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ENTROPY SOLUTIONS TO THE BUCKLEY–LEVERETT EQUATIONS

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Introduction

We consider a mathematical model of flow of two immiscible fluids of different mobility in Hele–Shaw cells [1] and a porous medium. The motion of fluids is described by the Buckley–Leverett equations which can be written as follows [2, 3]:

$$s_t + \mathbf{v} \cdot \nabla A(s) = 0, \tag{1.1}$$

$$\operatorname{div} \mathbf{v} + f = 0, \quad \mathbf{v} = -k(s)\nabla p. \tag{1.2}$$

Here $s(x,t)$ is the saturation of one of the fluids, $\mathbf{v}(x,t)$ is the seepage velocity of the mixture, and $p(x,t)$ is the pressure. The fractional flow function A and the mobility k are given smooth functions of the phase saturation. We suppose that

$$A, k \in C^\infty(R), \quad 0 < C^{-1} < k(s) < C < \infty, \quad |A''(s)| > 0. \tag{1.3}$$

Observe that (1.1) and (1.2) constitute an elliptic-hyperbolic system of nonlinear PDE's. Boundary value problems for these systems are studied rather poorly. It is well known that the equations can be simplified in the case of symmetry. Typical examples here are traveling waves and the 2–D Riemann problem.

We consider the following boundary value problem which can be regarded as a generalization of the boundary value problems for self-similar solutions. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\mathbf{b}(x)$ be a given vector field of class $C^2(\Omega)$. The vector $-|\mathbf{b}|^{-1}\mathbf{b}$ defines the direction of wave propagation and $|\mathbf{b}|$ is the wave speed. We denote by $\partial\Omega^+$ the set of all points $x \in \partial\Omega$ such that

$$\partial\Omega^+ : \quad \mathbf{b} \cdot \mathbf{n} > 0,$$

where $\mathbf{n}$ is the unit outward normal to Ω. The problem is to find functions $s \in L_\infty(\Omega)$ and $p \in H_1(\Omega)$ and a vector-function $\mathbf{v}(x)$ satisfying the following equations:

$$\Omega : \begin{array}{c} \mathbf{v} \cdot \nabla A(s) - \mathbf{b} \cdot \nabla s = 0, \\ \operatorname{div}(k(s)\nabla p) = f, \quad \mathbf{v} = -k(s)\nabla p, \end{array} \tag{1.4}$$

$$\partial\Omega : \quad \mathbf{v} \cdot \mathbf{n} = 0, \quad \partial\Omega^+ : \quad s = s_0(x).$$

We suppose that f and s_0 are subject to the conditions

$$f \in L_\infty(\Omega), \quad s_0 \in \operatorname{Lip}(\partial\Omega), \quad \int\limits_\Omega f\,dx = 0.$$

We consider two kinds of generalized solutions to (1.4). The first is an entropy solution.

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DEFINITION 1.1. A vector field $\mathbf{v} \in L_2(\Omega)$ and a function $s \in L_\infty(\Omega)$ are called an *entropy solution* to (1.4) if for arbitrary functions $\xi, \eta : \Omega \to \mathbb{R}^1$ and $\varphi, \Phi, \Psi : \mathbb{R}^1 \to \mathbb{R}^1$ satisfying the conditions

$$\xi \in C^\infty(\Omega), \quad \eta \in \overset{\circ}{C}{}^\infty(\mathbb{R}^2), \quad \eta \geq 0, \quad \operatorname{spt} \eta \cap (\partial\Omega \setminus \partial\Omega^+) = \varnothing,$$
$$\varphi \in C^1(\mathbb{R}^1), \quad \varphi' \geq 0, \quad \Psi' = a(s)\varphi(s), \quad \Phi' = \varphi(s), \quad a = A', \tag{1.5}$$

the following relations are valid:

$$\int_\Omega ((\Psi(s) \cdot \mathbf{v} - \Phi(s) \cdot \mathbf{b}) \cdot \nabla\eta - (\Phi(s) \operatorname{div} \mathbf{b} + \Psi(s) f)\eta)\, dx + \int_{\partial\Omega^+} \Phi(s_0)\eta \mathbf{b} \cdot \mathbf{n} \geq 0, \tag{1.6}$$

$$\int_\Omega (\nabla p \nabla \xi \cdot k(s) - f\xi)\, dx = 0, \quad \mathbf{v} = -k(s)\nabla p.$$

Entropy solutions to scalar conservation laws were studied by many mathematicians. We only note that existence and uniqueness of entropy solutions to the Cauchy problem were proven in [4, 5]. Well-posedness of boundary value problems in bounded domains was established in [6].

To define a measure-valued solution to (1.4), we introduce some notations. We denote by ν_x a family of probability Radon measures ν_x on $\mathbb{R}^1 \times \mathbb{R}^2$ depending on $x \in \Omega$ [7]. We suppose that

(a) the mapping $x \to \nu_x$ is weakly measurable from Ω into the space of Radon measures;

(b) there exist constants M_0 and M_1 and an exponent $r > 2$ such that

$$\operatorname{spt} \nu_x \subset \{s \in \mathbb{R}^1, q \in \mathbb{R}^2 : |s| \leq M_0\}, \quad \int_{\mathbb{R}^3} (1 + |q|)^r\, d\nu_x \leq M_1.$$

These conditions imply that the function

$$x \to \int_{\mathbb{R}^3} f(s, q)\, d\nu_x \equiv f^*(x)$$

is measurable on Ω for a Borel function $f(s, q)$ satisfying the inequality $|f(s, q)| \leq c(s)(1 + |q|^2)$. Define

$$\mathbf{V}_x(\lambda) = \int_{[\lambda,\infty)} \int_{\mathbb{R}^2} q\, d\nu_x, \quad \Lambda_x(\lambda) = \int_{[\lambda,\infty)} \int_{\mathbb{R}^2} d\nu_x. \tag{1.7}$$

These functions are left-continuous and have bounded variations in λ almost everywhere in Ω. It is easily seen that the following identities are valid for a bounded Borel function φ:

$$\int_{\mathbb{R}^3} \varphi(s) q\, d\nu_x = -\int_{\mathbb{R}^1} \varphi(\lambda) d\mathbf{V}_x(\lambda), \quad \int_{\mathbb{R}^3} \varphi(s)\, d\nu_x = -\int_{\mathbb{R}^1} \varphi(\lambda) d\Lambda_x(\lambda). \tag{1.8}$$

Now, we are ready to define a measure-valued solution to (1.4).

DEFINITION 1.2. A Young measure ν_x is called a *measure-valued solution* to (1.4) if the relations

$$\begin{gathered}\int_\Omega (\mathbf{P}^*_\varphi \cdot \nabla\eta - (\Phi^* \operatorname{div} \mathbf{b} + \Psi^* f)\eta)\, dx + \int_{\partial\Omega^+} \Phi(s_0)\eta \cdot \mathbf{b} \cdot \mathbf{n}\, ds \geq 0, \\ \operatorname{div} \int_{\mathbb{R}^1} d\mathbf{V}_x(\lambda) = f \quad \operatorname{rot} \int_{\mathbb{R}^1} k^{-1}(\lambda)\, d\mathbf{V}_x(\lambda) = 0\end{gathered} \tag{1.9}$$

hold for all functions φ, η, ξ, Φ, and Ψ satisfying (1.5), where

$$\mathbf{P}^*_\varphi \equiv -\int_{\mathbb{R}^1} \Psi(\lambda) d\mathbf{V}_{x,\lambda} + \int_{\mathbb{R}^1} \Phi(\lambda)\, d\Lambda_x(\lambda) \cdot \mathbf{b}(x),$$

$$\Phi^* = -\int_{\mathbb{R}^1} \Phi(\lambda)\, d\Lambda_x(\lambda), \quad \Psi^* = -\int_{\mathbb{R}^1} \Psi(\lambda)\, d\Lambda_x(\lambda).$$

The concept of measure-valued solutions to conservation laws was introduced in [7] and developed in [8]. Observe that our definition differs from that of [7].

We also consider the following elliptic regularization of (1.4):

$$\Omega: \quad \begin{array}{l} -\varepsilon\Delta s + \mathbf{v}\cdot\nabla A(s) - \mathbf{b}\nabla s = 0, \\ \mathbf{v} = -k(s)\nabla p, \quad \operatorname{div}\mathbf{v} + f = 0, \end{array} \tag{1.10}$$

$$\varepsilon\nabla s\cdot\mathbf{n} + \gamma(s - s_0) = 0, \quad \mathbf{v}\cdot\mathbf{n} = 0. \tag{1.11}$$

Here the nonnegative Lipschitz continuous function γ is defined by the equalities

$$\partial\Omega^+ : \gamma = \mathbf{b}\cdot\mathbf{n}, \quad \partial\Omega\setminus\partial\Omega^+ : \gamma = 0.$$

The main results of the present article are the following theorems of existence and structure of measure-valued solutions to (1.4).

Theorem 1.1. *Suppose that the above conditions are satisfied. Then*

(i) for an arbitrary $\varepsilon > 0$ the problem (1.10), (1.11) has a solution $s, \mathbf{v} \in H_\alpha(\Omega)$, $\alpha > 2$, which satisfies the inequalities

$$\|s\|_{L_\infty(\Omega)} + \|\mathbf{v}\|_{L_{r_0}(\Omega)} < M, \quad \varepsilon^{1/2}\|\nabla s\|_{L_2(\Omega)} < M,$$

where the constant M is independent of ε and $r_0 := r_0(k) > 2$;

(ii) there exist a sequence $(s_\varepsilon, \mathbf{v}_\varepsilon)$ of solutions to the problem (1.10), (1.11) and a Young measure ν_x such that, for an arbitrary function $f : \mathbb{R}^3 \to \mathbb{R}$, $|f(s,q)| \le c(s)(1 + |q|^2)$, the sequence $f(s_\varepsilon, \mathbf{v}_\varepsilon)$ converges weakly in $L_{r_0/2}(\Omega)$ to the function

$$f^*(x) = \int_{\mathbb{R}^3} f(s,\mathbf{v})\, d\nu_x$$

as $\varepsilon \to 0$. The measure ν_x is a measure-valued solution to (1.4).

To state the results on the structure of measure-valued solutions to (1.4), we introduce some notations.

Definition 1.3. Given $\varphi \in C^1(R)$, the pair of the smooth functions

$$\Phi(s) = \int_0^s \varphi(\lambda)\, d\lambda + \mathrm{const}, \quad \Psi = \int_0^s a(\lambda)\varphi(\lambda)\, d\lambda + \mathrm{const}$$

is the *entropy pair* corresponding to φ. Given s and $\mathbf{v}$, the vector field $\mathbf{P}_\varphi = \Psi(s)\mathbf{v} - \Phi(s)\mathbf{b}$ is the *flow* corresponding to φ.

