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EXISTENCE AND CONSTRUCTION OF ANISOTROPIC SOLUTIONS TO THE MULTIDIMENSIONAL EQUATION OF NONLINEAR DIFFUSION. I

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1. Introduction. Consider the following multidimensional nonlinear heat equation:

$$u_t = \nabla \cdot (K(u)\nabla u) \equiv \Delta\Phi(u), \quad u \stackrel{\Delta}{=} u(\mathbf{x},t) : \Omega \times \overline{\mathbb{R}}^+ \to \mathbb{R}^+, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain; $\mathbb{R}^+ = (0,\infty)$; $\overline{\mathbb{R}}^+ = \{t : 0 \leq t < +\infty\}$; $u(\mathbf{x},t) \geq 0$ is the temperature of a medium; $K(u)$ is a function defined for all $u \in \mathbb{R}^+$; $K(u) > 0$ for $u > 0$; $K(0) \geq 0$; $K(u)$ is the nonlinear heat conductivity of the medium; and $\Phi(u) = \int_0^u K(\xi)\,d\xi$. Henceforth we make the following assumptions about $\Phi(u)$: $\Phi(u) \in C^\infty(\mathbb{R}^+) \cap C^1(\overline{\mathbb{R}}^+)$, $\Phi'(u) > 0$ for $u \neq 0$, $\Phi(0) = 0$, $\Phi'(0) \geq 0$ or $\Phi(u) \in C^\infty(\mathbb{R}^+) \cap C(\overline{\mathbb{R}}^+)$, $\Phi(0) = 0$, $\Phi'(u) > 0$ for $u \neq 0$, and $\Phi'(+0) = +\infty$. Equations like (1.1) appear in many models of mathematical physics and their study is actual for the modern theory of nonlinear partial differential equations and its applications. Moreover, (1.1) belongs to the class of the so-called implicitly degenerate parabolic equations whose rigorous mathematical theory was laid down in rather recent investigations. We indicate, for instance, the articles [1–7] studying some special properties of solutions to (1.1) which relate to degeneration. Thus, the nonlinear equation (1.1) is parabolic for $u > 0$ and degenerates into a first-order nonlinear evolution equation at $u = 0$.

Many publications are devoted to constructing exact nonnegative solutions to (1.1) [1, 6, 8–26], wherein urgency of this topic is indicated. In this article, which adheres to [8–11, 14–16, 19, 20, 27–35], we obtain new exact nonselfsimilar anisotropic (in the space variables) explicit nonnegative solutions to (1.1) for $K(u) = u^\lambda, \lambda \in \mathbb{R}$. Depending on the parameter $\lambda \in \mathbb{R}$, we examine the cases in which the integral $\int_0^1 K(u)u^{-1}\,du$ is finite or equals $+\infty$; i.e.,

$$\int_0^1 u^{\lambda-1}\,du \leq +\infty. \tag{1.2}$$

Below we propose and study an original construction [33–35] for solving the multidimensional nonlinear diffusion equation

$$u_t = \nabla \cdot (u^\lambda \nabla u), \quad u \stackrel{\Delta}{=} u(\mathbf{x},t) : \Omega \times \overline{\mathbb{R}}^+ \to \mathbb{R}^+, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ is a parameter of a nonlinear medium whose values differ for different heat conduction processes [3]. We show that, under certain assumptions, the proposed construction enables us to obtain exact nonnegative solutions both for the class of porous medium equations (nonstationary filtration) with $\lambda > 0$ and for the class of equations (1.3) with a negative exponent λ in the nonlinear heat conductivity. In particular, this class contains the so-called equations of fast diffusion ($-1 < \lambda < 0$) and limit diffusion ($\lambda = -1$, $n = 2$). The above-obtained exact nonnegative solutions are mostly noninvariant under point transformation groups and Lie–Bäcklund groups [36, 37].

The closest results were obtained in [8–10, 12–16, 18–20]. In particular, in [19] there was proposed a method for constructing an n-parameter family of exact nonnegative solutions $u(\mathbf{x},t)$ to the Cauchy problem for the nonlinear diffusion equation

$$u_t = \Delta u^m, \quad u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad t > 0,$$

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with initial data given in the form of a finite or infinite measure. Here $m \in \mathbb{R}$, $m > 0$, and $n \in \mathbb{N}$, $n \geq 2$. If $0 < m < 1$ then the support of the measure is a hypersurface in $\mathbb{R}^n$, while for $m > 1$ the initial measure is concentrated in a domain bounded by a second-order surface in $\mathbb{R}^k$, $k < n$. Moreover, in [19] new exact nonselfsimilar anisotropic (in the space variables) explicit nonnegative solutions were given for the equation $u_t = \Delta \log u$, $u = u(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$, which is the limit case of the fast diffusion equation. The cases were of $n = 2$ (the limit diffusion equation) and $n = 3$. A well-known property of nonstationary filtration type equations is finiteness of the speed with which the supports of their solutions change. The first general results on finiteness of the speed of the change of supports of solutions to nonstationary filtration type equations were established in [38–40]. Moreover, it was proven therein that convergence of the integral (1.2) is a necessary and sufficient condition for finiteness of the velocity of propagation of perturbations in the processes described by (1.3). In other words, if the integral (1.2) diverges then $u(\mathbf{x}, t) > 0$ in $\mathbb{R}^n$ for all $t \in \mathbb{R}^+$.

2. Derivation of the resolving system for the multidimensional equation of nonlinear diffusion. Introduce the functions

$$Z_k(\mathbf{x}, t) = \frac{1}{2}(\mathbf{x}, A_k(t)\mathbf{x}) + (\mathbf{x}, \mathbf{B}_k(t)) + C_k(t), \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$; $A_k(t) = [a_{kij}(t)]$ are $(n \times n)$-matrices; $\mathbf{B}_k(t) = (b_{k1}(t), \dots, b_{kn}(t))'$ is a column vector; $C_k(t)$ is a scalar function; $a_{kij}(t), b_{ki}(t), C_k(t) \in C^1(\overline{\mathbb{R}}^+)$ are real functions; $k = 1, 2$; $i, j = 1, 2, \dots, n$; and $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^n$. We search for a solution to (1.3) in the form

$$u(\mathbf{x}, t) = \left[\lambda [Z_1(\mathbf{x}, t)]_+^p + \lambda [Z_2(\mathbf{x}, t)]_+^q\right]_+^{1/\lambda}, \tag{2.2}$$

where $\lambda, p, q \in \mathbb{R}$, $\lambda \neq 0$, and $[\cdot]_+ = \max\{[\cdot], 0\}$. Inserting (2.2) in (1.3), after simple transformations we arrive at

$$\begin{aligned} pZ_1^{p-1}\frac{\partial}{\partial t}Z_1 + qZ_2^{q-1}\frac{\partial}{\partial t}Z_2 &= \left(\lambda p Z_1^{2p-1}\Delta Z_1 + p[p(\lambda+1) - \lambda]Z_1^{2p-2}|\nabla Z_1|^2\right) \\ &+\left\{\lambda q Z_2^{2q-1}\Delta Z_2 + q[q(\lambda+1) - \lambda]Z_2^{2q-2}|\nabla Z_2|^2\right\} \\ &+\left[\lambda p Z_1^{p-1} Z_2^q \Delta Z_1 + \lambda p(p-1) Z_1^{p-2} Z_2^q |\nabla Z_1|^2 + \lambda q Z_1^p Z_2^{q-1}\Delta Z_2 \right. \\ &\left. +\lambda q(q-1) Z_1^p Z_2^{q-2}|\nabla Z_2|^2 + 2pq Z_1^{p-1} Z_2^{q-1}(\nabla Z_1, \nabla Z_2)\right]. \end{aligned} \tag{2.3}$$

To separate (2.3) with respect to the functions Z_1 and Z_2 of (2.1), we use the following reasons based on the order of homogeneity [41, p. 178] of each summand in (2.3). Observe that $\Delta Z_k(\mathbf{x}, t) = \operatorname{tr} A_k(t)$ and introduce the scalar functions $Z_3 = |\nabla Z_1|^2$, $Z_4 = |\nabla Z_2|^2$, and $Z_5 = (\nabla Z_1, \nabla Z_2)$. The functions $Z_k(\mathbf{x}, t)$, $k = 1, 2, \dots, 5$, in (2.3) have the same structure. Indeed, each of them has three summands: a quadratic form $\sum_{i,j=1}^n r_{ij}(t) x_i x_j$, a linear form $\sum_i^n s_i(t) x_i$, and a scalar function $h(t)$. Consider the order of homogeneity of each summand in (2.3). The first summand on the left-hand side of (2.3) has order p; the second, q; each of the summands in parentheses, $(2p-1)$; each of the summands in braces, $(2q-1)$; and, finally, each of the summands in brackets, $(p+q-1)$. Looking at these orders, we easily see that Z_1 and Z_2 in (2.3) can be separated, for instance, at $q = 1$. In this case (2.2) takes the form

$$u(\mathbf{x}, t) = \left[\lambda [Z_1(\mathbf{x}, t)]_+^p + \lambda Z_2(\mathbf{x}, t)\right]_+^{1/\lambda}, \tag{2.4}$$

and (2.3) becomes

$$\begin{aligned} pZ_1^{p-1}\frac{\partial}{\partial t}Z_1 + \frac{\partial}{\partial t}Z_2 &= \left(\lambda p Z_1^{2p-1}\Delta Z_1 + p[p(\lambda+1) - \lambda]Z_1^{2p-2}|\nabla Z_1|^2\right) \\ &+\left\{\lambda Z_2 \Delta Z_2 + |\nabla Z_2|^2\right\} + \left[\lambda p Z_1^{p-1} Z_2 \Delta Z_1 + \lambda p(p-1) Z_1^{p-2} Z_2 |\nabla Z_1|^2 \right. \\ &\left. +\lambda Z_1^p \Delta Z_2 + 2p Z_1^{p-1}(\nabla Z_1, \nabla Z_2)\right]. \end{aligned} \tag{2.5}$$

Thereby, equating the summands with the same homogeneity order in (2.5), we arrive at a system of three equations in Z_1 and Z_2. Thus, the following holds:

