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Representation of the Space of Polyanalytic Functions as a Direct Sum of Orthogonal Subspaces. Application to Rational Approximations

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ABSTRACT. Suppose that $D = \{z : |z| < 1\}$, $L_2(D)$ is the space of functions square-integrable over area in D, $A_k(D)$ is the set of all k-analytic functions in D, ($A_1(D) = A(D)$ is the set of all analytic functions in D), $A_kL_2(D) = L_2(D) \cap A_k(D)$, $A_1L_2(D) = AL_2(D)$,

$$A_kL_2^0(D) = \left\{f : f(z) = \frac{\partial^{k-1}}{\partial z^{k-1}}((1 - z\bar{z})^{k-1}F(z)), \quad F \in A(D), \ f \in A_kL_2(D)\right\}.$$

It is proved that the subspaces $A_kL_2^0(D)$, $k = 1, 2, \dots$, are orthogonal to one another and the space $A_mL_2(D)$ is the direct sum of such subspaces for $k = 1, 2, \dots, m$. The kernel of the orthogonal projection operator from the space $A_mL_2(D)$ onto its subspaces $A_kL_2^0(D)$ is obtained. These results are applied to the study of the properties of polyrational functions of best approximation in the metric $L_2(D)$.

KEY WORDS: polyanalytic function, direct sum of orthogonal subspaces, rational approximation, extremum problem, Bessel's inequality, polyrational function.

A function $f(z)$ that has continuous partial derivatives with respect to x and y up to order $m \geq 1$ inclusive in the domain G is called a *polyanalytic function of order* m (m*-analytic*) in the domain G if in this domain it satisfies the generalized Cauchy–Riemann equation $\partial^m f/\partial\bar{z}^m = 0$.

It is well known that any m-analytic function in the domain G can be uniquely expressed as (see [1])

$$f(z) = \varphi_0(z) + \bar{z}\varphi_1(z) + \dots + \bar{z}^{m-1}\varphi_{m-1}(z), \tag{1}$$

where the φ_k are holomorphic in G. For the case in which $G = D := \{z : |z| < 1\}$, relation (1) can be reduced to the form (see [2])

$$f(z) = P(z, \bar{z}) + g_0(z) + (1 - |z|^2)g_1(z) + \dots + (1 - |z|^2)^{m-1}g_{m-1}(z), \tag{2}$$

where the g_k are holomorphic in D, $P(z, \bar{z}) = P_0 + \bar{z}P_1(z) + \dots + \bar{z}^{m-1}P_{m-1}(z)$, $P_0 = \text{const}$, $P_k(z)$ for $k \geq 1$ is a polynomial in z of degree at most $k - 1$.

In what follows, the functions φ_k and g_k will be called *holomorphic components* of the polyanalytic function $f(z)$.

Polyanalytic functions were treated in [1]. In the papers by Dolzhenko and Danchenko [2–6], the boundary behavior of polyanalytic functions was studied using the representation (2) and integral representations for polyanalytic functions were obtained. In this paper we study various properties of polyanalytic functions from the space $L_2(D)$.

§1. Representation of polyanalytic functions as a direct sum

Let us introduce some notation. $L_2(D)$ denotes the space of complex-valued functions f with the ordinary norm

$$\|f\|_2 = \left\{\int\int_D |f(z)|^2 \, dx \, dy\right\}^{1/2}; \tag{3}$$

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$A_k(D)$ is the set all k-analytic functions in D ($A_1(D) = A(D)$ is the set all analytic functions in D); $A_kL_2(D) = L_2(D) \cap A_k(D)$, $A_1L_2(D) = AL_2(D)$;

$$A_kL_2^0(D) = \left\{ f : f(z) = \frac{\partial^{k-1}}{\partial z^{k-1}}\left((1 - z\bar{z})^{k-1}F(z)\right),\ F \in A(D),\ f \in A_kL_2(D) \right\};$$

$C^{(n)}(\overline{D})$ denotes the class of n-times continuously differentiable functions on $\overline{D}$ For $n = \infty$, this class will be denoted by $C^\infty(\overline{D})$.

In what follows, we use the notation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad \binom{n}{k} = 0 \quad \text{for } k > n \text{and for } k < 0.$$

We shall also use Green's formula

$$\iint_D f(z)\frac{\partial}{\partial \bar{z}}\overline{g(z)}\,dx\,dy = \frac{1}{2i}\int_{\partial D} f(z)\overline{g(z)}\,dz - \iint_D \overline{g(z)}\frac{\partial}{\partial \bar{z}} f(z)\,dx\,dy,$$

which is valid for all functions $f, g \in C^1(\overline{D})$.

Theorem 1. *The polyanalytic function*

$$f(z) = \frac{\partial^{k-1}}{\partial z^{k-1}}\left((1 - z\bar{z})^{k-1}F(z)\right)$$

belongs to the space $A_kL_2^0(D)$ *if and only if* $F \in AL_2(D)$.

Corollary 1. a) *The system of functions*

$$\frac{1}{(k-1)!}\sqrt{\frac{n+k}{\pi}}\,\frac{\partial^{k-1}}{\partial z^{k-1}}\left\{(1 - z\bar{z})^{k-1}z^n\right\}, \quad n = 0, 1, \dots,$$

forms an orthonormal basis of polynomials in the space $A_kL_2^0(D)$.

b) *If for some positive integer* k *the polyanalytic function*

$$\frac{\partial^{k-1}}{\partial z^{k-1}}\left\{(1 - z\bar{z})^{k-1}g(z)\right\}$$

belongs to the space $A_kL_2^0(D)$ *and*

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad z \in D,$$

then

$$\iint_D \left|\frac{\partial^{k-1}}{\partial z^{k-1}}\left\{(1 - z\bar{z})^{k-1}g(z)\right\}\right|^2 dx\,dy = \pi((k-1)!)^2 \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+k}. \tag{4}$$

Theorem 2. *The subspaces* $A_kL_2^0(D)$ $(k = 1, 2, \dots)$ *are pairwise orthogonal and the space* $A_mL_2(D)$ *is the direct sum of the first* m *of them, i.e.,*

$$A_mL_2(D) = A_1L_2(D) \oplus A_2L_2^0(D) \oplus \dots \oplus A_mL_2^0(D).$$

Remark 1. Theorem 2 was announced without proof in [7].

Corollary 2. *If $f \in A_m L_2(D)$, then the functions $(1-|z|^2)^{m-1}\varphi_j(z)$, $j=0,1,\dots,m-1$, belong to the space $L_2(D)$, where φ_j is the corresponding component of the polyanalytic function f in the representation* (1).

