is a bundle of rank 3 on P. (We have used Proposition 2 and the equality $p_{1*}\mathcal{O}(-1,1) = \mathcal{O}_{\mathbf{P}}(-1) \otimes p_{1*}p_2^*\mathcal{O}_{\check{\mathbf{P}}}(1) = \Omega_{\mathbf{P}}(1)$). Accordingly, $\widetilde{X}_3 = \bigcup_{l\in\mathbf{P}} S_l$, and by virtue of (18), the morphism f_l is globalized to the forgetful morphism

$$f\colon \widetilde{X}_3 \to X_3 : (Z_2, Z_3, Y) \mapsto (Z_2, Z_3),$$

which is determined by the relative (over $\mathbf{P}$) complete linear series of the sheaf $\mathcal{O}_{\mathbb{P}(\tilde{\mathcal{E}}_3)}(1)$ and contracts the divisor

$$W = \{(Z_2, Z_3, Y) \in \widetilde{X}_3 \mid Z_3 \text{ is three collinear points}\}.$$

Finally, the forgetful morphism $g\colon X_3 = \mathbb{P}(\mathcal{F}_3) \to H_3(0)$: $(Z_2, Z_3) \mapsto Z_3$ is nothing but the contraction of the divisor D on X_3, which is the isomorphic image under the morphism f of the divisor

$$\widetilde{D} = \{(Z_2, Z_3, Y) \in \widetilde{X}_3 \mid Z_3 = 0^{(1)} \text{ is the first infinitesimal neighborhood of } 0 \text{ in the plane } Y\}.$$

Clearly, $g(D) = \operatorname{Sing} H_3(0)$. In addition, it is easy to verify that $\mathcal{O}_{X_3}(D) = \mathcal{O}_{\mathbb{P}(\mathcal{F}_3)}(1) \otimes \nu^*\mathcal{O}_{\mathbf{P}}(-2)$, where $\nu\colon \mathbb{P}(\mathcal{F}_3) \to \mathbf{P}$ is the structural morphism and $\mathcal{O}_{\mathbb{P}(\mathcal{F}_3)}(1) = \mathcal{O}_{X_3/\mathbf{P}}(1)$ is a Grothendieck sheaf such that $\nu_*\mathcal{O}_{X_3/\mathbf{P}}(1) = \mathcal{F}_3$; we also have $h^0(\mathcal{O}_{X_3}(D)) = 1$. Thus the following theorem is valid.

Theorem 2. *The punctual Hilbert scheme $H_3(0) = \operatorname{Hilb}^3 k[[x, y, z]]$ is the image of the variety*

$$X_3 = \{(Z_2, Z_3) \in H_2(0) \times H_3(0) \mid Z_2 \subset Z_3\}$$

of complete punctual flags (which is isomorphic to the smooth irreducible variety $\mathbb{P}(\mathcal{O}_{\mathbf{P}}(2) \oplus \Omega_{\mathbf{P}}(1))$) under the birational forgetful morphism $g\colon X_3 \to H_3(0)$ contracting the divisor

$$D = \{(Z_2, Z_3, Y) \in \widetilde{X}_3 \mid Z_3 \textit{ is the first infinitesimal neighborhood of } 0 \textit{ in some plane, i.e., } Z_3 \textit{ is not a locally complete intersection}\}.$$

The divisor D is uniquely determined as the unique divisor of the linear series $|\mathcal{O}_{X_3/\mathbf{P}}(1) \otimes \nu^*\mathcal{O}_{\mathbf{P}}(-2)|$, where $\nu\colon \mathbb{P}(\mathcal{O}_{\mathbf{P}}(2) \oplus \Omega_{\mathbf{P}}(1)) \to \mathbf{P}$ is the structural morphism. In addition, $g(D) = \operatorname{Sing} H_3(0) \simeq \check{\mathbf{P}}$, and the scheme $H_3(0)$ is analytically isomorphic to the direct product of $\check{\mathbf{P}}$ and the cone over a cubic normcurve in a neighborhood of any point $Z_3 \in \operatorname{Sing} H_3(0)$.

Remark 1. Obviously, the divisor D on X_3 is isomorphic to the variety $\Gamma_{0,2} \subset \mathbf{P} \times \check{\mathbf{P}}$ of flags "(point, plane)", and the morphism $g|D\colon D \to \operatorname{Sing} H_3(0)$ coincides with the projection map $\mathrm{pr}_2\,|\Gamma_{0,2}\colon \Gamma_{0,2} \to \check{\mathbf{P}}$.