Theorem 1. *The multidimensional nonlinear diffusion equation* (1.3) *has an exact nonnegative solution of the form*

$$u(\mathbf{x},t) = \left[\lambda\left[\frac{1}{2}(\mathbf{x}, A_1(t)\mathbf{x}) + (\mathbf{x}, \mathbf{B}_1(t)) + C_1(t)\right]_+^p + \lambda\left[\frac{1}{2}(\mathbf{x}, A_2(t)\mathbf{x}) + (\mathbf{x}, \mathbf{B}_2(t)) + C_2(t)\right]\right]_+^{1/\lambda}, \tag{2.6}$$

provided that the matrices $A_k(t)$ with entries $a_{kij}(t) \in C^1(\overline{\mathbb{R}^+})$, the column vectors $\mathbf{B}_k(t)$ with components $b_{ki}(t) \in C^1(\overline{\mathbb{R}^+})$, and the scalar functions $C_k(t) \in C^1(\overline{\mathbb{R}^+})$, $i, j = 1, 2, \dots, n$, $k = 1, 2$, are connected by the relations

$$\frac{\partial}{\partial t}Z_2 = \lambda Z_2 \Delta Z_2 + |\nabla Z_2|^2,$$

$$pZ_1\frac{\partial}{\partial t}Z_1 = p\lambda Z_1 Z_2 \Delta Z_1 + \lambda Z_1^2 \Delta Z_2 + \lambda p(p-1) Z_2 |\nabla Z_1|^2 + 2pZ_1(\nabla Z_1, \nabla Z_2),$$

$$\lambda Z_1 \Delta Z_1 + [p(\lambda+1) - \lambda]|\nabla Z_1|^2 = 0, \tag{2.7}$$

where Z_1 and Z_2 are defined by (2.1); $\lambda, p \in \mathbb{R}$; *and* $\lambda \neq 0$.

Relation (2.7) is referred to as the *resolving system* for the nonlinear diffusion equation (1.3). Recalling the definition of Z_3, Z_4, and Z_5, we see that the first and third equations of the resolving system (2.7) have the homogeneity order 1 and the second equation, 2.

Finally, note that equations like (2.3) in which all summands have the same homogeneity order appear and are separated by means of Hirota's method [42] which is an effective tool for constructing exact solutions to one-dimensional nonlinear evolution equations. In particular, this method (with minor modifications and the Pade approximation) was used in [41, pp. 177–209] for construction of exact one- and two-phase solutions to a broad class of homogeneous semilinear parabolic equations.

We can call the solution (2.6) to (1.3) a solution of "finite-sum" form. In [12], a method of generalized separation of variables was proposed which enables us to construct particular exact solutions

$$v(x,t) = \sum_{i=1}^{k} a_i(t) f_i(x) \tag{S}$$

for a broad class of nonlinear partial differential equations of the form

$$T^p(v) = X^q(v), \tag{E}$$

where $T^p(v)$ is a polynomial of degree p in the function $v(x,t)$ and its derivatives with respect to $t \in \mathbb{R}^1$; $X^q(v)$ is a polynomial of degree q in the function $v(x,t)$ and its derivatives with respect to $x \in \mathbb{R}^1$; and $a_i(t)$ and $f_i(x)$ are sought sufficiently smooth functions. Eventually, constructions (S) and (E) necessitate studying compatibility of two systems of ordinary differential equations (ODE) one of which involves only functions of t and the other, only functions of x. In other words, the systems of ODE resulting from inserting (S) in (E) are overdetermined [43] (the number of equations exceeds the number of sought functions). A representation for particular exact solutions in "finite-sum" form was used by some authors [1, 8–12, 14, 22, 25, 44–46] for analysis of various classes of nonlinear equations.

3. Study of the resolving system of equations. In the general case, study of the resolving system (2.7) causes great difficulties. Therefore, we consider a particular case in which (2.7) reduces to an overdetermined system (the number of equations is greater than the number of sought functions) of algebraic-differential equations (ADE) which is solvable under certain assumptions.

Thus, put $\xi = p(\lambda+1) - \lambda$ and $\xi \neq 0$. Then (2.7) takes the form

$$\frac{\partial}{\partial t}Z_2 = \lambda Z_2 \Delta Z_2 + |\nabla Z_2|^2, \tag{3.1}$$

$$\frac{\partial}{\partial t}Z_1 = \sigma Z_2 \Delta Z_1 + \tau Z_1 \Delta Z_2 + 2(\nabla Z_1, \nabla Z_2), \tag{3.2}$$

$$\lambda Z_1 \Delta Z_1 + \xi |\nabla Z_1|^2 = 0, \tag{3.3}$$

where $\sigma = p\lambda/\xi$; $\tau = \lambda/p$; $p, \lambda \in \mathbb{R}$; $\lambda \neq 0$; and $p \neq 0$.

Theorem 2. *Suppose that $A_k(t)$ are symmetric matrices with entries $a_{kij}(t) \in C^1(\overline{\mathbb{R}^+})$, $\mathbf{B}_k(t)$ are column vectors with components $b_{ki}(t) \in C^1(\overline{\mathbb{R}^+})$, and $C_k(t) \in C^1(\overline{\mathbb{R}^+})$ are scalar functions. Then the functions Z_1 and Z_2 defined by (2.1) satisfy (3.1)–(3.3) if and only if $A_k(t)$, $\mathbf{B}_k(t)$, and $C_k(t)$ satisfy the system of* ADE

$$\dot{A}_2 = 2A_2^2 + \lambda(\operatorname{tr} A_2)A_2, \tag{3.4.1}$$

$$\dot{\mathbf{B}}_2 = 2A_2\mathbf{B}_2 + \lambda(\operatorname{tr} A_2)\mathbf{B}_2, \tag{3.4.2}$$

$$\dot{C}_2 = |\mathbf{B}_2|^2 + \lambda(\operatorname{tr} A_2)C_2, \tag{3.4.3}$$

$$\dot{A}_1 = 4A_1A_2 + \tau(\operatorname{tr} A_2)A_1 + \sigma(\operatorname{tr} A_1)A_2, \tag{3.4.4}$$

$$\dot{\mathbf{B}}_1 = 2(A_1\mathbf{B}_2 + A_2\mathbf{B}_1) + \tau(\operatorname{tr} A_2)\mathbf{B}_1 + \sigma(\operatorname{tr} A_1)\mathbf{B}_2, \tag{3.4.5}$$

$$\dot{C}_1 = 2(\mathbf{B}_1, \mathbf{B}_2) + \tau(\operatorname{tr} A_2)C_1 + \sigma(\operatorname{tr} A_1)C_2, \tag{3.4.6}$$

$$\lambda(\operatorname{tr} A_1)A_1 + 2\xi A_1^2 = 0, \tag{3.4.7}$$

$$\lambda(\operatorname{tr} A_1)\mathbf{B}_1 + 2\xi A_1\mathbf{B}_1 = 0, \tag{3.4.8}$$

$$\lambda(\operatorname{tr} A_1)C_1 + \xi|\mathbf{B}_1|^2 = 0, \tag{3.4.9}$$

where $\sigma = p\lambda/\xi$; $\tau = \lambda/p$; $p, \lambda \in \mathbb{R}$; $\lambda \neq 0$; $p \neq 0$; $\xi = p(\lambda+1) - \lambda$; $\xi \neq 0$; and $\operatorname{tr} A_k = \sum_{i=1}^n a_{kii}(t)$ is the trace of the matrix $A_k(t)$; $k = 1, 2$.

PROOF. Suppose that Z_1 and Z_2 defined by (2.1) satisfy (3.1)–(3.3). From symmetry of $A_k(t)$ we then obtain

$$|\nabla Z_k|^2 = (\mathbf{x}, A_k^2\mathbf{x}) + 2(\mathbf{x}, A_k\mathbf{B}_k) + |\mathbf{B}_k|^2, \tag{3.5}$$

$$(\nabla Z_1, \nabla Z_2) = (\mathbf{x}, A_1A_2\mathbf{x}) + (\mathbf{x}, A_1\mathbf{B}_2 + A_2\mathbf{B}_1) + (\mathbf{B}_1, \mathbf{B}_2);$$

moreover,

$$\nabla Z_k = A_k\mathbf{x} + \mathbf{B}_k, \Delta Z_k = \nabla \cdot (\nabla Z_k) = \operatorname{tr} A_k, \tag{3.6}$$

$$\frac{\partial}{\partial t} Z_k = \frac{1}{2}(\mathbf{x}, \dot{A}_k\mathbf{x}) + (\mathbf{x}, \dot{\mathbf{B}}_k) + \dot{C}_k; \quad \mathbf{x} \in \mathbb{R}^n; \quad k = 1, 2. \tag{3.7}$$

In view of (3.5)–(3.7), from (3.1)–(3.3) we derive (3.4).

Show that (3.4.1)–(3.4.3) in (3.4) imply (3.1). Assume that $k = 2$. Then, starting from (3.7) and using (3.5) and (3.6), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} Z_2 &= \frac{1}{2}(\mathbf{x}, [2A_2^2 + \lambda(\operatorname{tr} A_2)A_2]\mathbf{x}) + (\mathbf{x}, 2A_2\mathbf{B}_2 + \lambda(\operatorname{tr} A_2)\mathbf{B}_2) \\ &+ |\mathbf{B}_2|^2 + \lambda(\operatorname{tr} A_2)C_2 = \lambda(\operatorname{tr} A_2)\left[\frac{1}{2}(\mathbf{x}, A_2\mathbf{x}) + (\mathbf{x}, \mathbf{B}_2) + C_2\right] \\ &+ [(\mathbf{x}, A_2^2\mathbf{x}) + 2(\mathbf{x}, A_2\mathbf{B}_2) + |\mathbf{B}_2|^2] = \lambda Z_2 \Delta Z_2 + |\nabla Z_2|^2. \end{aligned}$$

Similarly, we can prove that (3.4.4)–(3.4.6) of (3.4) yield (3.2) and that (3.4.7)–(3.4.9) yields (3.3). The theorem is proven.

