Siberian Mathematical Journal, Vol. 40, *No.* 5, 1999

STRONGLY SUBEXPONENTIAL DISTRIBUTIONS AND BANACH ALGEBRAS OF MEASURES[†)]

B. A. Rogozin and M. S. Sgibnev

UDC 517.986.225

1. Introduction. An extensive bibliography is devoted to studies of various properties of subexponential and related distributions (the so-called $\mathcal{S}(\gamma)$*-distributions*).

DEFINITION 1. A probability distribution G *belongs to the class* $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if

(1) G is concentrated on $[0,\infty)$ and $G([x,\infty)) > 0$ for all $x \geq 0$;

(2) for every $y \in \mathbb{R}$, there is a limit

$$\lim_{x\to\infty} \frac{G([x+y,\infty))}{G([x,\infty))} = \exp(-\gamma y);$$

(3) there is a finite limit

$$\lim_{x\to\infty} \frac{G * G([x,\infty))}{G([x,\infty))} = c.$$

The class $\mathcal{S}(\gamma)$ with $\gamma = 0$ was introduced in the article [1], wherein the basic properties of distributions of this class were studied. These distributions were later called *subexponential* [2]. In [1], it was postulated that the constant c in (3) equals 2. The classes $\mathcal{S}(\gamma)$ for $\gamma > 0$ were introduced in [3, 4] in a somewhat different but equivalent way. It was claimed in [3], with a reference to [4], that the constant c in (3) must equal $2\int_0^\infty e^{\gamma x}\, G(dx)$ by necessity. We make some remarks on this question at the end of the article.

The "tails" of distributions in $\mathcal{S}(\gamma)$ can be used as norming functions in the construction of some Banach algebras of measures with the exact asymptotic behavior of tails [4–7]. The scheme of these constructions is as follows: Fix some distribution $G \in \mathcal{S}(\gamma)$ and put $\tau(x) = 1 - G(x)$. Now, in a Banach algebra, for instance, the Banach algebra of finite measures defined on the σ-algebra $\mathcal{B}$ of Borel subsets of the real axis $\mathbb{R}$, select a collection $\mathfrak{Sl}(\tau)$ of measures ν such that

$$Q(\nu) \stackrel{\text{def}}{=} \sup_{x\geq 0} \frac{|\nu|([x,\infty))}{\tau(x)} < \infty$$

and there is a limit

$$l(\nu) \stackrel{\text{def}}{=} \lim_{x\to\infty} \frac{\nu([x,\infty))}{\tau(x)} \in \mathbb{C};$$

here $|\nu|(A)$ is the total variation of a measure ν on $A \in \mathcal{B}$ and $\mathbb{C}$ is the field of complex numbers. Defining the multiplication of elements in $\mathfrak{Sl}(\tau)$ as convolution, we make the collection $\mathfrak{Sl}(\tau)$ into a Banach algebra with some norm equivalent to the norm $|\nu|(\mathbb{R}) + Q(\nu)$ [6].

In the present article, we consider new classes of probability distribution, calling them *strong* $\mathcal{S}(\gamma)$*-distributions* (see Definition 2), and consider the corresponding Banach algebras of measures with exact asymptotic behavior of tails. If ν is a measure and $y \in \mathbb{R}$ then we denote by ν_y the translation of ν by y: $\nu_y(A) \stackrel{\text{def}}{=} \nu(A-y)$, $A \in \mathcal{B}$; here $A + x \stackrel{\text{def}}{=} \{u \in \mathbb{R} : u - x \in A\}$.

†) The research was financially supported by the Russian Foundation for Basic Research (Grants 96–01–00091, 99–01–01130, and 99–01–00502 for the first author and Grants 96–01–01939 and 96–15–96295 for the second author).

Omsk, Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 40, No. 5, pp. 1137–1146, September–October, 1999. Original article submitted June 2, 1998.

DEFINITION 2. Suppose that $\gamma \geq 0$ is a number. A probability distribution G is called a *strong $\mathcal{S}(\gamma)$-distribution* (symbolically, $G \in \mathcal{S}_{\text{str}}(\gamma)$) if

(a) G is concentrated on $[0,\infty)$ and $G([x,\infty)) > 0$ for all $x \geq 0$;

(b) for every $y \in \mathbb{R}$, there is a limit

$$\lim_{x\to\infty} \frac{|G_y - e^{\gamma y}G|([x,\infty))}{G([x,\infty))} = 0;$$

(c) there is a constant c such that

$$\lim_{x\to\infty} \frac{|G * G - cG|([x,\infty))}{G([x,\infty))} = 0.$$

The distributions of the class $\mathcal{S}_{\text{str}}(\gamma)$ for $\gamma = 0$ are called *strongly subexponential*. Clearly, $\mathcal{S}_{\text{str}}(\gamma) \subset \mathcal{S}(\gamma)$.