3. The punctual Hilbert scheme $\operatorname{Hilb}^4 k[[x, y, z]]$

3.1. Preliminary evaluations. Let X_3 be the variety of punctual flags of zero-dimensional subschemes of length 3 in $\mathbf{P}^3$ that is mentioned in the statement of Theorem 2. We use the standard notation $\mathcal{O}_{X_3}(m, n) := \mathcal{O}_{X_3/\mathbf{P}}(1)^{\otimes m} \otimes \nu^*\mathcal{O}_{\mathbf{P}}(n)$ for $m, n \in \mathbb{Z}$. Consider the universal flag of punctual families $\{0\} \times X_3 = \mathbf{T}_1 \subset \mathbf{T}_2 \subset \mathbf{T}_3 \in \mathbf{P}^3 \times X_3$, where $\mathbf{T}_3$ is the universal three-point space with $\operatorname{Supp} \mathbf{T}_3 = \{0\} \times X_3$. Since $\mathbf{T}_1 \simeq X_3$, we can put $\mathcal{O}_{\mathbf{T}_1}(m, n) := \mathcal{O}_{X_3}(m, n)$ for $m, n \in \mathbb{Z}$. We have the exact triples

$$0 \to \mathcal{O}_{\mathbf{T}_1}(0,1) \to \mathcal{O}_{\mathbf{T}_2} \to \mathcal{O}_{\mathbf{T}_1} \to 0, \qquad 0 \to \mathcal{O}_{\mathbf{T}_1}(a,b) \to \mathcal{O}_{\mathbf{T}_3} \to \mathcal{O}_{\mathbf{T}_2} \to 0. \tag{20}$$

The first triple is evident. To find a and b in the second triple, consider $\mathcal{S} = \ker(\mathrm{res} : \mathcal{O}_{\mathbf{T}_3} \to \mathcal{O}_{\mathbf{T}_1})$. The triples (20) give the exact triple

$$0 \to \mathcal{O}_{\mathbf{T}_1}(a,b) \to \mathcal{S} \to \mathcal{O}_{\mathbf{T}_1}(0,1) \to 0, \tag{21}$$

and the description of D as the set $\{(Z_2, Z_3) \in X_3 \mid \text{scheme } Z_3 \text{ is not a locally complete intersection in } \mathbf{P}^3\}$ (see Theorem 2) implies

$$D = \{z = (Z_2, Z_3) \in X_3 \mid \dim(\mathcal{S} \otimes k(0, z)) = 2\}. \tag{22}$$

Let us apply the functor $\mathcal{E}xt^{\bullet}_{\mathcal{O}_{\mathbf{T}_1}}(\mathcal{O}_{\mathbf{T}_1}(0,1), -)$ to (21) and consider the first connecting homomorphism δ in the resulting long exact sequence; taking into account the obvious equality

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbf{T}_1}}(\mathcal{O}_{\mathbf{T}_1}(0,1), \mathcal{O}_{\mathbf{T}_1}(a,b)) = T_0\mathbf{P}^3 \otimes \mathcal{O}_{\mathbf{T}_1}(a, b-1) \simeq 3\mathcal{O}_{\mathbf{T}_1}(a, b-1),$$

we obtain $\delta\colon \mathcal{O}_{\mathbf{T}_1} \to 3\mathcal{O}_{\mathbf{T}_1}(a, b-1)$. By virtue of (22), δ vanishes along D being a section (as previously, we identify $\mathbf{T}_1$ with X_3). Taking into account (21) and Theorem 2, we see that $(a, b-1) = (1, 0)$, i.e.,

$$a = b = 1. \tag{23}$$

Let us denote the projection $\mathbf{P}^3 \times X_3 \to X_3$ by p_2 and apply the functor $\mathcal{E}xt^i_{p_2}(-, \mathcal{O}_{\mathbf{T}_1})$ to the sheaves $\mathcal{O}_{\mathbf{T}_i}$, $i = 1, 2, 3$, and the second triple in (20). Using Lemma 2 and the properties of base change for relative $\mathcal{E}xt$-sheaves [8], we obtain the following lemma.

Lemma 3. (i) $\operatorname{rk}\mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) = 3$ *for* $i = 1, 2$, *and* $\operatorname{rk}\mathcal{E}xt^3_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) = 1$.

(ii) *The* $\mathcal{O}_{X_3}$*-sheaves* $\mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1})$ *and* $\mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1})$, *where* $i \geq 0$, *are locally free, and for an arbitrary point* $z = (\{0\}, Z_2, Z_3) \in X_3$, *the corresponding base change homomorphisms*

$$\mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1) \quad \textit{and} \quad \mathcal{O}_{\mathbf{T}_1}) \otimes k(z) \to \operatorname{Ext}^i(k(0), k(0)),$$

where $i \geq 0$, *are isomorphisms. In particular, the* $\mathcal{O}_{X_3}$*-sheaves*

$$\mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}), \qquad \mathcal{E}xt^i_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1})$$

are invertible for $i = 0$ *and* 3, *and the natural morphism*

$$\alpha_0\colon \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1})$$

is an isomorphism. Similarly, the sheaf $\mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1})$ *is invertible.*

(iii) *In the base change diagram*

$$\begin{array}{ccc} \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}) \otimes k(z) & \xrightarrow{\partial_1 \otimes k(z)} & \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \otimes k(z) \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{Ext}^0(k(0), k(0)) & \longrightarrow & \operatorname{Ext}^1(\mathcal{O}_{Z_2}, k(0)) \end{array},$$

the lower horizontal map is injective for any $z \in X_3$, *and hence,* $\operatorname{coker}\partial_1$ *is a locally free sheaf of rank* 2.