Theorems 1 and 2 yield

Assertion 1. *If symmetric matrices $A_k(t)$ with entries $a_{kij}(t) \in C^1(\overline{\mathbb{R}^+})$, column vectors $\mathbf{B}_k(t)$ with components $b_{ki}(t) \in C^1(\overline{\mathbb{R}^+})$, and scalar functions $C_k(t) \in C^1(\overline{\mathbb{R}^+})$ satisfy the overdetermined system (3.4) then (2.6) is an exact nonnegative solution to the multidimensional nonlinear diffusion equation (1.3).*

Consider a solution $u(\mathbf{x}, t)$ to (1.3) of the form (2.4) for $Z_1(\mathbf{x}, t) \equiv 0$; i.e.,

$$u(\mathbf{x}, t) = \left[\lambda\left[\frac{1}{2}(\mathbf{x}, A_2(t)\mathbf{x}) + (\mathbf{x}, \mathbf{B}_2(t)) + C_2(t)\right]\right]_+^{1/\lambda}. \tag{3.8}$$

In this case we have $A_1(t) \equiv 0, \mathbf{B}_1(t) \equiv 0$, and $C_1(t) \equiv 0$ in (3.4) and this system is reduced to the system of ODE

$$\dot{A}_2 = 2A_2^2 + \lambda(\operatorname{tr} A_2)A_2,\ \dot{\mathbf{B}}_2 = 2A_2\mathbf{B}_2 + \lambda(\operatorname{tr} A_2)\mathbf{B}_2,\ \dot{C}_2 = |\mathbf{B}_2|^2 + \lambda(\operatorname{tr} A_2)C_2 \tag{3.9}$$

which was derived and partially studied in [30]. Assume given, at $t = 0$, a real symmetric matrix $A_2(0) \in M_n(\mathbb{R})$, a column vector $\mathbf{B}_2(0) \in M_{n,1}(\mathbb{R})$, and a scalar $C_2(0) \in \mathbb{R}$, where $M_n(\mathbb{R})$ is the set of $(n \times n)$-matrices with entries in $\mathbb{R}$ and $M_{n,k}(\mathbb{R})$ is the set of $(n \times k)$-matrices with entries in $\mathbb{R}$ [47]. Write down $A_2(0)$ as $A_2(0) = SD(0)S'$, where $S \in M_n(\mathbb{R})$ is an orthogonal matrix, $SS' = S'S = I$, I is the identity matrix, and $D(0) = \operatorname{diag}[d_1(0), \dots, d_n(0)]$ is a diagonal matrix, $d_l(0) \in \mathbb{R}$, $l = 1, 2, \dots n$, are the eigenvalues of $A_2(0)$. It is well known that every real symmetric matrix can be written in this form [47]. Show that if $A_2(0)$, $\mathbf{B}_2(0)$, and $C_2(0)$ are defined then solving the Cauchy problem for (3.9) is reduced to solving the Cauchy problem for some scalar nonlinear ODE. More precisely, prove one of the main results of the article:

Theorem 3. *Suppose that $A_2(0)$, $S \in M_n(\mathbb{R})$ are real symmetric matrices, $\mathbf{B}_2(0) \in M_{n,1}(\mathbb{R})$ is a column vector, and $C_2(0) \in \mathbb{R}$ is a scalar. Let $z(t)$ be a real solution of the Cauchy problem*

$$\dot{z}(t) = \prod_{l=1}^{n} [1 - 2d_l(0)z(t)]^{-\lambda/2}, \quad z(0) = 0, \quad \dot{z}(t) = \frac{d}{dt}z(t). \tag{3.10}$$

Then a solution to the Cauchy problem

$$\dot{A}_2(t) = 2A_2^2(t) + \lambda[\operatorname{tr} A_2(t)]A_2(t), \quad A_2(t)|_{t=0} = A_2(0), \tag{3.11}$$

$$\dot{\mathbf{B}}_2(t) = 2A_2(t)\mathbf{B}_2(t) + \lambda[\operatorname{tr} A_2(t)]\mathbf{B}_2(t), \quad \mathbf{B}_2(t)|_{t=0} = \mathbf{B}_2(0), \tag{3.12}$$

$$\dot{C}_2(t) = |\mathbf{B}_2(t)|^2 + \lambda[\operatorname{tr} A_2(t)]C_2(t), \quad C_2(t)|_{t=0} = C_2(0) \tag{3.13}$$

has the form

$$A_2(t) = \dot{z}(t)SQ(t)S'A_2(0), \tag{3.14}$$

$$\mathbf{B}_2(t) = \dot{z}(t)SQ(t)S'\mathbf{B}_2(0), \tag{3.15}$$

$$C_2(t) = \dot{z}(t)[C_2(0) + z(t)(Q(t)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0))]. \tag{3.16}$$

Moreover, $A_2(t)$ is a symmetric matrix for all t in its domain and

$$Q(t) = \operatorname{diag}[[1 - 2d_1(0)z(t)]^{-1}, \dots, [1 - 2d_n(0)z(t)]^{-1}]; \tag{3.17}$$

$$A_2(0) = SD(0)S'; \quad \lambda, d_l(0) \in \mathbb{R}; \quad \lambda \neq 0, \quad d_l(0) \neq 0; \quad l = 1, 2, \dots, n.$$

PROOF. Let $A_2(t)$, $\mathbf{B}_2(t)$, and $C_2(t)$ be defined by (3.14)–(3.17). Show that these functions give a solution to the Cauchy problem (3.11)–(3.13). For convenience, introduce the notation $v(t) = \operatorname{tr} A_2(t)$ and calculate the trace of $A_2(t)$. From (3.14) we easily see that

$$\begin{aligned} v(t) = \operatorname{tr} A_2(t) &= \operatorname{tr}(\dot{z}(t)SQ(t)S'A_2(0)) = \operatorname{tr}(\dot{z}(t)SQ(t)D(0)S') \\ &= \dot{z}(t)\operatorname{tr}(Q(t)D(0)) = \dot{z}(t)\sum_{k=1}^{n} \frac{d_k(0)}{1 - 2d_k(0)z(t)}, \end{aligned} \tag{3.18}$$

where $Q(t)$ is the diagonal matrix (3.17). On the other hand, since $1 - 2d_l(0)z(t) \neq 0$ for $l = 1, 2, \dots, n$; involving the condition $z(0) = 0$ and differentiating (3.10) with respect to t, we obtain

$$\ddot{z}(t) = \lambda \left[\sum_{k=1}^{n} \frac{d_k(0)}{1 - 2d_k(0)z(t)}\right] \dot{z}^2(t) = \lambda v(t)\dot{z}(t); \tag{3.19}$$

moreover, $\dot{z}(0) = 1$ and $z(0) = 0$. Furthermore, differentiating (3.17) by directly, we can easily demonstrate that $Q(t)$ satisfies the Cauchy problem

$$\dot{Q}(t) = 2\dot{z}(t)D(0)Q^2(t) = 2\dot{z}(t)Q^2(t)D(0), \quad Q(t)|_{t=0} = I. \tag{3.20}$$

Show that the matrix $A_2(t)$ of (3.14) satisfies (3.11). Indeed, differentiating (3.14) and using (3.18)–(3.20), we obtain

$$\begin{aligned}
\dot{A}_2(t) &= \ddot{z}(t)SQ(t)S'A_2(0) + \dot{z}(t)S\dot{Q}(t)S'A_2(0) \\
&= \lambda v(t)\dot{z}(t)SQ(t)S'A_2(0) + 2\dot{z}^2(t)SQ^2(t)D(0)S'A_2(0) \\
&= \lambda v(t)A_2(t) + 2\dot{z}^2(t)SQ(t)D(0)S'SQ(t)S'A_2(0) \\
&= \lambda v(t)A_2(t) + 2\dot{z}^2(t)\left(SQ(t)D(0)S'\right)\left(SQ(t)D(0)S'\right) = \lambda(\operatorname{tr} A_2(t))A_2(t) + 2A_2^2(t).
\end{aligned}$$

Verify that the column vector $\mathbf{B}_2(t)$ of (3.15) is a solution to (3.12). Differentiating (3.15) and using (3.18)–(3.20), we obtain

$$\begin{aligned}
\dot{\mathbf{B}}_2(t) &= \ddot{z}(t)SQ(t)S'\mathbf{B}_2(0) + \dot{z}(t)S\dot{Q}(t)S'\mathbf{B}_2(0) \\
&= \lambda v(t)\dot{z}(t)SQ(t)S'\mathbf{B}_2(0) + 2\dot{z}^2(t)SQ^2(t)D(0)S'\mathbf{B}_2(0) \\
&= \lambda v(t)\mathbf{B}_2(t) + 2(\dot{z}(t)SQ(t)D(0)S')(\dot{z}(t)SQ(t)S'\mathbf{B}_2(0)) \\
&= \lambda(\operatorname{tr} A_2(t))\mathbf{B}_2(t) + 2A_2(t)\mathbf{B}_2(t).
\end{aligned}$$

According to (3.17), $Q(t) = (I - 2z(t)D(0))^{-1}$; i.e., $(I - 2z(t)D(0))Q(t) = I$ or, what is the same,

$$Q(t) = I + 2z(t)Q(t)D(0). \tag{3.21}$$

Finally, using (3.18)–(3.21), we easy see that $C_2(t)$ defined by (3.16) satisfies (3.13). Indeed, differentiating (3.16) and using (3.18)–(3.21), we find

$$\begin{aligned}
\dot{C}_2(t) &= \ddot{z}(t)C_2(0) + \dot{z}(t)(\mathbf{B}_2(0), \mathbf{B}_2(t)) + z(t)(\mathbf{B}_2(0), \dot{\mathbf{B}}_2(t)) \\
&= \lambda v(t)\dot{z}(t)C_2(0) + \dot{z}(t)(\mathbf{B}_2(0), \mathbf{B}_2(t)) + z(t)(\mathbf{B}_2(0), 2A_2(t)\mathbf{B}_2(t) + \lambda v(t)\mathbf{B}_2(t)) \\
&= \lambda v(t)[\dot{z}(t)C_2(0) + z(t)(\mathbf{B}_2(0), \mathbf{B}_2(t))] + (\mathbf{B}_2(0), [\dot{z}(t)I + 2z(t)A_2(t)]\mathbf{B}_2(t)) \\
&= \lambda v(t)C_2(t) + (\mathbf{B}_2(0), \dot{z}(t)S[I + 2z(t)Q(t)D(0)]S'\mathbf{B}_2(t)) \\
&= \lambda v(t)C_2(t) + (\mathbf{B}_2(0), \dot{z}(t)SQ(t)S'\mathbf{B}_2(t)) \\
&= \lambda v(t)C_2(t) + (\dot{z}(t)SQ(t)S'\mathbf{B}_2(0), \mathbf{B}_2(t)) = \lambda(\operatorname{tr} A_2(t))C_2(t) + |\mathbf{B}_2(t)|^2.
\end{aligned}$$