The object of our consideration is the collections of measures similar to the Banach algebras $\mathfrak{Sl}(\tau)$ which are constructed by means of the norming functions $\tau(x) = 1 - G(x)$ with a strong $\mathcal{S}(\gamma)$-distribution G.

2. Banach algebras and their properties. Fix a distribution $G \in \mathcal{S}(\gamma)$, $\gamma \geq 0$. Put $\tau(x) = 1 - G(x)$ and suppose that $0 \leq \gamma' \leq \gamma$. Consider the following collection of complex-valued σ-finite measures [7]:

$$S(\gamma',\gamma) = \left\{\nu : \int_{\mathbb{R}} \max(e^{\gamma' x}, e^{\gamma x})|\nu|(dx) < \infty\right\},$$

$$\mathfrak{S}(\gamma',\tau) = \{\nu \in S(\gamma',\gamma) : Q(\nu) < \infty\},$$

$$\mathfrak{So}(\gamma',\tau) = \left\{\nu \in \mathfrak{S}(\gamma',\tau) : \lim_{x\to\infty} \frac{|\nu|([x,\infty))}{\tau(x)} = 0\right\},$$

$$\mathfrak{Sl}(\gamma',\tau) = \left\{\nu \in \mathfrak{S}(\gamma',\tau) : \exists \lim_{x\to\infty} \frac{\nu([x,\infty))}{\tau(x)} \stackrel{\text{def}}{=} l(\nu) \in \mathbb{C}\right\}.$$

Given $\nu \in \mathfrak{S}(\gamma',\tau)$, put

$$\|\nu\|'_\tau = \int_{\mathbb{R}} \max(e^{\gamma' x}, e^{\gamma x})|\nu|(dx) + Q(\nu).$$

If $\gamma' = \gamma > 0$ then we additionally suppose that the function $\tau(x)e^{\gamma x}$ satisfies the condition

$$\tau(y)e^{\gamma y} \leq C_0\tau(x)e^{\gamma x} \quad \forall y \geq x \geq 0 \tag{1}$$

for some constant $C_0 \geq 1$.

Denote by $\hat{\nu}(s)$ the Laplace transform of a measure $\nu \in S(\gamma',\gamma)$: $\hat{\nu}(s) = \int_{\mathbb{R}} \exp(sx)\,\nu(dx)$. This integral converges absolutely with respect to the measure $|\nu|$ in the strip $\{\gamma' \leq \operatorname{Re} s \leq \gamma\}$.

The collection $\mathfrak{S}(\gamma',\tau)$ is a Banach algebra with some norm $\|\nu\|$ equivalent to the norm $\|\nu\|'_\tau$, and the collections $\mathfrak{So}(\gamma',\tau)$ and $\mathfrak{Sl}(\gamma',\tau)$ are Banach subalgebras of $\mathfrak{S}(\gamma',\tau)$. If ν, $\mu \in \mathfrak{Sl}(\gamma',\tau)$ then

$$l(\nu * \mu) = l(\nu)\hat{\mu}(\gamma) + l(\mu)\hat{\nu}(\gamma) + (c - 2\widehat{G}(\gamma))l(\nu)l(\mu) \tag{2}$$

(see [7, Propositions 1 and 2 and Remark 2] and [6, Proof of Proposition 2]). The needed changes in the proof of Propositions 2 in [6] and [7] for establishing equality (2) are connected with the fact that in this case

$$\lim_{n\to\infty}\lim_{x\to\infty} \frac{G_n * G_n([x,\infty))}{\tau(x)} = c - 2\widehat{G}(\gamma),$$

where $G_n(A) = G(A \cap [n, \infty))$, $n \geq 0$. Observe also that if $0 \leq \gamma' < \gamma$ then we always have

$$\tau(y)e^{\gamma' y} \leq C_0 \tau(x) e^{\gamma' x} \quad \forall y \geq x \geq 0 \tag{3}$$

for an arbitrary $G \in \mathcal{S}(\gamma)$ and some $C_0 \geq 1$ [7, Lemma 2]. For $\gamma = \gamma'$, inequality (3) transforms into condition (1).

