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ON δ-DERIVATIONS OF LIE ALGEBRAS

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Let A be an algebra over a unital commutative associative ring Φ containing $\frac{1}{2}$.

An *antiderivation* of A is a Φ-linear mapping $\phi : A \mapsto A$ satisfying the equality $(xy)\phi = -(x\phi)y - x(y\phi)$ for arbitrary $x, y \in A$. The author used this notion in [1] while studying Lie algebras that satisfy a certain identity of degree 5. In particular, it was proven in [1] that if $\frac{1}{6} \in \Phi$ then every prime Φ-algebra Lie possessing a nonzero antiderivation satisfies a standard identity of degree 5. It was also proven in [1] that 3-dimensional simple Lie algebras over a field serve as examples of simple algebras with a nonzero antiderivation. Antiderivations of Lie algebras were also considered by N. C. Hopkins [2] who proved that if L is a central simple finite-dimensional Lie algebra over a field of characteristic $p \neq 3, 5$ with a nondegenerate trace form then L has no antiderivations whenever $\dim L \geq 4$. In the present article we prove in particular that this theorem admits a broad generalization.

Given an arbitrary $\delta \in \Phi$, a δ-*derivation* of A is defined to be a Φ-linear mapping $\phi : A \mapsto A$ satisfying the identity

$$(xy)\phi = \delta(x\phi)y + \delta x(y\phi), \tag{1}$$

where x and y are arbitrary elements of A.

If $\delta = 1$ then ϕ is a derivation, and if $\delta = -1$ then ϕ is an antiderivation of A.

Construction of some integrable dynamical systems and their solutions uses the so-called R-matrix method (see, for instance, [3]) which is connected with the notion of double Lie algebra (see the definition below). In the present article we introduce the notion of the mutant A_ϕ of a Φ-algebra A which is determined by a δ-derivation ϕ, and prove in Theorem 1 that, under some constraints on the additive group of the ring Φ, the mutant A_ϕ is also a Lie algebra. In this case the δ-derivation ϕ determines a double Lie algebra structure on the Φ-module of the algebra A (Corollary 1).

We prove that every prime Lie Φ-algebra with a nondegenerate symmetric invariant bilinear form has no δ-derivations whenever $\delta \neq -1, 0, \frac{1}{2}, 1$ (Theorem 3). By the way we prove that if A is an arbitrary $\delta(\delta-1)$-torsion free Lie Φ-algebra with a symmetric invariant bilinear form then the restriction $\tilde{\phi}$ of the δ-derivation ϕ to the square of this algebra is a symmetric mapping (Theorem 2).

We next consider the case of $\delta = -1$. We prove that every prime Lie Φ-algebra A ($\frac{1}{6} \in \Phi$) with a nondegenerate antiderivation is a 3-dimensional simple algebra over the field of fractions of the center $Z_R(A)$ of the algebra $R(A)$ of right multiplications (Theorem 5). This assertion ensues from Theorem 4, claiming that the algebra A is an h-algebra (see the definition below), and the description for h-algebras in [1, Theorem 2]. Theorem 5 implies in particular that if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ with a nondegenerate symmetric invariant bilinear form then A has no antiderivations whenever $\dim A \geq 4$ (Corollary 2).

We give a complete description for the $\frac{1}{2}$-derivations of an arbitrary prime Φ-algebra A ($\frac{1}{6} \in \Phi$) with a nondegenerate symmetric invariant bilinear form. We prove that a linear mapping $\phi : A \mapsto A$ is a $\frac{1}{2}$-derivations if and only if $\phi \in \Gamma(A)$, where $\Gamma(A)$ is the centroid of the A (Theorem 6). This result implies in particular that if A is a central simple Lie algebra over a field Φ of characteristic $p \neq 2, 3$ with a nondegenerate symmetric invariant bilinear form then every $\frac{1}{2}$-derivations ϕ has the shape $x\phi = \alpha x$, where $\alpha \in \Phi$ (Corollary 3).

Observe that if $\delta = 0$ and $A^2 \neq A$ then by (1) every nonzero endomorphism ϕ of the Φ-module of A such that $(A^2)\phi = 0$ is a nonzero 0-derivation. If $A^2 = A$ and, in particular, A is a simple algebra then A has no nonzero 0-derivations. Henceforth we assume that $\delta \neq 0$. The case of $\delta = 1$ is not inspected either, since the definition of 1-derivation coincides with the standard definition of derivation of a Lie algebra.

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In what follows, unless otherwise stated, A is assumed to be an arbitrary Lie Φ-algebra. To simplify notations, we write words of the shape $w = (\dots(x_1x_2)\dots)x_n$ without parentheses: $w = x_1x_2\dots x_n$.

We denote by R_x the operator of the right multiplication by x, $x \in A$, and denote by $R(A)$ the algebra of right multiplications of A, i.e., the associative algebra generated by the identity operator and all operators of right multiplications of A. We denote the center of $R(A)$ by $Z_R(A)$ and denote the annihilator of A by $\operatorname{Ann} A$.

We recall that the operators of right multiplication in A are derivations.

Given an arbitrary function $f(x,y,z)$, we define the operator $\sigma(x,y,z)$ as follows:

$$\sigma(x,y,z)f(x,y,z) = f(x,y,z) + f(z,x,y) + f(y,z,x).$$

Observe that the Jacobi identity is written down in our notations as follows: $\sigma(x,y,z)xyz = 0$.

Let $xs(y,z,t)$ be a standard (Lie) polynomial of degree 4. Using the operator σ, we can write this polynomial in an arbitrary algebra as follows: $xs(y,z,t) = \sigma(y,z,t)(xyzt - xytz)$, $xs(y,z,t) = \sigma(y,z,t)(xyzt - xzyt)$. From here and the Jacobi identity we obtain the identities

$$xs(y,z,t) = \sigma(y,z,t)xy(zt), \tag{2}$$

$$xs(y,z,t) = \sigma(y,z,t)x(yz)t. \tag{3}$$

Observe that the operator $s(y,z,t)$ is skew-symmetric in y,z,t.