Show that $A_2(t)$ is a symmetric matrix for all t in its domain. Let $G(t) = SQ(t)S'$, where $Q(t)$ is defined by (3.17). Clearly, $G(t)$ is a nondegenerate symmetric matrix. Then $A_2(t) = \dot{z}(t)G(t)A_2(0)$, where $A_2(0)$ is a real symmetric matrix. First, verify that $A_2(0)$ and $G(t)$ commute. Indeed, in view of

$$\begin{aligned}
A_2(0)G^{-1}(t) &= A_2(0)[SQ(t)S']^{-1} = A_2(0)SQ^{-1}(t)S' \\
&= A_2(0)S[I - 2z(t)D(0)]S' = A_2(0)[SS' - 2z(t)SD(0)S'] \\
&= A_2(0)[I - 2z(t)A_2(0)] = A_2(0) - 2z(t)A_2^2(0) = [I - 2z(t)A_2(0)]A_2(0) = G^{-1}(t)A_2(0),
\end{aligned}$$

we have $G(t)A_2(0) = A_2(0)G(t)$. Moreover, $[G(t)A_2(0)]' = A_2'(0)G'(t) = A_2(0)G(t)$; i.e., $G(t)A_2(0)$ is a symmetric matrix. Therefore, so is $A_2(t)$. The theorem is proven.

EXAMPLE 1. Suppose that $n = 3$ and $d_l = d_l(0) \in \mathbb{R}$, $l = 1,2,3$. Then for $\lambda = -1$ a solution to the Cauchy problem (3.10) is expressed in terms of the Jacobi elliptic functions [48]. Consider the case in which $d_1 d_2 d_3 < 0$. If $z(t) > \frac{1}{2d_1} > \frac{1}{2d_2} > \frac{1}{2d_3}$ then a solution to the Cauchy problem (3.10) has the form

$$z(t) = \frac{d_2 - d_1 \operatorname{sn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{d_2/d_1}, k), k)}{2d_1 d_2 \operatorname{cn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{d_2/d_1}, k), k)}; \tag{1}$$

moreover, in this case $k = \sqrt{\frac{d_1(d_3-d_2)}{d_2(d_3-d_1)}}$. From the chain of the inequalities $z(t) \geq \frac{1}{2d_1} > \frac{1}{2d_2} > \frac{1}{2d_3}$ we obtain

$$z(t) = \frac{d_3 - d_1 \operatorname{cn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{(d_1-d_3)/d_1}, k), k)}{2d_1 d_3 \operatorname{sn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{(d_1-d_3)/d_1}, k), k)}. \tag{2}$$

If $\frac{1}{2d_1} > \frac{1}{2d_2} \geq z(t) > \frac{1}{2d_3}$ then

$$z(t) = \frac{1}{2d_3} + \frac{d_3-d_2}{2d_2 d_3}\operatorname{sn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{d_2/(d_2-d_3)}, k), k). \tag{3}$$

Finally, for $\frac{1}{2d_1} > \frac{1}{2d_2} > z(t) \geq \frac{1}{2d_3}$ we find that

$$z(t) = \frac{1}{2}(d_1 - d_3 + (d_3-d_2)\operatorname{sn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{(d_3-d_1)/(d_3-d_2)}, k), k))$$

$$\times \frac{1}{d_2(d_1-d_3) + d_1(d_3-d_2)\operatorname{sn}^2(\sqrt{d_2(d_1-d_3)}t + \operatorname{sn}^{-1}(\sqrt{(d_3-d_1)/(d_3-d_2)}, k), k)}.$$

Examine the cases of degeneration of elliptic functions. For $d_2 = d_3$, from (1) and (2) we obtain the following solutions with trigonometric functions:

$$z(t) = \frac{d_2 - d_1 \sin^2(\sqrt{d_2(d_1-d_2)}t + \sin^{-1}(\sqrt{d_2/d_1}))}{2d_1 d_2 \cos^2(\sqrt{d_2(d_1-d_2)}t + \sin^{-1}(\sqrt{d_2/d_1}))},$$

$$z(t) = \frac{d_2 - d_1 \cos^2(\sqrt{d_2(d_1-d_2)}t + \sin^{-1}(\sqrt{(d_1-d_2)/d_1}))}{2d_1 d_2 \sin^2(\sqrt{d_2(d_1-d_2)}t + \sin^{-1}(\sqrt{(d_1-d_2)/d_1}))}.$$

If $d_1 = d_2$ then the two-dimensional problem has the following solutions in hyperbolic functions which result from (2) and (3):

$$z(t) = \frac{d_3 - d_1 \operatorname{sech}^2(\sqrt{d_1(d_1-d_3)}t + \operatorname{arth}(\sqrt{(d_1-d_3)/d_1}))}{2d_1 d_3 \operatorname{th}^2(\sqrt{d_1(d_1-d_3)}t + \operatorname{arth}(\sqrt{(d_1-d_3)/d_1}))},$$

$$z(t) = \frac{1}{2d_3} + \frac{d_3-d_1}{2d_1 d_3}\operatorname{th}^2(\sqrt{d_1(d_1-d_3)}t + \operatorname{arth}(\sqrt{d_1/(d_1-d_3)})).$$

Thus, summing the results of Theorem 3 and Example 1, we easily obtain exact nonselfsimilar anisotropic (in the space variables) explicit nonnegative solutions to the equation

$$u_t = \Delta \log u, \quad u \stackrel{\Delta}{=} u(\mathrm{x}, t) : \Omega \times \overline{\mathbb{R}}^+ \to \mathbb{R}^+, \quad \mathrm{x} \in \mathbb{R}^n,$$

for $n = 2$ and $n = 3$.

REMARK 1. If we introduce the matrices

$$D(t) = \operatorname{diag}[d_1(t), \dots, d_n(t)], \quad d_l(t) = \frac{d_l(0)}{1 - 2d_l(0)z(t)}\dot{z}(t) \tag{3.22}$$

that are connected with $Q(t)$ of (3.17) by the relation

$$D(t) = \dot{z}(t)Q(t)D(0), \quad D(0) = S'A_2(0)S \tag{3.23}$$

then the solution (3.14)–(3.16) to the Cauchy problem (3.11)–(3.13) takes the form

$$A_2(t) = SD(t)S', \tag{3.24}$$

$$\mathbf{B}_2(t) = SD^{-1}(0)D(t)S'\mathbf{B}_2(0), \tag{3.25}$$

$$C_2(t) = \dot{z}(t)C_2(0) + z(t)(\mathbf{B}_2(0), SD^{-1}(0)D(t)S'\mathbf{B}_2(0)), \tag{3.26}$$

where $d_l(t)$ are the real eigenvalues of the matrix $A_2(t)$; $l = 1, 2, \dots, n$.

REMARK 2. It follows from Theorem 3 and the uniqueness theorem of [49] that (3.14)–(3.16) determine all solutions to the Cauchy problem (3.11)–(3.13) with a real symmetric initial matrix $A_2(0)$. Indeed, the only nonlinear equation in the Cauchy problem (3.11)–(3.13) under study is (3.11). Uniqueness of a solution to (3.11) ensues from the fact that the right-hand side of this equation satisfies the Lipschitz condition in every bounded subset of $\mathbb{R}^{n\times n}$ and so the classical theorem applies of uniqueness of a solution to a normal system of ODE.

From Assertion 1 and Theorem 3 we derive the following

Assertion 2. *If a symmetric matrix $A_2(t)$, a column vector $\mathbf{B}_2(t)$, and a scalar function $C_2(t)$ have the respective forms* (3.14), (3.15), *and* (3.16) *then the nonlinear diffusion equation* (1.3) *possesses the exact nonselfsimilar anisotropic (in the space variables) explicit nonnegative solution* (3.8).

Since $A_2(t)$, $\mathbf{B}_2(t)$, and $C_2(t)$ are determined (see (3.14)–(3.16) or (3.24)–(3.26)), we turn to studying the system of ODE (3.4.4)–(3.4.6).