Corollary 3. *The system of functions*

$$\frac{1}{(k-1)!}\sqrt{\frac{n+k}{\pi}}\,\frac{\partial^{k-1}}{\partial z^{k-1}}\{(1-z\bar z)^{k-1}z^n\},\quad k=1,2,\dots,m;\quad n=0,1,\dots,$$

forms an orthonormal basis of polynomials in the space $A_m L_2(D)$.

Remark 2. In [8], the proof of the completeness of the system from Corollary 3 by means of the orthogonalization method is outlined. Apparently, the proof given in the present paper is simpler.

For the statement of the next theorem, let us introduce the following additional notation:

$$\mathcal{R}_k^0(z,\zeta)=\frac{1}{\pi((k-1)!)^2}\frac{\partial^{2(k-1)}}{\partial z^{k-1}\partial\bar\zeta^{k-1}}\left\{(1-z\bar z)^{k-1}(1-\zeta\bar\zeta)^{k-1}\frac{(k-1)(1-z\bar\zeta)+1}{(1-z\bar\zeta)^2}\right\},$$
$$\mathcal{R}_m(z,\zeta)=\sum_{k=1}^{m}\mathcal{R}_k^0(z,\zeta),\quad z,\zeta\in D.$$

Obviously, $\mathcal{R}_k^0(z,\zeta)=\overline{\mathcal{R}_k^0(\zeta,z)}$ and for fixed $\zeta\in D$ the polyanalytic function $\mathcal{R}_k^0(z,\zeta)$ belongs to the space $A_k L_2^0(D)$.

Let us introduce the following integral operator:

$$P_k^0(f)(\zeta)=\iint_D f(z)\overline{\mathcal{R}_k^0(z,\zeta)}\,dx\,dy,\qquad f\in L_2(D),\quad \zeta\in D,\quad z=x+iy. \tag{5}$$

Denote by $P_m(f)$ the operator obtained by substituting $\mathcal{R}_m(z,\zeta)$ for the kernel $\mathcal{R}_k^0(z,\zeta)$ in (5). Then the following theorem holds.

Theorem 3. *$P_k^0(f)$ $(P_m(f))$ is the orthogonal projection operator from the space $L_2(D)$ onto its subspace $A_k L_2^0(D)$ $(A_m L_2(D))$.*

Theorem 2 implies that any polyanalytic function f from $A_m L_2(D)$ can be uniquely expressed as

$$f(z)=\sum_{k=1}^{m}\frac{\partial^{k-1}}{\partial z^{k-1}}\{(1-z\bar z)^{k-1}F_{k-1}(z)\},\quad F_{k-1}\in AL_2(D).$$

Then, using Theorem 3, we obtain the following result.

Corollary 4. *If $f\in A_m L_2(D)$, then*

$$\iint_D f(z)\overline{\mathcal{R}_k^0(z,\zeta)}\,dx\,dy=\frac{\partial^{k-1}}{\partial\zeta^{k-1}}\{(1-\zeta\bar\zeta)^{k-1}F_{k-1}(\zeta)\}.$$

Corollary 5. *Suppose that $\varphi(z)=\frac{\partial^k}{\partial z^k}\{(1-z\bar z)^k F(z)\}$ belongs to the space $A_{k+1}L_2^0(D)$. Then*

$$F^{(n)}(\zeta)=\frac{n!}{\pi(k!)^2}\iint_D \varphi(z)\frac{\partial^k}{\partial\bar z^k}\left\{(1-z\bar z)^k\bar z^n\frac{k(1-\bar z\zeta)+n+1}{(1-\bar z\zeta)^{n+2}}\right\}dx\,dy,$$
$$\zeta\in D,\quad n=0,1,\dots\,.$$

Remark 3. From Corollary 4 it follows that $\mathcal{R}_k^0(z,\zeta)$ is the kernel of the orthogonal projector from the space $A_mL_2(D)$ onto its subspace $A_kL_2^0(D)$. For $k=1$, this kernel coincides with the well-known Bergmann kernel for the disk D. In [8], another representation for $\mathcal{R}_m(z,\zeta)$ is given (without proof), namely

$$\mathcal{R}_m(z,\zeta)=\frac{m}{\pi(1-z\bar{\zeta})^{2m}}\sum_{j=0}^{m-1}(-1)^j\binom{m}{j+1}\binom{m+j}{m}|1-z\bar{\zeta}|^{2(m-1-j)}|z-\zeta|^{2j}. \tag{6}$$

Corollary 6. *We have*

$$\|\mathcal{R}_k^0\|_2=\sqrt{\mathcal{R}_k^0(z,z)}=\sqrt{\frac{2k-1}{\pi}}\,\frac{1}{1-|z|^2},\qquad z\in D.$$

Theorem 3 can be used to obtain growth estimates of polyanalytic functions as the point z approaches the boundary of the disk. Indeed, the following theorem is valid.

Theorem 4a. 1) *If the function φ belongs to $A_kL_2^0(D)$, then*

$$|\varphi(z)|\le\sqrt{\frac{2k-1}{\pi}}\,\frac{1}{1-|z|^2}\|\varphi\|_2,\qquad z\in D. \tag{7}$$

2) *If the function f belongs to $A_mL_2(D)$, then*

$$|f(z)|\le\frac{m}{\sqrt{\pi}}\,\frac{1}{1-|z|^2}\|f\|_2,\qquad z\in D. \tag{8}$$

Inequalities (7) and (8) are exact. The equality sign is attained, respectively, for

$$\varphi(z)=\mathcal{R}_k^0(z,\zeta)\qquad\text{and}\qquad f(z)=\mathcal{R}_m(z,\zeta)=\sum_{k=1}^{m}\mathcal{R}_k^0(z,\zeta).$$

Now let us apply Theorem 3 to extremum problems. Suppose that $\zeta\in D$ is a fixed point and

$$M_\zeta^0(k)=\{f\in A_kL_2^0(D):\ f(\zeta)=1\},\qquad M_\zeta(m)=\{f\in A_mL_2(D):\ f(\zeta)=1\}.$$

Then, using inequalities (7) and (8), we obtain the following theorem.