We put

$$q(x,y,z,t,u) = xs(y,z,t)u - xus(y,z,t).$$

By definition, $xs(y,z,t)$ is a skew-symmetric function in y,z,t. Therefore, $q(x,y,z,t,u)$ too is a skew-symmetric function in y,z,t.

In line with [1], we understand an *h-algebra* to be a Lie algebra that satisfies the identity $yz(tx)x + yx(zx)t = 0$.

It is well known (see, for instance, [1, p. 688]) that an algebra A is an h-algebra if and only if the identity $q(x,y,z,t,u) = 0$ holds in A.

Let α be a fixed nonzero element of the ring Φ. We call an algebra A an *α-torsion free* algebra if, for every $a \in A$, from $\alpha a = 0$ it follows that $a = 0$.

Henceforth ϕ stands for an arbitrary δ-derivation of an algebra A, where δ is a fixed nonzero element of the ring Φ. Unless otherwise stated, A is assumed $\delta(\delta-1)$-torsion free.

Observe that if Φ is a field then the last assumption is equivalent to the condition $\delta \neq 0,1$. In this case ϕ is not a derivation of A and its restriction to A^2 is nonzero.

If D is an arbitrary derivation of A and $[\phi, D] = \phi D - D\phi$ then it is easy to show that

$$(xy)[\phi, D] = \delta(x[\phi,D])y + \delta x(y[\phi,D])$$

for all $x, y \in A$; i.e., $[\phi, D]$ is a δ-derivation of A. In particular, $[\phi, R_u]$ is a δ-derivation of A for every $u \in A$.

Define the new multiplication $(\circ)$ on the Φ-module of the algebra A by setting $x \circ y = (xy)\phi$ for arbitrary $x, y \in A$. By bilinearity and anticommutativity of the multiplication of the algebra A, the Φ-module of A becomes an anticommutative Φ-algebra under the multiplication $(\circ)$. We call this algebra the *mutant* of A and denote it by A_ϕ.

Theorem 1. *If A is a $\delta(\delta-1)$-torsion free Lie Φ-algebra then the mutant A_ϕ of A is a Lie Φ-algebra.*

PROOF. We first prove that

$$\sigma(x,y,z)xy(z\phi) = 0 \tag{4}$$

for all $x, y, z \in A$. By anticommutativity, the Jacobi identity, the definition of σ, and (1) we have

$$\begin{aligned}
\delta(\delta-1)\sigma(x,y,z)xy(z\phi) &= \sigma(x,y,z)[\delta^2 xy(z\phi) - \delta xy(z\phi)] \\
= \sigma(x,y,z)[-\delta^2(z\phi)(xy) - \delta xy(z\phi)] &= \sigma(x,y,z)[-\delta^2(z\phi)xy + \delta^2(z\phi)yx - \delta xy(z\phi)] \\
&= -\sigma(x,y,z)[\delta^2(z\phi)xy + \delta^2 y(z\phi)x + \delta xy(z\phi)] \\
&= -\sigma(x,y,z)[\delta^2(x\phi)yz + \delta^2 x(y\phi)z + \delta xy(z\phi)] \\
&= -\sigma(x,y,z)[\delta(xy)\phi z + \delta xy(z\phi)] = -\sigma(x,y,z)(xyz)\phi = 0.
\end{aligned}$$

Since A is $\delta(\delta-1)$-torsion free, (4) follows.

By (1), (4), and the Jacobi identity we have

$$\delta\sigma(x,y,z)(xy)\phi z = \sigma(x,y,z)(xyz)\phi - \delta\sigma(x,y,z)xy(z\phi) = \sigma(x,y,z)(xyz)\phi = 0.$$

This implies the equality

$$\sigma(x,y,z)(xy)\phi z = 0. \tag{5}$$

By the definition of the multiplication ($\circ$) and (5) we have

$$\sigma(x,y,z)(x \circ y) \circ z = \sigma(x,y,z)[(xy)\phi z]\phi = 0.$$

Therefore, A_Φ satisfies the Jacobi identity; i.e., A_Φ is a Lie algebra, which completes the proof of the theorem.

If $\mathrm{End}_\Phi A$ is the endomorphism algebra of the Φ-module of A and $R \in \mathrm{End}_\Phi A$ then, in line with [3], we say that R determines a *double Lie algebra structure* on A if the Φ-module of A is also a Lie algebra under the new multiplication

$$(x * y)_R = \frac{1}{2}[(xR)y + x(yR)]. \tag{6}$$

Let ϕ be a δ-derivation of A such that $\frac{1}{2\delta(\delta-1)} \in \Phi$. Setting $R = \phi$ in (6), by (1) we have

$$(x * y)_\phi = \frac{1}{2}[(x\phi)y + x(y\phi)] = \frac{1}{2\delta}(xy)\phi = \frac{1}{2\delta} x \circ y.$$

This equality and Theorem 1 imply the following

Corollary 1. *If $\frac{1}{2\delta(\delta-1)} \in \Phi$ then the δ-derivation ϕ determines a double Lie algebra structure on the Lie Φ-algebra A.*

A *bilinear form* on an algebra A is a Φ-bilinear mapping $\gamma : A \times A \mapsto \Phi$, where $A \times A$ is the Cartesian square of A.

Henceforth we write (x,y) in place of $\gamma(x,y)$.

We call a bilinear form (x,y) *symmetric* if

$$(x,y) = (y,x), \tag{7}$$

and *invariant* (or *associative*) if

$$(xy,z) = (x,yz) \tag{8}$$

for all $x,y,z \in A$.

For $z = y$, from (8) and anticommutativity of A we obtain the equality

$$(xy,y) = 0. \tag{9}$$

We call a symmetric form (x,y) *nondegenerate* if there is no nonzero $a \in A$ such that $(A,a) = 0$.