4. Existence of Solutions to the Cauchy Problem for (3.4.4)–(3.4.6).

Assertion 3. *Suppose that the matrix $A_2(t)$ has the form*

$$A_2(t) = \dot{z}(t)SQ(t)S'A_2(0), \tag{4.1}$$

$u(t) = \operatorname{tr} A_1(t)$, $v(t) = \operatorname{tr} A_2(t)$; *and $A_1(0), A_2(0) \in M_n(\mathbb{R})$ are real symmetric matrices. Then the Cauchy problem*

$$\dot{A}_1(t) = 4A_1(t)A_2(t) + \tau v(t)A_1(t) + \sigma u(t)A_2(t), \quad A_1(t)|_{t=0} = A_1(0), \tag{4.2}$$

has the following solution:

$$A_1(t) = [\dot{z}(t)]^{\tau/\lambda}\left[\sigma\int\limits_0^t [\dot{z}(\eta)]^{1-\frac{\tau}{\lambda}}u(\eta)G^{-1}(\eta)A_2(0)\,d\eta + A_1(0)\right]G^2(t), \tag{4.3}$$

where $G(t) = SQ(t)S'$ and $u(t)$ is a function satisfying the linear Volterra integral equation

$$u(t) = \sigma\int\limits_0^t K(t,\eta)u(\eta)\,d\eta + f(t) \tag{4.4}$$

of the second kind with the kernel

$$K(t,\eta) = \left[\frac{\dot{z}(t)}{\dot{z}(\eta)}\right]^{\tau/\lambda}\dot{z}(\eta)\operatorname{tr}[Q^2(t)Q^{-1}(\eta)D(0)] \tag{4.5}$$

and free term

$$f(t) = [\dot{z}(t)]^{\tau/\lambda}\operatorname{tr}[A_1(0)G^2(t)]; \tag{4.6}$$

here $\tau = \lambda/p$, $\sigma = \lambda p/\xi$, and $\xi = p(\lambda+1) - \lambda$. Moreover, for $A_1(t)$ to be symmetric for all t in its domain, it is necessary and sufficient that $A_1(0)$ and $A_2(0)$ commute:

$$A_1(0)A_2(0) = A_2(0)A_1(0). \tag{4.7}$$

PROOF. First of all, observe that, on using (3.17) and the relation $A_2(0) = SD(0)S'$, from (3.20) we obtain

$$\dot{G}(t) = 2\dot{z}(t)G^2(t)A_2(0), \quad G(t)|_{t=0} = I. \tag{4.8}$$

It is easily seen that $G(t)A_2(0) = A_2(0)G(t)$ and $G(t)\dot{G}(t) = \dot{G}(t)G(t)$. Finally, recall that, alongside (3.10), the function $z(t)$ satisfies the ODE (3.19). Using this, demonstrate that $A_1(t)$ is a solution to the Cauchy problem (4.2). Indeed, differentiating (4.3), using (4.1), (4.2), and (4.8), and recalling that $\tau/\lambda = 1/p$, $p \in \mathbb{R}$, and $p \neq 0$, we obtain

$$\begin{aligned}
\dot{A}_1(t) = \frac{1}{p}[\dot{z}(t)]^{\frac{1}{p}-1}\ddot{z}(t)\left[\sigma\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}}u(\eta)G^{-1}(\eta)A_2(0)\,d\eta + A_1(0)\right]G^2(t) \\
+\sigma\dot{z}(t)u(t)G^{-1}(t)A_2(0)G^2(t) \\
+2[\dot{z}(t)]^{\frac{1}{p}}\left[\sigma\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}}u(\eta)G^{-1}(\eta)A_2(0)\,d\eta + A_1(0)\right]G(t)\dot{G}(t) \\
= \tau v(t)[\dot{z}(t)]^{\frac{1}{p}}\left[\sigma\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}}u(\eta)G^{-1}(\eta)A_2(0)\,d\eta + A_1(0)\right]G^2(t) \\
+\sigma u(t)\dot{z}(t)G(t)A_2(0) \\
+4[\dot{z}(t)]^{\frac{1}{p}}\left[\sigma\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}}u(\eta)G^{-1}(\eta)A_2(0)\,d\eta + A_1(0)\right]G^2(t)\dot{z}(t)G(t)A_2(0) \\
= \tau v(t)A_1(t) + \sigma u(t)A_2(t) + 4A_1(t)A_2(t).
\end{aligned}$$

Grounding on (4.3), we now calculate the trace $u(t)$ of $A_1(t)$. The following chain of equalities is valid:

$$\begin{aligned}
u(t) = \operatorname{tr} A_1(t) = \operatorname{tr}\left[\sigma\int_0^t [\dot{z}(t)]^{\frac{1}{p}}[\dot{z}(\eta)]^{1-\frac{1}{p}}u(\eta)G^{-1}(\eta)A_2(0)G^2(t)\,d\eta + [\dot{z}(t)]^{\frac{1}{p}}A_1(0)G^2(t)\right] \\
= \sigma\int_0^t \left[\frac{\dot{z}(t)}{\dot{z}(\eta)}\right]^{\frac{1}{p}}\dot{z}(\eta)\operatorname{tr}[G^{-1}(\eta)A_2(0)G^2(t)]u(\eta)\,d\eta + [\dot{z}(t)]^{\frac{1}{p}}\operatorname{tr}[A_1(0)G^2(t)] \\
= \sigma\int_0^t \left[\frac{\dot{z}(t)}{\dot{z}(\eta)}\right]^{\frac{1}{p}}\dot{z}(\eta)\operatorname{tr}[Q^2(t)Q^{-1}(\eta)D(0)]u(\eta)\,d\eta + f(t) = \sigma\int_0^t K(t,\eta)u(\eta)\,d\eta + f(t).
\end{aligned}$$

It follows that $u(t)$ satisfies the linear Volterra integral equation of the second kind (4.4) with kernel (4.5) and free term (4.6). Thereby we have demonstrated that $A_1(t)$ is a solution to the Cauchy problem (4.2). To complete the proof, we have to validate that (4.7) is a necessary and sufficient condition for symmetry of $A_1(t)$ for all t in its domain.

Suppose that (4.7) holds. Observe that $A_2(0) = SD(0)S'$ and $G(t) = SQ(t)S'$ is a nondegenerate symmetric matrix. We easily see that the matrices $A_1(0)$ and $G^{-1}(t)$ commute. Indeed, the following chain of equalities holds:

$$
\begin{aligned}
A_1(0)G^{-1}(t) &= A_1(0)[SQ(t)S']^{-1} = A_1(0)SQ^{-1}(t)S' \\
&= A_1(0)S[I - 2z(t)D(0)]S' = A_1(0)[SS' - 2z(t)SD(0)S'] \\
&= A_1(0)[I - 2z(t)A_2(0)] = A_1(0) - 2z(t)A_1(0)A_2(0) \\
&= A_1(0) - 2z(t)A_2(0)A_1(0) = [I - 2z(t)A_2(0)]A_1(0) = G^{-1}(t)A_1(0).
\end{aligned}
$$

Consequently, $A_1(0)G(t) = G(t)A_1(0)$ and $A_1(0)G^2(t) = G^2(t)A_1(0)$. Since

$$
[A_1(0)G^2(t)]' = [G^2(t)]'[A_1(0)]' = [G'(t)]^2[A_1(0)]' = G^2(t)A_1(0) = A_1(0)G^2(t),
$$

the matrix $A_1(0)G^2(t)$ is symmetric. The relation $A_2(0)G(t) = G(t)A_2(0)$ was validated above. Thereby $A_2(0)G^2(t) = G^2(t)A_2(0)$. Hence, symmetry of $A_2(0)G^2(t)$ is immediate. Indeed, we have

$$
[A_2(0)G^2(t)]' = [G^2(t)]'[A_2(0)]' = [G'(t)]^2[A_2(0)]' = G^2(t)A_2(0) = A_2(0)G^2(t).
$$

Using these results and rewriting $A_1(t)$ in the form

$$
A_1(t) = \sigma[\dot{z}(t)]^{\frac{1}{p}} \left[\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta) Q^{-1}(\eta)\, d\eta \right] A_2(0)G^2(t) + [\dot{z}(t)]^{\frac{1}{p}} A_1(0)G^2(t), \tag{4.9}
$$

we easily see that $A_1(t)$ is symmetric for all t in its domain.

Prove the converse. Suppose that $A_1(t)$ of (4.9) is symmetric. Since $A_2(0)G^2(t)$ is a symmetric matrix, the matrix $A_1(0)G^2(t)$ as well is symmetric. By symmetry of $A_1(0)$ and $G^2(t)$, this amounts to their commutativity $A_1(0)G^2(t) = G^2(t)A_1(0)$. Hence, we have the relation $G^{-2}(t)A_1(0) = A_1(0)G^{-2}(t)$; i.e., $G^{-2}(t)$ and $A_1(0)$ commute. Recall that $G^{-1}(t) = SQ^{-1}(t)S' = I - 2z(t)A_2(0)$. Hence, $G^{-2}(t) = I - 4z(t)A_2(0) + 4z^2(t)A_2^2(0)$. We thus arrive at the relation

$$
\left[I - 4z(t)A_2(0) + 4z^2(t)A_2^2(0)\right]A_1(0) = A_1(0)\left[I - 4z(t)A_2(0) + 4z^2(t)A_2^2(0)\right].
$$

Removing the brackets and recalling that $z(t) \neq 0$, we obtain

$$
A_2(0)A_1(0) - z(t)A_2^2(0)A_1(0) = A_1(0)A_2(0) - z(t)A_1(0)A_2^2(0).
$$

Since $z(0) = 0$; putting $t = 0$, from the last relation we derive (4.7). The assertion is proven.

Theorem 3 and Assertion 3 yield

Theorem 4. *Suppose that $A_2(t)$ and $A_1(t)$ are real symmetric matrices of the form (4.1) and (4.3) satisfying (3.4.1) and (3.4.4) and defined at $t = 0$; i.e., $A_2(t)|_{t=0} = A_2(0) \in M_n(\mathbb{R})$ and $A_1(t)|_{t=0} = A_1(0) \in M_n(\mathbb{R})$. Then there exist real diagonal matrices $D(0) = \operatorname{diag}[d_1(0), \dots, d_n(0)]$ and $\Lambda(0) = \operatorname{diag}[\lambda_1(0), \dots, \lambda_n(0)]$ and an orthogonal matrix $S \in M_n(\mathbb{R})$ such that*

$$
A_2(t) = \dot{z}(t)SQ(t)D(0)S', \tag{4.10}
$$

$$
A_1(t) = [\dot{z}(t)]_+^{\frac{1}{p}} S \left[\sigma \int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta) Q^{-1}(\eta) D(0)\, d\eta + \Lambda(0) \right] Q^2(t)S', \tag{4.11}
$$

where $u(t)$ is a solution to the linear Volterra integral equation (4.4) of the second kind with kernel (4.5) and free term

$$
f(t) = [\dot{z}(t)]^{\frac{1}{p}} \operatorname{tr}[\Lambda(0)Q^2(t)]; \tag{4.12}
$$

here $Q(t)$ is the diagonal matrix of (3.17).