The following theorem yields a relation between the functions $\mathbf{V}_x(\lambda)$ and $\Lambda_x(\lambda)$ which define measure-valued solutions to (1.4).

Denote by $\mathbf{v}^*(x)$ the weak limit of the sequence $\mathbf{v}_\varepsilon(x)$. We can represent the domain Ω as the union of two disjoint sets

$$\Omega_0 = \{x : \mathbf{v}^*(x)\times\mathbf{b}(x) = 0\}, \quad \Omega_1 = \{x : \mathbf{v}^*(x)\times\mathbf{b}(x) \ne 0\}.$$

Theorem 1.2. *Under the conditions of Theorem 1.1, there exist a measurable function $s^*(x)$, $x \in \Omega_1$, and a family of functions $\rho_x(\lambda)$, $x \in \Omega_0$, such that*

(i) the following relations hold:

$$\mathbf{V}_x(\lambda) = \left(\frac{1}{a(\lambda)}\Lambda_x(\lambda) - \frac{1}{a(\lambda)}H(s^*(x) - \lambda)\right)\mathbf{b}(x) + H\left(s^*(x) - \lambda\right)\mathbf{v}^*(x), \quad x \in \Omega_1, \tag{1.12}$$

$$\mathbf{V}_x(\lambda) = \rho_x(\lambda)\mathbf{b}(x), \quad x \in \Omega_0.$$

Here $H(s)$ is the Heaviside function: $H(s) = 0$ for $s < 0$ and $H(s) = 1$ for $s \geq 0$.

(ii) The sequence of flows $\mathbf{P}_{\varepsilon,\varphi} = \Psi(s_\varepsilon)\mathbf{v}_\varepsilon - \Phi(s_\varepsilon)\mathbf{b}$ converges weakly in $L_2(\Omega_1)$ to $\mathbf{P}^*_\varphi = \Psi(s^*)\mathbf{v}^* - \Phi(s^*)\mathbf{b}$ if $\varphi \in C^1(R)$.

Relations (1.12) have some symmetry property. Define

$$\mathbf{U}_x(\lambda) = \int\limits_{(-\infty,\lambda]}\int\limits_{\mathbb{R}^2} q\,d\nu_x, \quad \chi_x(\lambda) = \int\limits_{(-\infty,\lambda]}\int\limits_{\mathbb{R}^2} d\nu_x.$$

It follows from (1.7) that

$$\mathbf{U}_x(\lambda) = \mathbf{v}^* - \lim_{\tau\to\lambda+0}\mathbf{V}_x(\tau), \quad \chi_x(\lambda) = 1 - \lim_{\tau\to\lambda+0}\Lambda_x(\tau),$$

$$H(\lambda - s^*) = 1 - \lim_{\tau\to\lambda+0} H(s^* - \tau).$$

Inserting these relations into (1.12), we obtain

$$\mathbf{U}_x(\lambda) = \left(\frac{1}{a(\lambda)}\chi_x(\lambda) - \frac{1}{a(\lambda)}H(\lambda - s^*(x))\right)\mathbf{b}(x) + H\left(\lambda - s^*(x)\right)\mathbf{v}^*(x). \tag{1.13}$$

It does not follow from Theorem 1.2 that the function s^* is the weak limit of the sequence s_ε. Our next proposition shows that, under some additional assumptions on the functions A and k, the solutions to the regularized problem converge strongly on Ω_1 to an entropy solution to (1.4). To state these additional assumptions, we introduce some notations. We denote by Σ the family of parabolas given by the formulas

$$y = p(z), \quad p(z) = z^2 + q_1 z + q_2, \quad q_i \in \mathbb{R}^1.$$

We say that a function $f : [c, d] \to \mathbb{R}^1$ is *strictly p-concave at a point* $z_0 \in (c, d)$ if it satisfies the following condition:

Condition P. *Fix a parabola $p_0 \in \Sigma$ such that $f(z_0) = p_0(z_0)$ and $f'(z_0) = p_0'(z_0)$. Let $z^* \in (c, d)$ be an arbitrary point. If a line $l \parallel \mathrm{Tan}_{z^*} p_0$ crosses the graph of the function $f(z)$ at points $(z_1, f(z_1))$ and $(z_2, f(z_2))$ such that*

$$z^* = \lambda z_1 + (1 - \lambda)z_2, \quad \lambda \in [0, 1], \quad z_0 \notin (z_1, z_2),$$

then

$$\lambda f(z_1) + (1 - \lambda)f(z_2) \leq p_0(z^*).$$

The equality holds if and only if $z^* = z_1 = z_2 = z_0$.

Consider the family of the functions $\Gamma, N : [-M, M] \to \mathbb{R}^1$ depending on a parameter α and given by the formulas

$$\Gamma(\lambda) = \frac{K(\alpha)}{a(\alpha)} + \int\limits_\alpha^\lambda \frac{K'(s)}{a(s)}\,ds,$$
$$N(\lambda) = \int\limits_\alpha^\lambda a^{-1}(s)(K(s)\Gamma(s))'\,ds, \quad K(s) = k^{-1}(s). \tag{1.14}$$

Denote by $\lambda = Z(z)$ the inverse of $z = \Gamma(\lambda)$ and set

$$f_\alpha(z) = N(Z(z)), \quad z_\alpha = \Gamma(\alpha) = \frac{K(\alpha)}{a(\alpha)}.$$

Condition S. *There is $M > \sup_{\varepsilon>0} \|s_\varepsilon\|_{L_\infty(\Omega)}$ such that the function $\Gamma : (-M, M) \to \mathbb{R}^1$ is strictly monotone and the function $f_\alpha(z)$ is strictly p-concave at the point z_α for every $\alpha \in (-M, M)$.*

EXAMPLE. Simple calculations show that

$$f'_\alpha(z) = \Gamma(\lambda) + a^{-1}(\lambda)K(\lambda), \quad f''_\alpha(z) = 2 - \frac{a'(\lambda)K(\lambda)}{a(\lambda)K'(\lambda)} = -\frac{(\log(K^{-2}(\lambda)a(\lambda))'}{(\log K(\lambda))'},$$

where $\lambda = Z(z)$. Therefore, the inequalities $K' > 0$ and $(aK^{-2})' > 0$ imply Condition S.

We note that this condition is close to the stability condition $k'(s) < 0$ ($K'(s) > 0$) [6] but does not coincide with the latter.

Theorem 1.3. *If the conditions of Theorem 1.1 are satisfied and the functions a and K are subject to Condition S then*

$$s_\varepsilon \to s^* \text{ strongly in } L_2(\Omega_1), \quad \mathbf{v}_\varepsilon \to \mathbf{v}^* \text{ strongly in } L_2(\Omega_1).$$

Corollary 1.1. *Under the conditions of Theorem 1.3, the function s^* and the vector field $\nabla p^* = -k(s^*)^{-1}\mathbf{v}^*$ represent an entropy solution to (1.4) on an open set $G \subset \Omega_1$.*

2. Proof of Theorem 1.1

We begin with proving solvability of the problem (1.10), (1.11). Consider the following family of boundary value problems depending on a parameter $\tau \in [0, 1]$:

$$\begin{aligned} \Omega:\ & \varepsilon\Delta s = -\tau(a(s)k\tau(s)\cdot\nabla s\nabla p + \mathbf{b}\cdot\nabla s), \\ \partial\Omega:\ & \varepsilon\nabla s\cdot\mathbf{n} + \gamma(s - \tau s_0) = 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \Omega:\ & \Delta p = \Pi[k_\tau(s)^{-1}(\tau f - \nabla k_\tau(s)\nabla p)], \\ \partial\Omega:\ & \nabla p\cdot\mathbf{n} = 0, \quad \langle p, 1\rangle = 0. \end{aligned} \tag{2.2}$$

Here the function k_τ and the operator Π are defined by the formulas

$$\Pi f = f - (\operatorname{mes}\Omega)^{-1}\langle f, 1\rangle, \quad k_\tau(s) = 1 + \tau(k(s) - 1),$$

and the given functions γ, s_0, and f satisfy the following conditions:

$$\gamma, s_0 \in \operatorname{Lip}(\partial\Omega), \quad f \in L_\infty(\Omega), \quad \int_\Omega f\,dx = 1.$$

First we prove a priori estimates for solutions to the problem (2.1), (2.2).

Lemma 2.1. *Assume that $(s, p) \in W^2_r(\Omega)$, $r > 2$, is a solution to (2.1). Then there exist a constant c, independent of ε and τ, and an exponent $r_0 > 2$ depending on k such that*

$$\|\nabla p\|_{L_{r_0}(\Omega)} + \varepsilon^{1/2}\|\nabla s\|_{L_2(\Omega)} + \|s\|_{L_\infty(\Omega)} \le c. \tag{2.3}$$

PROOF. Multiplying both sides of (2.2) by k_τ, we obtain the equality

$$\operatorname{div}(k_\tau\nabla s) = \tau f - k_\tau(\operatorname{mes}\Omega)^{-1}\langle k_\tau^{-1}(\tau f - \nabla k_\tau\nabla p), 1\rangle.$$

Integrating this equations over Ω yields $\langle k_\tau^{-1}(\tau f - \nabla k_\tau \nabla p), 1\rangle = 0$. Hence, we conclude that p is a solution to the boundary value problem

$$\Omega: \ \operatorname{div}(k_\tau(s)\nabla p) = \tau f, \quad \partial\Omega: \ \nabla p \cdot \mathbf{n} = 0.$$

The estimate for p is a consequence of the a priori estimates for solutions to second-order elliptic equations with bounded coefficients.

The function s is a solution to the following boundary value problem for an elliptic equation:

$$\begin{aligned} \Omega: &\quad \varepsilon\Delta s + \tau(a(s)k(s)\nabla p + \mathbf{b}) \cdot \nabla s = 0, \\ \partial\Omega: &\quad \varepsilon\nabla s \cdot \mathbf{n} + \gamma(s - s_0\tau) = 0. \end{aligned} \tag{2.4}$$

It follows from the conditions of the lemma that s, ∇p, $b \in C^\beta(\Omega)$, $0 < \beta < 1$, and $0 \leq \gamma \in \mathrm{Lip}(\partial\Omega)$. From here and the maximum principle we obtain the inequality $\min s_0 \leq s(x) \leq \max s_0$ which implies boundedness of $\|s\|_{L_\infty(\Omega)}$.