3.2. The properties of the sheaf $\mathcal{E} = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1})$ and the variety $X_3 = \mathbb{P}(\check{\mathcal{E}})$ of punctual flags of lengths 2, 3, and 4. Let us apply the functor $\mathcal{E}xt^{\bullet}_{p_2}(-, \mathcal{O}_{\mathbf{T}_1})$ to the second triple in (20); taking into account (23), we obtain the long exact sequence

$$\begin{aligned} 0 &\to \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\alpha_0} \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\beta_0} \mathcal{E}xt^0_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}) \\ &\xrightarrow{\partial_1} \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\alpha_1} \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\beta_1} \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}) \\ &\xrightarrow{\partial_2} \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\alpha_2} \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\beta_2} \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}) \\ &\xrightarrow{\partial_3} \mathcal{E}xt^3_{p_2}(\mathcal{O}_{\mathbf{T}_2}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\alpha_3} \mathcal{E}xt^3_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1}) \xrightarrow{\beta_3} \mathcal{E}xt^3_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1), \mathcal{O}_{\mathbf{T}_1}) \to 0. \end{aligned} \tag{24}$$

Consider the morphism ∂_2 in this sequence. Lemma 3 directly implies $\operatorname{rk}\ker\partial_2 = 1$; therefore, $\ker\partial_2$ is invertible as the kernel of a morphism of locally free sheaves. Note that, by virtue of (20), (21), and (23), the morphism ∂_2 is included in the commutative diagram

$$\begin{array}{ccc} 3\mathcal{O}_{X_3}(-1,-1) & \longrightarrow & 3\mathcal{O}_{X_3}(0,-1) \\ \| & & \| \\ \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1),\mathcal{O}_{\mathbf{T}_1}) & \xrightarrow{e} & \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_1}(0,1),\mathcal{O}_{\mathbf{T}_1}) \\ \| & & \uparrow{\scriptstyle\lambda} \\ \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1),\mathcal{O}_{\mathbf{T}_1}) & \xrightarrow{\partial_2} & \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \end{array}$$

Now, Theorem 2 implies that e is decomposed as

$$3\mathcal{O}_{X_3}(-1,-1) \stackrel{\cdot D}{\to} 3\mathcal{O}_{X_3}(0,-1) \stackrel{e'}{\to} 3\mathcal{O}_{X_3}(0,-1),$$

where e' is a bundle morphism (i.e., $e'\otimes k(z)$ has the same rank at all points $z\in X_3$). Therefore, the sheaf $\operatorname{coker} e$ has homological dimension ≤ 1. On the other hand, the first triple in (20) gives

$$\operatorname{coker}(\lambda\colon \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_1}(0,1),\mathcal{O}_{\mathbf{T}_1})) = \mathcal{E}xt^3_{p_2}(\mathcal{O}_{\mathbf{T}_1},\mathcal{O}_{\mathbf{T}_1}) \simeq \mathcal{O}_{X_3},$$

and assertion (ii) of Lemma 3 implies the invertibility of the sheaf

$$\mathcal{L} := \ker(\lambda\colon \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1}) \to \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_1}(0,1),\mathcal{O}_{\mathbf{T}_1})),$$

which together with the preceding diagram gives the exact sequence

$$0 \to \mathcal{L} \to \operatorname{coker}\partial_2 \to \operatorname{coker} e \to \mathcal{O}_{X_3} \to 0.$$

This and the condition $hd(\operatorname{coker} e) \le 1$ implies $hd(\operatorname{coker}\partial_2) \le 1$; therefore,

$$\mathcal{T}or_2^{\mathcal{O}_{X_3}}(\operatorname{coker}\partial_2, k(z)) = 0, \qquad z \in X_3. \tag{25}$$

Now, by (24) and Lemma 3,

$$\mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1),\mathcal{O}_{\mathbf{T}_1}) \stackrel{\partial_2}{\to} \mathcal{E}xt^2_{p_2}(\mathcal{O}_{\mathbf{T}_2},\mathcal{O}_{\mathbf{T}_1})$$

is a morphism of locally free sheaves of rank 3, and $\operatorname{rk}\partial_2 = 2$; hence the sheaf $\ker\partial_2$ is invertible. Moreover, by (25) this sheaf is a subbundle in $\mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}(1,1),\mathcal{O}_{\mathbf{T}_1})$. This, (24), and assertion (iii) of Lemma 3 imply that the sheaf $\mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3},\mathcal{O}_{\mathbf{T}_1})$ is locally free and has rank 3. Thus, we have proved the following proposition.