Proof. It is clear that the real symmetric matrices $A_2(t)$ and $A_1(t)$ of (4.1) and (4.3) satisfy the Cauchy problems (3.11) and (4.2). Moreover, $A_2(0)$ and $A_1(0)$ commute; i.e., (4.7) holds. On the other hand, under certain assumptions two real symmetric commuting matrices $A_2(0)$ and $A_1(0)$ can be reduced to a diagonal form simultaneously. Indeed [47], a necessary and sufficient condition for existence of a real orthogonal matrix S such that $S'A_1(0)S = \Lambda(0)$ and $S'A_2(0)S = D(0)$ is the commutation of $A_1(0)$ and $A_2(0)$. Hence, $A_1(0) = S\Lambda(0)S'$ and $A_2(0) = SD(0)S'$. Inserting these expressions in (4.1) and (4.3), we arrive at (4.10) and (4.11). Moreover, (4.6) implies (4.12). The theorem is proven.

Since the functions $\mathbf{B}_2(t)$, $A_2(t)$, $A_1(t)$, $v(t) = \operatorname{tr} A_2(t)$, and $u(t) = \operatorname{tr} A_1(t)$ are now determined, we prove the following

Assertion 4. *Suppose that a column vector $\mathbf{B}_2(t)$ and matrices $A_2(t)$ and $A_1(t)$ are defined by* (3.15), (4.10), *and* (4.11). *Moreover, suppose that $v(t)$ has the form* (3.18) *and $u(t)$ is a solution to the linear Volterra integral equation* (4.4) *of the second kind with kernel* (4.5) *and free term* (4.12). *Then the Cauchy problem*

$$\dot{\mathbf{B}}_1(t) = [2A_2(t) + \tau v(t)I]\mathbf{B}_1(t) + [2A_1(t) + \sigma u(t)I]\mathbf{B}_2(t), \quad \mathbf{B}_1(t)|_{t=0} = \mathbf{B}_1(0) \tag{4.13}$$

possesses the solution

$$\mathbf{B}_1(t) = [\dot{z}(t)]^{\frac{1}{p}} SQ(t)\Bigg[\sigma\left(\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)\,d\eta\right)Q(t)S'\mathbf{B}_2(0)$$
$$+2z(t)Q(t)\Lambda(0)S'\mathbf{B}_2(0) + S'\mathbf{B}_1(0)\Bigg]. \tag{4.14}$$

PROOF. Introduce the column vector

$$\mathbb{B}_1(t) = \sigma\left(\int_0^t [\dot{z}(t)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)\,d\eta\right) Q(t)S'\mathbf{B}_2(0) + 2z(t)Q(t)\Lambda(0)S'\mathbf{B}_2(0) + S'\mathbf{B}_1(0). \tag{4.15}$$

Then (4.14) takes the form

$$\mathbf{B}_1(t) = [\dot{z}(t)]^{\frac{1}{p}} SQ(t)\mathbb{B}_1(t). \tag{4.16}$$

Differentiating (4.16) with respect to time, we arrive at the relation

$$\dot{\mathbf{B}}_1(t) = \frac{1}{p}[\dot{z}(t)]^{\frac{1}{p}-1}\ddot{z}(t)SQ(t)\mathbb{B}_1(t) + [\dot{z}(t)]^{\frac{1}{p}}(S\dot{Q}(t)\mathbb{B}_1(t) + SQ(t)\dot{\mathbb{B}}_1(t)).$$

Using (3.19) and (3.20), we obtain

$$\dot{\mathbf{B}}_1(t) = [2A_2(t) + \tau v(t)I]\mathbf{B}_1(t) + [\dot{z}(t)]^{\frac{1}{p}} SQ(t)\dot{\mathbb{B}}_1(t). \tag{4.17}$$

Grounding on (4.15), we calculate $\dot{\mathbb{B}}_1(t)$:

$$\dot{\mathbb{B}}_1(t) = \sigma[\dot{z}(t)]^{1-\frac{1}{p}} u(t)S'\mathbf{B}_2(0) + 2\sigma\left(\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)\,d\eta\right)\dot{z}(t)Q^2(t)D(0)S'\mathbf{B}_2(0)$$
$$+2\dot{z}(t)Q(t)\Lambda(0)S'\mathbf{B}_2(0) + 4z(t)\dot{z}(t)Q^2(t)D(0)\Lambda(0)S'\mathbf{B}_2(0).$$

By (3.21), we have

$$\dot{\mathbb{B}}_1(t) = \sigma[\dot{z}(t)]^{1-\frac{1}{p}} u(t)S'\mathbf{B}_2(0) + 2\dot{z}(t)Q^2(t)\Lambda(0)S'\mathbf{B}_2(0)$$
$$+2\sigma\left(\int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)\,d\eta\right)\dot{z}(t)Q^2(t)D(0)S'\mathbf{B}_2(0).$$

Now, we easily see that

$$[\dot z(t)]^{\frac{1}{p}} SQ(t)\dot{\mathbb{B}}_1(t) = \sigma u(t)\dot z(t)SQ(t)S'\mathbf{B}_2(0)$$

$$+2[\dot z(t)]^{\frac{1}{p}} S\left(\sigma\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)D(0)\,d\eta\right) Q^2(t)S' \times \dot z(t)SQ(t)S'\mathbf{B}_2(0)$$

$$+2[\dot z(t)]^{\frac{1}{p}} S\Lambda(0)Q^2(t)S' \times \dot z(t)SQ(t)S'\mathbf{B}_2(0) = \sigma u(t)\dot z(t)SQ(t)S'\mathbf{B}_2(0)$$

$$+2[\dot z(t)]^{\frac{1}{p}} S\left[\sigma\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}} u(\eta)Q^{-1}(\eta)D(0)\,d\eta + \Lambda(0)\right] Q^2(t)S' \times \dot z(t)SQ(t)S'\mathbf{B}_2(0)$$

$$= \sigma u(t)I\mathbf{B}_2(t) + 2A_1(t)\mathbf{B}_2(t) = [2A_1(t) + \sigma u(t)I]\mathbf{B}_2(t).$$

Hence, (4.17) takes the form

$$\dot{\mathbf{B}}_1(t) = [2A_2(t) + \tau v(t)I]\mathbf{B}_1(t) + [2A_1(t) + \sigma u(t)I]\mathbf{B}_2(t).$$

Thereby the function $\mathbf{B}_1(t)$ of (4.14) is a solution to the above equation. Finally, recalling that $z(0) = 0$, $\dot z(0) = 1$, and $Q(0) = I$, we can easily verify that the solution (4.14) satisfies the initial condition. So, $\mathbf{B}_1(t)$ is a solution to the Cauchy problem (4.13). The assertion is proven.

Thus, we are ready to turn to studying solvability of the Cauchy problem for ODE (3.4.6). We prove

Assertion 5. *Suppose that the column vectors $\mathbf{B}_2(t)$ and $\mathbf{B}_1(t)$ and the scalar function $C_2(t)$ are defined by* (3.15), (4.14), *and* (3.16). *Moreover, suppose that the function $v(t) = \operatorname{tr} A_2(t)$ has the form* (3.18) *and $u(t) = \operatorname{tr} A_1(t)$ is a solution to the linear Volterra equation* (4.4) *of the second kind with kernel* (4.5) *and free term* (4.12). *Then the Cauchy problem*

$$\dot C_1(t) = \tau v(t)C_1(t) + \sigma u(t)C_2(t) + 2(\mathbf{B}_1(t), \mathbf{B}_2(t)), \quad C_1(t)|_{t=0} = C_1(0), \tag{4.18}$$

has the solution

$$C_1(t) = [\dot z(t)]^{\frac{1}{p}}\Bigg[C_1(0) + 2z(t)(Q(t)S'\mathbf{B}_1(0), S'\mathbf{B}_2(0))$$

$$+2z^2(t)(Q^2(t)\Lambda(0)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) + \sigma\left(\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}} u(\eta)\,d\eta\right)$$

$$\times[C_2(0) + z(t)(Q(t)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) + z(t)|Q(t)S'\mathbf{B}_2(0)|^2]$$

$$-\sigma\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}} u(\eta)\,d\eta\right)|Q(t)S'\mathbf{B}_2(0)|^2\Bigg], \tag{4.19}$$

where $z(t)$ is defined by (3.10) *and satisfies* (3.19).

Proof. To simplify (4.19), introduce the notation

$$\mathbb{C}_1(t) = C_1(0) + 2z(t)(Q(t)S'\mathbf{B}_1(0), S'\mathbf{B}_2(0))$$

$$+2z^2(t)(Q^2(t)\Lambda(0)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) + \sigma\left(\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}} u(\eta)\,d\eta\right)$$

$$\times[C_2(0)+z(t)(Q(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))+z(t)|Q(t)S'\mathbf{B}_2(0)|^2]$$

$$-\sigma\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)|Q(t)S'\mathbf{B}_2(0)|^2. \tag{4.20}$$

Then (4.19) takes the form

$$C_1(t)=[\dot z(t)]^{\frac{1}{p}}\mathbb{C}_1(t). \tag{4.21}$$

First of all, observe that $z(0)=0$ and $\dot z(t)=1$. Thus, we easily see that the function (4.19) satisfies the initial condition $C_1(t)|_{t=0}=C_1(0)$. Differentiating (4.21) with respect to time and using (3.19) and (4.20), we obtain

$$\dot C_1(t)=\tau v(t)C_1(t)+[\dot z(t)]^{\frac{1}{p}}\dot{\mathbb{C}}_1(t). \tag{4.22}$$

Grounding on (4.20), we now calculate $\dot{\mathbb{C}}_1(t)$. Using (3.20), we obtain

$$\begin{aligned}\dot{\mathbb{C}}_1(t)&=2\dot z(t)(Q(t)S'\mathbf{B}_1(0),S'\mathbf{B}_2(0))+4z(t)\dot z(t)(Q^2(t)D(0)S'\mathbf{B}_1(0),S'\mathbf{B}_2(0))\\&+4z(t)\dot z(t)(Q^2(t)\Lambda(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+8z^2(t)\dot z(t)(Q^3(t)\Lambda(0)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+\sigma[\dot z(t)]^{1-\frac{1}{p}}u(t)[C_2(0)+z(t)(Q(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))]+\sigma\left(\int_0^t[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)\\&\times\{\dot z(t)(Q(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))+2z(t)\dot z(t)(Q^2(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+\dot z(t)|Q(t)S'\mathbf{B}_2(0)|^2+4z(t)\dot z(t)(Q^3(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\}\\&-4\sigma\dot z(t)\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0)).\end{aligned}$$