Multiplying both sides of (2.4) by s and integrating over Ω, we obtain the following estimates:

$$\varepsilon\|\nabla s\|^2_{L_2(\Omega)} \leq \|\operatorname{div}\mathbf{b}s^2\|_{L_1(\Omega)} + \|f\psi(s)\|_{L_1(\Omega)} + \max_{\partial\Omega}\{|\gamma s_0|\} \leq c, \quad \psi(s)' = sa(s).$$

The lemma is proven.

Lemma 2.2. *Under the conditions of Lemma 2.1, for an arbitrary $r > 2$ there is a constant $c(r, \varepsilon)$ such that $\|(s,p)\|_{W^2_r(\Omega)} \leq c(r,\varepsilon)$.*

PROOF. Introduce the sequences λ_k and n_k, $k \geq 1$, of positive numbers as follows:

$$\lambda_{k+1} = 2^{-1}n_k; \quad n_k = \begin{cases} 2\lambda_k(2-\lambda_k)^{-1} & \text{if } 2 > \lambda_k, \\ n_{k-1} + 1 & \text{if } 2 \leq \lambda_k, \end{cases}$$

$$\lambda_1 = (2^{-1} + r_0^{-1})^{-1}, \quad n_1 = r_0.$$

Since $\lambda_k \to \infty$, $k \to \infty$, it suffices to prove the estimates

$$\|(s,p)\|_{W^2_{\lambda_k}(\Omega)} \leq c(\lambda_k, \varepsilon), \quad k \geq 1. \tag{2.5}$$

Observe that a solution to the Neumann problem

$$\Omega: \ \Delta u = f, \quad \partial\Omega: \ \nabla u \cdot \mathbf{n} = g$$

satisfies the following inequality [9]:

$$\|u\|_{W^2_\lambda(\Omega)} \leq c(\Omega,\lambda)(\|f\|_{L_\lambda(\Omega)} + \|g\|_{W^1_\lambda(\Omega)} + \|u\|_{L_1(\Omega)}), \quad 1 < \lambda < \infty.$$

From here and Lemma 2.1 we conclude that a solution to (2.1) satisfies the inequalities

$$\|(s,p)\|_{W^2_{\lambda_k}(\Omega)} \leq c(k)(\|\nabla s\nabla p\|_{L_{\lambda_k}(\Omega)} + \|s\|_{W^1_{\lambda_k}(\Omega)} + 1). \tag{2.6}$$

It follows from Lemma 2.1 that

$$\|\nabla s\nabla p\|_{L_{\lambda_1}(\Omega)} \leq \|\nabla s\|_{L_2(\Omega)}\|\nabla p\|_{L_{r_0}(\Omega)} \leq c, \quad 1 < \lambda_1 < 2.$$

Therefore, (2.5) holds for $k = 1$. On assuming that (2.6) holds for k, we prove the inequality for $k+1$. From the embedding theorem we obtain

$$\|(s,p)\|_{W^1_{n_k}(\Omega)} \leq c(k)\|(s,p)\|_{W^2_{\lambda_k}(\Omega)} \leq c(k), \quad \|\nabla s\|_{L_{\lambda_{k+1}}(\Omega)} \leq c(k)\|s\|_{W^2_{\lambda_k}(\Omega)} \leq c(k).$$

Hölder's inequality implies

$$\|\nabla s \nabla p\|_{L_{\lambda_{k+1}}(\Omega)} \le c(k)\|\nabla s\|_{L_{n_k}(\Omega)}\|\nabla p\|_{L_{n_k}(\Omega)}.$$

Now, from the previous estimates and (2.6) we obtain (2.5) for $k+1$. The lemma is proven.

Fix a number $r > 2$ and consider the nonlinear operator $\Phi : [0,1] \times W_r^2(\Omega)^2 \to W_R^2(\Omega)^2$ defined by the following relations. Given $\tilde{s}, \tilde{p} \in W_r^2(\Omega)^2$, $\tau \in [0,1]$, the pair $(s,p) = \Phi(\tau, \tilde{s}, \tilde{p})$ is a solution to the following linear boundary value problem:

$$\begin{aligned} \Omega :\ & \varepsilon \Delta s = -\tau(a(\tilde{s})k\tau(\tilde{s}) \cdot \nabla\tilde{s}\nabla\tilde{p} + \mathbf{b}\cdot\nabla\tilde{s}), \\ \partial\Omega :\ & \varepsilon\nabla s \cdot \mathbf{n} + \gamma(s - \tau s_0) = 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Omega :\ & \Delta\tilde{p} = \Pi[k_\tau(s)^{-1}(\tau f - \nabla k_\tau(\tilde{s})\nabla\tilde{p})], \\ & \partial\Omega : \nabla p \cdot \mathbf{n} = 0, \\ & \langle p, 1\rangle = 0. \end{aligned} \tag{2.8}$$

Denote by $\Sigma \subset W_r^2(\Omega)^2$ the closed ball that consists of the couples of s and p satisfying the inequality $\|(s,p)\|_{W_r^2(\Omega)} \le c(r) + 1$. Consider a sequence $(\tau_n, s_n, p_n) \in [0,1] \times \Sigma$, $n \ge 1$. Since the embedding $W_r^2(\Omega) \to C^1(\Omega)$ is compact; dropping down to a subsequence, we may assume that it converges strongly in $C^1(\Omega)^2 \times [0,1]$. Therefore, the sequences $\tau_n(k(s_n)\nabla p_n + \mathbf{b})\nabla s_n$ and $\tau_n \operatorname{div}\big((1 - k(s_n))\nabla p_n\big)$ converge strongly in some space $L_\alpha(\Omega)$, $\alpha > 1$. From here and the a priori estimates for solutions to the Poisson equation we conclude that the sequence $\Phi(\tau_n, s_n, p_n)$ converges in $W_r^2(\Omega)$. Hence, the operator Φ is compact and continuous on $[0,1] \times \Sigma$.

Since $\Phi(0,s,p) = 0$, the mapping $I - \Phi(0,\cdot) : W_r^2(\Omega)^2 \to W_r^2(\Omega)^2$ is a homeomorphism.

If $(s,p) = \Phi(\tau, s, p)$ is a fixed point then (s,p) is a solution to the problem (2.1), (2.2). By Lemma 2.2, the pair (s,p) satisfies the inequality $\|(s,p)\|_{W_r^2(\Omega)^2} \le c(r)$ and $(s,p) \in \operatorname{int}\Sigma$. Therefore, the operator $\Phi(\tau,\cdot)$, $\tau \in [0,1]$, has no fixed point on the boundary of Σ.

By the Leray–Schauder fixed point theorem, the operator $\Phi(1,\cdot,\cdot)$ has a fixed point $(s_\varepsilon, p_\varepsilon) \in \Sigma$. It is clear that $(s_\varepsilon, p_\varepsilon)$ is a solution to the problem (1.10), (1.11).

To complete the proof of Theorem 1.1, we show that a weak limit point of the set of solutions to the problem (1.10), (1.11) is a measure-valued solution to (1.4). Consider a sequence $(s_\varepsilon, v_\varepsilon)$ of solutions to the problem (1.10), (1.11). Dropping down to a subsequence, we may assume that, for an arbitrary function $F : \mathbb{R}^3 \to \mathbb{R}$ satisfying the inequality $|F(s,q)| \le C(s)(1 + |q|^2)$, the sequence $F(s_\varepsilon, v_\varepsilon)$ converges weakly in $L_{r0/2}(\Omega)$ to some $F^* \in L_2(\Omega)$.

The version in [10] of the fundamental theorem on Young measures implies that there is a weakly measurable family of probability measures ν_x in $\mathbb{R}^3$ such that the equality

$$F^*(x) = \int_{\mathbb{R}^3} f(s, \mathbf{v})\, d\nu_x$$

holds almost everywhere in Ω. It follows that weak limits of the sequences $g(s_\varepsilon)$ and $g(s_\varepsilon)\mathbf{v}_\varepsilon$, $g \in C(\mathbb{R}^1)$, have representations (1.7) and (1.8).

Since the sequence $(s_\varepsilon, \mathbf{v}_\varepsilon)$ is uniformly bounded in $L_\infty(\Omega) \times L_{r_0}(\Omega)^2$, this measure satisfies conditions (a) and (b).

It remains to prove that ν_x is a measure-valued solution to (1.4). To this end, we choose an arbitrary smooth nondecreasing function $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ and a nonnegative function $\eta \in C^\infty(\Omega)$ with $\operatorname{spt}\eta \cap \partial\Omega \subset \partial\Omega^+$. Multiplying both sides of (1.10) by $\varphi(s_\varepsilon)$ and integrating over Ω, we obtain the following identity:

$$-\int_{\Omega} \mathbf{P}_\varepsilon \cdot \nabla\eta\, dx + \int_{\Omega} (\Phi(s_\varepsilon)\operatorname{div}\mathbf{b} + \Psi(s_\varepsilon)f)\,\eta\, dx - \int_{\partial\Omega^+} \left(\varepsilon\varphi(s_\varepsilon)\frac{\partial s_\varepsilon}{\partial n} + \Phi(s_\varepsilon)\mathbf{b}\cdot\mathbf{n}\right)\eta\, ds$$
$$= -\varepsilon\int_{\Omega} \varphi(s_\varepsilon)\nabla s_\varepsilon\cdot\nabla\eta\, dx - \varepsilon\int_{\Omega}\varphi'(s_\varepsilon)|\nabla s_\varepsilon|^2\eta\, dx,$$

$$\mathbf{P}_\varepsilon = \Psi(s_\varepsilon)\mathbf{v}_\varepsilon - \Phi(s_\varepsilon)\mathbf{b}.$$

Since φ' is a nonnegative function and η vanishes on $\partial\Omega\setminus\partial\Omega^+$, we conclude that

$$-\int_{\Omega} \mathbf{P}_\varepsilon\cdot\nabla\eta\, dx + \int_{\Omega}(\Phi(s_\varepsilon)\operatorname{div}\mathbf{b} + \Psi(s_\varepsilon)f)\cdot\eta\, dx$$
$$-\int_{\partial\Omega^+}\left(\varepsilon\varphi(s_\varepsilon)\frac{\partial s_\varepsilon}{\partial n} + \gamma\Phi(s_\varepsilon)\right)\eta\, ds \le -\varepsilon\int_{\Omega}\varphi(s_\varepsilon)\nabla s_\varepsilon\cdot\nabla\eta\, dx.$$