Proposition 3. (i) *The sheaf* $\mathcal{E} := \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3},\mathcal{O}_{\mathbf{T}_1})$ *is a locally free sheaf of rank* 3, *and the exact sequence of bundles on* X_3

$$0 \to \operatorname{im}\alpha_1 \to \mathcal{E} \to \operatorname{im}\beta_1 \to 0, \tag{26}$$

where $\operatorname{rk}\operatorname{im}\beta_1 = 1$, *holds.*

(ii) *The base change morphism*

$$b(z)\colon \mathcal{E}\otimes k(z) \to \operatorname{Ext}^1(\mathcal{O}_{Z_3}, k(0))$$

is injective for arbitrary $z = (\{0\}, Z_2, Z_3) \in X_3$.

This proposition and the irreducibility of the variety $H_4(0)$ (see [1, 2]) imply the following assertion.

Corollary 3. (i) *The scheme* $X_4 := P(\mathcal{E}) = \mathbb{P}(\check{\mathcal{E}})$ *parametrizing the punctual flags* $z = (Z_2, Z_3, Z_4)$, $\{0\} \in Z_2 \subset Z_3 \subset Z_4$, *is a smooth irreducible variety, and the projection*

$$\pi_4 \colon X_4 \to X_3 : (Z_2, Z_3, Z_4) \to (Z_2, Z_3)$$

coincides with the structural morphism $\mathbb{P}(\check{\mathcal{E}}) \to X_3$.

(ii) *The forgetful morphism*

$$\sigma \colon X_4 \to H_4(0) : \quad (Z_2, Z_3, Z_4) \to Z_4 \tag{27}$$

is surjective, and it is a desingularization of the variety $H_4(0)$.

3.3. The properties of the sheaf $\mathcal{E}|D$ and the varieties X_D and $\mathcal{Y}$. Consider an arbitrary point $z = (\{0\}, Z_2, Z_3) \in D$. Description (22) implies that the triple

$$0 \to k(0)^2 \to \mathcal{O}_{Z_3} \to k(0) \to 0 \tag{28}$$

is exact. Applying the functor $\mathrm{Ext}^\bullet(-, k(0))$ to (28), we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(k(0), k(0)) \xrightarrow{\gamma_{0z}} \mathrm{Hom}(\mathcal{O}_{Z_3}, k(0)) \longrightarrow \mathrm{Hom}(k(0)^2, k(0))$$
$$\xrightarrow{\partial_{1z}} \mathrm{Ext}^1(k(0), k(0)) \xrightarrow{\varepsilon_z} \mathrm{Ext}^1(\mathcal{O}_{Z_3}, k(0)).$$

Here, obviously, γ_{0z} is an isomorphism, and by (28) and the identification $\mathrm{Ext}^1(k(0), k(0)) = T_0\mathbf{P}^3$, we have $\mathrm{im}\,\partial_{1z} = T_0 Z_3 \simeq k^2$; therefore, $\mathrm{im}\,\varepsilon_z \simeq k$, and for any vector $\xi \in \mathrm{Ext}^1(k(0), k(0)) \setminus \mathrm{im}\,\partial_{1z}$, the corresponding extension $\xi : 0 \to k(0) \to \mathcal{O}_{Z_2} \to k(0) \to 0$ determines a scheme $Z_2(\xi)$ such that

$$\mathrm{Span}(T_0 Z_3, T_0 Z_2) = \mathrm{Ext}^1(k(0), k(0)) = T_0\mathbf{P}^3. \tag{29}$$

For this vector ξ, the nonzero vector $\varepsilon_z(\xi) \in \mathrm{Ext}^1(\mathcal{O}_{Z_3}, k(0))$ determines a nonzero extension

$$\varepsilon_z(\xi) : 0 \to k(0) \to \mathcal{O}_{Z_4} \to \mathcal{O}_{Z_3} \to 0,$$

which, together with (28) and the last triple, is included in the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & k(0)^2 & \longrightarrow & \mathcal{O}_{Z_3} & \longrightarrow & k(0) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & k(0)^2 & \longrightarrow & \mathcal{O}_{Z_4} & \longrightarrow & \mathcal{O}_{Z_2} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & k(0) & = & k(0) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{30}$$

Here $Z_3 \subset Z_4 \supset Z_2$ by construction. By virtue of (29), we have $l(\mathbf{P}^1 \cap Z_4) \geq 2$ for any line $\mathbf{P}^1 \subset \mathbf{P}^3$ through the point 0. The condition $l(Z_4) = 4$ makes the last inequality into an equality; hence $\mathcal{O}_{Z_4} = \mathcal{O}^3_{\mathbf{P}}/\boldsymbol{m}^2$, where $\boldsymbol{m} = \mathcal{I}_{0,\mathbf{P}^3}$ is the sheaf of ideals of the (reduced) point 0. In other words,

$$Z_4 = \mathrm{Spec}(\mathcal{O}^3_{\mathbf{P}}/\boldsymbol{m}^2) = \mathrm{Spec}(k[[x, y, z]]/(x, y, z)^2) = T_0\mathbf{P}^3. \tag{31}$$