Theorem 4b. 1) *The extremum problem $\|f\|_2\to\inf$, $f\in M_\zeta^0(k)$, has the unique solution f_0:*

$$f_0(z)=\frac{\mathcal{R}_k^0(z,\zeta)}{\mathcal{R}_k^0(\zeta,\zeta)},\qquad \|f_0\|_2=\sqrt{\frac{\pi}{2k-1}}(1-\zeta\bar{\zeta}),\quad z,\zeta\in D.$$

2) *The extremum problem $\|f\|_2\to\inf$, $f\in M_\zeta(m)$, has the unique solution f_0:*

$$f_0(z)=\frac{\mathcal{R}_m(z,\zeta)}{\mathcal{R}_m(\zeta,\zeta)},\qquad \|f_0\|_2=\frac{\sqrt{\pi}}{m}(1-\zeta\bar{\zeta}),$$

where $\mathcal{R}_m(z,\zeta)=\sum_{k=1}^{m}\mathcal{R}_k^0(z,\zeta)$, $z,\zeta\in D$.

Note that inequality (8) and section 2) of Theorem 4b were obtained in [8], using the representation (6) for the kernel $\mathcal{R}_m(z,\zeta)$.

For the proof of these theorems, some lemmas are required.

Lemma 1. *If* $h_{k,n}(z)=\frac{\partial^{k-1}}{\partial z^{k-1}}\{(1-z\bar{z})^{k-1}z^n\}$, $k=1,2,\dots,m$; $n=0,1,\dots$, *then*

$$\iint_D h_{k,n}(z)\overline{h_{l,j}(z)}\,dx\,dy=0,\quad k\neq l,\qquad \iint_D h_{k,n}(z)\overline{h_{k,j}(z)}\,dx\,dy=\begin{cases}0, & n\neq j,\\ \frac{\pi((k-1)!)^2}{n+k}, & n=j.\end{cases}\tag{9}$$

Proof of Lemma 1. Let $k<l$. Since $h_{k,n}\in A_kL_2^0(D)$, we have

$$\frac{\partial^{l-1}}{\partial\bar{z}^{l-1}}\{h_{k,n}(z)\}=0.$$

Therefore, applying Green's formula to the scalar product on the left-hand side of relation (9) $l-1$ times, we obtain (9).

Now let $k=l$. Then, similarly, using Green's formula once again, we obtain

$$\begin{aligned}\iint_D h_{k,n}(z)\overline{h_{l,j}(z)}\,dx\,dy&=(k-1)!\iint_D\frac{\partial^{k-1}}{\partial z^{k-1}}\{z^{n+k-1}\}\left(1-|z|^2\right)^{k-1}\bar{z}^j\,dx\,dy\\&=(k-1)!\frac{(n+k-1)!}{n!}\iint_D\left(1-|z|^2\right)^{k-1}z^n\bar{z}^j\,dx\,dy.\end{aligned}\tag{10}$$

For $n\neq j$, the last integral is zero. But if $n=j$, then, passing to polar coordinates, from (10) we obtain

$$\iint_D h_{k,n}(z)\overline{h_{l,j}(z)}\,dx\,dy=(k-1)!\frac{(n+k-1)!}{n!}\pi B(k,n+1),\tag{11}$$

where $B(k,n+1)=n!(k-1)!/(n+k)!$ is the Euler integral of the first kind (see [9, p. 750]). Substituting the value $B(k,n+1)$ into relation (11), we conclude the proof of Lemma 1. □

Lemma 2. a) *If* $n\geq k$, *then*

$$\sum_{m=0}^{2k}(-1)^m\left(\sum_{j=0}^{m}\binom{k}{j}\binom{n+j}{k}\binom{k}{m-j}\binom{n+m-j}{k}\right)=1,\tag{12}$$

$$\sum_{m=0}^{2k}(-1)^m m\left(\sum_{j=0}^{m}\binom{k}{j}\binom{n+j}{k}\binom{k}{m-j}\binom{n+m-j}{k}\right)=2k(n+1).\tag{13}$$

b) *If* $n\leq k$, *then*

$$\sum_{m=0}^{2k}(-1)^m\left(\sum_{j=0}^{m}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{n-m+j}\binom{k+m-j}{k}\right)=1,\tag{14}$$

$$\sum_{m=0}^{2k}(-1)^m m\left(\sum_{j=0}^{m}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{n-m+j}\binom{k+m-j}{k}\right)=2n(k+1).\tag{15}$$

Proof of Lemma 2. Consider the generating function

$$g(t)=\frac{1}{(k!)^2}\left\{\left((1-t)^kt^n\right)^{(k)}\right\}^2,\qquad\text{where}\quad t\in(-\infty,\infty).$$

Suppose that $n\geq k$. Then we have

$$g(t)=\frac{1}{(k!)^2}\left\{\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\frac{k!}{j!}(1-t)^j\frac{n!}{(n-j)!}t^{n-j}\right\}^2,\tag{16}$$

$$g(t)=t^{2(n-k)}\sum_{m=0}^{2k}(-1)^m\left(\sum_{j=0}^{m}\binom{k}{j}\binom{n+j}{k}\binom{k}{m-j}\binom{n+m-j}{k}\right)t^m.\tag{17}$$

Assuming $t=1$, from relations (16) and (17) we obtain (12). Next, calculating the derivative $g'(t)$ from relations (16) and (17) and equating the resulting values of $g'(1)$, in view of relation (12), we obtain (13). The proof of relations (14) and (15) is similar if we consider the same function $g(t)$ for $n\leq k$. □

Proof of Theorem 1. Consider a polyanalytic function $f \in A_k L_2^0(D)$, and let

$$F(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then, using Lemma 1 and Bessel's inequality, we obtain

$$\iint_D |f(z)|^2 \, dx \, dy \geq \pi((k-1)!)^2 \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+k},$$

i.e., $F \in AL_2(D)$.

Let us prove the sufficiency.

It is readily shown that if $F \in AL_2(D)$, then for any positive integer j the function $(1-|z|^2)^j F^{(j)}(z)$ belongs to $L_2(D)$. Then, writing the polyanalytic function

$$f(z) = \frac{\partial^{k-1}}{\partial z^{k-1}}\big((1-z\bar{z})^{k-1} F(z)\big)$$

as

$$f(z) = \sum_{j=0}^{k-1} (-1)^{(k-1-j)} \binom{k-1}{j} \frac{(k-1)!}{j!} \bar{z}^{k-1-j} (1-|z|^2)^j F^{(j)}(z),$$

we conclude the proof Theorem 1. □

Proof of Theorem 2. The orthogonality of the subspaces $A_k L_2^0(D)$ follows from Lemma 1. We prove the theorem by induction. Suppose that it is proved for $m = l$. Let us express the space $A_{l+1}L_2(D)$ as the direct sum of the subspace $A_l L_2(D)$ and its orthogonal complement B. By the induction hypothesis, we obtain

$$A_{l+1}L_2(D) = A_1 L_2(D) \oplus A_2 L_2^0(D) \oplus \cdots \oplus A_l L_2^0(D) \oplus B.$$

Let us prove that the subspace B coincides with $A_{l+1}L_2^0(D)$. To do this, it suffices to prove that any function φ from $B \cap C^\infty(\overline{D})$ orthogonal to all subspaces $A_k L_2^0(D)$, $k = 1, 2, \ldots, l+1$, is identically zero in D.