It follows from the boundary conditions

$$\varepsilon\frac{\partial s_\varepsilon}{\partial n} = -\gamma(s_\varepsilon - s_0)$$

and convexity of Φ that

$$\partial\Omega^+ :\ \varphi(s_\varepsilon)\varepsilon\frac{\partial s_\varepsilon}{\partial n} + \gamma\Phi(s_\varepsilon) = \gamma(\Phi(s_\varepsilon) - \varphi(s_\varepsilon)(s_\varepsilon - s_0))$$
$$= \gamma\left(\Phi(s_0) + \int_{s_0}^{s_\varepsilon}(\varphi(t) - \varphi(s_\varepsilon))\, dt\right) \le \gamma\Phi(s_0).$$

From here and the previous inequality we obtain

$$-\int_{\Omega}\mathbf{P}_\varepsilon\nabla\eta\, dx + \int_{\Omega}(\Phi(s_\varepsilon)\operatorname{div}\mathbf{b} + \Psi(s_\varepsilon)f)\,\eta\, dx - \int_{\partial\Omega^+}\gamma\Phi(s_0)\eta\, ds \le -\varepsilon\int_{\Omega}\nabla s_\varepsilon\nabla\eta\varphi\, dx.$$

It follows from the definition of a Young measure and the estimates for solutions to the problem (1.10), (1.11) that

$$\int_{\Omega}(\Phi(s_\varepsilon)\operatorname{div}\mathbf{b} + \Psi(s_\varepsilon)f)\eta\, dx \to -\int_{\Omega}\left(\int_{\mathbb{R}^1}(\Phi(\lambda)\operatorname{div}\mathbf{b}(x) + \Psi(\lambda)f(x))\, d\Lambda_x(\lambda)\right) dx,$$

$$\int_{\Omega} P_\varepsilon\cdot\nabla\eta\, dx \to \int_{\Omega}\left(-\int_{\mathbb{R}^1}\Psi(\lambda)\, d\mathbf{V}_{x,\lambda} + \int_{\mathbb{R}^1}\Phi(\lambda)\, d\Lambda_x(\lambda)\cdot\mathbf{b}(x)\right)\cdot\nabla\eta\, dx$$
$$= \int_{\Omega} P^*_\varphi\cdot\nabla\eta\, dx,\ \varepsilon\int_{\Omega}\nabla s_\varepsilon\cdot\nabla\eta\varphi\, dx \to 0$$

as $\varepsilon \to 0$. Passing to the limit in the previous inequality, we see that ν_x is a measure-valued solution to the first equation of (1.4).

Multiplying the second equation of (1.10) by an arbitrary smooth function $\xi(x)$ and integrating over Ω, we obtain the identity

$$\int_{\Omega} (\mathbf{v}_{\varepsilon} \cdot \nabla \xi + f\xi)\, dx = 0.$$

It remains to note that the weak limit of the sequence $\nabla p_{\varepsilon} = k^{-1}(s_{\varepsilon})\mathbf{v}_{\varepsilon}$ coincides with the function

$$x \to -\int_{\mathbb{R}^1} k^{-1}(\lambda) d\mathbf{V}_x(\lambda), \quad x \in \Omega,$$

and Theorem 1.1 is proven.

3. Proof of Theorem 1.2

1. Preliminaries. The proof of Theorem 1.2 bases on the compensated compactness principle and splits naturally into several parts. Take an arbitrary function $\varphi \in C^1$ and consider the sequence $\mathbf{P}_{\varepsilon,\varphi}$ of the flows defined by the equalities

$$\mathbf{P}_{\varepsilon,\varphi} = \Psi(s_{\varepsilon})\mathbf{v}_{\varepsilon} - \Phi(s_{\varepsilon})\mathbf{b}. \tag{3.1}$$

Here $(s_{\varepsilon},\ \mathbf{v}_{\varepsilon})$ are solutions to the problem (1.10), (1.11) and $(\Psi,\ \Phi)$ is the entropy pair corresponding to φ.

Lemma 3.1. *Under the conditions of Theorem 1.2, the set of the functions* $\operatorname{div} \mathbf{P}_{\varepsilon,\varphi}$, $\varepsilon > 0$, *is compact in* $H_2^{-1}(\Omega)$.

PROOF. Consider the sequence of the functionals $F_{\varepsilon} : \overset{\circ}{W}{}_r^1(\Omega) \to \mathbb{R}$ given by the formula

$$\langle F_{\varepsilon,\eta} \rangle = \int_{\Omega} \varepsilon\varphi'(s_{\varepsilon})|\nabla s_{\varepsilon}|^2 \eta\, dx + \int_{\Omega} (\Phi(s_{\varepsilon}) \operatorname{div} \mathbf{b} + \Psi(s_{\varepsilon}) f)\eta\, dx.$$

The estimates of Theorem 1.1 for the solutions to the problem (1.10), (1.11) imply the inequality

$$|\langle F_{\varepsilon}, \eta \rangle| \le C\|\eta\|_{L_{\infty}(\Omega)}.$$

Therefore, $\{F_{\varepsilon}\} \subset B$, where B is a bounded subset of the space $C^*(\Omega)$. From (1.10) we obtain the following identity which is valid for a smooth compactly-supported function η:

$$\int_{\Omega} \eta \operatorname{div} \mathbf{P}_{\varepsilon,\varphi}\, dx + \langle F_{\varepsilon,\varphi}, \eta \rangle = -\varepsilon \int_{\Omega} \varphi(s_{\varepsilon}) \nabla s_{\varepsilon} \nabla \eta\, dx.$$

Now, the estimates of Theorem 1.1 imply that the right-hand side satisfies the inequality

$$\left| \varepsilon \int_{\Omega} \varphi \nabla s_{\varepsilon} \nabla \eta\, dx \right| \le C\sqrt{\varepsilon}\|\eta\|_{H_1(\Omega)}.$$

Hence, the sequence $\operatorname{div} \mathbf{P}_{\varepsilon,\varphi} + F_{\varepsilon}$ converges to 0 in $H_1(\Omega)$. Therefore, $\{\operatorname{div} \mathbf{P}_{\varepsilon,\varphi} + F_{\varepsilon}\} \subset A$, where A is a compact subset of $H_{-1}(\Omega)$. On the other hand, the set $\{\mathbf{P}_{\varepsilon,\varphi}\}$ is bounded in $L_r(\Omega)$, $r > 2$. Hence, the functions $\operatorname{div} \mathbf{P}_{\varepsilon,\varphi}$ belong to some bounded set $C \subset W_r^{-1}(\Omega)$. Thus, we have $\{\operatorname{div} \mathbf{P}_{\varepsilon,\varphi}\} \subset (A - B) \cap C$. By Murat's lemma [11], the family of the functions $\operatorname{div} \mathbf{P}_{\varepsilon,\varphi}$ is precompact in $H_{-1}(\Omega)$ and the lemma is proven.

Take two arbitrary functions $\varphi_i \in C^1(\mathbb{R}^1)$, $i = 1, 2$. Now, Theorem 1.1 shows that the flows $\mathbf{P}_{\varepsilon,\varphi_i}$ converge weakly in $L_2(\Omega)$ to some vector-functions $\mathbf{P}_i^*(x)$ which have the representations

$$\mathbf{P}_i^*(x) = -\int_{\mathbb{R}} \Psi_i(\lambda) d\mathbf{V}_x(\lambda) + \int_{\mathbb{R}} \Phi_i(\lambda) d\Lambda_x(\lambda)\ \mathbf{b}(x). \tag{3.2}$$

The functions

$$Q_\varepsilon = \mathbf{P}_{\varepsilon,\varphi_1} \times \mathbf{P}_{\varepsilon,\varphi_2} = (\Psi_2(s_\varepsilon)\Phi_1(s_\varepsilon) - \Psi_1(s_\varepsilon)\Phi_2(s_\varepsilon))\mathbf{v}_\varepsilon \times \mathbf{b}$$

converge weakly in $L_{r/2}(\Omega)$, $r > 2$, to the function $Q^*(x)$ given by the formula

$$Q^*(x) = \int_R (\Phi_2(\lambda)\Psi_1(\lambda) - \Phi_1(\lambda)\Psi_2(\lambda))\, d\mathbf{V}_x(\lambda) \times \mathbf{b}(x). \tag{3.3}$$

Lemma 3.2. *Under the conditions of Theorem 1.2, the following equalities hold at almost all points of Ω:*

$$Q_*(x) = \mathbf{P}_1^*(x) \times \mathbf{P}_2^*(x), \quad w\text{-}\lim_{\varepsilon\to 0}(\nabla p_\varepsilon \cdot \mathbf{P}_{\varepsilon,\varphi_i}) = w\text{-}\lim_{\varepsilon\to 0} \nabla p_\varepsilon \cdot \mathbf{P}_i^*. \tag{3.4}$$

PROOF. It follows from Lemma 2.1 that the set of the functions $\operatorname{rot} \mathbf{P}_{\varepsilon,\varphi_1}^{\perp} = \operatorname{div} \mathbf{P}_{\varepsilon,\varphi_1}$ is compact in $H_{-1}(\Omega)$. Theorem 1.1 implies that the sequences ∇p_ε and $\mathbf{P}_{\varepsilon,\varphi_i}$ are bounded in $L_r(\Omega)$. From the curl-divergence lemma we obtain

$$w\text{-}\lim_{\varepsilon\to 0}(\mathbf{P}_{\varepsilon,\varphi_1} \times \mathbf{P}_{\varepsilon,\varphi_2}) = w\text{-}\lim_{\varepsilon\to 0}(\mathbf{P}_{\varepsilon,\varphi_1}^{\perp} \cdot \mathbf{P}_{\varepsilon,\varphi_2})$$
$$= w\text{-}\lim_{\varepsilon\to 0} \mathbf{P}_{\varepsilon,\varphi_1}^{\perp} \cdot w\text{-}\lim_{\varepsilon\to 0} \mathbf{P}_{\varepsilon,\varphi_2} = w\text{-}\lim_{\varepsilon\to 0} \mathbf{P}_{\varepsilon,\varphi_1} \times w\text{-}\lim_{\varepsilon\to 0} \mathbf{P}_{\varepsilon,\varphi_2},$$
$$w\text{-}\lim_{\varepsilon\to 0}(\nabla p_\varepsilon \cdot \mathbf{P}_{\varepsilon,\varphi_i}) = w\text{-}\lim_{\varepsilon\to 0} \nabla p_\varepsilon \cdot w\text{-}\lim_{\varepsilon\to 0} \mathbf{P}_{\varepsilon,\varphi_i}$$

which completes the proof.