Since

$$\mathrm{Ext}^1(k(0), k(0)) = \mathrm{Ext}^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}, \mathcal{O}_{\mathbf{T}_1}) \otimes k(z) \simeq 3\mathcal{O}_{X_3} \otimes k(z) \simeq 3\mathcal{O}_D \otimes k(z),$$

we can easily show that the image $\mathrm{im}\,\varepsilon_D$ of the morphism

$$\varepsilon_D \colon 3\mathcal{O}_D = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}, \mathcal{O}_{\mathbf{T}_1})|D \to \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1})|D = \mathcal{E}|D$$

of $\mathcal{O}_D$-sheaves, which is a globalization of the homomorphism $\mathrm{Ext}^1(k(0), k(0)) \xrightarrow{\varepsilon_z} \mathrm{Ext}^1(\mathcal{O}_{Z_3}, k(0))$, is the canonical quotient sheaf of $3\mathcal{O}_D$, i.e., $\mathrm{im}\,\varepsilon_D$ is isomorphic to $g_D^*\mathcal{O}_{\check{\mathbf{P}}}(1)$, where $g_D := g|D \colon D \to \mathrm{Sing}\, H_3(0)$ is the projection (see Remark 1). Taking into account Corollary 3, we obtain the following proposition.

Proposition 4. (i) *The morphism of locally free $\mathcal{O}_D$-sheaves*

$$\varepsilon_D\colon 3\mathcal{O}_D = \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_1}, \mathcal{O}_{\mathbf{T}_1})|D \to \mathcal{E}|D$$

induced by the surjection $\mathcal{O}_{\mathbf{T}_3} \twoheadrightarrow \mathcal{O}_{\mathbf{T}_1}$ is a morphism of rank 3 bundles on D and has rank 1, and

$$\operatorname{im}\varepsilon_D = g_D^*\mathcal{O}_{\check{\mathbf{P}}}(1). \tag{32}$$

Thus the embedding of the subvariety $\mathcal{Y} := \mathbb{P}((\operatorname{im}\varepsilon_D)^\vee)$ into the variety $X_D := X_4 \times_{X_3} D$ is the section

$$D \hookrightarrow X_D\colon\ (Z_2, Z_3) \mapsto (Z_2, Z_3, Z_4),$$

where $Z_4 = T_0\mathbf{P}^3$, of the projection $\pi_D = \pi_4|X_D\colon X_D \to D$.

(ii) *Let $\sigma\colon X_4 \to H_4(0)\colon\ (Z_2, Z_3, Z_4) \mapsto Z_4$ be the forgetful morphism* (27). *Then $\sigma(\mathcal{Y}) = \{T_0\mathbf{P}^3\}$ is a point.*

3.4. The variety X_{pnc} of plane noncurvilinear flags. Consider the set of *plane noncurvilinear* punctual flags

$$X_{\mathrm{pnc}} := \{(Z_2, Z_3, Z_4) \in H_2(0) \times H_3(0) \times H_4(0) \mid \dim T_0Z_4 = 2$$

and Z_4 lies in some plane passing through $\{0\}$, i.e., $\dim\operatorname{Span}(Z_4) = 2\}$. From Corollary 3, by continuity we see that X_{pnc} is a subvariety in X_4. In addition, it is easy to see that $X_{\mathrm{pnc}} \simeq P(\mathcal{F}) = \mathbb{P}(\check{\mathcal{F}})$, where $\mathcal{F}$ is some subbundle of rank 2 in the bundle $\mathcal{E}|D$, so the natural projection (forgetful morphism)

$$\pi_1 = \pi_D|X_{\mathrm{pnc}}\colon X_{\mathrm{pnc}} \to D\colon\ (Z_2, Z_3, Z_4) \mapsto Z_4$$

coincides with the structure morphism $\mathbb{P}(\check{\mathcal{F}}) \to D$.

Consider the projection $g_D := g|D\colon D \to \operatorname{Sing} H_3(0)$. As follows from [6, Proposition 2.7], for an arbitrary plane $Y \in \check{\mathbf{P}}$, the fiber $Q_Y = (g_D\pi_1)^{-1}(\{Y\})$ is a quadric, and the morphism $\sigma|Q_Y$, where σ is the forgetful morphism (27), coincides with the double covering $\sigma_Y\colon Q_Y \to P(S^2(T_0Y))$ branched in the conic–Veronese image of $P(T_0Y) \hookrightarrow P(S^2(T_0Y))$. Note that $P(S^2(T_0Y))$ is the fiber of the projection $\tau\colon \operatorname{Hilb}^2\mathbf{P} \to \check{\mathbf{P}}\colon\ z \mapsto \operatorname{Span} Z$, which coincides with the structural morphism $\mathbb{P}(S^2\Omega_{\check{\mathbf{P}}}) \to \check{\mathbf{P}}$ under the natural isomorphism

$$\operatorname{Hilb}^2\mathbf{P} \simeq P(\mathcal{A}d(T_{\check{\mathbf{P}}}(-1))) \simeq \mathbb{P}(S^2\Omega_{\check{\mathbf{P}}}).$$