We say that a Φ-linear mapping $\phi : A \mapsto A$ is *symmetric* (with respect to the form (x,y)) if

$$(x\phi, y) = (x, y\phi) \tag{10}$$

for all $x,y \in A$.

From now on, we assume that the algebra A is furnished with a symmetric invariant bilinear form (x,y).

Theorem 2. *If A is an arbitrary Lie Φ-algebra ($\frac{1}{2} \in \Phi$) with a symmetric invariant bilinear form (x, y) and ϕ is a δ-derivation of A such that A is $\delta(\delta - 1)$-torsion free, then the restriction $\tilde{\phi}$ of ϕ to the square A^2 of A is symmetric. In particular, if $A = A^2$ then the mapping ϕ is symmetric.*

PROOF. It suffices to show that

$$((tz)\phi, xy) = (tz.(xy)\phi) \tag{11}$$

for arbitrary $x, y, z, t \in A$.

Successively using (1), (8), (4), (8), and the Jacobi identity, we obtain

$$\begin{aligned}\sigma(x,y,z)((tz)\phi, xy) &= \sigma(x,y,z)(\delta(t\phi)z + \delta t(z\phi), xy)\\ &= \delta\sigma(x,y,z)((t\phi)z, xy) + \delta\sigma(x,y,z)(t(z\phi), xy)\\ &= \delta\sigma(x,y,z)((t\phi)z, xy) + \delta\sigma(x,y,z)(t, (z\phi)(xy))\\ &= \delta\sigma(x,y,z)((t\phi)z, xy) - \delta(t, \sigma(x,y,z)xy(z\phi))\\ = \delta\sigma(x,y,z)((t\phi)z, xy) &= \delta\sigma(x,y,z)(t\phi, z(xy)) = -\delta(t\phi, \sigma(x,y,z)xyz) = 0.\end{aligned}$$

This implies the equalities

$$\begin{aligned}((tz)\phi, xy) + ((ty)\phi, zx) + ((tx)\phi, yz) &= 0,\\ -((zt)\phi, xy) - ((zy)\phi, tx) - ((zx)\phi, yt) &= 0,\\ ((yt)\phi, zx) + ((yx)\phi, tz) + ((yz)\phi, xt) &= 0,\\ ((xt)\phi, yz) + ((xz)\phi, ty) + ((xy)\phi, zt) &= 0.\end{aligned}$$

Add these equalities together and collect similar terms by using commutativity of A to obtain $2((tz)\phi, xy) - 2((xy)\phi, tz) = 0$, which implies (11) after cancellation by 2. The theorem is proven.

Lemma 1. *The following equalities hold for all $x, y, z, t, v, w \in A$:*

$$(xs(y,z,t), w) = (ws(y,z,t), x), \tag{12}$$

$$(q(x,y,z,t,v), w) = (q(w,y,z,t,v), x), \tag{13}$$

$$\sigma(x,v,w)(q(x,y,z,t,v), w) = 0. \tag{14}$$

PROOF. Using (2), (8) twice, (7), the Jacobi identity, and again (2), we obtain

$$\begin{aligned}(xs(y,z,t), w) &= \sigma(y,z,t)(xy(zt), w) = \sigma(y,z,t)(xy, ztw) = \sigma(y,z,t)(x, y(ztw))\\ &= \sigma(y,z,t)(x, w(zt)y) = \sigma(y,z,t)(w(zt)y, x) = \sigma(y,z,t)(wy(zt) + w(zty), x)\\ &= \sigma(y,z,t)(wy(zt), x) + \sigma(y,z,t)(w(yzt), x)\\ &= \sigma(y,z,t)(wy(zt), x) = (ws(y,z,t), x),\end{aligned}$$

and (12) is established.

Using the definition of the function q, (8), (12), again (12), (8), and again the definition of q, we obtain

$$\begin{aligned}(q(x,y,z,t,v), w) &= (xs(y,z,t)v, w) - (xvs(y,z,t), w)\\ &= (xs(y,z,t), vw) - (ws(y,z,t), xv) = (vws(y,z,t), x) + (ws(y,z,t), vx)\\ &= -(wvs(y,z,t), x) + (ws(y,z,t)v, x) = (q(w,y,z,t,v), x),\end{aligned}$$

and (13) is established.

By the definition of q we have

$$q(x,y,z,t,x) = xs(y,z,t)x. \tag{15}$$

From here and (9) we infer the equality

$$(q(x,y,z,t,x),x) = (xs(y,z,t)x,x) = 0. \tag{16}$$

Linearize (16) in x:

$$\begin{aligned}&[(q(x,y,z,t,v),w) + (q(w,y,z,t,v),x)] + [(q(w,y,z,t,x),v)\\ &+(q(v,y,z,t,x),w)] + [(q(v,y,z,t,w),x) + (q(x,y,z,t,w),v)] = 0.\end{aligned}$$

Apply (13) to the last equality:

$$2(q(x,y,z,t,v),w) + 2(q(w,y,z,t,x),v) + 2(q(v,y,z,t,w),x) = 0.$$

After cancellation by 2 we arrive at (14), finishing the proof of the lemma.