Consider a collection of functions $f_{km}(\alpha, s, \mathbf{v})$, $k = 1, 2$, $m \geq 1$, $\alpha, s \in \mathbb{R}^1$, $\mathbf{v} \in \mathbb{R}^2$, which belong to the first Baire class, are left- or right-continuous in α, and satisfy the inequalities $|f_{km}(\alpha, s, \mathbf{v})| \leq c(\alpha, s)(1 + |\mathbf{v}|^k)$. Suppose that $g(x, f_{k,m})$ is a continuous function such that $|g(f_{k,m})| \leq c(1 + |f_{1m}|^2 + |f_{2m}|)$. Put

$$\eta_{k,m}(\alpha, x) = \int_{\mathbb{R}^1} f_{km}(\alpha, s, q)\, d\nu_x, \quad G(\alpha, x) = g(\eta_{k,m}(\alpha, x)).$$

Lemma 3.3. *Under the above assumptions, the mapping $x \to G(\alpha, x)$ is measurable for every $\alpha \in \mathbb{R}^1$. If the inequality*

$$\int_\Omega \xi(x) G(\alpha, x)\, dx \leq 0$$

holds for arbitrary $\alpha \in \mathbb{R}^1$ and $0 \leq \xi \in C(\Omega)$, then there is a Borel set $E \subset \Omega$ such that $\operatorname{mes}(\Omega \setminus E) = 0$ *and $G(\alpha, x) \leq 0$ for an arbitrary $\alpha \in \mathbb{R}^1$ and $x \in E$.*

PROOF. Since $f_{k,m}$ are Baire functions, there is a sequence f_{kmj} of continuous functions having the same bounds as $f_{k,m}$ and such that $f_{kmj} \to f_{k,m}$ pointwise on $\mathbb{R}^2 \times \mathbb{R}^2$. From the Lebesgue theorem we obtain

$$\eta_{k,m}(\alpha, x) = \lim_{j\to\infty} \int_{\mathbb{R}^1} f_{km}(\alpha, s, q)\, d\nu_x.$$

Therefore, $\eta_{k,m}(\alpha, \cdot)$ and $G(\alpha, \cdot)$ are measurable functions, for they are pointwise limits of sequences of measurable functions. Take a countable everywhere dense set $\{\alpha_i\}_{i\geq 1} \subset \mathbb{R}^1$. By the Luzin theorem, for every α_i, there is a Borel set $\Omega_{im} \subset \Omega$ such that the function $G(\alpha_i, \cdot)$ is continuous on Ω_{im} and $\text{mes}(\Omega \setminus \Omega_{im}) < m^{-1}2^{-i}$. From the conditions of the lemma we obtain $G(\alpha_i, x) \le 0$ for $x \in \Omega_{im}$. It follows that $G(\alpha_i, x) \le 0$ for arbitrary α_i and $x \in E = \cup_m \cap_i \Omega_{im}$. Since G is left- or right-continuous in α, this equality holds for every $\alpha \in \mathbb{R}^1$ and the lemma is proven.

2. Proof of Theorem 1.2. We begin with proving item (*i*) of the theorem. Fix two arbitrary numbers $\alpha < \beta$ and a function $\omega \in C^\infty(\mathbb{R})$ satisfying the conditions

$$\omega(-s) = \omega(s), \quad \omega(s) = 0 \text{ for } |s| > 1, \quad \int_{-\infty}^{\infty} \omega\, dx = 1,$$

and set

$$\varphi_{1,n}(s) = n\omega(n(s-\alpha)+1), \quad \varphi_{2,n}(s) = n\omega(n(s-\beta)+1).$$

Let Ψ_{in}, Φ_{in} be the entropy pair corresponding to the function φ_{in}. Observe that the sequences Ψ_{in} and Ψ_{2n} converge pointwise to the functions $a(\alpha)H(s-\alpha)$ and $a(\beta)H(s-\beta)$ and the sequences Φ_{1n} and Φ_{2n} converge pointwise to the functions $H(s-\alpha)$ and $H(s-\beta)$.

We denote by $\mathbf{P}_{\varepsilon,in}$ the flow defined by (3.1) with Ψ and Φ replaced with Φ_{in} and Ψ_{in}. We also denote by $\mathbf{P}^*_{in}$ and Q^*_n the weak limits of $\mathbf{P}_{\varepsilon,in}$ and their vector products given by (3.1) and (3.2). Inserting Φ_{in} and Ψ_{in} into the identity

$$Q^*_n(x) = \mathbf{P}^*_{1n}(x) \times \mathbf{P}^*_{2n}(x)$$

and passing to the limit as $n \to \infty$, we obtain the following equality:

$$\int_\Omega \Big(\int_{[\beta,\infty)} (a(\alpha)-a(\beta))d\mathbf{V}_x(\lambda) \times \mathbf{b}(x) - \mathbf{R}_x(\alpha) \times \mathbf{R}_x(\beta) \Big) \xi(x)\, dx = 0.$$

Here ξ is an arbitrary continuous function and the vector-function $\mathbf{R}_x(s)$ depends on $x \in \Omega$ according to the formula

$$\mathbf{R}_x(s) = -a(s) \int_{[s,\infty)} d\mathbf{V}_x(\lambda) + \int_{[s,\infty)} d\Lambda_x(\lambda)\mathbf{b}(x).$$

Putting

$$\pm G = \int_{\mathbb{R}^3} H(s-\beta)(a(\alpha)-a(\beta))\mathbf{v} \times \mathbf{b}(x) d\nu_x - \mathbf{R}_x(\alpha) \times \mathbf{R}_x(\beta)$$

and applying Lemma 3.3, we conclude that the equalities

$$\int_{[\beta,\infty)} (a(\alpha)-a(\beta))d\mathbf{V}_x(\lambda) \times \mathbf{b}(x) - \mathbf{R}_x(\alpha) \times \mathbf{R}_x(\beta) = 0$$

hold at almost every point of Ω for arbitrary α and β.

We simply write $\mathbf{R}(s)$, $\mathbf{V}(s)$, and $\Lambda(s)$ instead of $R_x(s)$, $\mathbf{V}_x(s)$, and $\Lambda_x(s)$, unless confusion is possible. We can rewrite the previous relations in the short form

$$(a(\beta)-a(\alpha))\mathbf{V}(\beta) \times \mathbf{b} = \mathbf{R}(\alpha) \times \mathbf{R}(\beta), \quad \mathbf{R}(s) = a(s)\mathbf{V}(s) - \Lambda(s)\mathbf{b}. \tag{3.5}$$

Denote by E_x the set that consists of all points β such that $\mathbf{V}_x(\beta) \times \mathbf{b}(x) \neq 0$.

Suppose that $E_x \neq \varnothing$. Equalities (3.5) imply $\mathbf{R}(\alpha) \times \mathbf{R}(\beta) \neq 0$ for all $\alpha < \beta$, $\beta \in E_x$. Next, we prove that $(-\infty, \beta] \subset E_x$ for all $\beta \in E_x$.

Since the vector-function $\mathbf{V}(\alpha)$ is left-continuous, we can choose $\gamma < \beta$ such that $\mathbf{R}(\gamma) \times R(\beta) \neq 0$. It follows from (3.5) that the equalities

$$\mathbf{R}(\alpha) \times \mathbf{R}(\beta) = (a(\beta) - a(\alpha))\mathbf{V}(\beta) \times \mathbf{b}, \quad \mathbf{R}(\alpha) \times \mathbf{R}(\gamma) = (a(\gamma) - a(\alpha))\mathbf{V}(\gamma) \times \mathbf{b}$$

hold for all $\alpha < \gamma < \beta$. We may consider these relations as a system of linear algebraic equations in the components of the vector $\mathbf{R}(\alpha)$. Since $\mathbf{R}(\beta) \times \mathbf{R}(\gamma) \neq 0$, this system is nondegenerate. It is clear that its unique solution has the form $\mathbf{R}(\alpha) = a(\alpha)\mathbf{f} + \mathbf{g}$, where the vectors $\mathbf{f}$ and $\mathbf{g}$ are independent of α and satisfy the equations

$$\mathbf{f} \times \mathbf{R}(\alpha) = -\mathbf{V}(\beta) \times \mathbf{b}, \quad \mathbf{f} \times \mathbf{R}(\gamma) = -\mathbf{V}(\gamma) \times \mathbf{b},$$
$$\mathbf{g} \times \mathbf{R}(\beta) = a(\beta)\mathbf{V}(\beta) \times \mathbf{b}, \quad \mathbf{g} \times \mathbf{R}(\gamma) = a(\gamma)\mathbf{V}(\gamma) \times \mathbf{b}.$$

Since the function a is monotone and $\mathbf{V}(\beta) \times \mathbf{b}$, $\mathbf{V}(\gamma) \times \mathbf{b}$ differ from zero, we conclude that the vectors $\mathbf{f}$ and $\mathbf{g}$ are linearly independent.

Therefore, the vectors $\mathbf{R}(\alpha)$ are linearly independent for $\alpha < \beta$. From here and the equality

$$(a(\alpha) - a(\alpha'))\mathbf{V}(\alpha') \times \mathbf{b} = \mathbf{R}(\alpha') \times \mathbf{R}(\alpha), \quad \alpha' < \alpha < \beta,$$

we obtain $\mathbf{R}(\alpha') \times \mathbf{b} \neq 0$. Hence, we conclude that $\alpha' \in E_x$; therefore, $(-\infty, \beta) \subset E_x$ for every $\beta \in E_x$. This yields the inclusion $(-\infty, s^*) \subset E_x$, $s^* = \sup E_x$.

Our next claim is that the function $\mathbf{V}(\alpha) \times \mathbf{b}$ is constant on the set E_x. Take arbitrary elements $\alpha_i \in \mathbb{R}^1$ satisfying the inequalities $\alpha_1 < \alpha_2 < \gamma < \beta$. Relations (3.5) imply

$$(a(\alpha_1) - a(\alpha_2))\mathbf{f} \times \mathbf{g} = -(a(\alpha_1) - a(\alpha_2))\mathbf{V}(\alpha_2) \times \mathbf{b}.$$

Since the vectors $\mathbf{f}$ and $\mathbf{g}$ are linearly independent and the function a is strictly monotone, we conclude that $\mathbf{V}(\alpha_2) \times \mathbf{b}$ is independent of α_2. Therefore, the function $\mathbf{V}(\alpha_2) \times \mathbf{b} \neq 0$ is constant on E_x. Due to left-continuity of $\mathbf{V}(\alpha) \times \mathbf{b}$, we have $s^*(x) \equiv \sup E_x \in E_x$ and $E_x = (-\infty, s^*(x)]$. It follows from the definition of a Young measure and Theorem 1.1 that

$$\mathbf{V}(\alpha) = \mathbf{v}^* \equiv w\text{-}\lim_{\varepsilon \to 0} \mathbf{v}_\varepsilon \text{ for } \alpha < -M,$$

where $M = \sup_\varepsilon \|s_\varepsilon\|_{L_\infty(\Omega)}$. From here and what was proven above we obtain

$$\mathbf{V}(\alpha) \times \mathbf{b} = \mathbf{v}^* \times \mathbf{b} \text{ for } \alpha \leq s^*, \quad \mathbf{V}(\alpha) \times \mathbf{b} = 0 \text{ for } \alpha > s^*. \tag{3.6}$$

Our next goal is to derive an explicit formula for the vector-function $\mathbf{V}(\alpha)$.