Therefore, the morphism $\sigma|X_{\mathrm{pnc}}$ coincides with the double covering $\sigma_1\colon X_{\mathrm{pnc}} \to \operatorname{Hilb}^2\mathbf{P}$ branched in a divisor of the diagonal $\Delta = \{z \in \operatorname{Hilb}^2\mathbf{P} \mid \operatorname{Supp} Z = \{pt\}\}$. Thus we have proved the following assertion.

Proposition 5. *The forgetful morphism $\sigma|X_{\mathrm{pnc}}$ coincides with the double covering $X_{\mathrm{pnc}} \to \operatorname{Hilb}^2\mathbf{P}$ branched in a divisor of the diagonal $\Delta = \{z \in \operatorname{Hilb}^2\mathbf{P} \mid \operatorname{Supp} Z = \{pt\}\}$.*

3.5. A description of the morphism $\sigma|X_D$. Consider the complement $X_D^* := X_D \setminus \{\mathcal{Y} \cup X_{\mathrm{pnc}}\}$ of the union $\mathcal{Y} \cup X_{\mathrm{pnc}}$ in X_D. Proposition 4 shows that

$$X_D^* = \{(Z_2, Z_3, Z_4) \in X_D \mid \dim T_0Z_4 = 2,\ \dim\operatorname{Span}(Z_4) = 3\}. \tag{33}$$

Take an arbitrary point $(Z_2, Z_3, Z_4) \in X_D^*$. It is easy to see that the conditions

$$\dim T_0Z_4 = 2, \qquad \dim\operatorname{Span}(Z_4) = 3$$

on the scheme Z_4 mean that Z_4 lies on the germ $(Q_{Z_4}, 0)$ of some quadric passing through 0 and specified by the equation

$$Q_{Z_4} = \{z = ax^2 + bxy + cy^2,\quad a, b, c \in k\} \tag{34}$$

in suitable local coordinates x, y, z in a neighborhood of the point 0; $Y_{Z_4} := \mathcal{P}T_0Z_4 = \{z = 0\} \in \check{\mathbf{P}}$ is a projective plane in $\mathbf{P}^3$ such that $Y_{Z_4} \cap Z_4 = Y_{Z_4} \cap Q_{Z_4} = Z_3$. Note that the projection $(x, y, z) \mapsto (0, y, z)$

implements the analytic isomorphism $(Q_{Z_4}, 0) \xrightarrow{\simeq} (Y_{Z_4}, 0)$. This and Proposition 5 readily imply that the morphism $\sigma|X_{\text{pnc}}$ can be extended to the double covering $\sigma|(X_D^* \cup X_{\text{pnc}}) = \sigma|(X_D \setminus \mathcal{Y})$. Taking into account Proposition 4 (ii), we see that $\sigma|X_D$ is factored through the double covering σ_D in the commutative diagram that extends the double covering $\sigma_1 \colon X_{\text{pnc}} \to \operatorname{Hilb}^2 \mathbf{P} \simeq P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)))$:

$$\begin{array}{ccc} X_{\text{pnc}} & \hookrightarrow & X_D \\ \big\downarrow{\scriptstyle \sigma_1} & & \big\downarrow{\scriptstyle \sigma_D} \\ P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1))) & \overset{j}{\hookrightarrow} & P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)) \oplus \mathcal{O}_{\check{\mathbf{P}}}(m)) \end{array}, \tag{35}$$

where the embedding j is induced by the embedding of the first term in the direct sum

$$\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)) \oplus \mathcal{O}_{\check{\mathbf{P}}}(m)$$

and m is an integer. Thus $\sigma|X_D$ decomposes as

$$\sigma|X_D = \sigma_Y \cdot \sigma_D, \tag{36}$$

where, by construction, $\sigma_Y \colon \sigma_D(X_D) = P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)) \oplus \mathcal{O}_{\check{\mathbf{P}}}(m)) \to \sigma(X_D)$ is the contraction into the point $\{T_0\mathbf{P}^3\}$ of the section $P(\mathcal{O}_{\check{\mathbf{P}}}(m))$ of the structural projection

$$P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)) \oplus \mathcal{O}_{\check{\mathbf{P}}}(m)).$$

Therefore, $\sigma(X_D)$ is the cone with vertex at the point $\{T_0\mathbf{P}^3\}$ over the variety

$$P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1))) \simeq \operatorname{Hilb}^2 \mathbf{P}.$$

