Mathematical Notes, Vol. 67, No. 6, 2000

Rational Approximations to Certain Numbers

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ABSTRACT. The exact order of approximation to certain numbers by rational numbers is established. The basic tool for this purpose is an expansion in regular continued fractions. Some new such expansions are also derived.

KEY WORDS: rational approximation, continued-fraction expansion, Euler expansion, Fibonacci sequence.

In 1978 Davis [1] proved that for any $\varepsilon > 0$ there exist infinitely many rational numbers p/q satisfying the inequality

$$\left|e - \frac{p}{q}\right| < \left(\frac{1}{2} + \varepsilon\right)\frac{\ln\ln q}{q^2 \ln q},$$

while the inequality

$$\left|e - \frac{p}{q}\right| < \left(\frac{1}{2} - \varepsilon\right)\frac{\ln\ln q}{q^2 \ln q}$$

has only finitely many solutions. The proof of this assertion was carried out by using the continued-fraction expansion of the number e

$$e = [2\,;\,1,2,1,1,4,1,1,6,1,\dots] = [2\,;\,\overline{1,2+2\lambda,1}],$$

obtained by Euler, as well as the integral representations for the numerators and denominators of the convergents of this continued fraction. Here we have used the symbol $\overline{1,2+2\lambda,1}$ to denote the infinite sequence of numbers obtained by successively joining the blocks $1, 2+2\lambda, 1$ for $\lambda = 0,1,2,\dots$. Such abbreviated notation will be used throughout the paper.

The continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

will be written as

$$b_0 + \frac{a_1}{b_1} \, \frac{}{+} \, \frac{a_2}{b_2} \, \frac{}{+} \, \frac{a_3}{b_3} \, \frac{}{+} \, \cdots \tag{1}$$

(see [2]) and in the case $a_1 = a_2 = \dots = 1$ the conventional notation $[b_0\,;\,b_1,b_2,b_3,\dots]$ is used.

The main result of the paper is the following assertion.

Theorem 1. *Suppose that a number α is defined by the continued fraction*

$$\alpha = [b_0\,;\,b_1,\dots,b_s,\overline{c_1+\lambda d_1,\dots,c_m+\lambda d_m}],$$

where all the b_i, c_i, d_i belong to $\mathbb{Z}$ and the d_i are nonnegative, while b_i, c_i, $i \ge 1$, are positive. Suppose also that Ω is the number of nonzero numbers d_i, $\Omega > 0$, and

$$d = \max_{1\le i\le m} d_i.$$

Translated from *Matematicheskie Zametki*, Vol. 67, No. 6, pp. 931–937, June, 2000.
Original article submitted September 22, 1999.

 0001–4346/2000/6756–0786\$25.00

Then for $c=\Omega/d$ *and any* $\varepsilon>0$ *the following assertions are valid:*

1) *the inequality*

$$\left|\alpha-\frac{p}{q}\right|<(c+\varepsilon)\frac{\ln\ln q}{q^2\ln q}$$

has infinitely many solutions in rational numbers p/q;

2) *the inequality*

$$\left|\alpha-\frac{p}{q}\right|<(c-\varepsilon)\frac{\ln\ln q}{q^2\ln q}$$

has only finitely many solutions in rational numbers p/q.

If in the continued fraction for α all the numbers d_i are zero, then, as is well known, in this case α is a quadratic irrationality and does not admit rational approximations of order better than $1/q^2$.

The Davis assertion follows from Theorem 1, since in our case $\Omega = 1$ and $d = 2$. The proof of Theorem 1 uses the elementary properties of continued fractions and is much simpler than the proofs of the Davis theorem. Below we consider a number of other examples. Further, new expansions of some numbers in continued fractions of the type described above will be obtained. The properties of continued fractions are described in the books [2, 3].

The following lemma is concerned with arbitrary regular continued fractions, not necessarily of the form described in Theorem 1.

Lemma 1. *Suppose that* $\alpha=[a_0,a_1,a_2,\dots]$ *is a regular continued fraction and* p_n/q_n, $n\geq 0$, *is a sequence of its convergents. Then the following inequalities are valid:*

$$\frac{1}{(a_{n+1}+2)q_n^2}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{a_{n+1}q_n^2},\qquad n\geq 1, \tag{2}$$

$$a_1\cdots a_n\leq q_n\leq F_n\cdot a_1\cdots a_n,\qquad n\geq 1, \tag{3}$$

where F_n *is the Fibonacci sequence.*

Proof. Essentially, all the inequalities of this lemma are well known. By [3, Chap. 24], the following relation is valid:

$$\left|\alpha-\frac{p_n}{q_n}\right|=\frac{1}{(\alpha_{n+1}q_n+q_{n-1})q_n},$$

where α_{n+1} is the complete quotient of the continued fraction. Hence from the inequalities

$$a_{n+1}\leq\alpha_{n+1}<a_{n+1}+1,\qquad 0\leq q_{n-1}\leq q_n$$

we obtain the estimates (2).