Lemma 2. *The following equalities hold in the algebra A:*

$$\sigma(y,z,t)xt[(yz)\phi] = \sigma(y,z,t)x(t\phi)(yz), \tag{17}$$

$$\delta(\delta+1)\sigma(y,z,t)x(t\phi)(yz) = -\delta^2(x\phi)s(y,z,t) + [xs(y,z,t)]\phi. \tag{18}$$

PROOF. Using (3), (1), again (3), twice the Jacobi identity, (5), and (4), we obtain

$$\begin{aligned}[xs(y,z,t)]\phi &= \sigma(y,z,t)[x(yz)t]\phi = \delta\sigma(y,z,t)[x(yz)]\phi t + \delta\sigma(y,z,t)x(yz)(t\phi)\\ &= \delta^2\sigma(y,z,t)(x\phi)(yz)t + \delta^2\sigma(y,z,t)x[(yz)\phi]t + \delta\sigma(y,z,t)x(yz)(t\phi)\\ &= \delta^2(x\phi)s(y,z,t) + \delta^2\sigma(y,z,t)x[(yz)\phi]t + \delta\sigma(y,z,t)x(yz)(t\phi)\\ &= \delta^2(x\phi)s(y,z,t) + \delta^2\sigma(y,z,t)xt[(yz)\phi] + \delta^2\sigma(y,z,t)x[(yz)\phi t]\\ &\quad+\delta\sigma(y,z,t)x(t\phi)(yz) + \delta\sigma(y,z,t)x[(yz)(t\phi)]\\ &= \delta^2(x\phi)s(y,z,t) + \delta^2\sigma(y,z,t)xt[(yz)\phi] + \delta\sigma(y,z,t)x(t\phi)(yz).\end{aligned} \tag{19}$$

On the other hand, by (2) and (1) we have

$$\begin{aligned}[xs(y,z,t)]\phi &= \sigma(y,z,t)[xy(zt)]\phi = \sigma(y,z,t)[xt(yz)]\phi\\ &= \delta\sigma(y,z,t)(xt)\phi(yz) + \delta\sigma(y,z,t)xt[(yz)\phi]\\ &= \delta^2\sigma(y,z,t)(x\phi)t(yz) + \delta^2\sigma(y,z,t)x(t\phi)(yz) + \delta\sigma(y,z,t)xt[(yz)\phi]\\ &= \delta^2(x\phi)s(t,y,z) + \delta\sigma(y,z,t)xt[(yz)\phi] + \delta^2\sigma(y,z,t)x(t\phi)(yz).\end{aligned}$$

Subtract the last equality from (19) and collect similar terms by using skew symmetry of s to obtain the equality

$$\delta(\delta-1)\sigma(y,z,t)xt[(yz)\phi] - \delta(\delta-1)\sigma(y,z,t)x(t\phi)(yz) = 0.$$

After cancellation by $\delta(\delta-1)$, we arrive at (17).

Using (5), twice the Jacobi identity, (1), (3), (2), twice (1), (3), (2), skew symmetry of s, the Jacobi identity, and (4), we obtain

$$
\begin{aligned}
0 = \delta^2\sigma(y,z,t)x[(yz)\phi t] &= \delta^2\sigma(y,z,t)x[(yz)\phi]t - \delta^2\sigma(y,z,t)xt[(yz)\phi] \\
&= -\delta^2\sigma(y,z,t)(x\phi)(yz)t + \delta\sigma(y,z,t)[x(yz)]\phi t \\
&\quad +\delta^2\sigma(y,z,t)[(xt)\phi](yz) - \delta\sigma(y,z,t)[xt(yz)]\phi \\
&= -\delta^2(x\phi)s(y,z,t) + \delta\sigma(y,z,t)[x(yz)]\phi t + \delta^2\sigma(y,z,t)[(xt)\phi](yz) - \delta[xs(t,y,z)]\phi \\
&= -\delta^2(x\phi)s(y,z,t) - \delta\sigma(y,z,t)x(yz)(t\phi) + \sigma(y,z,t)[x(yz)t]\phi \\
&\quad +\delta^3\sigma(y,z,t)(x\phi)t(yz) + \delta^3\sigma(y,z,t)x(t\phi)(yz) - \delta[xs(t,y,z)]\phi \\
&= -\delta^2(x\phi)s(y,z,t) - \delta\sigma(y,z,t)x(yz)(t\phi) + [xs(y,z,t)]\phi \\
&\quad +\delta^3(x\phi)s(t,y,z) + \delta^3\sigma(y,z,t)x(t\phi)(yz) - \delta[xs(y,z,t)]\phi \\
&= (\delta^3-\delta^2)(x\phi)s(y,z,t) - (\delta-1)[xs(y,z,t)]\phi \\
&\quad -\delta\sigma(y,z,t)x(yz)(t\phi) + \delta^3\sigma(y,z,t)x(t\phi)(yz) \\
&= (\delta^3-\delta^2)(x\phi)s(y,z,t) - (\delta-1)[xs(y,z,t)]\phi \\
&\quad -\delta\sigma(y,z,t)x(t\phi)(yz) - \delta\sigma(y,z,t)x[yz(t\phi)] + \delta^3\sigma(y,z,t)x(t\phi)(yz) \\
&= (\delta^3-\delta^2)(x\phi)s(y,z,t) - (\delta-1)[xs(y,z,t)]\phi + (\delta^3-\delta^2)\sigma(y,z,t)x(t\phi)(yz) \\
&= (\delta-1)[\delta^2(x\phi)s(y,z,t) - [xs(y,z,t)]\phi + \delta(\delta+1)\sigma(y,z,t)x(t\phi)(yz)].
\end{aligned}
$$

After cancellation by $\delta - 1$, we arrive at (18), finishing the proof of the lemma.

We recall that a nonzero algebra is said to be *prime* if the product of arbitrary two nonzero ideals of the algebra is nonzero.

It is well known that the operator ring Φ of an arbitrary prime Φ-algebra B is an integral domain and B is Φ-torsion free; i.e., for arbitrary $\alpha \in \Phi$ and $b \in B$, it follows from $\alpha b = 0$ that $\alpha = 0$ or $b = 0$.

Furthermore, it is well known that every prime Lie algebra has no nonzero solvable ideals; in particular, the algebra is not solvable itself.

The proof of the following lemma is analogous to that of Lemma 5 in [1] and we expose it just for the sake of completeness.