Suppose that the polyanalytic function $\varphi \in B \cap C^\infty(\overline{D})$ has the representation

$$\varphi(z) = \sum_{p=0}^{l} \bar{z}^p g_p(z) = \sum_{p=0}^{l} \bar{z}^p \sum_{n=0}^{\infty} a_{n,p} z^n.$$

Taking into account the orthogonality of the function φ to the subspaces $A_k L_2^0(D)$, $k = 1, 2, \ldots, l+1$, we obtain

$$\iint_D \frac{\partial^{k-1}}{\partial \bar{z}^{k-1}} \{(1-z\bar{z})^{k-1} \bar{z}^\nu\} \varphi(z) \, dx \, dy = 0, \qquad \nu = 0, 1, \ldots. \tag{18}$$

For $k = 1$, relation (18) acquires the form

$$\iint_D \bar{z}^\nu \sum_{p=0}^{l} \bar{z}^p \sum_{n=0}^{\infty} a_{n,p} z^n \, dx \, dy = 0, \qquad \nu = 0, 1, \ldots.$$

Hence, passing to the polar coordinates and calculating the last integral, we obtain

$$\sum_{p=0}^{l} \frac{1}{\nu+p+1} a_{\nu+p,p} = 0, \qquad \nu = 0, 1, \ldots. \tag{19}$$

For $k \geq 2$, using Green's formula, from relation (18) we obtain

$$\iint_D (1 - z\bar{z})^{k-1} \bar{z}^{\nu} \sum_{p=k-1}^{l} p \dots (p-k+2) \bar{z}^{p-k+1} \left(\sum_{n=0}^{\infty} a_{n,p} z^n \right) dx\, dy = 0,$$
$$k = 2, 3, \dots, l+1; \quad \nu = 0, 1, \dots .$$

Just as above, passing to polar coordinates and calculating the integral on the left-hand side of the last relation, we obtain the following system of equations, together with relation (19), for the coefficients of the components of the polyanalytic function φ:

$$\sum_{p=k-1}^{l} \frac{p \dots (p-k+2)}{(\nu+p-k+2) \dots (\nu+p+1)} a_{\nu+p-k+1,p} = 0, \qquad k = 1, 2, \dots, l+1; \quad \nu = 0, 1, \dots . \tag{20}$$

For each fixed ν, system (20) is of triangular form. Solving this system, we obtain $a_{n,p} = 0$ for $p = 0, 1, \dots, l$ and $n = 0, 1, \dots$, which proves the statement. □

Proof of Theorem 3. Consider the following δ-functional in the space $A_k L_2^0(D)$:

$$\delta(\varphi(z)) = \varphi(\zeta), \qquad \zeta \in D, \quad \varphi \in A_k L_2^0(D).$$

Using Corollary 1, it is easy to prove the continuity of the δ-functional in the metric defined by relation (3) for functions from the space $A_k L_2^0(D)$. Therefore, the theorem on the general form of a linear continuous functional in Hilbert space implies that there exists a function from the space $A_k L_2^0(D)$ that determines this functional. Denote this function by $\mathcal{R}_k^0(z, \zeta)$. Then we obtain

$$\iint_D \varphi(z) \overline{\mathcal{R}_k^0(z, \zeta)}\, dx\, dy = \varphi(\zeta), \qquad \zeta \in D, \tag{21}$$

for any function $\varphi \in A_k L_2^0(D)$.

The function $\mathcal{R}_k^0(z, \zeta)$ is called the *reproducing kernel* of the space $A_k L_2^0(D)$. Taking into account relation (21), let us expand the kernel $\mathcal{R}_k^0(z, \zeta)$ in its Fourier series with respect to the orthonormal system of functions (see Corollary 1) $e_{k,n}(z)$ from the space $A_k L_2^0(D)$:

$$\mathcal{R}_k^0(z, \zeta) = \sum_{n=0}^{\infty} e_{k,n}(z) \overline{e_{k,n}(\zeta)}. \tag{22}$$

After simple calculations, from relation (22) we obtain the required representation for the kernel $\mathcal{R}_k^0(z, \zeta)$, which, together with relation (21), concludes the proof of Theorem 3. □

Proof of Corollary 2. It follows from Theorem 2 that for any function $f \in A_m L_2(D)$ there exist functions $F_{k-1} \in AL_2(D)$, $k = 1, 2, \dots, m$, for which the following relation is valid:

$$f(z) = \sum_{k=1}^{m} \frac{\partial^{k-1}}{\partial z^{k-1}} \{(1 - z\bar{z})^{k-1} F_{k-1}(z)\} = \sum_{j=0}^{m-1} \bar{z}^j (-1)^j \sum_{k=j+1}^{m} \binom{k-1}{j} \left(z^j F_{k-1}(z)\right)^{(k-1)}.$$

Hence, using the representation (1) for the polyanalytic function f, we obtain

$$\varphi_j(z) = (-1)^j \sum_{k=j+1}^{m} \binom{k-1}{j} \left(z^j F_{k-1}(z)\right)^{(k-1)}, \quad j = 0, 1, \dots, m-1. \tag{23}$$

It is readily verified that the condition $g \in AL_2(D)$ is a necessary and sufficient condition for the function $(1 - |z|^2)^{k-1} g^{(k-1)}(z)$ to belong to the space $A_k L_2(D)$ for any fixed positive integer k. Therefore, relation (23) implies the assertion of Corollary 2. □

Proof of Corollary 6. It is more convenient to find the norm of $\mathcal{R}^0_{k+1}(z,\zeta)$. Using relation (22), we obtain

$$\pi\mathcal{R}^0_{k+1}(\zeta,\zeta) = \pi\sum_{n=0}^{\infty} e_{k,n}(\zeta)\overline{e_{k,n}(\zeta)} = \sum_{n=0}^{k}(n+k+1)$$
$$\times\left\{\sum_{m=0}^{2n}(-1)^m\left(\sum_{j=0}^{m}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{n-m+j}\binom{k+m-j}{k}\right)t^m\right\}t^{k-n}$$
$$+\sum_{n=k+1}^{\infty}(n+k+1)$$
$$\times\left\{\sum_{m=0}^{2k}(-1)^m\left(\sum_{j=0}^{m}\binom{k}{j}\binom{n+j}{k}\binom{k}{m-j}\binom{n+m-j}{k}\right)t^m\right\}t^{n-k},$$

where $t=|\zeta|^2$, $\zeta\in D$.