Given a distribution G in the class $\mathcal{S}_{\mathrm{str}}(\gamma)$, we consider the following collection of measures:

$$\mathfrak{S}\mathfrak{L}(\gamma', \tau) = \{\nu \in \mathfrak{S}(\gamma', \tau) : \nu = aG + \omega,\ a \in \mathbb{C},\ \omega \in \mathfrak{S}\mathfrak{o}(\gamma', \tau)\}.$$

Obviously, $\mathfrak{S}\mathfrak{o}(\gamma', \tau) \subset \mathfrak{S}\mathfrak{L}(\gamma', \tau) \subset \mathfrak{S}\mathfrak{l}(\gamma', \tau)$, and $l(\nu) = a$ for $\nu = aG + \omega \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$. Henceforth, given an element $\nu = aG + \omega \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$, the notation $L(\nu)$ means that $\nu = L(\nu)G + \omega \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$, where $\omega \in \mathfrak{S}\mathfrak{o}(\gamma', \tau)$. The distinction between the elements of $\mathfrak{S}\mathfrak{L}(\gamma', \tau)$ and those of the algebra $\mathfrak{S}\mathfrak{l}(\gamma', \tau)$ is as follows: for $\nu \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ the *total variation* $|\nu - aG|([x, \infty))$ is "o-small" of $\tau(x)$ as $x \to \infty$, whereas for $\nu \in \mathfrak{S}\mathfrak{l}(\gamma', \tau)$ the difference $\nu([x, \infty)) - l(\nu)G([x, \infty))$ is "o-small" of $\tau(x)$.

The proofs of Propositions 1 and 2 and Lemma 1 exhibited below are modifications of the corresponding arguments in the proofs of Theorems 1–4 of [4] as applied to the "tail" algebras in question; moreover, in the proof of Proposition 1 we use σ-finiteness of the measures under consideration.

Proposition 1. *Take $G \in \mathcal{S}_{\mathrm{str}}(\gamma)$. Then the collection $\mathfrak{S}\mathfrak{L}(\gamma', \tau)$ of measures is a Banach subalgebra of the algebra $\mathfrak{S}(\gamma', \tau)$. The relations $G * \omega \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ and $L(G * \omega) = \widehat{\omega}(\gamma)$ hold for $\omega \in \mathfrak{S}\mathfrak{o}(\gamma', \tau)$.*

PROOF. Completeness of the normed space $\mathfrak{S}\mathfrak{L}(\gamma', \tau)$ can be proven routinely. Show that $G * \omega \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ if $\omega \in \mathfrak{S}\mathfrak{o}(\gamma', \tau)$. We have

$$\frac{|G * \omega - \widehat{\omega}(\gamma)G|([x, \infty))}{\tau(x)} \leq \frac{1}{\tau(x)} \int_{-\infty}^{\infty} |G_y - e^{\gamma y} G|([x, \infty))|\omega|(dy)$$

$$\leq \int_{-N}^{N} \frac{|G_y - e^{\gamma y} G|([x, \infty))}{\tau(x)} |\omega|(dy) + \int_{-\infty}^{-N} + \int_{N}^{\infty} = I_1 + I_2 + I_3. \tag{4}$$

By the dominated convergence theorem, $I_1 \to 0$ as $x \to \infty$, since the integrand is dominated by the quantity

$$\frac{G([x - N, \infty)) + e^{\gamma N} G([x, \infty))}{\tau(x)}$$

which, in turn, is bounded by the number $C + e^{\gamma N} < \infty$, where $C = \sup_{x \geq 0} \frac{\tau(x-N)}{\tau(x)}$. Estimate I_2. For $\gamma' < \gamma$ we use inequality (3) and for $\gamma' = \gamma$, condition (1):

$$I_2 \leq \int_{-\infty}^{-N} \frac{G([x - y, \infty))}{\tau(x)} |\omega|(dy) + \int_{-\infty}^{-N} e^{\gamma y} |\omega|(dy)$$

$$\leq C_0 \int_{-\infty}^{-N} e^{\gamma' y} |\omega|(dy) + \int_{-\infty}^{-N} e^{\gamma y} |\omega|(dy) \leq (C_0 + 1) \int_{-\infty}^{-N} e^{\gamma' y} |\omega|(dy). \tag{5}$$

Estimate I_3:

$$I_3 \leq \int_{N}^{\infty} \frac{G([x - y, \infty))}{\tau(x)} |\omega|(dy) + \int_{N}^{\infty} e^{\gamma y} |\omega|(dy) = I_4 + I_5. \tag{6}$$

Change the order of integration in

$$I_4 = \int_N^\infty \int_{x-y}^\infty \frac{G(dz)|\omega|(dy)}{\tau(x)}$$

to obtain

$$I_4 = \int_0^\infty \int_{\max\{N,x-z\}}^\infty \frac{|\omega|(dy)G(dz)}{\tau(x)}$$

$$= \int_0^\infty \frac{|\omega|([\max\{N,x-z\},\infty))}{G([\max\{N,x-z\},\infty))} \cdot \frac{G([\max\{N,x-z\},\infty))}{\tau(x)} G(dz)$$

$$\le \sup_{u\ge N} \frac{|\omega|([u,\infty))}{\tau(u)} \int_0^\infty \frac{G([\max\{N,x-z\},\infty))}{\tau(x)} G(dz).$$