Thus we have the commutative diagram

$$\begin{array}{ccccc} D & \simeq & \mathcal{Y} & \hookrightarrow & X_D \\ \big\downarrow{\scriptstyle g_D} & & \big\downarrow{\scriptstyle \sigma_D} & & \big\downarrow{\scriptstyle \sigma_D} \\ \check{\mathbf{P}} & \simeq & P(\mathcal{O}_{\check{\mathbf{P}}}(m)) & \hookrightarrow & P(\mathcal{A}d\,(T_{\check{\mathbf{P}}}(-1)) \oplus \mathcal{O}_{\check{\mathbf{P}}}(m)) \\ & & \big\downarrow{\scriptstyle \sigma_Y} & & \big\downarrow{\scriptstyle \sigma_Y} \\ & & \{T_0\mathbf{P}^3\} & \in & \sigma(X_D) \end{array}. \tag{37}$$

Remark 2. The cone $\sigma(X_D)$ over $\operatorname{Hilb}^2 \mathbf{P}$ contains a divisor subcone K_Δ over the divisor Δ, and by construction, $K_\Delta = \{Z_4 \in H_4(0) \mid Z_4 \text{ is not a locally complete intersection in } \mathbf{P}^3\}$.

3.6. The exceptional divisor W on X_4 and the contraction of W under the morphism σ. Let us consider the variety $X_3^* := X_3 \setminus D$ and an arbitrary point $z = (Z_2, Z_3) \in X_3^*$. By definition, Z_3 is a curvilinear scheme (i.e., it lies on a smooth curve); thus the triple

$$\xi \colon \quad 0 \to \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_3} \xrightarrow{\otimes k(0)} k(0) \to 0 \tag{38}$$

is exact. Applying the functor $\operatorname{Ext}^\bullet(-, k(0))$ to this triple, we obtain a long exact sequence. The first connecting homomorphism

$$\partial_1 \colon \operatorname{Hom}(\mathcal{O}_{Z_2}, k(0)) \to \operatorname{Ext}^1(k(0), k(0))$$

in this sequence is injective by Lemma 2; therefore, the image $\operatorname{im}\alpha_1$ of the homomorphism

$$\alpha_1 \colon \operatorname{Ext}^1(k(0), k(0)) \to \operatorname{Ext}^1(\mathcal{O}_{Z_3}, k(0))$$

that follows ∂_1 is two-dimensional. By Lemma 2 and Proposition 3 (ii),

$$P(\operatorname{Ext}^1(\mathcal{O}_{Z_2}, k(0))) = \pi_4^{-1}(z), \tag{39}$$

where $\pi_4\colon X_4 \to X_3$ is the projection; hence $W_z := P(\operatorname{im}\alpha_1) \simeq \mathbf{P}^1$ is a divisor in $\pi_4^{-1}(z)$. We put $W^* := \bigcup_{z\in X_3^*} W_z$; let $W := \overline{W^*}$ be the closure of W^* in X_4. By construction, W is a divisor on X_4.

Let us describe an arbitrary point $w = (Z_2, Z_3, Z_4) \in W^*$. For this purpose, we identify the points in $\operatorname{Ext}^1(k(0), k(0)) = T_0\mathbf{P}^3$ with zero-dimensional schemes of length 2 supported at 0 and, taking into account identification (39), consider an arbitrary scheme (of length 2) $Z_2' \in \operatorname{Ext}^1(k(0), k(0)) \setminus (\operatorname{im}\partial_1)$ (i.e., $Z_2' \neq Z_2$) such that $P(k\alpha_1(Z_2')) = w$. By the definitions of the maps ∂_1 and α_1, we have the commutative diagram of extensions

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
\xi : 0 & \longrightarrow & \mathcal{O}_{Z_2} & \longrightarrow & \mathcal{O}_{Z_3} & \longrightarrow & k(0) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{Z_2} & \longrightarrow & \mathcal{O}_{Z_4} & \longrightarrow & \mathcal{O}_{Z_2'} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & k(0) & = & k(0) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{40}$$

Remark 3. Diagram (40) and the condition $Z_2' \neq Z_2$, where we have $Z_2, Z_2' \subset Z_4$ by construction, imply $\dim T_0 Z_4 = 2$. Therefore, Z_4 lies on the germ $(Q_{Z_4}, 0)$ of some quadric of form (34) (or, in a special case, of the plane), and it is not a locally complete intersection in $(Q_{Z_4}, 0)$. The last condition and the middle horizontal triple in (40) (or, equivalently, the pair (Q_{Z_4}, Z_2)) uniquely determines the scheme Z_4 (see [6, Secs. 2.5–6]).