Introducing the notation $\nu = m+k-n$ for $n\le k$ and $\nu = m+n-k$ for $n>k$ and rearranging the summation signs, we have

$$\pi\mathcal{R}^0_{k+1}(\zeta,\zeta) = \sum_{\nu=0}^{\infty}a_\nu t^\nu = \sum_{n=0}^{k}(n+k+1)\left\{\sum_{\nu=k-n}^{k+n}(-1)^{\nu+n-k}\right.$$
$$\left.\times\left(\sum_{j=0}^{\nu+n-k}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{k-\nu+j}\binom{\nu+n-j}{k}\right)t^\nu\right\}$$
$$+\sum_{m=0}^{2k}(-1)^m\left\{\sum_{\nu=m+1}^{\infty}(\nu-m+2k+1)\left(\sum_{j=0}^{m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{m-j}\binom{\nu+k-m+j}{k}\right)t^\nu\right\}$$
$$=\sum_{\nu=0}^{k}\left\{\sum_{n=k-\nu}^{k}(-1)^{\nu+n-k}(n+k+1)\left(\sum_{j=0}^{\nu+n-k}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{k-\nu+j}\binom{\nu+n-j}{k}\right)\right\}t^\nu$$
$$+\sum_{\nu=k+1}^{2k}\left\{\sum_{n=\nu-k}^{k}(-1)^{\nu+n-k}(n+k+1)\left(\sum_{j=0}^{\nu+n-k}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{k-\nu+j}\binom{\nu+n-j}{k}\right)\right\}t^\nu$$
$$+\sum_{\nu=1}^{2k}\left\{\sum_{m=0}^{\nu-1}(-1)^m(\nu-m+2k+1)\left(\sum_{j=0}^{m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{m-j}\binom{\nu+k-m+j}{k}\right)\right\}t^\nu$$
$$+\sum_{\nu=2k+1}^{\infty}\left\{\sum_{m=0}^{2k}(-1)^m(\nu-m+2k+1)\left(\sum_{j=0}^{m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{m-j}\binom{\nu+k-m+j}{k}\right)\right\}t^\nu. \tag{24}$$

For $\nu\ge 2k+1$, from (24) we obtain

$$a_\nu = \sum_{m=0}^{2k}(-1)^m(\nu-m+2k+1)\left(\sum_{j=0}^{m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{m-j}\binom{\nu+k-m+j}{k}\right). \tag{25}$$

Hence, taking into account the simplest properties of binomial coefficients, we obtain

$$a_\nu = \sum_{m=0}^{2k}(-1)^m(\nu+m+1)\left(\sum_{j=0}^{2k-m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{2k-m-j}\binom{\nu-k+m+j}{k}\right)$$
$$=\sum_{m=0}^{k}(-1)^m(\nu+m+1)\left(\sum_{j=k-m}^{k}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{2k-m-j}\binom{\nu-k+m+j}{k}\right)$$

$$+\sum_{m=k+1}^{2k}(-1)^m(\nu+m+1)\left(\sum_{j=0}^{2k-m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{2k-m-j}\binom{\nu-k+m+j}{k}\right).$$

Further, setting $n=j-(k-m)$, we have

$$a_\nu=\sum_{m=0}^{k}(-1)^m(\nu+m+1)\left(\sum_{n=0}^{m}\binom{k}{n}\binom{\nu+n}{k}\binom{k}{m-n}\binom{\nu+m-n}{k}\right)$$
$$+\sum_{m=k+1}^{2k}(-1)^m(\nu+m+1)\left(\sum_{n=m-k}^{k}\binom{k}{n}\binom{\nu+n}{k}\binom{k}{m-n}\binom{\nu+m-n}{k}\right).$$

Since $\binom{n}{j}=0$ for $j>n$ and $j<0$, it follows that for $m\geq k$ we have

$$\sum_{n=0}^{m}\binom{k}{n}\binom{\nu+n}{k}\binom{k}{m-n}\binom{\nu+m-n}{k}=\sum_{n=m-k}^{k}\binom{k}{n}\binom{\nu+n}{k}\binom{k}{m-n}\binom{\nu+m-n}{k}.$$

Therefore, for the coefficients a_ν, $\nu\geq 2k+1$, we obtain

$$a_\nu=\sum_{m=0}^{2k}(-1)^m(\nu+m+1)\left(\sum_{n=0}^{m}\binom{k}{n}\binom{\nu+n}{k}\binom{k}{m-n}\binom{\nu+m-n}{k}\right).$$

Hence, in view of relations (12) and (13), for $\nu\geq 2k+1$ we obtain

$$a_\nu=\nu+1+2k(\nu+1)=(2k+1)(\nu+1).$$

Now let $k<\nu\leq 2k$. Then from (24) we obtain

$$a_\nu=\sum_{n=\nu-k}^{k}(-1)^{\nu+n-k}(n+k+1)\left(\sum_{j=0}^{\nu+n-k}\binom{k}{n-j}\binom{k+j}{k}\binom{k}{k-\nu+j}\binom{\nu+n-j}{k}\right)$$
$$+\sum_{m=0}^{\nu-1}(-1)^m(\nu-m+2k+1)\left(\sum_{j=0}^{m}\binom{k}{j}\binom{\nu+k-j}{k}\binom{k}{m-j}\binom{\nu+k-m+j}{k}\right). \qquad (26)$$

Introducing the notation $n+k=m$ and writing the summands in reverse order with respect to m, we can transform the first sum in (26) to the form

$$A=\sum_{m=\nu}^{2k}(-1)^{\nu+m}(m+1)\left(\sum_{j=0}^{\nu+m-2k}\binom{k}{m-k-j}\binom{k+j}{j}\binom{k}{\nu-j}\binom{\nu+m-k-j}{k}\right)$$
$$=\sum_{m=\nu}^{2k}(-1)^m(\nu-m+2k+1)\left(\sum_{j=0}^{2\nu-m}\binom{k}{\nu+k-m-j}\binom{k+j}{j}\binom{k}{\nu-j}\binom{2\nu-m+k-j}{k}\right)$$
$$=\sum_{m=\nu}^{2k}(-1)^m(\nu-m+2k+1)\left(\sum_{j=\nu-k}^{\nu+k-m}\binom{k}{\nu+k-m-j}\binom{k+j}{j}\binom{k}{\nu-j}\binom{2\nu-m+k-j}{k}\right).$$

Setting $n=j+m-\nu$ and using the simplest properties of binomial coefficients, we obtain $(m\geq\nu>k)$

$$A=\sum_{m=\nu}^{2k}(-1)^m(\nu-m+2k+1)\left(\sum_{n=m-k}^{k}\binom{k}{n}\binom{\nu+k-n}{k}\binom{k}{m-n}\binom{\nu+k-m+n}{k}\right)$$

$$= \sum_{m=\nu}^{2k} (-1)^m (\nu - m + 2k + 1) \left(\sum_{n=0}^{m} \binom{k}{n} \binom{\nu + k - n}{k} \binom{k}{m-n} \binom{\nu + k - m + n}{k} \right).$$

Substituting this expression into (26), we obtain the same expression for a_ν as given by relation (25). Therefore, also for $k < \nu \leq 2k$ we have $a_\nu = (2k+1)(\nu+1)$.