Change again the order of integration:

$$I_4 \le \sup_{u\ge N} \frac{|\omega|([u,\infty))}{\tau(u)} \int_N^\infty \frac{G([x-y,\infty))}{\tau(x)} G(dy) \le M \sup_{u\ge N} \frac{|\omega|([u,\infty))}{\tau(u)}, \tag{7}$$

where

$$M = \sup_{x\ge 0} \frac{G*G([x,\infty))}{\tau(x)} < \infty.$$

Assume that $\varepsilon > 0$ is arbitrary. Take N so large that the right-hand side of (5) be less than $\varepsilon/4$, the integral I_5 in (6) be less than $\varepsilon/4$, and $\sup_{u\ge N} \frac{|\omega|([u,\infty))}{\tau(u)} < \frac{\varepsilon}{4M}$ in (7). Afterwards, take x_0 so large that $I_1 < \varepsilon/4$ for $x \ge x_0$. Combining the above estimates, we find that the left-hand side of (4) is less than ε for $x \ge x_0$. Hence, $G*\omega - \widehat{\omega}(\gamma)G \in \mathfrak{So}(\gamma',\tau)$. It follows from condition (c) that $G*G \in \mathfrak{SL}(\gamma',\tau)$ and $L(G*G) = c$. Finally, we conclude that $\mathfrak{SL}(\gamma',\tau)$ is a Banach subalgebra of the algebra $\mathfrak{S}(\gamma',\tau)$.

Now, we show that the constant c in condition (c) equals $2\widehat{G}(\gamma)$. To prove that $\mathfrak{So}(\gamma',\tau)$ is a Banach subalgebra of $\mathfrak{S}(\gamma',\tau)$, we do not use the equality $c = 2\widehat{G}(\gamma)$ (see [7]). We put $d = \widehat{G}(\gamma)$. If $G \in \mathcal{S}_{\rm str}(\gamma)$ then conditions (2) and (3) of Definition 1 are satisfied. Therefore, the equality

$$G*G([x,\infty)) = 2\int_0^{x/2} G([x-y,\infty))\,G(dy) + G([x/2,\infty))^2$$

and Fatou's lemma yield the inequality $c \ge 2\widehat{G}(\gamma)$.

Lemma 1. *Assume that $G \in \mathcal{S}_{\rm str}(\gamma)$. Then the spectrum σ_G of the element $G \in \mathfrak{SL}(\gamma',\tau)$ lies in the set $\{z \in \mathbb{C} : |z| \le d\} \cup \{c-d\}$.*

PROOF. Let $m : \mathfrak{SL}(\gamma',\tau) \to \mathbb{C}$ be an arbitrary homomorphism and let m_0 be the restriction of m to $\mathfrak{So}(\gamma',\tau)$. If $m_0(\nu) \ne \hat{\nu}(\gamma)$ for some $\nu \in \mathfrak{So}(\gamma',\tau)$ then we represent $G*\nu$ as

$$G*\nu = \hat{\nu}(\gamma)G + \nu_1, \tag{8}$$

where $\nu_1 \in \mathfrak{So}(\gamma',\tau)$ by Proposition 1. If $m_0(\nu) = \hat{\nu}(\gamma)$ for every $\nu \in \mathfrak{So}(\gamma',\tau)$ then we write

$$G*G = cG + \nu_G, \quad \nu_G \in \mathfrak{So}(\gamma',\tau). \tag{9}$$

By Theorem 1 and Remark 2 of [7], there is a homomorphism m_1, an extension of m_0 to the basis algebra $S(\gamma',\gamma)$. Acting by the homomorphisms m_1 and m on (8) and (9), we conclude that both quantities $m_1(G)$ and $m(G)$ satisfy the equation

$$x[m_0(\nu)-\hat{\nu}(\gamma)] = m_0(\nu_1)$$

or the equation

$$x^2 = cx + d(d-c). \tag{10}$$

In the first case we have

$$m(G) = m_1(G) = \frac{m_0(\nu_1)}{m_0(\nu)-\hat{\nu}(\gamma)};$$

and the integral representation for the homomorphism m_1 (see [8, Theorem 1]) implies that $|m_1(G)| \le d$ (put $\varphi(x) = \max(e^{\gamma' x}, e^{\gamma x})$ in the indicated theorem). Thus, $|m(G)| \le d$. In the second case equation (10) has the roots $x_1 = d$ and $x_2 = c-d$. If $c = 2d$ then $x = d$ is a unique root, $m(G) = d$, and the lemma is valid again. If $c \ne 2d$ then $m(G) = d$ or $m(G) = s - d$ and the functional m has the form

$$m(\nu) = (c-2d)L(\nu) + \hat{\nu}(\gamma), \quad \nu \in \mathfrak{SL}(\gamma',\tau). \tag{11}$$

(We verify immediately that the continuous functional defined by (11) is a homomorphism.) Lemma 1 is proven.