We can simplify the expression in braces. Indeed, using (3.21), we easily see that

$$\begin{aligned}&\dot z(t)(Q(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))+2z(t)\dot z(t)(Q^2(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+\dot z(t)|Q(t)S'\mathbf{B}_2(0)|^2+4z(t)\dot z(t)(Q^3(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&=2\dot z(t)(Q^3(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0)).\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\mathbb{C}}_1(t)&=2\dot z(t)(Q(t)S'\mathbf{B}_1(0),S'\mathbf{B}_2(0))+4z(t)\dot z(t)(Q^2(t)D(0)S'\mathbf{B}_1(0),S'\mathbf{B}_2(0))\\&+4z(t)\dot z(t)(Q^2(t)\Lambda(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+8z^2(t)\dot z(t)(Q^3(t)\Lambda(0)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&+\sigma[\dot z(t)]^{1-\frac{1}{p}}u(t)[C_2(0)+z(t)(Q(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))]\\&+2\sigma\dot z(t)\left(\int_0^t[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0))\\&-4\sigma\dot z(t)\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)D(0)S'\mathbf{B}_2(0),S'\mathbf{B}_2(0)).\end{aligned}$$

Multiplying the last relation by $[\dot z(t)]^{\frac{1}{p}}$ and using (3.21), we easily see that

$$
\begin{aligned}
[\dot z(t)]^{\frac{1}{p}}\dot C_1(t) &= 2[\dot z(t)]^{\frac{1}{p}+1}(Q(t)[I+2z(t)Q(t)D(0)]S'\mathbf{B}_1(0), S'\mathbf{B}_2(0)) \\
&\quad +4z(t)[\dot z(t)]^{\frac{1}{p}+1}(Q^2(t)\Lambda(0)[I+2z(t)Q(t)D(0)]S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&\quad +\sigma u(t)C_2(t) + 2\sigma[\dot z(t)]^{\frac{1}{p}+1}\left(\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&\quad -4\sigma[\dot z(t)]^{\frac{1}{p}+1}\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)D(0)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&= \sigma u(t)C_2(t) + 2[\dot z(t)]^{\frac{1}{p}+1}(Q^2(t)S'\mathbf{B}_1(0), S'\mathbf{B}_2(0)) \\
&\quad +4z(t)[\dot z(t)]^{\frac{1}{p}+1}(Q^3(t)\Lambda(0)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&\quad +2\sigma[\dot z(t)]^{\frac{1}{p}+1}\left(\int_0^t [\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&\quad -4\sigma[\dot z(t)]^{\frac{1}{p}+1}\left(\int_0^t z(\eta)[\dot z(\eta)]^{1-\frac{1}{p}}u(\eta)\,d\eta\right)(Q^3(t)D(0)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)) \\
&= \sigma u(t)C_2(t) + 2(\mathbf{B}_1(t), \mathbf{B}_2(t)).
\end{aligned}
$$

Inserting this expression in (4.22), we arrive at the ODE

$$\dot C_1(t) = \tau v(t)C_1(t) + \sigma u(t)C_2(t) + 2(\mathbf{B}_1(t), \mathbf{B}_2(t)).$$

Thus, $C_1(t)$ of (4.19) is a solution to the Cauchy problem (4.18). The assertion is proven.

REMARK 3. It is easy to verify that if we consider the function

$$u_0(t) = u(t)[\dot z(t)]^{-\frac{1}{p}} \tag{4.23}$$

instead of $u(t)$ then $u_0(t)$ satisfies the linear Volterra integral equation of the second kind

$$u_0(t) = \sigma\int_0^t K_0(t,\eta)u_0(\eta)\,d\eta + f_0(t) \tag{4.24}$$

with the kernel

$$K_0(t,\eta) = \dot z(\eta)\,\mathrm{tr}[Q^2(t)Q^{-1}(\eta)D(0)] \tag{4.25}$$

and free term

$$f_0(t) = \mathrm{tr}[Q^2(t)\Lambda(0)]. \tag{4.26}$$

Moreover, the matrices $A_k(t)$, the column vectors $\mathbf{B}_k(t)$, and the scalar functions $C_k(t)$, $k = 1, 2$, take the form

$$A_2(t) = \dot z(t)SQ(t)D(0)S', \tag{4.27}$$

$$\mathbf{B}_2(t) = \dot z(t)SQ(t)S'\mathbf{B}_2(0), \tag{4.28}$$

$$C_2(t) = \dot z(t)\left[C_2(0) + z(t)\left(Q(t)S'\mathbf{B}_2(0), S'\mathbf{B}_2(0)\right)\right], \tag{4.29}$$

$$A_1(t) = [\dot{z}(t)]^{\frac{1}{p}} S \left[\sigma \int_0^t \dot{z}(\eta) u_0(\eta) Q^{-1}(\eta) D(0)\, d\eta + \Lambda(0) \right] Q^2(t) S', \tag{4.30}$$

$$\mathbf{B}_1(t) = [\dot{z}(t)]^{\frac{1}{p}} S Q(t) \Bigg[\sigma \left(\int_0^t \dot{z}(\eta) u_0(\eta) Q^{-1}(\eta)\, d\eta \right) Q(t) S' \mathbf{B}_2(0)$$
$$+ 2z(t) Q(t) \Lambda(0) S' \mathbf{B}_2(0) + S' \mathbf{B}_1(0) \Bigg], \tag{4.31}$$

$$C_1(t) = [\dot{z}(t)]^{\frac{1}{p}} \Bigg[C_1(0) + 2z(t)(Q(t) S' \mathbf{B}_1(0), S' \mathbf{B}_2(0))$$
$$+ 2z^2(t)(Q^2(t) \Lambda(0) S' \mathbf{B}_2(0), S' \mathbf{B}_2(0))$$
$$+ \sigma \left(\int_0^t \dot{z}(\eta) u_0(\eta)\, d\eta \right) [C_2(0) + z(t)(Q(t) S' \mathbf{B}_2(0), S' \mathbf{B}_2(0))$$
$$+ z(t) |Q(t) S' \mathbf{B}_2(0)|^2] - \sigma \left(\int_0^t z(\eta) \dot{z}(\eta) u_0(\eta)\, d\eta \right) |Q(t) S' \mathbf{B}_2(0)|^2 \Bigg]. \tag{4.32}$$

REMARK 4. It is obvious that if we introduce the matrix $\Lambda(t) = \operatorname{diag}[\lambda_1(t), \dots, \lambda_n(t)]$ with the real eigenvalues

$$\lambda_k(t) = \left[\sigma d_k(t) \int_0^t [1 - 2d_k(0) z(\eta)] [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta)\, d\eta + \lambda_k(0) \right]$$
$$\times [1 - 2d_k(0) z(t)]^{-2} [\dot{z}(t)]^{\frac{1}{p}}, \quad k = 1, 2, \dots, n,$$

then (4.30) becomes

$$A_1(t) = S \Lambda(t) S', \tag{4.30$'$}$$

$$\Lambda(t) = [\dot{z}(t)]^{\frac{1}{p}} \left[\sigma \int_0^t [\dot{z}(\eta)]^{1-\frac{1}{p}} u(\eta) Q^{-1}(\eta) D(0)\, d\eta + \Lambda(0) \right] Q^2(t).$$

Finally, note that diagonalization of square matrices $A(\chi) = [a_{ij}(\chi)]$, $i, j = 1, 2, \dots, n$, whose entries are holomorphic functions in the complex variable χ is discussed in § 2 of [50, Chapter II].

5. Properties of solutions to (3.4.7)–(3.4.9). The results of Sections 3 and 4 enable us to continue studying solvability of the overdetermined system (3.4). This is a rather complicated problem. Therefore, we subordinate the study of (3.4) to examining solvability of equations (3.4.1)–(3.4.9) in a certain order. It is well known [43] that overdetermined systems of equations may have no solutions at all. For this reason, we show that (3.4), which is a overdetermined system, has solutions other than the trivial solution $A_k(t) = 0$, $\mathbf{B}_k(t) = \mathbf{0}$, $C_k(t) = 0$, $k = 1, 2$.

Our nearest purpose consists in studying solvability of the algebraic equation (3.4.7) in the class of diagonal matrices of the form (4.30′). It follows from (3.4.7) and (4.30′) that

$$\lambda_k(t)[\lambda u(t) + 2\xi \lambda_k(t)] = 0,$$

where

$$u(t) = \operatorname{tr} A_1(t) = \sum_{i=1}^{n} \lambda_i(t); \quad \xi = p(\lambda+1) - \lambda; \quad \xi \neq 0; \quad \lambda \neq 0.$$

Thereby for every $k = 1, 2, \dots, n$ we have $\lambda_k(t) = 0$ or $\lambda_k(t) = -\frac{\lambda}{2\xi} u(t)$. Consequently, all nonzero $\lambda_k(t)$ are equal to one another. Put $\mathscr{K} = \{k : \lambda_k(t) \neq 0\} = m \leq n$. Then the relation $\lambda u(t) + 2\xi u(t) = 0$ implies the dependence

$$\lambda m \sum_{i=1}^{n} \lambda_i(t) + 2\xi \sum_{\mathscr{K}} \lambda_k(t) = 0.$$

Since

$$\sum_{i=1}^{n} \lambda_i(t) = \sum_{\mathscr{K}} \lambda_k(t) = \operatorname{tr} A_1(t) \neq 0, \quad m = \operatorname{rank} A_1(t),$$

we have $\lambda \operatorname{rank} A_1(t) + 2\xi = 0$. If we consider the function

$$\varphi(t) = -\frac{\lambda}{2\xi} u(t) = \frac{\operatorname{tr} A_1(t)}{\operatorname{rank} A_1(t)}$$

then it is clear that $\Lambda(t) = \varphi(t) E_m$ and $A_1(t) = \varphi(t) S E_m S'$, where $E_m = \operatorname{diag}[e_1, \dots, e_n]$ and $e_k \in \{0, 1\}$.