To prove inequalities (3), let us use the induction method. For $n=1$ and $n=2$, inequalities (3) obviously hold. For $n\geq 2$, it follows from the relation $q_{n+1}=a_{n+1}q_n+q_{n-1}$ that $a_{n+1}q_n\leq q_{n+1}\leq a_{n+1}(q_n+q_{n-1})$. By the induction assumption and the recurrence equation $F_{n+1}=F_n+F_{n-1}$ for Fibonacci numbers, from these inequalities we obtain inequalities (3). □

We shall apply this lemma to the number α from Theorem 1. In this case $a_n=b_n$ for $n\leq s$. But if $n>s$, we define integers t and r by the relation $n-s=mt+r$, $1\leq r\leq m$. Then $a_n=c_r+td_r$.

Lemma 2. *Under the assumptions of Theorem* 1, *the following asymptotic formula holds as* $n\to\infty$:

$$\ln(a_1\cdots a_n)=\frac{\Omega}{m}\cdot n\ln n+O(n). \tag{4}$$

Proof. If on the left-hand side of (4) we discard all factors corresponding to the numbers b_i as well as to the numbers $c_i + \lambda d_i$ with $d_i = 0$, then the resulting error will be $O(n)$. Therefore, it suffices to prove Lemma 2 under the assumption that $s = 0$ and for any j the condition $d_j \neq 0$ holds, i.e., $\Omega = m$. Now the assertion is easy to prove with the help of the relation

$$c_i(c_i + d_i)(c_i + 2d_i) \cdot (c_i + td_i) \cdot \Gamma\left(\frac{c_i}{d_i}\right) = d_i^{t+1}\Gamma\left(\frac{c_i}{d_i} + t + 1\right)$$

and the asymptotic formula for the logarithm of the gamma function. □

Proof of Theorem 1. First, let us prove the second assertion of Theorem 1. If the rational number p/q is not a convergent to the number α, then the following inequality is valid: (see [3, Chap. 25])

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{2q^2},$$

and therefore, the inequality from assertion 2) of the theorem no longer holds for large q.

If $p/q = p_n/q_n$, $n \geq s$, is a convergent to the number α, then we define integers t and r so that $n + 1 - s = mt + r$, $1 \leq r \leq m$. Then $a_{n+1} = c_r + td_r$.

If $d_r = 0$, then it follows from the left-hand inequality (2) that

$$\left|\alpha - \frac{p_n}{q_n}\right| \geq \frac{1}{(c_r + 2)q_n^2}.$$

This contradicts the inequality from assertion 2) of Theorem 1 for large n.

But if $d_r \neq 0$, then from the left-hand inequality (2) we obtain

$$\left|\alpha - \frac{p_n}{q_n}\right| \geq \frac{1}{(td_r + c_r + 2)q_n^2}. \tag{5}$$

In view of the fact that $\ln F_n = O(n)$, from (3) and Lemma 2 we obtain

$$\ln q_n = \ln(a_1 \cdots a_n) + O(n) = \frac{\Omega}{m} \cdot n \ln n + O(n)$$

and

$$t = \frac{1}{\Omega} \cdot \frac{\ln q_n}{\ln \ln q_n} \cdot \big(1 + o(1)\big) \tag{6}$$

as $t \to \infty$. Now it follows from (5) that

$$\left|\alpha - \frac{p_n}{q_n}\right| \geq \frac{\Omega}{d_r} \cdot \frac{\ln \ln q_n}{q_n^2 \ln q_n} \cdot \big(1 + o(1)\big). \tag{7}$$

Since $d_r \leq d$, it follows that inequality (7) contradicts the inequality from assertion 2) of Theorem 1 for sufficiently large t.

Thus any solution p/q of the inequality from assertion 2) has a denominator bounded by some constant depending on α. The proof of assertion 2) is complete.