Proposition 2. *The constant c in condition* (c) *equals* $2\widehat{G}(\gamma)$.

PROOF. Take a rectifiable Jordan contour Γ surrounding the set $\sigma_G \cap \{|z| \le d\}$ so that the point $c-d$ lies outside Γ; for instance, let Γ be a circle centered at the origin with radius $r \in (d, c-d)$. We have

$$\kappa \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_\Gamma (\lambda E - G)^{-1}\lambda\, d\lambda \in \mathfrak{SL}(\gamma',\tau),$$

since κ is the integral along Γ of the continuous function $(\lambda E - G)^{-1}\lambda$ with values in $\mathfrak{SL}(\gamma',\tau)$. Show that $\kappa = G$. Take the continuous homomorphism $h : \mathfrak{SL}(\gamma',\tau) \to \mathbb{C}$ to be the value of the Laplace transform at an arbitrarily fixed point s with $\operatorname{Re} s = 0$; i.e., $h(\nu) \overset{\text{def}}{=} \hat{\nu}(s)$, $\nu \in \mathfrak{SL}(\gamma',\tau)$. Apply h to both sides of the defining equality for κ. By continuity of h, we obtain

$$\widehat{\kappa}(s) = h(\kappa) = \frac{1}{2\pi i}\oint_\Gamma [\lambda - h(G)]^{-1}\lambda\, d\lambda = \frac{1}{2\pi i}\oint_\Gamma [\lambda - \widehat{G}(s)]^{-1}\lambda\, d\lambda = \widehat{G}(s).$$

Thus, the measures κ and G have the same Laplace transform. Hence, $\kappa = G$.

Now, suppose that m is the homomorphism (11). Apply m to both sides of the just established equality

$$G = \frac{1}{2\pi i}\oint_\Gamma (\lambda E - G)^{-1}\lambda\, d\lambda$$

to obtain

$$m(G) = \frac{1}{2\pi i}\oint_\Gamma \frac{\lambda}{\lambda - m(G)} d\lambda = \frac{1}{2\pi i}\oint_\Gamma \frac{\lambda}{\lambda-(c-d)} d\lambda = 0.$$

This contradicts the fact that $m(G) = c - d > 0$. Consequently, $c = 2d$.

Fix $G \in \mathcal{S}_{\text{str}}(\gamma)$. Recall that $\mathcal{S}_{\text{str}}(\gamma) \subset \mathcal{S}(\gamma)$ and $\mathfrak{So}(\gamma',\tau) \subset \mathfrak{SL}(\gamma',\tau) \subset \mathfrak{S}(\gamma',\tau)$. As demonstrated in [7], the homomorphisms from the Banach algebras $\mathfrak{So}(\gamma',\tau)$ and $\mathfrak{S}(\gamma',\tau)$ into the field $\mathbb{C}$ of complex numbers are the restrictions of homomorphisms of the basis algebra $S(\gamma',\gamma)$ to $\mathfrak{So}(\gamma',\tau)$ and $\mathfrak{S}(\gamma',\tau)$.

We turn to describing the homomorphisms $\mathfrak{SL}(\gamma',\tau) \to \mathbb{C}$.

Theorem 1. *Take $G \in S_{\mathrm{str}}(\gamma)$. Then every homomorphism from the Banach algebra $\mathfrak{S}\mathfrak{L}(\gamma', \tau)$ into $\mathbb{C}$ is the restriction of some homomorphism from $S(\gamma', \gamma)$ into $\mathbb{C}$.*

PROOF. Put

$$\varphi_1(x) = \begin{cases} \exp(\gamma' x), & x < 0, \\ \exp(\gamma x) + \frac{1}{\tau(x)}, & x \geq 0. \end{cases}$$

Note that the constant c in condition (c) equals $2\widehat{G}(\gamma)$ by Proposition 2. Now, the arguments are the same as in [6].

We have the following assertion for the values of an analytic function at elements of the Banach algebra $\mathfrak{S}\mathfrak{L}(\gamma', \tau)$ (the terminology and general results we use below are contained in [9, § 11], see also [7]).