Let $(\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{R}^2$ such that $\mathbf{e}_1 = |\mathbf{b}|^{-1}\mathbf{b}$ and $\mathbf{e}_1 \times \mathbf{e}_2 = 1$. Denote by $V_i(\alpha)$ the components of the vector $\mathbf{V}(\alpha)$. From the above remark we obtain

$$V_2(\alpha) = v_2^* \equiv -\frac{\mathbf{v}^* \times \mathbf{b}}{|\mathbf{b}|} \neq 0 \text{ for } \alpha \leq s^*,$$
$$V_2(\alpha) = 0 \text{ for } \alpha > s^*.$$

From here and the definition of $\mathbf{R}(\alpha)$ we infer that this vector-function has the representation $\mathbf{R}(\alpha) = q(\alpha)\mathbf{e}_1 + a(\alpha)v_2^*\mathbf{e}_2$ for all $\alpha \leq s^*$. Inserting this representation into (3.5), we obtain the equality

$$(q(\alpha)a(\beta) - q(\beta)a(\alpha))v_2^* = (a(\beta) - a(\alpha))(-|\mathbf{b}|v_2^*),$$

which is valid for all $\alpha < \beta \leq s^*$. Hence, we conclude that $q(\alpha) = -|\mathbf{b}| + Ca(\alpha)$, where C is independent of $\alpha < s^*$. Since $\mathbf{R}(\alpha) = a(\alpha)\mathbf{v}^* - \mathbf{b}$ for $\alpha < -M$, it follows that $C = v_1^*$. We have thus obtained the formula

$$\mathbf{R}(\alpha) = a(\alpha)\mathbf{v}^* - \mathbf{b} \text{ for } \alpha \leq s^*, \quad \mathbf{R}(\alpha) = 0 \text{ for } s > s^*,$$

which, together with (3.5), implies

$$\mathbf{V}(\alpha) = \mathbf{v}^* + \frac{\Lambda(\alpha) - 1}{a(\alpha)}\mathbf{b} \;\text{ for }\; \alpha \le s^*,$$
$$\mathbf{V}(\alpha) = \frac{\Lambda(\alpha)}{a(\alpha)}\mathbf{b} \;\text{ for }\; \alpha > s^*. \tag{3.7}$$

If $E_x = \varnothing$ then $\mathbf{V}_x(\lambda) \times \mathbf{b} = 0$ for all $\lambda \in \mathbb{R}^1$. It follows that there exists a function $\rho_x(\lambda)$ such that

$$\mathbf{V}_x(\lambda) = \rho_x(\lambda)\mathbf{b}. \tag{3.8}$$

To complete the proof of item (i) of Theorem 1.2, we show that $E_x = \varnothing$ for $x \in \Omega_0$ and $E_x \neq \varnothing$ for $x \in \Omega_1$. If $E_x = \varnothing$ then $\mathbf{V}_x(\lambda) \times \mathbf{b}(x) = 0$ for all $\lambda \in \mathbb{R}^1$. Since $\mathbf{V}_x(\lambda) = v^*(x)$ for every $\lambda \le -M$, we have $\mathbf{v}^*(x) \times \mathbf{b}(x) = 0$ and $x \in \Omega_0$. If $E_x \neq \varnothing$ then $\mathbf{v}^*(x) \times \mathbf{b}(x) = \mathbf{V}_x(-M) \times \mathbf{b}(x) \neq 0$ by (3.6). Thus, we obtain $x \in \Omega_1$ and come to the desired assertion.

PROOF OF ITEM (ii). By Theorem 1.1, the definition of measure-valued solutions, and item (i) of Theorem 1.2, the following equalities hold almost everywhere in Ω_1:

$$w\text{-}\lim_{\varepsilon \to 0} \mathbf{P}_{\varepsilon,\varphi} = \int_{\mathbb{R}^1} \Phi(\lambda)\, d\Lambda_x(\lambda) \cdot \mathbf{b}(x) - \int_{\mathbb{R}^1} \Psi(\lambda)\, d\mathbf{V}_{x,\lambda}, \tag{3.9}$$

where

$$\mathbf{V}_x(\lambda) = \frac{\Lambda_x(\lambda) - H(s^*(x) - \lambda)}{a(\lambda)}\mathbf{b}(x) + H(s^*(x) - \lambda)\mathbf{v}^*(x). \tag{3.10}$$

Since $\Lambda_x(\lambda) = 0$ for $\lambda > M$ and $\Lambda_x(\lambda) = 1$ for $\lambda < -M$, we have

$$\Lambda_x(\lambda) - H(s^*(x) - \lambda) = 0 \quad \text{for } |\lambda| > M.$$

Integration by parts yields

$$\int_{\mathbb{R}^1} \Psi(\lambda)\, d\left(\frac{\Lambda_x(\lambda)}{a(\lambda)} - \frac{H(s^*(x) - \lambda)}{a(\lambda)}\right)$$
$$= -\int_{\mathbb{R}^1} \varphi(\lambda)(\Lambda_x(\lambda) - H(s^*(x) - \lambda))\, d\lambda = \Phi(s^*(x)) + \int_{\mathbb{R}^1} \Phi(\lambda)\, d\Lambda_x(\lambda). \tag{3.11}$$

It is easy to see that

$$\int_{\mathbb{R}^1} \Psi(\lambda)\, dH(s^*(x) - \lambda) = -\Psi(s^*(x)). \tag{3.12}$$

Combining (3.10), (3.11), and (3.12) we arrive at the relation

$$-\int_{\mathbb{R}^1} \Psi(\lambda)\, d\mathbf{V}_{x,\lambda} = \Psi(s^*(x))\mathbf{v}^*(x) - \Phi(s^*(x))\mathbf{b}(x) - \int_{\mathbb{R}^1} \Phi(\lambda)\, d\Lambda_x(\lambda)\, \mathbf{b}.$$

Inserting the right-hand side of the above equality into (3.9), we obtain

$$w\text{-}\lim_{\varepsilon \to 0} \mathbf{P}_{\varepsilon,\varphi} = \Psi(s^*(x))\mathbf{v}^*(x) - \Phi(s^*(x))\mathbf{b}(x)$$

and the assertion is proven.

4. Proof of Theorem 1.3

1. Preliminaries. First we give an equivalent statement of Condition S. Given $\alpha \in (-M, M)$, we denote by $\mathscr{P}_\alpha^+$ ($\mathscr{P}_\alpha^-$) the sets of probability measures concentrated on the intervals $[\alpha, M]$ ($[-M, \alpha]$). We consider the functions $\Gamma(\lambda)$ and $N(\lambda)$ defined by (1.14) and denote by $\lambda = Z(z)$ the inverse of $z = \Gamma(\lambda)$. Recall that $\Gamma(\lambda)$ and Z are smooth and strictly monotone functions.

Proposition 4.1. *The following assertions are equivalent:*
(a) *The function $f_\alpha(z) = N(Z(z))$ is strictly p-concave at the point $z_\alpha = \Gamma(\alpha)$.*
(b) *The following inequality holds:*

$$\sup_{\mu \in \mathscr{P}_\alpha^\pm} \left[(\Gamma(\alpha))^2 + \int_{[-M,M]} N\, d\mu - \left(\int_{[-M,M]} \Gamma\, d\mu \right)^2 \right] \leq 0. \tag{4.1}$$

REMARK. Put

$$J(\mu) = \Gamma(\alpha)^2 + \int_{[-M,M]} N\, d\mu - \left(\int_{[-M,M]} \Gamma\, d\mu \right)^2.$$

For $\mu \in \mathscr{P}_\alpha^-$ with $m^-(\lambda) = \mu((-\infty, \lambda])$ we have $m^-(\lambda) = 0$ for $\lambda < -M$ and $m^-(\lambda) = 1$ for $\lambda \geq \alpha$; and integration by parts yields

$$\int_{(-\infty,\alpha]} K(\lambda)\Gamma(\lambda)\, d\left(\frac{m^-(\lambda)}{a(\lambda)} \right) - \left(\int_{(-\infty,\alpha]} \Gamma\, d\mu \right)^2 = J(\mu). \tag{4.2}$$

For $\mu \in \mathscr{P}_\alpha^+$ with $m^+(\lambda) = -\mu([\lambda, \infty))$ we have

$$\int_{[\alpha,\infty)} K(\lambda)\Gamma(\lambda)\, d\left(\frac{m^+(\lambda)}{a(\lambda)} \right) - \left(\int_{[\alpha,\infty)} \Gamma\, d\mu \right)^2 = J(\mu).$$

The proof of Proposition 4.1 bases on the following

Lemma 4.1. *Each of the variational problems*

$$J(\mu^-) = \sup_{\mu \in \mathscr{P}_\alpha^-} J(\mu), \ \mu^- \in \mathscr{P}_\alpha^-, \quad J(\mu^+) = \sup_{\mu \in \mathscr{P}_\alpha^-} J(\mu), \ \mu^- \in \mathscr{P}_\alpha^-$$

has at least one solution which satisfies the following extremal relations:

$$T^\pm(\lambda) \equiv N(\lambda) - 2\rho^\pm \Gamma(\lambda) = \text{const}, \quad \lambda \in \operatorname{spt} \mu^\pm, \tag{4.3}$$

where

$$\rho^\pm = \int_{[-M,M]} \Gamma\, d\mu^\pm.$$

PROOF. We give the proof only in the case of $\mu \in \mathscr{P}_\alpha^-$, leaving the other case to the reader. The existence of a measure μ^- is a consequence of weak continuity of the functional J and weak compactness of the set $\mathscr{P}_\alpha^- \subset C^*[-M, M]$. If the support of the measure μ^- consists of a single point then (4.3) is trivial. Assume that $\operatorname{spt} \mu^-$ contains more than one point and prove that the equality $T(\lambda) = \text{const}$ holds μ^--a.e. Suppose to the contrary that $T^-(\lambda) \neq \text{const}$ μ^--a.e. on the segment $[-M, M]$. It follows that there exist compact sets B_i, $i = 1, 2$, such that

$$\mu^-(B_i) > 0, \quad \sup_{\lambda \in B_1} T^-(\lambda) < \inf_{\lambda \in B_2} T^-(\lambda). \tag{4.4}$$