Lemma 3. *If A is a prime Lie algebra then, for an arbitrary nonzero ideal I and every δ-derivation ϕ ($\delta \neq 0$), the restriction $\tilde{\phi}$ of ϕ to I is nonzero as well.*

PROOF. Suppose that there is a nonzero ideal of A and a nonzero δ-derivation ϕ such that $I\phi = 0$. By the definition of δ-derivation, for arbitrary $a \in I$ and $x \in A$ we then have $\delta a(x\phi) = (ax)\phi - \delta(a\phi)x = 0$. Since $\delta \neq 0$, it follows that $a(x\phi) = 0$. Therefore, $I(A\phi) = 0$; i.e., $A\phi \subseteq \operatorname{Ann} I$, where $\operatorname{Ann} I$ is the annihilator of the ideal I. It is easy to see that the annihilator of an ideal in any Lie algebra is again an ideal. Since $A\phi \subseteq \operatorname{Ann} I$, the ideal N of A generated by all elements of $A\phi$ lies therefore in $\operatorname{Ann} I$. Hence, $IN = 0$ and, since A is a prime algebra and $I \neq 0$, we have $N = 0$ and $A\phi = 0$. Therefore, $\phi = 0$, which contradicts our supposition. The lemma is proven.

Henceforth, unless otherwise stated, we assume that A is a prime Lie Φ-algebra furnished with a nondegenerate symmetric invariant bilinear form (x, y).

Let us prove that if $a \in A$ then

$$(A^2, a) = 0 \Rightarrow a = 0. \tag{20}$$

Indeed, by invariance of the form, (7), and the condition $(A^2, a) = 0$ we have $(ay, x) = (a, yx) = (yx, a) = 0$ for all $x, y \in A$. By nondegeneracy of the form, we infer the equality $ay = 0$; i.e., $a \in \operatorname{Ann} A$. Since A is a prime algebra, it follows that $\operatorname{Ann} A = 0$ and so $a = 0$; i.e., (20) holds.

Lemma 4. *If A is a prime algebra, A is endowed with a nondegenerate symmetric invariant bilinear form (x, y), and ϕ is a δ-derivation of A ($\delta \neq 0, 1$), then*

$$(\delta^2 + \delta - 1)[xs(y,z,t)]\phi = -\delta^2(x\phi)s(y,z,t) \tag{21}$$

for all $x, y, z, t \in A$.

PROOF. Let x, y, z, and t be arbitrary elements in A and let v be an arbitrary element in A^2. Using (17), (8), (7), (11), twice (1), trice linearized (9), the definition of σ, (11), and twice (2), we obtain

$$\begin{aligned}
\sigma(y,z,t)(x(t\phi)(yz),v) &= \sigma(y,z,t)(xt[(yz)\phi],v) = -\sigma(y,z,t)((yz)\phi(xt),v) \\
&= -\sigma(y,z,t)((yz)\phi,xtv) = -\sigma(y,z,t)(xtv,(yz)\phi) = -\sigma(y,z,t)((xtv)\phi,yz) \\
&= -\delta^2\sigma(y,z,t)((x\phi)tv,yz) - \delta^2\sigma(y,z,t)(x(t\phi)v,yz) - \delta\sigma(y,z,t)(xt(v\phi),yz) \\
&= \delta^2\sigma(y,z,t)((x\phi)t(yz),v) + \delta^2\sigma(y,z,t)(x(t\phi)(yz),v) + \delta\sigma(y,z,t)(xt(yz),v\phi) \\
&= \delta^2\sigma(y,z,t)((x\phi)y(zt),v) + \delta^2\sigma(y,z,t)(x(t\phi)(yz),v) + \delta\sigma(y,z,t)(xy(zt),v\phi) \\
&= \delta^2\sigma(y,z,t)((x\phi)y(zt),v) + \delta^2\sigma(y,z,t)(x(t\phi)(yz),v) + \delta\sigma(y,z,t)([xy(zt)]\phi,v) \\
&= \delta^2((x\phi)s(y,z,t),v) + \delta^2\sigma(y,z,t)(x(t\phi)(yz),v) + \delta([xs(y,z,t)]\phi,v).
\end{aligned}$$

Collecting similar terms, we arrive at the equality

$$(\delta^2(x\phi)s(y,z,t) + (\delta^2-1)\sigma(y,z,t)x(t\phi)(yz) + \delta[xs(y,z,t)]\phi, v) = 0,$$

which by (20) implies

$$\delta^2(x\phi)s(y,z,t) + (\delta^2-1)\sigma(y,z,t)x(t\phi)(yz) + \delta[xs(y,z,t)]\phi = 0.$$

Hence,

$$-(\delta+1)(\delta-1)\sigma(y,z,t)x(t\phi)(yz) = \delta^2(x\phi)s(y,z,t) + \delta[xs(y,z,t)]\phi.$$

Multiply the last equality by δ:

$$-\delta(\delta+1)(\delta-1)\sigma(y,z,t)x(t\phi)(yz) = \delta^3(x\phi)s(y,z,t) + \delta^2[xs(y,z,t)]\phi.$$

Multiply (18) by $\delta - 1$ to obtain

$$\delta(\delta+1)(\delta-1)\sigma(y,z,t)x(t\phi)(yz) = -\delta^2(\delta-1)(x\phi)s(y,z,t) + (\delta-1)[xs(y,z,t)]\phi.$$

Adding the last two equalities and collecting similar terms, we find that

$$\delta^2(x\phi)s(y,z,t) + (\delta^2+\delta-1)[xs(y,z,t)]\phi = 0.$$

This implies (21). The lemma is proven.