Theorem 2. *Suppose that an analytic function $f(z)$ applies to an element $\nu \in S(\gamma', \gamma)$ and that $f(\nu) \in S(\gamma', \gamma)$, where $f(\nu)$ is the value of $f(z)$ at $\nu \in S(\gamma', \gamma)$. If $\nu \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ then $f(\nu) \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ and*

$$L(f(\nu)) = f'(\hat{\nu}(\gamma)) \cdot L(\nu). \tag{12}$$

PROOF. The membership $f(\nu) \in \mathfrak{S}\mathfrak{L}(\gamma', \tau)$ is guaranteed by the preceding theorem and the general theory [9, § 11]. Since $L(f(\nu)) = l(f(\nu))$ and the functional l is defined on the ambient algebra $\mathfrak{S}\mathfrak{l}(\gamma', \tau)$, by Theorem 3 of [7] we have $l(f(\nu)) = f'(\hat{\nu}(\gamma)) \cdot l(\nu)$ for $\nu \in \mathfrak{S}\mathfrak{L}(\gamma', \tau) \subset \mathfrak{S}\mathfrak{l}(\gamma', \tau)$, which proves (12).

3. Corollaries. We give some consequences of the above results.

Corollary 1. *Suppose that G is a distribution with density g and that $g(x) \equiv 0$ for $x < 0$. If*

$$\lim_{x \to \infty} \frac{g(x-y)}{g(x)} = e^{\gamma y}, \quad \lim_{x \to \infty} \frac{1}{g(x)} \int_0^x g(x-z) g(z)\, dz = b < \infty$$

for every $y \in \mathbb{R}$ and some $\gamma \geq 0$ then $G \in S_{\mathrm{str}}(\gamma)$ and $b = 2\widehat{G}(\gamma)$.

PROOF. The distribution G with density g belongs to the class $S_{\mathrm{str}}(\gamma)$ with the constant $c = b$. Applying Proposition 2, we obtain $b = 2\widehat{G}(\gamma)$.

Suppose that G is an arithmetic distribution concentrated at the points $0, 1, 2, \ldots$. If we take the domain of variation in Definitions 1 and 2 to be the set $\mathbb{Z}$ of integers rather than $\mathbb{R}$, then we arrive at distributions for which all corresponding assertions remain valid with obvious changes of statements. In particular, if $g(n) = G(\{n\})$, $n = 0, 1, 2, \ldots$, $\gamma \geq 0$, and

$$\lim_{n \to \infty} \frac{g(n-1)}{g(n)} = e^{\gamma}, \quad \lim_{n \to \infty} \frac{1}{g(n)} \sum_{k=0}^{n} g(n-k) g(k) = b < \infty$$

then $b = 2\widehat{G}(\gamma)$.

Corollary 1 is contained in [4] and its analog given above for arithmetic distributions, in [4, 10, 11]. For instance, putting $f(n) = g(n)e^{\gamma n}$ and $\Phi(z) = z^2$ and applying Theorems 3 and 4 of [10], we obtain $b = 2\widehat{G}(\gamma)$.

Lemma 2. *Suppose that $G \in S(\gamma)$, $\gamma \geq 0$, and $a = \int_0^\infty \tau(x)\, dx < \infty$. Then*

$$\int_0^x \tau(x-y)\tau(y)\, dy = (c-2) \int_x^\infty \tau(y)\, dy + o\left(\int_x^\infty \tau(y)\, dy \right)$$

as $x \to \infty$.

PROOF. We have

$$J = \int_x^\infty G * G([z,\infty))\,dz = \int_x^\infty \int_0^z \tau(z-y)\,G(dy)dz + \int_x^\infty \tau(z)\,dz.$$

If we change the order of integration in the double integral then we come to

$$J = \int_0^x \left(\int_{x-y}^\infty \tau(z)\,dz \right) G(dy) + a\tau(x) + \int_x^\infty \tau(z)\,dz.$$

Integrating by parts in the first integral, we obtain

$$J = \int_0^x \tau(x-y)\tau(y)\,dy + 2\int_x^\infty \tau(z)\,dz.$$

If we use condition (2) of Definition 1 for G then we arrive at

$$J = c\int_x^\infty \tau(z)\,dz + o\left(\int_x^\infty \tau(z)\,dz \right)$$

as $x \to \infty$; whence the assertion of the lemma follows.