We put $g(\lambda) = (-1)^i \mu^-(B_i)^{-1}$ for $\lambda \in B_i$ and $g(\lambda) = 0$ for $\lambda \in [-M, M] \setminus B_i$. Note that the equalities

$$\langle \mu_\varepsilon, \varphi \rangle = \int\limits_{[-M,M]} \varphi(\lambda)(1 + \varepsilon g(\lambda))\, d\mu^-, \quad \varphi \in \overset{\circ}{C}[-M, M],\ |\varepsilon| \leq \{\sup_{\lambda \in [-M,M]} |g(\lambda)|\}^{-1}$$

define a family of probability measures $\mu_\varepsilon \in \mathscr{P}_\alpha^-$. It is clear that $J(\mu_\varepsilon) - J(\mu^-) \leq 0$, which implies the relations

$$\lim_{\varepsilon \to \pm 0} |\varepsilon|^{-1}(J(\mu_\varepsilon) - J(\mu^-)) \equiv \pm \int\limits_{[-M,M]} T^-(\lambda) g(\lambda)\, d\mu^- \leq 0.$$

From this fact and the definition of g we obtain the equality

$$\frac{1}{\mu^-(B_1)} \int\limits_{B_1} T^-(\lambda)\, d\mu^- = \frac{1}{\mu^-(B_2)} \int\limits_{B_2} T^-(\lambda)\, d\mu^-$$

which contradicts to (4.4). Therefore, the function $T^-(\lambda)$ is equal to a constant μ^--a.e. Since the function $T^-(\lambda)$ is continuous, the equality $T^-(\lambda) = \text{const}$ holds on the support of the measure μ^- and the lemma is proven.

We turn to proving Proposition 4.1 and establish the implication $(a) \Longrightarrow (b)$. Examine (4.1) in the case $\mu \in \mathscr{P}_\alpha^-$. Fix an arbitrary point $\alpha \in (-M, M)$ and suppose that the function $f_\alpha(z) = N(Z(z))$ is p-concave at $z_\alpha = \Gamma(\lambda)$. We set

$$\lambda_1 = \inf \operatorname{spt} \mu^-, \quad \lambda_2 = \sup \operatorname{spt} \mu^-, \quad z_j = \Gamma(\lambda_j), \quad z^* = \rho^- \equiv \int\limits_{[-M,M]} \Gamma(\lambda)\, d\mu^-.$$

Since Γ is a strictly monotone and μ^- is a probability measure concentrated on the segment $[-M, \alpha]$, we have

$$z^* = \tau z_1 + (1 - \tau) z_2, \ \tau \in [0, 1], \quad z_\alpha \notin (z_1, z_2).$$

Simple calculations show that

$$f_\alpha(z_\alpha) = N(\alpha) = 0, \quad f'_\alpha(z_\alpha) = N'(\alpha)(\Gamma'(\alpha))^{-1} = 2z_\alpha.$$

Thus,

$$f_\alpha(z_\alpha) = p_\alpha(z_\alpha), \quad f'_\alpha(z_\alpha) = p'_\alpha(z_\alpha),$$

where $p_\alpha(z) = z^2 - z_\alpha^2$. With this notations, we can rewrite $J(\mu^-)$ as

$$J(\mu^-) = \int\limits_{[-M,M]} N\, d\mu^- - p_\alpha(z^*). \tag{4.5}$$

It follows from Lemma 4.1 that

$$T^-(\lambda) \equiv N(\lambda) - 2z^*\Gamma(\lambda) = c = \text{const}, \quad \lambda \in \operatorname{spt} \mu^-. \tag{4.6}$$

From this and the equalities $N(\lambda_i) = f_\alpha(z_i)$, $z_i = \Gamma(\lambda_i)$, we obtain $f_\alpha(z_1) - f_\alpha(z_2) = 2z^*(z_1 - z_2)$. Therefore, the points $(z_i, f_\alpha(z_i))$ belong to some line l such that $l \parallel \operatorname{Tan}_{z^*} p_\alpha$.

On the other hand, the equalities $N(\lambda_i) = f_\alpha(z_i)$ imply

$$c = f_\alpha(z_i) - 2z^* z_i = \tau f_\alpha(z_1) + (1 - \tau) f_\alpha(z_2) - 2z^{*2}.$$

Inserting these relations into (4.6), we obtain

$$N(\lambda) = \tau f_\alpha(z_1) + (1-\tau) f_\alpha(z_2) + 2z^*\Gamma(\lambda) - 2z^{*2}.$$

Integrating both sides of the above equality with respect to the measure μ^-, we find that

$$\int\limits_{[-M,M]} N\, d\mu^- = \tau f_\alpha(z_1) + (1-\tau) f_\alpha(z_2).$$

Combining this relation with (4.5), we obtain

$$J(\mu^-) = \sup_{\mu\in\mathscr{P}_\alpha^-} J(\mu) = \tau f_\alpha(z_1) + (1-\tau) f_\alpha(z_2) - p_\alpha(z^*).$$

Since the function f_α is strictly p-concave at z_α, we conclude that $J(\mu^-) \le 0$ and the equality holds if and only if $z_1 = z_2 = z^*$. Note that $z_1 = z_2 = z^*$ implies $\operatorname{spt}\mu^- = \{\alpha\}$ and the assertion is proven. The same proof remains valid for the case $\mu \in \mathscr{P}_\alpha^+$.

Prove the implication $(b) \Longrightarrow (a)$. Fix an arbitrary point $\alpha \in (-M, M)$ and assume that inequalities (4.1) hold at this point. To obtain a contradiction, suppose that the function $f_\alpha(z)$ is not strictly p-concave at α. As was mentioned above, the parabola $p_\alpha : y = z^2 - z_\alpha^2$ meets the graph of the function f_α at the point $(z_\alpha f_\alpha(z_\alpha))$; and the graphs of the functions p_α and f_α have a common tangent at this point.

By hypothesis, there exist $z_1 \le z_2$ such that $z_\alpha \notin (z_1, z_2)$, $z^* = \tau z_1 + (1-\tau)z_2$, $\tau \in [0,1]$, and

$$f_\alpha(z_1) - f_\alpha(z_2) = p'_\alpha(z^*)(z_1 - z_2), \quad p_\alpha(z^*) < \tau f_\alpha(z_1) + (1-\tau) f_\alpha(z_2). \tag{4.7}$$

We put $\lambda_i = Z(z_i)$ and $\mu = \tau\delta(\lambda - \lambda_1) + (1-\tau)\delta(\lambda - \lambda_2)$. Since the function $Z(z)$ is strictly monotone, we have either $(\lambda_1, \lambda_2) \subset [-M, \alpha]$ or $(\lambda_1, \lambda_2) \subset [\alpha, M]$; hence $\mu \in \mathscr{P}_\alpha^- \cup \mathscr{P}_\alpha^+$. The equalities $f_\alpha(z_i) = N(\lambda_i)$, $z_\alpha = \Gamma(\alpha)$, and $z^* = \tau\Gamma(\lambda_1) + (1-\tau)\Gamma(\lambda_2)$ imply that

$$p_\alpha(z^*) = \Big(\int\limits_{[-M,M]} \Gamma\, d\mu\Big)^2 - \Gamma(\alpha)^2, \quad \tau f_\alpha(z_1) + (1-\tau) f_\alpha(z_2) = \int\limits_{[-M,M]} N(\lambda)\, d\mu.$$

From here and (4.7) we obtain

$$\Gamma(\alpha)^2 + \int\limits_{[-M,M]} N(\lambda)\, d\mu - \Big(\int\limits_{[-M,M]} \Gamma\, d\mu\Big)^2 > 0,$$

which contradicts (4.2).

2. Proof of Theorem 1.3. First we prove that the sequence s_ε converges strongly in the space $L_1(\Omega_1)$. It suffices to establish that the equalities $\Lambda_x(\lambda) = H(s^*(x) - \lambda)$ and $\chi_x(\lambda) = H(\lambda - s^*(x))$ hold almost everywhere on Ω_1. Since the variation of the function $\Lambda_x(\lambda)$ equals 1, we only need to show that the functions $\Lambda_x(\lambda)$ and $(\chi_x(\lambda))$ are constant on the intervals $(-\infty, s^*(x))$ and $(s^*(x), \infty)$.

Consider the first case. Introduce the sequence Ψ_n, Φ_n of entropy pairs as follows:

$$\Psi_n(\lambda) = -\int_\lambda^M (h(n(\alpha - s))K)'\, ds, \quad \Phi_n(\lambda) = -\int_\lambda^M a^{-1}(s)(h(n(\alpha - s))K)'\, ds.$$

Here $h \in C^1(\mathbb{R}^1)$ is a smooth nondecreasing function such that $h(s) = 0$ for $s < -1$ and $h(s) = 1$ for $s \ge 0$. Let $\mathbf{P}_{\varepsilon,n} = \Psi_n(s_\varepsilon)\mathbf{v}_\varepsilon - \Phi_n(s_\varepsilon)\mathbf{b}$ be the flow corresponding to this entropy pair.

Observe that the sequences Ψ_n and Φ_n converge pointwise to the functions $\Psi(\lambda) = K(\lambda)H(\alpha-\lambda)$ and $\Phi(\lambda) = \Gamma(\lambda)H(\alpha-\lambda)$.