Next, for the chosen point $w = (Z_2, Z_3, Z_4)$, consider the point $z = (Z_2, Z_3) = \pi_4(w)$. By construction, there is a one-to-one correspondence between the point $w \in \pi_4^{-1}(z)$ and the subspace $\operatorname{Span}(Z_2, Z_2')$ in $T_0\mathbf{P}^3$, which is a plane passing through the line $\mathcal{P}T_0 Z_2$, or, equivalently, a point $v(w)$ of the divisor D on X_3 lying in the fiber $\gamma_1^{-1}(Z_2)$, where $\gamma_1\colon D \to \mathbf{P}$ is the natural projection. In the plane $\nu^{-1}(Z_2)$, consider the projective line $l(w) = \operatorname{Span}(z, v(w))$ and its open subset $l^*(w) = l(w) \setminus \{v(w)\}$. Simple calculations involving equation (34) of the germ Q_{Z_4} show that

$$l^*(w) = \{(Z_2, Z_3', Z_4) \in W^* \mid Z_3' \subset Q_{Z_4} \cap Y,\ Z_2 \subset Y \in \check{\mathbf{P}}^3,\ T_0 Y \neq T_0 Q_{Z_4}\}. \tag{41}$$

Obviously, the condition $Z_3' \subset Q_{Z_4} \cap Y$ uniquely determines the scheme Z_3'. Therefore, the line

$$m^*(w) := \{(Z_2, Z_3', Z_4) \in W^* \mid (Z_2, Z_3') \in l^*(w)\}$$

through the point $w = (Z_2, Z_3, Z_4)$ is determined uniquely, and $\sigma(m^*(w)) = \{Z_4\} = Z_4$. Let $m(w)$ be the closure of $m^*(w)$ in W; then $m(w) \simeq \mathbf{P}^1$. We have $\sigma(m(w)) = Z_4$. By virtue of Remarks 2 and 3, $Z_4 \in K_\Delta$, and, as is easy to see, the map $\sigma\colon W \to K_\Delta$ is surjective. Thus the following proposition is valid.

Proposition 6. *The morphism $\sigma\colon X_4 \to H_4(0)$ contracts the divisor W on X_4 onto the four-dimensional cone $K_\Delta = \{Z_4 \in H_4(0) \mid Z_4$ is not a locally complete intersection in $\mathbf{P}^3\}$.*

Remark 4. It is easy to see that W contains the section $\mathcal{Y} = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_4 = T_0\mathbf{P}^3\}$ of the projection $\pi_D\colon X_D \to D$ (see Proposition 4), and the diagram

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & W \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ \{T_0\mathbf{P}^3\} & \in & K_\Delta \end{array} \tag{42}$$

is commutative.

Collecting Propositions 3, 4, and 6, Corollary 3, Remark 4, and the description of the morphism $\sigma|X_D$ (see (35)–(37)), we obtain the main result of this section.

Theorem 3. (i) *Let X_3 be the variety of punctual flags of length 3 in space. Consider the projection $p_2\colon \mathbf{P}^3 \times X_3 \to X_3$ and put $\mathcal{E} := \mathcal{E}xt^1_{p_2}(\mathcal{O}_{\mathbf{T}_3}, \mathcal{O}_{\mathbf{T}_1})$. Then $\mathcal{E}$ is a locally free sheaf of rank 3 and $X_4 = \mathbb{P}(\check{\mathcal{E}})$ is a smooth irreducible variety that parametrizes the punctual flags $z = (Z_2, Z_3, Z_4)$, $\{0\} \in Z_2 \subset Z_3 \subset Z_4$, of lengths 2, 3, and 4 and whose projection*

$$\pi_4\colon X_4 \to X_3\colon\ (Z_2, Z_3, Z_4) \to (Z_2, Z_3)$$

coincides with the structure morphism $\mathbb{P}(\check{\mathcal{E}}) \to X_3$.

(ii) *Consider the divisors*

$$X_D = \pi_4^{-1}(D) = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_3 \text{ is not a locally complete intersection}\},$$
$$W = \{(Z_2, Z_3, Z_4) \in X_4 \mid Z_4 \text{ is not a locally complete intersection}\}$$

on the variety X_4. If $\sigma\colon X_4 \to H_4(0)\colon\ (Z_2, Z_3, Z_4) \mapsto Z_4$ is the forgetful morphism, then σ is a birational morphism decomposed as $\sigma = \sigma_2 \cdot \sigma_1$, where σ_1 is the contraction of the divisor W and σ_2 is the normalization morphism (glueing along the divisor $\sigma_1(X_D)$), so that $\sigma|X_D\colon X_D \to \sigma(X_D)$ is a double covering at a generic point. In addition,

$$K_\Delta = \sigma(W) = \{Z_4 \in H_4(0) \mid Z_4 \text{ is not a locally complete intersection}\},$$
$$\sigma(X_D) = \{Z_4 \in H_4(0) \mid Z_4 \text{ is not a curvilinear scheme (i.e., does not lie on a smooth curve)}\}.$$

Moreover, $\sigma(X_D)$ is a cone over $\mathrm{Hilb}^2\mathbf{P}$ in which K_Δ is the subcone over the diagonal $\Delta \subset \mathrm{Hilb}^2\mathbf{P}$, and $\sigma(X_D)$ is the set of singularities of the variety $H_4(0)$: $\sigma(X_D) = \mathrm{Sing}\,H_4(0)$.

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