Theorem 3. *A prime Lie Φ-algebra A ($\frac{1}{2} \in \Phi$) with a nondegenerate symmetric invariant bilinear form has no nonzero δ-derivations whenever $\delta \neq -1, 0, \frac{1}{2}, 1$.*

PROOF. Let $\delta \in \Phi$, $\delta \neq -1, 0, \frac{1}{2}, 1$. Suppose that ϕ is a nonzero δ-derivation. Let y, z, t, and w be arbitrary elements in A and let x and v be arbitrary elements in A^2. By (21), (12), and (11) we have

$$\begin{aligned}
(\delta^2+\delta-1)([xs(y,z,t)]\phi,v) &= -\delta^2((x\phi)s(y,z,t),v) \\
&= -\delta^2(vs(y,z,t),x\phi) = -\delta^2([vs(y,z,t)]\phi,x).
\end{aligned} \tag{22}$$

Interchanging x and v in (22), we obtain

$$(\delta^2+\delta-1)([vs(y,z,t)]\phi,x) = -\delta^2([xs(y,z,t)]\phi,v).$$

From here and (22) we infer that

$$(\delta^2+\delta-1)^2([xs(y,z,t)]\phi,v) = -\delta^2(\delta^2+\delta-1)([vs(y,z,t)]\phi,x) = \delta^4([xs(y,z,t)]\phi,v).$$

Hence,

$$[(\delta^2+\delta-1)^2-\delta^4]([xs(y,z,t)]\phi, v)=0,$$

and by (20) we have

$$[(\delta^2+\delta-1)^2-\delta^4][xs(y,z,t)]\phi=0.$$

Since $(\delta^2+\delta-1)^2-\delta^4=2(\delta-1)(\delta+1)(\delta-\frac{1}{2})$, the preceding equality implies

$$2(\delta-1)(\delta+1)\left(\delta-\frac{1}{2}\right)[xs(y,z,t)]\phi=0. \tag{23}$$

Since $\frac{1}{2}\in\Phi$ and $\delta\neq -1,\frac{1}{2},1$, it follows that $2(\delta-1)(\delta+1)(\delta-\frac{1}{2})\neq 0$. Since A is Φ-torsion free, from here and (23) we obtain

$$[xs(y,z,t)]\phi=0. \tag{24}$$

Let I be a Φ-submodule of the Φ-module of A which is generated by all elements of the shape $xs(y,z,t)$, where $x\in A^2$ and $y,z,t\in A$. From (24) we find that $I\phi=0$. Since R_w is a derivation of A; therefore, $xs(y,z,t)w\in I$, I is an ideal of A. Therefore, by Lemma 3 from the equality $I\phi=0$ we obtain $I=0$. Hence, in A we have the identity $xs(y,z,t)=0$, where $x\in A^2$ and $y,z,t\in A$. From (12) and this identity we find that $(ws(y,z,t),x)=(xs(y,z,t),w)=0$. This fact and (20) imply the identity

$$ws(y,z,t)=0. \tag{25}$$

From (2) and anticommutativity of A we have

$$ws(y,z,t)-ys(w,z,t)=\sigma(y,z,t)wy(zt)-\sigma(w,z,t)yw(zt)=2wy(zt).$$

Using (25), we now deduce the identity $wy(zt)=0$. Hence, A is solvable, which contradicts the fact that it is prime. Thus, ϕ is the zero mapping, which completes the proof of the theorem.

EXAMPLE. Let L be a 3-dimensional simple Lie algebra over a field Φ of characteristic $p\neq 2,3$. The results of the present article imply that L can have nonzero δ-derivations if and only if $\delta=1,\frac{1}{2},-1$. If $\delta=1$ then, since L is perfect, all derivations are inner, i.e., they are exhausted by the operators of right multiplication. If $\delta=\frac{1}{2}$ then by straightforward calculations we check that all $\frac{1}{2}$-derivations are exhausted by the mappings $\phi_\alpha: x\mapsto \alpha x$, where $\alpha\in\Phi$. The antiderivations of the algebra L are described in [1, p. 694].

Henceforth we assume that $\frac{1}{6}\in\Phi$ and consider the case of $\delta=-1,\frac{1}{2}$.

Suppose that $\delta=-1$, i.e., ϕ is an antiderivation of the algebra A. The following equalities hold in A:

$$[xs(y,z,t)]\phi=(x\phi)s(y,z,t), \tag{26}$$

$$\sigma(y,z,t)xs(y\phi,z,t)=0 \tag{27}$$

for all $x,y,z,t\in A$ (a proof of (26) and (27) is given in [1, Lemma 3] and we expose it here for the sake of completeness).

Equality (26) follows from (21) with $\delta=-1$. Prove (27). By (2) and the definition of antiderivation, we have

$$\begin{aligned}
[xs(y,z,t)]\phi &= \sigma(y,z,t)[xy(zt)]\phi=-\sigma(y,z,t)[(xy)\phi(zt)+xy[(zt)\phi]]\\
&=\sigma(y,z,t)(x\phi)y(zt)+\sigma(y,z,t)[x(y\phi)(zt)+[xy(z\phi)t]+xy[z(t\phi)]]\\
&=(x\phi)s(y,z,t)+\sigma(y,z,t)[x(y\phi)(zt)+xt[(y\phi)z]+xz[t(y\phi)]]\\
&=(x\phi)s(y,z,t)+\sigma(y,z,t)[\sigma(y\phi,z,t)x(y\phi)(zt)]\\
&=(x\phi)s(y,z,t)+\sigma(y,z,t)xs(y\phi,z,t).
\end{aligned}$$

From here and (26) we get (27).

Lemma 5. *The following equalities hold in A:*

$$\sigma(y,z,t)q(x,y\phi,z,t,u)=0, \tag{28}$$
$$q(x,y,z,t,u)\phi=q(x\phi,y,z,t,u), \tag{29}$$
$$2q(x,y,z,t,u)\phi=-q(x,y,z,t,u\phi) \tag{30}$$

for all $x,y,z,t,u\in A$.

PROOF. Equality (28) follows from the definition of the function q and (27).