We conclude from the definition of Ψ_n that $\Psi_n(s) \geq K(s)H(\alpha - s)$. Since $\nabla p_\varepsilon = -K(s_\varepsilon)\mathbf{v}_\varepsilon$, we have

$$|H(\alpha - s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 = H(\alpha - s_\varepsilon)|\nabla p_\varepsilon|^2 \leq -\Psi_n(s_\varepsilon)\nabla p_\varepsilon v_\varepsilon,$$

hence

$$|H(\alpha - s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 \leq -\nabla p_\varepsilon \cdot \mathbf{P}_{\varepsilon,n} - \Phi_n(s_\varepsilon)\nabla p_\varepsilon \cdot \mathbf{b} = \Phi_n(s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon \cdot \mathbf{b} - \nabla p_\varepsilon \cdot \mathbf{P}_{\varepsilon,n}. \quad (4.8)$$

By Lemma 2.1 and item (*ii*) of Theorem 1.2, the relations

$$w\text{-}\lim_{\varepsilon\to 0}(\nabla p_\varepsilon \cdot \mathbf{P}_{\varepsilon,n}) = \nabla p^* \cdot w\text{-}\lim_{\varepsilon\to 0}\mathbf{P}_{\varepsilon,n} = \nabla p^* \cdot (\Psi_n(s^*)\mathbf{v}^* - \Phi_n(s^*)\mathbf{b})$$

hold almost everywhere on Ω_1. Passing to the limit in (4.8) we obtain the inequality

$$w\text{-}\lim_{\varepsilon\to 0}|H(\alpha - s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 \leq w\text{-}\lim_{\varepsilon\to 0}\Phi_n(s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon \cdot \mathbf{b} - \nabla p^* \cdot (\Psi_n(s^*)\mathbf{v}^* - \Phi_n(s^*)\mathbf{b}). \quad (4.9)$$

The definition of the vector-function $\mathbf{U}_x(\lambda)$ implies

$$\begin{aligned} w\text{-}\lim_{\varepsilon\to 0}|H(\alpha - s_\varepsilon)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 &= \int_{\mathbb{R}^3} K^2 H(\alpha - s)|q|^2\,d\nu_x \\ &= w\text{-}\lim_{\varepsilon\to 0}\Phi_n(s_\varepsilon)K(s_\varepsilon)v_\varepsilon = \int_{\mathbb{R}^1} K(\lambda)\Phi_n(\lambda)\,dU_x(\lambda). \end{aligned} \quad (4.10)$$

Inserting these relations into (4.9) and applying Lemma 3.3, we arrive at the inequality

$$\int_{(-\infty,\alpha]\times\mathbb{R}^2} K^2|q|^2\,d\nu_x \leq \int_{\mathbb{R}^1}\Phi_n(\lambda)K(\lambda)dU_x(\lambda)\cdot\mathbf{b} - \nabla p^*(\Psi_n(s^*)\mathbf{v}^* - \Phi_n(s^*)\mathbf{b}) \quad (4.11)$$

which holds at almost every point of Ω_1 for all α and n. Using Cauchy's inequality, we obtain

$$\left|\int_{(-\infty,\alpha]} K(\lambda)dU_x\right|^2 \equiv \left|\int_{(-\infty,\alpha]\times\mathbb{R}^2} K(\lambda)q\,d\nu_x\right|^2 \leq \nu_x((-\infty,\alpha]\times\mathbb{R}^2)\int_{(-\infty,\alpha]\times\mathbb{R}^2} K^2(\lambda)|q|^2\,d\nu_x.$$

From this identity, $\nu_x((-\infty,\alpha]\times\mathbb{R}^2) = \chi_x(\alpha)$, and (4.11) we obtain

$$\left|\int_{(-\infty,\alpha]} Kd\mathbf{U}_x(\lambda)\right|^2 \leq \left(\int_{\mathbb{R}^1}\Phi_n(\lambda)K(\lambda)dU_x(\lambda)\cdot\mathbf{b} - \nabla p^*(\Psi_n(s^*)v^* - \Phi_n(s^*)\mathbf{b})\right)\chi_x(\alpha). \quad (4.12)$$

Fix $x \in \Omega_1$ and choose $\alpha < s^*(x)$ such that the function $\chi_x(\lambda)$ is continuous at α. It is easy to see that

$$\lim_{n\to\infty}\Psi_n(s^*(x)) = \lim_{n\to\infty}\Phi_n(s^*(x)) = 0.$$

Passing to the limit in (4.12) as $n \to \infty$, we obtain

$$\left|\int_{(-\infty,\alpha]} K(\lambda)\,dU_x(\lambda)\right|^2 \leq \int_{(-\infty,\alpha]}\Gamma(\lambda)K(\lambda)dU_x(\lambda)\cdot\mathbf{b}(x)\chi_x(\alpha).$$

It follows from (1.13) and the inequality $\alpha < s^*(x)$ that the identity $U_x(\lambda) = \chi_x(\lambda)/a(\lambda)\mathbf{b}(x)$ holds for every $\lambda < \alpha$. We have thus obtained the inequality

$$\Big| \int_{(-\infty,\alpha]} Kd\Big(\frac{\chi_x(\lambda)}{a(\lambda)}\Big)\Big|^2 \le \chi_x(\alpha) \int_{(-\infty,\alpha]} K\Gamma\, d\Big(\frac{\chi_x}{a}\Big).$$

Using the identity

$$\int_{(-\infty,\alpha]} K(\lambda)\, d\Big(\frac{\chi_x(\lambda)}{a(\lambda)}\Big) = \int_{(-\infty,\alpha]} \Big(\frac{K(\alpha)}{a(\alpha)} + \int_\alpha^\lambda \frac{K(t)'}{a(t)}\,dt\Big)\, d\chi_x(\lambda) = \int_{(-\infty,\alpha]} \Gamma(\lambda)\, d\chi_x(\lambda),$$

rewrite the above inequality as

$$\Big| \int_{(-\infty,\alpha]} \Gamma(\lambda)\, d\chi_x(\lambda)\Big|^2 \le \chi_x(\alpha) \int_{(-\infty,\alpha]} K(\lambda)\Gamma(\lambda)\, d\Big(\frac{\chi_x(\lambda)}{a(\lambda)}\Big). \tag{4.13}$$

Suppose that $\chi_x(\alpha) \neq 0$. Under this assumption, the equalities

$$\int_E d\mu = \frac{1}{\chi_x(\alpha)} \int_{(-\infty,\alpha]\cap E} d\chi_x(\lambda), \quad E \subset \mathbb{R}^1,$$

define a probability measure μ such that $\operatorname{spt}\mu \subset [-M, \alpha]$. By the continuity of $\chi_x(\lambda)$ at α, we have $\mu(\{\alpha\}) = 0$. It follows from (4.13) that

$$\int_{(-\infty,\alpha]} K\Gamma\, d\Big(\frac{m^-(\lambda)}{a(\lambda)}\Big) - \Big(\int_{(-\infty,\alpha]} \Gamma\, d\mu\Big)^2 \ge 0.$$

Here $m^-(\lambda) = \mu(-\infty, \lambda]$. Observe that the functions $a(s)$ and $k(s)$ satisfy Condition S. Therefore, the corresponding function f_α is strictly p-concave at z_α. From this fact and the assertion (b) of Proposition 4.1 we conclude that $\mu = \delta(\lambda - \alpha)$ which contradicts the equality $\mu(\alpha) = 0$. Hence, $\chi_x(\alpha)$ vanishes at α. Since the function χ_x is monotone and continuous at almost every point of the real axis, it follows that $\chi_x(\alpha) = 1 - \Lambda_x(\alpha) = 0$ for every $\alpha < s^*(x)$ and the assertion is proven.

It remains to prove that $\Lambda_x(\alpha)$ is constant on the interval $(s^*(x), \infty)$. Consider the sequences Ψ_n, Φ_n of the entropy pairs defined as follows:

$$\Psi_n(\lambda) = \int_{-M}^{\lambda} (h(n(s-\alpha))K(s))'\, ds, \quad \Phi_n(\lambda) = \int_{-M}^{\lambda} a^{-1}(s)(h(n(s-\alpha))K)'\, ds.$$

It is easily seen that they converge pointwise to the functions

$$\Psi(s) = K(s)H(s-\alpha) \text{ and } \Phi(s) = \Gamma(s)H(s-\alpha).$$

As in the proof of the equality $\chi_x(\alpha) = 0$ for $\alpha < s^*(x)$, we show that the inequalities

$$w\text{-}\lim_{\varepsilon\to 0} |H(s_\varepsilon - \alpha)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 \le -\int_{\mathbb{R}^1} \Phi_n K d\mathbf{V}_x(\lambda) - \nabla p^*(\Psi_n(s^*)\mathbf{v}^* - \Phi_n(s^*)\mathbf{b})$$

and

$$\Lambda_x(\alpha)w\text{-}\lim_{\varepsilon\to 0}|H(s_\varepsilon-\alpha)K(s_\varepsilon)\mathbf{v}_\varepsilon|^2 \geq \Big|\int\limits_{[\alpha,\infty)} K(\lambda)d\mathbf{V}_x(\lambda)\Big|^2$$

hold at almost every point of Ω_1 for all α. Fix a point $x\in\Omega_1$ and choose $\alpha > s^*(x)$ such that $\Lambda_x(\lambda)$ is continuous at α. It is clear that

$$\lim_{n\to\infty}\Psi_n(s^*(x)) = \lim_{n\to\infty}\Phi_n(s^*(x)) = 0$$

and $\mathbf{V}_x(\lambda) = a^{-1}(\lambda)\Lambda_x(\lambda)\mathbf{b}$ for $\lambda \geq \alpha$. Due to the choice of α, we can rewrite the previous inequality as follows:

$$\Big|\int\limits_{[\alpha,\infty)} K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big)\Big|^2 \leq -\int\limits_{\mathbb{R}^1}\Phi_n(\lambda)K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big).$$

Since

$$\int\limits_{[\alpha,\infty)}\Gamma(\lambda)\,d\Lambda_x(\lambda) = \int\limits_{[\alpha,\infty)} K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big),$$

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^1}\Phi_n(\lambda)K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big) = \int\limits_{[\alpha,\infty)}\Gamma(\lambda)K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big),$$

we obtain

$$\Big(\int\limits_{[\alpha,\infty)}\Gamma(\lambda)\,d\Lambda_x(\lambda)\Big)^2 \leq -\int\limits_{[\alpha,\infty)}\Gamma(\lambda)K(\lambda)\,d\Big(\frac{\Lambda_x(\lambda)}{a(\lambda)}\Big).$$

If $\Lambda_x(\alpha)\neq 0$ then the relations

$$\int\limits_E d\mu = -\frac{1}{\Lambda_x(\alpha)}\int\limits_{[\alpha,\infty)\cap E} d\Lambda_x(\lambda), \quad E\subset\mathbb{R}^1$$

define a probability measure μ which is concentrated on the interval $[\alpha, M]$. We can rewrite the previous inequality as

$$\int\limits_{\mathbb{R}^1}\Gamma(\lambda)K(\lambda)\,d\Big(\frac{m^+}{a}\Big) - \Big(\int\limits_{\mathbb{R}^1}\Gamma(\lambda)\,d\mu\Big)^2 \geq 0.$$

From this fact and the assertion (b) of Proposition 4.1 we conclude that $\mu = \delta(\lambda-\alpha)$ which contradicts the equality $\mu(\alpha) = 0$. Therefore, $\Lambda_x(\alpha) = 0$ for all $\alpha > s^*(x)$, which proves Theorem 1.3.

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