To prove assertion 1), we choose r so that $d_r = d$, and set $n + 1 = s + mt + r$ for an arbitrarily chosen positive integer t. Then we have $a_{n+1} = c_r + td$, and from the right-hand inequality (2) we obtain

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{(td + c_r)q_n^2};$$

by (6), this leads to the inequality

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{\Omega}{d} \cdot \frac{\ln \ln q_n}{q_n^2 \ln q_n} \cdot \big(1 + o(1)\big)$$

valid for all positive integers t. This means that the convergents p_n/q_n with $n + 1 = s + mt + r$, where r is defined by $d_r = d$, and t is any sufficiently large integer, satisfy the inequality from assertion 1) of Theorem 1. □

Let us now derive corollaries of Theorem 1.

Corollary 1. *The number*

$$\tanh\frac{2}{a}=\frac{e^{2/a}-1}{e^{2/a}+1},\qquad a\in\mathbb{Z},\quad a>0,$$

satisfies the assertion of Theorem 1 *with constant* $c=1/(2a)$.

This assertion follows from the expansion $\tanh(2/a)=[0;\overline{(1+2\lambda)a}]$ proved by Euler in 1737. Note that the paper [4] contains only the proof of the existence of a constant c for which the second assertion of Theorem 1 for the number $\tanh(1/a)$ holds. For similar results, also see [5].

Corollary 2. *The numbers* e^2, $e^{1/a}$, *and* $e^{2/b}$, *where* a,b, $a\ge 2$, $b\ge 3$, *are integers and* b *is odd, satisfy the assertion of Theorem* 1 *with constants* c, *equal to* $1/4$, $1/(2a)$, *and* $1/(4b)$, *respectively.*

This assertion holds on the strength of the following classical expansions in continued fraction (see [6, Chap. 14]):

$$e^2=[7;\overline{2+3\lambda,1,1,3+3\lambda,18+12\lambda}],\qquad e^{1/a}=[1;\overline{(1+2\lambda)a-1,1,1}],$$

$$e^{2/b}=\left[1;\overline{\frac{b-1}{2}+3b\lambda,6b+12b\lambda,\frac{5b-1}{2}+3b\lambda,1,1}\right].$$

Note that the assertions on the numbers $e^{1/a}, e^{2/b}$ are proved in the paper by Davis [7].

In the next theorem, we present a number of expansions in regular continued fractions, from which, using Theorem 1, we can be obtain results on rational approximations.

Theorem 2. *The following expansions in continued fractions are valid:*

$$\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}}=[0;\overline{(4\lambda+1)u,(4\lambda+3)v}],\qquad u,v\in\mathbb{N},\tag{8}$$

$$ae^{1/a}=[a+1;\overline{2a-1,2\lambda+2,1}],\qquad a\in\mathbb{N},\tag{9}$$

$$a^{-1}e^{1/a}=[0;a-1,2a,\overline{1,2\lambda+2,2a-1}],\qquad a\in\mathbb{Z},\quad a>1,\tag{10}$$

$$2e=[5;2,\overline{3,2\lambda+2,3,1,2\lambda+2}],\tag{11}$$

$$3e=[8;6,2,\overline{5,2\lambda+2,5,1,2\lambda+2,5,1,2\lambda+2,1}],\tag{12}$$

$$4e=[10;1,6,1,7,2,\overline{7,\lambda+2,7,1,\lambda+1,1}],\tag{13}$$

$$\frac{1}{2}e=[1;2,\overline{2\lambda+1,3,1,2\lambda+1,1,3}],\tag{14}$$

$$\frac{1}{3}e=[0;1,9,\overline{1,1,2\lambda+1,5,1,2\lambda+1,1,1,26+18\lambda}],\tag{15}$$

$$\frac{1}{4}e=[0;1,2,8,3,\overline{1,1,1,\lambda+1,7,1,\lambda+1,2}],\tag{16}$$

$$\tan\frac{1}{a}=[0;a-1,\overline{1,(2\lambda+3)a-2}],\qquad a\in\mathbb{Z},\quad a>1,\tag{17}$$

$$\tan 1=[1;\overline{2\lambda+1,1}],\tag{18}$$

$$\sqrt{a}\tan\frac{1}{\sqrt{a}}=[1;\overline{(4\lambda+3)a-2,1,4\lambda+3,1}],\qquad a\in\mathbb{N},\tag{19}$$

$$\frac{1}{\sqrt{a}}\tan\frac{1}{\sqrt{a}}=[0;a-1,\overline{1,4\lambda+1,1,(4\lambda+5)a-2}],\qquad a\in\mathbb{Z},\quad a>1,\tag{20}$$

$$2\tan 1=[3;8,\overline{1,3\lambda+2,2,3\lambda+3,12\lambda+20}],\tag{21}$$

$$\frac{1}{2}\tan 1=[0;1,3,\overline{1+3\lambda,1,12+2\lambda,1,3+3\lambda,2}].\tag{22}$$

To prove this theorem, we use some classical expansions of the values of analytic functions in nonregular continued fractions, as well as the following assertion.