Since the commutator $[\phi,R_u]=\phi R_u-R_u\phi$ is an antiderivation for every operator R_u of right multiplication in A, by virtue of (26) we have the equality

$$xs(y,z,t)[\phi,R_u]=x[\phi,R_u]s(y,z,t).$$

Whence

$$[xs(y,z,t)]\phi u-[xs(y,z,t)u]\phi=(x\phi)us(y,z,t)-(xu)\phi s(y,z,t),$$

and in view of (26)

$$(x\phi)s(y,z,t)u-[xs(y,z,t)u]\phi=(x\phi)us(y,z,t)-[(xu)s(y,z,t)]\phi.$$

Therefore,

$$[xs(y,z,t)u]\phi-[(xu)s(y,z,t)]\phi=(x\phi)s(y,z,t)u-(x\phi)us(y,z,t).$$

This equality and the definition of q imply (29).

Using the definitions of q and antiderivation, twice (26), and again the definition of antiderivation, we obtain

$$\begin{aligned}q(x,y,z,t,u)\phi&=[xs(y,z,t)u]\phi-[(xu)s(y,z,t)]\phi\\&=-[xs(y,z,t)]\phi u-xs(y,z,t)(u\phi)-(xu)\phi s(y,z,t)\\&=-(x\phi)s(y,z,t)u-xs(y,z,t)(u\phi)+(x\phi)us(y,z,t)+x(u\phi)s(y,z,t)\\&=-q(x\phi,y,z,t,u)-q(x,y,z,t,u\phi).\end{aligned}$$

From (29) and the preceding equality we infer (30):

$$\begin{aligned}2q(x,y,z,t,u)\phi&=q(x,y,z,t,u)\phi+q(x,y,z,t,u)\phi\\&=q(x\phi,y,z,t,u)+q(x,y,z,t,u)\phi\\&=q(x\phi,y,z,t,u)-q(x\phi,y,z,t,u)-q(x,y,z,t,u\phi)=-q(x,y,z,t,u\phi).\end{aligned}$$

The lemma is proven.

Lemma 6. *If the form (x,y) is nondegenerate then the following equality holds in A:*

$$[q(w,y,z,t,v)-q(v,y,z,t,w)]\phi=0 \tag{31}$$

for all $y,z,t\in A$ and $v,w\in A^2$.

PROOF. Successively using (30), (14), (11), and (29), we obtain

$$\begin{aligned}2(q(x,y,z,t,v)\phi,w)&=-(q(x,y,z,t,v\phi),w)\\&=(q(w,y,z,t,x),v\phi)+(q(v\phi,y,z,t,w),x)\\&=(q(w,y,z,t,x)\phi,v)+(q(v\phi,y,z,t,w),x)\\&=(q(w,y,z,t,x)\phi,v)+(q(v,y,z,t,w)\phi,x).\end{aligned} \tag{32}$$

Interchange v and w in (32):

$$2(q(x,y,z,t,w)\phi, v) = (q(v,y,z,t,x)\phi, w) + (q(w,y,z,t,v)\phi, x).$$

Subtracting this equality from (32), we arrive at the equality

$$\begin{aligned} &2[(q(x,y,z,t,v)\phi, w) - (q(x,y,z,t,w)\phi, v)] \\ &= (q(w,y,z,t,x)\phi, v) - (q(v,y,z,t,x)\phi, w) - (q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi, x). \end{aligned} \tag{33}$$

In view of (11), (13), and (29) we have

$$\begin{aligned} &2[(q(x,y,z,t,v)\phi, w) - (q(x,y,z,t,w)\phi, v)] \\ &= 2[(q(x,y,z,t,v), w\phi) - (q(x,y,z,t,w), v\phi)] \\ &= 2[(q(w\phi,y,z,t,v), x) - (q(v\phi,y,z,t,w), x)] \\ &= 2(q(w\phi,y,z,t,v) - q(v\phi,y,z,t,w), x) \\ &= 2(q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi, x). \end{aligned} \tag{34}$$

Applying (34) to the left-hand side of (33) and collecting similar terms, we obtain

$$\begin{aligned} &3(q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi, x) \\ &= (q(w,y,z,t,x)\phi, v) - (q(v,y,z,t,x)\phi, w). \end{aligned} \tag{35}$$

Using (30), (13), and again (30), we deduce the equality

$$\begin{aligned} (q(w,y,z,t,x)\phi, v) &= -\frac{1}{2}(q(w,y,z,t,x\phi), v) \\ &= -\frac{1}{2}(q(v,y,z,t,x\phi), w) = (q(v,y,z,t,x)\phi, w). \end{aligned}$$

Therefore, the right-hand side of (35) vanishes; i.e.,

$$3(q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi, x) = 0$$

or

$$(q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi, x) = 0, \tag{36}$$

since $\frac{1}{6} \in \Phi$. By nondegeneracy of the form (x,y), from (36) we infer $q(w,y,z,t,v)\phi - q(v,y,z,t,w)\phi = 0$. Hence, (31) holds. The lemma is proven.

Theorem 4. *If a prime Lie Φ-algebra A $\left(\frac{1}{6} \in \Phi\right)$ with a nondegenerate symmetric invariant bilinear form has a nonzero antiderivation ϕ, then A satisfies the identity*

$$[(yz)(tx)]x + [(yx)(zx)]t = 0. \tag{37}$$

PROOF. Let a, b, x, y, z, t, and u be arbitrary elements of A and let v and w be arbitrary elements of A^2. Consider the Φ-submodule B of the Φ-module of A which is generated by all elements of the shape $q(w,y,z,t,v) - q(v,y,z,t,w)$. By Lemma 6

$$B\phi = 0. \tag{38}$$

Since R_u is a derivation of A, we have $[q(w,y,z,t,v) - q(v,y,z,t,w)]u \in B$. Hence, B is an ideal of A. Therefore, by Lemma 3 and (38) we have $B = 0$, and so the identity

$$q(w,y,z,t,v) - q(v,y,z,t,w) = 0 \tag{39}$$

holds in A. By the definition of antiderivation, we have the inclusion $(A^2)\phi \subseteq A^2$. Using (29), this inclusion, (39), (30), and again (39), we obtain

$$q(w,y,z,t,v)\phi = q(w\phi,y,z,t,v) = q(v,y,z,t,w\phi)$$
$$= -2q(v,y,z,t,w)\phi = -2q(w,y,z,t,v)\phi.$$