Thus, the following holds:

Assertion 6. *Suppose that $A_1(t) = \varphi(t) S E_m S' \neq 0$, $E_m = \operatorname{diag}[e_1, \dots, e_n]$, $e_k \in \{0, 1\}$, $k = 1, 2, \dots, n$, $\operatorname{rank} E_m = m \in \{1, 2, \dots, n\}$, $\varphi(t)$ is an arbitrary real function such that $\varphi(t) \neq 0$ for all t in the domain of $A_1(t)$, and $S \in M_n(\mathbb{R})$ is a real orthogonal matrix. If $m = -2\xi/\lambda$ then $A_1(t)$ is a solution to* (3.4.7) *and the following relation holds:*

$$\operatorname{rank} E_m = \operatorname{rank} A_1(t) = -\frac{2\xi}{\lambda}, \tag{5.1}$$

where $\xi = p(\lambda+1) - \lambda$, $\xi \neq 0$, and $\lambda, p \in \mathbb{R}$, $\lambda \neq 0$, $p \neq 0$.

PROOF. Clearly, each of the matrices E_m is equivalent [47] to the matrix $\operatorname{diag}[1, \dots, 1, 0, \dots, 0]$, where $e_k = 1$ for $k = 1, 2, \dots, m$ and $e_k = 0$ for $k = m+1, \dots, n$. Thus, without loss of generality we suppose below that $E_m = \operatorname{diag}[1, \dots, 1, 0, \dots, 0]$. We easily see that $E_m = E_m^2$; i.e., the matrix E_m is idempotent. Since $A_1(t)$ and E_m are connected by the relation $A_1(t) = \varphi(t) S E_m S'$ and, moreover, $A_1(t) \neq 0$ and $\varphi(t) \neq 0$; the matrices $A_1(t)$ and E_m are equivalent too. On the other hand, it is well known that for equivalence of two real $(n \times n)$-matrices, it is necessary and sufficient that they have the same rank. Observing that $\operatorname{rank} E_m = m$ and $m = -\frac{2\xi}{\lambda}$, we now derive (5.1). Finally, recalling that

$$m = -\frac{2\xi}{\lambda}, \quad E_m = E_m^2, \quad \varphi(t) \neq 0,$$

we easily verify that the matrix $A_1(t) = \varphi(t) S E_m S'$ satisfies (3.4.7). Indeed, we have

$$\lambda (\operatorname{tr} A_1) A_1 + 2\xi A_1^2 = \lambda m \varphi^2(t) S E_m S' + 2\xi \varphi^2(t) S E_m^2 S' = \lambda m \varphi^2(t) S (E_m - E_m^2) S' \equiv 0,$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $m \in \{1, 2, \dots, n\}$. It is clear that $A_1(t) \in M_n(\mathbb{R})$ is a real symmetric matrix. The assertion is proven.

Since, on the one hand, $\xi = -\frac{\lambda m}{2}$ and, on the other hand, $\xi = p(\lambda+1) - \lambda$; the parameters of the system of ADE (3.4) under study are connected by the relation $2p(\lambda+1) = \lambda(2-m)$. In this case $\tau = \frac{\lambda}{p}$, $\sigma = -\frac{2p}{m}$, $\lambda \neq 0$, and $p \neq 0$.

REMARK 5. If $\lambda = -\frac{2}{m}$ then the dependence $2p(\lambda+1) = \lambda(2-m)$ implies that $m = 2$ or $p = 1$, $m \in \{1, 2, \dots, n\}$. Thus, if $m = 2$ then $\lambda = -1$, $\tau = -\frac{1}{p}$, $\sigma = -p$, and $\xi = 1$; if $p = 1$ then $\lambda = \tau = \sigma = -\frac{2}{m}$ and $\xi = 1$.

In view of the dependence $m = -\frac{2\xi}{\lambda}$ and the shape of $A_1(t)$, equations (3.4.8) and (3.4.9) take the following form:

$$(I - E_m)S'\mathbf{B}_1(t) = 0, \tag{5.2}$$

$$|\mathbf{B}_1(t)|^2 = 2\varphi(t)C_1(t). \tag{5.3}$$

We return to studying validity of (5.2) and (5.3) after finding the column vector $\mathbf{B}_1(t)$ and the scalar function $C_1(t)$.

Below we search a solution to (3.4) for a fixed $m \in \{1, 2, \dots, n\}$, assuming that $A_1(t)=\varphi(t)SE_mS'$. Further study (3.4) splits into two independent cases: $p = 2$ and $p \neq 2$.

Before passing to the case $p \neq 2$, we state the following

Assertion 7. *Assume $p \neq 2$. Then for the matrices*

$$A_1(t) = \varphi(t)SE_mS', \tag{5.4}$$

$$A_2(t) = \psi(t)SE_mS' \tag{5.5}$$

to be a solution to the overdetermined system of equations (3.4.1), (3.4.4), (3.4.7), *it is sufficient that $\varphi(t)$ and $\psi(t)$ satisfy the system of ODE*

$$\dot{\varphi}(t) = (\tau m + \sigma m + 4)\varphi(t)\psi(t), \tag{5.6}$$

$$\dot{\psi}(t) = (\lambda m + 2)\psi^2(t), \tag{5.7}$$

where $\varphi(t) \neq 0$ for all t in the domain of $A_1(t)$; $\psi(t) \neq 0$ for all t in the domain of $A_2(t)$; $\tau = \frac{\lambda}{p}$; $\sigma = -\frac{2p}{m}$; $\lambda \in \mathbb{R}$; and $m \in \{1, 2, \dots, n\}$.

PROOF. By Assertion 6, $A_1(t)$ in (5.4) is a solution to (3.4.7). Inserting $A_1(t)$ in the matrix equation (3.4.4), using (3.24), and making simple transformations, we arrive at the equality

$$\left[\frac{\dot{\varphi}(t)}{\varphi(t)} - \tau \operatorname{tr} A_2(t)\right] E_m = \left[4E_m + \sigma m I\right] D(t),$$

where $D(t)$ is the matrix of (3.22). In view of the shape of E_m, the above relation splits into two equalities:

$$\frac{\dot{\varphi}(t)}{\varphi(t)} - \tau \operatorname{tr} A_2(t) = (\sigma m + 4)d_k(t); \quad k = 1, 2, \dots, m;$$

$$0 = \sigma m d_k(t); \quad k = m+1, \dots, n;$$

moreover, $\sigma m + 4 = 2(2-p) \neq 0$. Since $\sigma m = -2p \neq 0$, we have $d_k(t) \equiv 0$ for $k = m+1, \dots, n$. Thereby if $k = 1, 2, \dots, m$ then the eigenvalues $d_k(t)$ of $A_2(t)$ are independent of k and have the form

$$d_k(t) = \frac{1}{\sigma m + 4}\left[\frac{\dot{\varphi}(t)}{\varphi(t)} - \tau \operatorname{tr} A_2(t)\right] \triangleq \psi(t); \tag{5.8}$$

i.e., $D(t) = \psi(t)E_m$. Thus, if $A_2(t)$ is defined by (5.5) then $\operatorname{tr} A_2(t) = m\psi(t)$. Now, (5.8) enables us to obtain (5.6). On the other hand, it is easy to see that (3.4.1), together with (5.5), yields (5.7). The assertion is proven.

Corollary 1. *If $A_2(t)$ is defined by* (5.5) *then*

$$A_2(t) = \frac{d(0)\dot{z}(t)}{1-2d(0)z(t)} S E_m S'; \tag{5.9}$$

moreover,

$$\psi(t) = \psi(0)[1-2\psi(0)z(t)]_+^{-(\lambda m+2)/2}, \tag{5.10}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $m \in \{1,2,\dots,n\}$, $\psi(0) \neq 0$, and $d(0) \neq 0$.

PROOF. Introducing the notation $q_k(t) = [1-2d_k(0)z(t)]^{-1}$, we rewrite (3.17) as $Q(t) = \operatorname{diag}[q_1(t), \dots, q_n(t)]$. By Assertion 7, $d_k(0) \equiv 0$ for $k = m+1, \dots, n$ and $d_k(0) = \psi(0) \triangleq d(0)$ for $k = 1,2,\dots,m$. Therefore, $q_k(t) = [1-2d(0)z(t)]^{-1}$ for $k = 1,2,\dots,m$ and $q_k(t) = 1$ for $k = m+1,\dots,n$. Here we have used $z(0) = 0$. Thereby

$$Q(t) = [1-2d(0)z(t)]^{-1}E_m + (I - E_m). \tag{5.11}$$

On the other hand, using (3.22), from (5.8) we infer that

$$\psi(t) = d(0)[1-2d(0)z(t)]^{-1}\dot{z}(t), \tag{5.12}$$

where $\dot{z}(t)$ is defined by (3.10), taking in our case the form

$$\dot{z}(t) = [1-2d(0)z(t)]^{-\lambda m/2}, \quad z(0) = 0. \tag{5.13}$$

Thus, (5.5) and (5.12) imply (5.9). Moreover, (5.12) and (5.13) lead to (5.10). The corollary is proven.

REMARK 6. If $\lambda m + 2 \neq 0$ and $p \neq 2$ then the Cauchy problem (5.13) has the solution

$$z(t) = \frac{1}{2d(0)} - \frac{1}{2d(0)}[1-(\lambda m+2)d(0)t]_+^{2/(\lambda m+2)}; \tag{5.14}$$

moreover,

$$\psi(t) = \psi(0)[1-(\lambda m+2)\psi(0)t]^{-1}. \tag{5.15}$$

If $\lambda = -\frac{2}{m}$ and $p \neq 2$ then a solution to the Cauchy problem (5.13) is defined by the formula

$$z(t) = \frac{1}{2d(0)}[1-\exp(-2d(0)t)]; \tag{5.16}$$

moreover, $\psi(t) = \psi(0)$. Here $\lambda \in \mathbb{R}$; $\lambda \neq 0$; $\psi(0) = d(0) \neq 0$; and $m \in \{1,2,\dots,n\}$.

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