Lemma 3. *Suppose that* $r_0 = 1, r_1, r_2, \dots$ *are nonzero numbers and*

$$a_n^* = r_n r_{n-1} a_n, \quad n = 1, 2, 3, \dots, \qquad b_n^* = r_n b_n, \quad n = 0, 1, 2, \dots .$$

Then the continued fraction

$$b_0^* + \frac{a_1^*}{b_1^* +} \frac{a_2^*}{b_2^* +} \cdots$$

converges to the same number as the fraction (1).

Proof. For the proof, see [2, Theorem 2.6]. □

Proof of Theorem 2. To prove relation (8), set $x = \sqrt{uv}$ in the relation $\tanh(1/x) = [0; \overline{(2\lambda+1)x}]$ and use Lemma 3 with $r_0 = 1$, $r_{2k+2} = \sqrt{v/u}$, $r_{2k+1} = \sqrt{u/v}$, $k \geq 0$.

The expansions (9)–(16) are proved by similar methods. For example, to prove relation (9) we set $x = 1/a$ in the expansion [2, relation (6.1.37)] and then apply Lemma 3 with $r_0 = 1$, $r_{2k} = a$, $r_{2k-1} = 1$, $k \geq 1$. As a result, we obtain the expansion

$$ae^{1/a} = a + \frac{1}{1 -} \frac{1}{2a +} \frac{1}{3 -} \frac{1}{2a +} \frac{1}{5 -} \frac{1}{2a +} \frac{1}{7 -} \cdots . \tag{23}$$

Let P_n/Q_n and p_n/q_n denote the sequences of convergents for the right-hand sides of (23) and (9), respectively. By induction on n, it is easily shown that

$$\frac{P_{2n+1}}{Q_{2n+1}} = \frac{p_{3n}}{q_{3n}}, \quad \frac{P_{2n+2}}{Q_{2n+2}} = \frac{p_{3n+1}}{q_{3n+1}}, \qquad n \geq 0.$$

This proves relation (9).

Relations (17)–(22) are also proved by similar methods. For example, to prove relation (17) we can use the expansion

$$\tan x = \frac{x}{1 -} \frac{x^2}{3 -} \frac{x^2}{5 -} \frac{x^2}{7 -} \frac{x^2}{9 -} \cdots \tag{24}$$

(see [2, relation (6.1.55)]). The even part of the continued fraction (18) is of the form (see [2, Theorem 2.10])

$$1 + \frac{1}{2 -} \frac{1}{5 -} \frac{1}{7 -} \frac{1}{9 -} \cdots .$$

Comparing this fraction with (24) for $x = 1$, we can easily find that (24) and, simultaneously, the continued fraction (18) are equal to $\tan 1$. □

Note that all the expansions in Theorem 2 were not given in the literature. Let us indicate another theorem proved in the same way as Theorem 1.

Theorem 3. *Suppose that* a_0, $a > 1$, $m > 1$ *are integers and*

$$\alpha = \Big[a_0 ; \overline{\underbrace{a^\lambda, \dots, a^\lambda}_{m}}\Big]_{\lambda=1}^{\infty}.$$

Then for $c = 1/\sqrt{a}$ *and any* $\varepsilon > 0$ *the following assertions are valid:*

1) *the inequality*

$$\left| \alpha - \frac{p}{q} \right| < (c + \varepsilon) q^{-2-\sqrt{2 \ln a/(m \ln q)}}$$

has infinitely many solutions in rational numbers p/q;

2) *the inequality*

$$\left| \alpha - \frac{p}{q} \right| < (c - \varepsilon) q^{-2-\sqrt{2 \ln a/(m \ln q)}}$$

has only finitely many solutions in rational numbers p/q.

The example to Theorem 3 yields the following expansion proved in [8]:

$$\frac{\sum_{s=0}^{\infty} a^{-(s+1)^2} \cdot \prod_{m=0}^{s} (a^{2m}-1)^{-1}}{\sum_{s=0}^{\infty} a^{-s^2} \cdot \prod_{m=0}^{s} (a^{2m}-1)^{-1}} = [0\,; a, a^2, a^3, a^4, \dots].$$

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