Now consider the case $0 \leq \nu \leq k$.

Note that for $m \geq \nu$ the following relation holds:

$$\sum_{j=0}^{m} \binom{k}{\nu - j} \binom{k+j}{k} \binom{k}{\nu - m + j} \binom{k + m - j}{k} = \sum_{j=m-\nu}^{\nu} \binom{k}{\nu - j} \binom{k+j}{k} \binom{k}{\nu - m + j} \binom{k + m - j}{k}. \tag{27}$$

It follows from relation (24) that for $\nu \leq k$ we can write

$$\begin{aligned} a_\nu = &\sum_{n=k-\nu}^{k} (-1)^{\nu+n-k} (n+k+1) \left(\sum_{j=0}^{\nu+n-k} \binom{k}{n-j} \binom{k+j}{k} \binom{k}{k-\nu+j} \binom{\nu+n-j}{k} \right) \\ &+ \sum_{m=0}^{\nu-1} (-1)^m (\nu - m + 2k + 1) \left(\sum_{j=0}^{m} \binom{k}{j} \binom{\nu + k - j}{k} \binom{k}{m-j} \binom{\nu + k - m + j}{k} \right) \\ = &A_1 + A_2. \end{aligned}$$

Set $m = n - (k - \nu)$. Then the sum A_1 acquires the form

$$A_1 = \sum_{m=0}^{\nu} (-1)^m (m + 2k + 1 - \nu) \sum_{j=0}^{m} \binom{k}{\nu - j} \binom{k+j}{k} \binom{k}{\nu - m + j} \binom{k + m - j}{k}.$$

Let us transform the sum A_2. To do this, let us make the substitution $m = 2\nu - n$. Then we obtain

$$A_2 = \sum_{n=\nu+1}^{2\nu} (-1)^n (n + 2k + 1 - \nu) \left(\sum_{j=0}^{2\nu - n} \binom{k}{j} \binom{\nu + k - j}{k} \binom{k}{2\nu - n - j} \binom{k - \nu + n + j}{k} \right).$$

Set $j + n - \nu = i$. Then, taking into account relation (27) and returning to the old variables, we can express A_2 as

$$A_2 = \sum_{m=\nu+1}^{2\nu} (-1)^m (m + 2k + 1 - \nu) \sum_{j=0}^{m} \binom{k}{\nu - j} \binom{k+j}{k} \binom{k}{\nu - m + j} \binom{k + m - j}{k}.$$

Using the expressions for A_1 and A_2 and relations (14), (15), we finally obtain

$$\begin{aligned} a_\nu = A_1 + A_2 &= \sum_{m=0}^{2\nu} (-1)^m (m + 2k + 1 - \nu) \sum_{j=0}^{m} \binom{k}{\nu - j} \binom{k+j}{k} \binom{k}{\nu - m + j} \binom{k + m - j}{k} \\ &= (2k+1)(\nu+1). \end{aligned}$$

Thus

$$\pi \mathcal{R}^0_{k+1}(\zeta, \zeta) = \sum_{\nu=0}^{\infty} (2k+1)(\nu+1) t^\nu = \frac{2k+1}{(1-t)^2},$$

where $t = |\zeta|^2$.

Corollary 6 is proved. □

§2. Rational approximation

Let $\mathcal{R}_n$ denote the class of all rational functions of degree at most n with poles outside the disk D. A polyanalytic function of the form

$$r(z) = \sum_{k=0}^{m-1} \bar{z}^k r_{n_k}(z), \qquad r_{n_k} \in \mathcal{R}_{n_k}, \quad k = 0, 1 \dots, m-1,$$

is called a *polyrational function of order* m (or an m*-rational* function).

By Theorem 2, any polyanalytic function $f \in A_m L_2(D)$ can be uniquely expressed as

$$f(z) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k F_k(z)\}, \qquad F_k \in AL_2(D), \quad k = 0, 1, \dots, m-1. \tag{28}$$

Using Theorem 3, we can show that for a polyrational function $r(z)$ its components F_k in the representation (28) need not be rational functions. Therefore, for a given polyanalytic function $f \in A_m L_2(D)$, we seek its polyrational function of best approximation in the metric (3) in the form

$$r(z) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k r_{n_k}(z)\}, \qquad r_{n_k} \in \mathcal{R}_{n_k}, \quad k = 0, 1, \dots, m-1. \tag{29}$$

For a given multi-index $(n) = (n_0, n_1, \dots, n_{m-1})$ with nonnegative components, by

$$r_{(n)}(z\,;\, f) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k r_{n_k}(z\,;\, f)\}, \qquad r_{n_k} \in \mathcal{R}_{n_k}, \quad k = 0, 1, \dots, m-1, \tag{30}$$

we denote the m-rational function of best approximation of degree at most (n) for the m-analytic function (28) in the metric (3) among polyrational functions of the form (29), and by $L_2 \widetilde{R}_{(n)}(f, D, m)$ we denote the corresponding deviation of a polyanalytic function from a polyrational function of the form (29) in the same metric. Obviously,

$$\|f - r_{(n)}(\cdot, f)\|_2^2 = \sum_{k=0}^{m-1} \left\| \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;\, f))\} \right\|_2^2. \tag{31}$$

Remark 4. It follows from relation (31) that the component $r_{n_k}(z\,;\, f)$ of the m-rational function of best approximation $r_{(n)}(z\,;\, f)$ (30) possesses the following property:

$$\left\| \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;\, f))\} \right\|_2 = \inf \left\{ \left\| \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k (F_k(z) - \rho_{n_k}(z))\} \right\|_2 ;\ \rho_{n_k} \in \mathcal{R}_{n_k} \right\},$$

i.e., $r_{n_k}(z\,;\, f)$ is the rational function of best approximation of degree at most n_k for the component F_k of the function (28) in the metric defined by the following relation:

$$\left\| \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k F_k(z)\} \right\|_2 = \left\{ \iint_D \left| \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k F_k(z)\} \right|^2 dx\, dy \right\}^{1/2}. \tag{32}$$

This remark will be used later on.