Corollary 2. *If $G \in \mathcal{S}(\gamma)$ and $\gamma > 0$ then $c = 2\widehat{G}(\gamma)$ and*

$$\lim_{x\to\infty} \frac{1}{\tau(x)} \int_0^x \tau(x-y)\tau(y)\,dy = 2\int_0^\infty e^{\gamma y}\tau(y)\,dy. \tag{13}$$

PROOF. The idea of the proof is to reduce the problem of the value of the constant c for the class $\mathcal{S}(\gamma)$, $\gamma > 0$, to a similar problem for the class $\mathcal{S}_{\mathrm{str}}(\gamma)$ which has already been solved. We do this by choosing a distribution $G_1 \in \mathcal{S}_{\mathrm{str}}(\gamma)$ such that $G_1((x,\infty)) \sim c_1\tau(x)$, $x \to \infty$, where $c_1 > 0$.

Take $0 < \gamma' < \gamma$ and a nonnegative measure ν such that $\nu([0,\infty)) = 0$ and $\nu([x,0)) = -x/a$ for $x < 0$ and $a = \int_0^\infty \tau(x)\,dx$. The measure ν belongs to $\mathfrak{So}(\gamma',\tau)$; therefore, by (2) $\nu * G \in \mathfrak{Sl}(\gamma',\tau)$ and

$$l(\nu * G) = \hat{\nu}(\gamma) = \frac{1}{\gamma a}. \tag{14}$$

Define the distribution G_1 as follows: $G_1((-\infty,0)) = 0$ and $G_1([x,\infty)) = \nu * G([x,\infty))$ for $x \geq 0$. Its density $g_1(x)$ equals $\tau(x)/a$ for $x \geq 0$ and $g_1(x) = 0$ for $x < 0$. We have $G_1 \in \mathfrak{Sl}(\gamma',\tau)$ and $l(G_1) = 1/\gamma a$ by (14). Moreover,

$$\widehat{G}_1(\gamma) = \int_0^\infty e^{\gamma y} g_1(y)\,dy = \frac{\widehat{G}(\gamma) - 1}{\gamma a}. \tag{15}$$

Using Lemma 2 and (14) we infer that

$$\int_0^x g_1(x-y)g_1(y)\,dy = \frac{c-2}{a}\int_x^\infty g_1(y)\,dy + o\left(\int_x^\infty g_1(y)\,dy \right) = \frac{c-2}{\gamma a} g_1(x) + o(g_1(x)) \tag{16}$$

as $x \to \infty$. Moreover, for g_1 we have

$$\lim_{x\to\infty} \frac{g_1(x-y)}{g_1(x)} = \exp\{\gamma y\}$$

for every $y \in \mathbb{R}$. Therefore, Corollary 1 together with (15) and (16) yields the equality $(c-2)/\gamma a = 2(\widehat{G}(\gamma)-1)/\gamma a$; whence $c = 2\widehat{G}(\gamma)$ and (13) holds as well.

4. Remarks. In [3], there was considered the class of probability distributions G concentrated on $[0,\infty)$ for which the limits

$$\lim_{t\to\infty} \frac{1-G*G(t)}{1-G(t)} = c < \infty, \tag{17}$$

$$\lim_{t\to\infty} \frac{1-G(t-b)}{1-G(t)} = \psi(b) \quad \forall b \in \mathbb{R} \tag{18}$$

and the integral

$$\int_0^\infty e^{\gamma t} dG(t) = d < \infty \tag{19}$$

exist (from (18) we easily infer that $\psi(b) \equiv \exp(\gamma b)$ for some $\gamma \geq 0$). Also, it was emphasized in [3, p. 664] that the following equality holds by necessity:

$$c = 2d. \tag{20}$$

Moreover, it was indicated that, in the article [4] by the same authors, this equality was proven in the case of $d = 1$ and in the case of $d > 1$ only when G is a latticed or absolutely continuous distribution. Finally, the authors of [3] claimed that in fact the methods of the proof of Theorems 1 and 4 of [4] extend without changes to an arbitrary G concentrated on $[0,\infty)$ and satisfying the conditions (17)–(19) and to arbitrary values $d \geq 1$, thereby establishing (20) in full generality. This opinion is shared by the authors of some other articles (see, for instance, [12, Lemma 2.1; 13, 14, 6, 5]).

Remark 1. However, we have to say that, in Theorem 4 of [4], a somewhat different class of probability distributions was considered as compared to the class of distributions satisfying the conditions (17)–(19) for $d = 1$. We explain the difference. Put $T_t = (-\infty, -t] \cup (t, \infty)$ for $t > 0$ and $T_0 = \mathbb{R}$. Theorem 4 of [4] deals with the class of probability measures μ on $\mathcal{B}$ such that $\mu(T_t) > 0$ for every $t \geq 0$ and there exist limits

$$\lim_{t\to\infty} \frac{\mu * \mu(T_t)}{\mu(T_t)} = c < \infty, \tag{21}$$

$$\lim_{t\to\infty} \frac{|\mu - \mu_\tau|(T_t)}{\mu(T_t)} = 0 \quad \forall \tau \in \mathbb{R}. \tag{22}$$