Theorem 5. *If for some integer k from the closed interval $[0, m-1]$ the component F_k of the m-analytic function* (28) *from $A_m L_2(D)$ is not a rational function of degree at most n_k, then*

$$\deg\{r_{n_k}(z\,;\, f)\} = n_k.$$

Remark 5. For $m = 1$ Theorem 5 was proved in [11], and for $m > 1$ it was announced in [10].

Corollary 7. *Under the assumptions of Theorem* 5, $L_2\widetilde{R}_{(n)}(f, D, m)$ *is strictly monotone decreasing with respect to the* $k+1$ *components of the multi-index* $(n) = (n_0, n_1, \dots, n_{m-1})$.

Corollary 8. *Under the assumptions of Theorem* 5, $\|r_{(n)}(\cdot\,; f)\|_2$ *is strictly monotone increasing with respect to the* $k+1$ *components of the multi-index* $(n) = (n_0, n_1, \dots, n_{m-1})$. *In particular,*

$$\|r_{(n-1)}(\cdot\,; f)\|_2 < \|r_{(n)}(\cdot\,; f)\|_2, \qquad \{L_2\widetilde{R}_{(n)}(f, D, m)\}^2 = \|f\|_2^2 - \|r_{(n)}(\cdot\,; f)\|_2^2.$$

Corollary 9. *If* $f \in A_m L_2(D)$, *then*

$$\{L_2\widetilde{R}_{(n)}(f, D, m)\}^2 \le \sum_{k=0}^{m-1} (k!)^2 \{L_2 R_{n_k}(F_k, D)\}^2 \le m\{L_2\widetilde{R}_{(n)}(f, D, m)\}^2,$$

where $L_2 R_n(g, D)$ *is the least deviation of the function* $g \in AL_2(D)$ *from rational functions belonging to* $\mathcal{R}_n$ *in the metric* (3).

To state the following theorem, we need some additional notation.
Let $r_{n_k}(z\,; f) = P_{n_k}(z)/Q_{n_k}(z)$, where P_{n_k} is a polynomial of degree at most n_k, and

$$Q_{n_k}(z) = \prod_{j=1}^{l_k} (1 - z\bar{\alpha}_{j,k})^{\nu_{j,k}}, \qquad |\alpha_{j,k}| < 1, \quad \sum_{j=1}^{l_k} \nu_{j,k} = \nu_k \le n_k, \quad k = 0, 1, \dots, m-1. \tag{33}$$

If $g \in AL_2(D)$, then $g^{(-1,k)}$ will denote a primitive for $z^k g(z)$ in the disk D; for $z = 0$ this primitive is zero, i.e.,

$$g^{(-1,k)}(z) = \int_0^z t^k g(t)\, dt, \qquad z \in D.$$

Theorem 6. *If the polyanalytic function*

$$f(z) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k} \{(1 - z\bar{z})^k F_k(z)\}$$

belongs to the space $A_m L_2(D)$, *then the following interpolation relations are valid*:

$$F_k^{(s)}(0) = r_{n_k}^{(s)}(0\,; f), \qquad s = 0, 1, \dots, 2(n_k - \nu_k), \tag{34}$$

$$F_k^{(-1,k)}(\alpha_{j,k}) = r_{n_k}^{(-1,k)}(\alpha_{j,k}\,; f), \qquad j = 1, \dots, l_k, \tag{35}$$

$$F_k^{(p)}(\alpha_{j,k}) = r_{n_k}^{(p)}(\alpha_{j,k}\,; f), \qquad j = 1, \dots, l_k, \quad p = 0, 1, \dots, 2\nu_{j,k} - 2, \quad k = 0, 1, \dots, m-1. \tag{36}$$

For a given multi-index $(n) = (n_0, n_1, \dots, n_{m-1})$, let $M_{(n)}(Q)$ denote the subspace of polyrational functions the form

$$r_{(n)}(z) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k} \left\{ (1 - z\bar{z})^k \frac{P_{n_k}(z)}{Q_{n_k}(z)} \right\},$$

where the polynomials $Q_{n_k}(z)$ are defined by relations (33) and fixed, and the $P_{n_k}(z)$ belong to the space of polynomials of degree at most n_k, $k = 0, 1, \dots, m-1$.

In the case of the approximation of a polyanalytic function $f \in A_m L_2(D)$ by polyrational functions from the space $M_{(n)}(Q)$ in the metric of the space $L_2(D)$, we obtain the following criterion of the element of best approximation.

Theorem 7. *Suppose we fix the multi-index* $(n) = (n_0, n_1, \dots, n_{m-1})$*, the function* $f \in A_m L_2(D)$*, and the polynomials* $Q_{n_k}(z)$ *defined by relations* (33) *and satisfying the condition* $\deg\{Q_{n_k}\} = \nu_k \le n_k$*,* $k = 0, 1, \dots, m-1$*. Then in order that the polyrational function*

$$r_{(n)}(z) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k r_{n_k}(z)\}, \quad r_{n_k}(z) = \frac{P_{n_k}(z)}{Q_{n_k}(z)},$$

be the polyrational function of best approximation from the space $M_{(n)}(Q)$ *for the polyanalytic function* $f \in A_m L_2(D)$ *in the metric* $L_2(D)$*, it is necessary and sufficient to have the following relations:*

$$F_k^{(s)}(0) = r_{n_k}^{(s)}(0), \quad s = 0, 1, \dots, n_k - \nu_k, \qquad F_k^{(-1,k)}(\alpha_{j,k}) = r_{n_k}^{(-1,k)}(\alpha_{j,k}), \quad j = 1, \dots, l_k,$$
$$F_k^{(p)}(\alpha_{j,k}) = r_{n_k}^{(p)}(\alpha_{j,k}), \quad j = 1, \dots, l_k, \quad p = 0, 1, \dots, \nu_{j,k} - 2, \quad k = 0, 1, \dots, m-1.$$

Remark 6. For $m = 1$, Theorems 6 and 7 were proved in [12].