One of the conclusions of Theorem 4 of [4] is that the constant c in (21) must obey the equality $c = 2$. Clearly, (22) implies (18) with $\psi(b) \equiv 1$. Therefore, if a probability measure μ is concentrated on $[0,\infty)$ and satisfies (21) and (22) then it meets (17)–(19) for $d = 1$. Condition (22) is essentially used in the proof of Theorem 4 of [4] but it does not follow from (17)–(19) for $\psi(b) \equiv 1$ for distributions G concentrated at integer points or their convolutions with the uniform distribution on $[0, 1/2]$. Therefore, the assertion of [3] that equality (20) was proven in [4] for $d = 1$ does not correspond to reality.

Remark 2. In the proof of the equality $c = 2$ [4, Theorem 4], the authors introduced the collection $\mathcal{A}_L = \{\nu = a\mu + \omega,\ a \in \mathbb{C},\ \omega \in \mathcal{A}_0\}$ of measures, where $\mathcal{A}_0$ is the set of finite measures ν such that

$$\sup_{t>0} \frac{|\nu|(T_t)}{\mu(T_t)} < \infty, \quad \lim_{t\to\infty} \frac{|\nu|(T_t)}{\mu(T_t)} = 0.$$

They defined the functional

$$L(\nu) = L(a\mu + \omega) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{\nu(T_t)}{\mu(T_t)} = a$$

on the set $\mathcal{A}_L$ and claimed that the conditions (21) and (22) imply the relations $\mu * \mu \in \mathcal{A}_L$ and $L(\mu * \mu) = c$. In our opinion, this assertion has to be proven or one should introduce the following condition instead of (21):

$$\lim_{t\to\infty} \frac{|\mu * \mu - c\mu|(T_t)}{\mu(T_t)} = 0.$$

It is the last condition that we use in Definition 2, Propositions 1 and 2, and Lemma 1.

Thus, there is a class distributions for which conditions (17)–(19) hold for $d = 1$ and conditions (21) and (22) do not hold for $d = 1$; therefore, the assertion of [3] that equality (20) with $d = 1$ was established in [4] under the conditions (17)–(19) is incorrect. Preserving the Banach-algebraic methods of [4], one can prove that $c = 2d$ for distributions in $\mathcal{S}(\gamma)$ with $\gamma > 0$ and for strongly subexponential distributions in the class $\mathcal{S}_{\text{str}}(0)$.

References

1. V. P. Chistyakov, "A theorem on sums of positive random variables and its application to random branching processes," Teor. Veroyatn. Primen., **9**, No. 4, 710–718 (1964).
2. K. B. Athreya and P. E. Ney, Branching Processes, Springer, Berlin (1972).
3. J. Chover, P. Ney, and S. Wainger, "Degeneracy properties of subcritical branching processes," Ann. Probab., **1**, 663–673 (1973).
4. J. Chover, P. Ney, and S. Wainger, "Functions of probability measures," J. Anal. Math., **26**, 255–302 (1973).
5. J. B. G. Frenk, On Banach Algebras, Renewal Measures and Regenerative Processes, Centre for Mathematics and Computer Science, Amsterdam (1987).
6. M. S. Sgibnev, "On Banach algebras of measures of the class $\mathcal{S}(\gamma)$," Sibirsk. Mat. Zh., **29**, No. 5, 162–171 (1988).
7. B. A. Rogozin and M. S. Sgibnev, "Banach algebras of measures on the real axis with the given asymptotics of distributions at infinity," Sibirsk. Mat. Zh., **40**, No. 3, 660–672 (1999).
8. B. A. Rogozin and M. S. Sgibnev, "Banach algebras of measures on the real axis," Sibirsk. Mat. Zh., **21**, No. 2, 160–169 (1980).
9. M. A. Naĭmark, Normed Rings [in Russian], Nauka, Moscow (1968).
10. W. Rudin, "Limits of ratios of tails of measures," Ann. Probab., **1**, 982–994 (1973).
11. P. Embrechts, "The asymptotic behaviour of series and power series with positive coefficients," Med. Konink. Acad. Wetensch. België, **45**, 41–61 (1983).
12. P. Embrechts and C. M. Goldie, "On convolution tails," Stochastic Process. Appl., **13**, 263–278 (1982).
13. D. B. H. Cline, "Convolutions of distributions with exponential and subexponential tails," J. Austral. Math. Soc. Ser. A, **43**, 347–365 (1987).
14. C. Klüppelberg, "Subexponential distributions and characterizations of related classes," Probab. Theory Related Fields, **82**, 259–269 (1989).