Proof of Theorem 5. Let $\deg\{r_{n_k}(z\,;f)\} < n_k$. As in [12], consider the function ($|\alpha| < 1$)

$$\Phi_k(\lambda) = \iint_D \left| \frac{\partial^k}{\partial z^k}\left\{(1 - z\bar{z})^k \left(F_k(z) - r_{n_k}(z\,;f) - \frac{\lambda}{1 - \bar{\alpha} z}\right)\right\}\right|^2 dx\,dy.$$

By Remark 4, we have $\Phi_k(0) \le \Phi_k(\lambda)$. Therefore, $\partial\Phi_k/\partial\bar{\lambda} = 0$ for $\lambda = 0$, i.e.,

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;f))\} \overline{\frac{\partial^k}{\partial z^k}\left\{(1 - z\bar{z})^k \frac{1}{1 - \bar{\alpha} z}\right\}}\, dx\,dy = 0.$$

Hence, expanding the function $1/(1 - \bar{\alpha} z)$ ($|z| < 1$, $|\alpha| < 1$) in the power series, we obtain

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;f))\} \overline{\frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k z^j\}}\, dx\,dy = 0, \qquad j = 0, 1, \dots. \tag{37}$$

It follows from relation (37) and Corollary 1 that the polyanalytic function

$$\frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;f))\}$$

from the space $A_{k+1} L_2^0(D)$ is orthogonal to the same space; but since the space $A_{k+1}^0 L_2(D)$ is complete, this yields $F_k(z) = r_{n_k}(z\,;f)$, $z \in D$. The obtained contradiction proves Theorem 5. □

Proof of Theorem 6. In what follows, let k be an integer from the closed interval $[0, m-1]$. Consider the function (also see [12])

$$\Psi_k(\lambda) = \iint_D \left| \frac{\partial^k}{\partial z^k}\left\{(1 - z\bar{z})^k \left(F_k(z) - r_{n_k}(z\,;f) - \frac{P_{n_k}(z) + \lambda u(z)}{Q_{n_k}(z) + \lambda v(z)}\right)\right\}\right|^2 dx\,dy,$$

where u and v are arbitrary polynomials of degree at most n_k.

Since $\Psi_k(0) \le \Psi_k(\lambda)$, we have $\partial\Psi_k/\partial\bar{\lambda} = 0$ for $\lambda = 0$. By Theorem 5, $\deg\{r_{n_k}(z\,;f)\} = n_k$. Therefore, since the polynomials u and v (of degree at most n_k) are arbitrary, it follows from the relation $\partial\Psi_k/\partial\bar{\lambda} = 0$ that

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1 - z\bar{z})^k (F_k(z) - r_{n_k}(z\,;f))\} \overline{\frac{\partial^k}{\partial z^k}\left\{(1 - z\bar{z})^k \frac{T(z)}{Q_{n_k}^2(z)}\right\}}\, dx\,dy = 0 \tag{38}$$

for any polynomial $T(z)$ of degree at most $2n_k$. Choosing the polynomial $T(z)$ from relation (38), we obtain

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^k(F_k(z)-r_{n_k}(z\,;\,f))\}\overline{\frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^k z^s\}}\,dx\,dy=0, \tag{39}$$
$$s=0,1,\dots,2(n_k-\nu_k),$$

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^k(F_k(z)-r_{n_k}(z\,;\,f))\}\overline{g_1(z)}\,dx\,dy=0, \tag{40}$$

$$\iint_D \frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^k(F_k(z)-r_{n_k}(z\,;\,f))\}\overline{g_2(z)}\,dx\,dy=0, \tag{41}$$
$$j=1,2,\dots,l_k,\quad p=0,1,\dots,2\nu_{j,k}-2,$$

where

$$g_1(z)=\frac{\partial^k}{\partial z^k}\left\{(1-z\bar{z})^k\frac{1}{1-z\bar{\alpha}_{j,k}}\right\},\qquad g_2(z)=\frac{\partial^k}{\partial z^k}\left\{(1-z\bar{z})^k z^p\frac{k(1-z\overline{\alpha_{j,k}})+p+1}{(1-z\bar{\alpha}_{j,k})^{p+2}}\right\}.$$

Using Corollary 5, from relations (39) and (41) we obtain relations (34) and (36) of Theorem 6. To prove relation (35), let us calculate the integral on the left-hand side of relation (40).

Set

$$F_k(z)-r_{n_k}(z\,;\,f)=\sum_{j=0}^{\infty}a_jz^j,\qquad \alpha_{j,k}=\alpha.$$

Expanding the function $1/(1-z\bar{\alpha})$, $|z|<1$, in its power series, let us express relation (40) as

$$\iint_D \sum_{j=0}^{\infty}a_j\frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^kz^j\}\overline{\sum_{j=0}^{\infty}\bar{\alpha}^j\frac{\partial^k}{\partial z^k}\{(1-z\bar{z})^kz^j\}}\,dx\,dy=0.$$

Hence, using Lemma 1, we obtain

$$\sum_{j=0}^{\infty}a_j\frac{\alpha^j}{j+k+1}=0,$$

which is equivalent to relation (35). □

Proof of Theorem 7. The proof is similar to that of Theorem 4 from [12] with the use of Remark 4. □

Proof of Corollary 9. Suppose that the component $F_k(z)$ of the function (28) is given by $F_k(z)=\sum_{j=0}^{\infty}a_{j,k}z^j$ and

$$r_{n_k}(z\,;\,F_k)=\sum_{j=0}^{\infty}b_{j,k}z^j,\qquad \rho_{n_k}(z\,;\,F_k)=\sum_{j=0}^{\infty}c_{j,k}z^j$$

are its rational functions of best approximation of degree at most n_k in the metrics defined by relations (32) and (3), respectively. Then, using Remark 4 and relation (4), we obtain

$$\begin{aligned}\{L_2\widetilde{R}_{(n)}(f,D,m)\}^2&=\pi\sum_{k=0}^{m-1}(k!)^2\left\{\sum_{j=0}^{\infty}\frac{|a_{j,k}-b_{j,k}|^2}{j+k+1}\right\}\le\pi\sum_{k=0}^{m-1}(k!)^2\left\{\sum_{j=0}^{\infty}\frac{|a_{j,k}-c_{j,k}|^2}{j+k+1}\right\}\\&\le\sum_{k=0}^{m-1}(k!)^2\{L_2R_{n_k}(F_k,D)\}^2.\end{aligned} \tag{42}$$

Further, we have

$$\{L_2R_{n_k}(F_k,D)\}^2=\pi\left\{\sum_{j=0}^{\infty}\frac{|a_{j,k}-c_{j,k}|^2}{j+1}\right\}\le\pi\left\{\sum_{j=0}^{\infty}\frac{j+k+1}{j+1}\,\frac{|a_{j,k}-b_{j,k}|^2}{j+k+1}\right\}. \tag{43}$$

Relations (42) and (43) yield the assertion of Corollary 9. □

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