Hence, $3q(w,y,z,t,v)\phi = 0$, $q(w,y,z,t,v)\phi = 0$. Again using Lemma 3, by analogy to the proof of (39) from the last equality we deduce the identity

$$q(w,y,z,t,v) = 0. \tag{40}$$

Using (13) and (40), we obtain $(q(x,y,z,t,v),w) = (q(w,y,z,t,v),x) = 0$. Since w is an arbitrary element of A^2, from here and (20) we deduce the identity

$$q(x,y,z,t,v) = 0. \tag{41}$$

In view of (14),

$$(q(v,y,z,t,x),u) + (q(u,y,z,t,v),x) + (q(x,y,z,t,u),v) = 0.$$

Using (41), we then obtain

$$(q(v,y,z,t,x),u) + (q(x,y,z,t,u),v) = 0. \tag{42}$$

Applying (13) to the first summand in this equality, we find that

$$(q(u,y,z,t,x) + q(x,y,z,t,u),v) = 0,$$

whence by (20) we infer the identity $q(u,y,z,t,x) + q(x,y,z,t,u) = 0$. Put in this identity $u = v$ with $v \in A^2$. In view of (41) we obtain $q(v,y,z,t,x) = 0$. From here and (42) we deduce the equality $(q(x,y,z,t,u),v) = 0$. Again applying (20) to this equality, we come to the identity $q(x,y,z,t,u) = 0$. Thus, A is an h-algebra; i.e., A satisfies (37). The theorem is proven.

From Theorem 4 and Theorem 2 of [1], describing the prime Lie algebras satisfying (37), we deduce the following

Theorem 5. *If a prime Lie Φ-algebra A $\left(\frac{1}{6} \in \Phi\right)$ with a nondegenerate symmetric invariant bilinear form has a nonzero antiderivation, then the center $Z_R(A)$ of the algebra $R(A)$ of right multiplications of A is an integral domain and the algebra A has no zero divisors*[1)] *over $Z_R(A)$. If Λ is the field of fractions of the center $Z_R(A)$ then $A_\Lambda = \Lambda \otimes_{Z_R(A)} A$ is a 3-dimensional central simple algebra over Λ.*

Corollary 2. *Let A be a central simple Lie algebra over a field Φ of characteristic $p \neq 2,3$ with a nondegenerate symmetric invariant bilinear form. If $\dim_\Phi A \geq 4$ then A has no nonzero antiderivations.*

Observe that if the ring Φ has characteristic $p = 2$ or $p = 3$ then every nonzero Lie Φ-algebra A possesses a nonzero antiderivation, since in the case of $p = 2$ the notions of antiderivation and derivation coincide and so every operator of right multiplication is an antiderivation, while in the case of $p = 3$, even for an arbitrary nonassociative algebra, the mapping $\phi_\alpha : x \mapsto \alpha x$, where α is an arbitrary element of the ring Φ, is an antiderivation.

Henceforth we assume that $\delta = \frac{1}{2}$. By definition, ϕ is a $\frac{1}{2}$-derivation if the equality

$$(xy)\phi = \frac{1}{2}[(x\phi)y + x(y\phi)] \tag{43}$$

holds for arbitrary $x, y \in A$. We recall that the *centroid* of an anticommutative algebra B is the centralizer $\Gamma(B)$ of the algebra $R(B)$ of right multiplications with respect to the endomorphism algebra of the Φ-module of the algebra B.

1) See the definition of zero divisors over $Z_R(A)$ in [1].

Theorem 6. *A linear mapping $\phi : A \mapsto A$ is a $\frac{1}{2}$-derivation of a prime Lie Φ-algebra A ($\frac{1}{6} \in \Phi$) with a nondegenerate symmetric invariant bilinear form if and only if $\phi \in \Gamma(A)$.*

PROOF. We begin with proving that if $\phi \in \Gamma(A)$ then ϕ is a $\frac{1}{2}$-derivation. Suppose that x and y are arbitrary elements in A. If $\phi \in \Gamma(A)$ then $R_y\phi = \phi R_y$ by the definition of $\Gamma(A)$. Therefore, $(xy)\phi = (x\phi)y$. Interchanging x and y in the last equality, we obtain $(yx)\phi = (y\phi)x$. From these equalities and anticommutativity of A we infer that

$$(xy)\phi = \frac{1}{2}[(xy)\phi - (yx)\phi] = \frac{1}{2}[(x\phi)y - (y\phi)x] = \frac{1}{2}[(x\phi)y + x(y\phi)].$$

Hence, ϕ is a $\frac{1}{2}$-derivation.

Prove the converse. Let ϕ be a $\frac{1}{2}$-derivation. We first prove that

$$(x\phi)x = 0 \tag{44}$$

for every $x \in A$.

Suppose that t and v are arbitrary elements in A^2. Using (43), (7), (11), (8), (7), (11), (43), (8), anticommutativity of A, (7), (11), and again (7) and anticommutativity of A, we obtain

$$\begin{aligned}((x\phi)t, xv) - ((x\phi)v, xt) &= -(x(t\phi), xv) + 2((xt)\phi, xv) + (x(v\phi), xt) - 2((xv)\phi, xt)\\
&= ((t\phi)x, xv) - ((v\phi)x, xt) + 2[((xt)\phi, xv) - (xt, (xv)\phi)]\\
&= ((t\phi)x, xv) - ((v\phi)x, xt) = (t\phi, x(xv)) - (v\phi, x(xt))\\
&= (vx^2, t\phi) - (tx^2, v\phi) = ((vx^2)\phi, t) - ((tx^2)\phi, v)\\
&= \frac{1}{2}[((vx)\phi x, t) + (vx(x\phi), t) - ((tx)\phi x, v) - (tx(x\phi), v)]\\
&= \frac{1}{2}[((vx)\phi, xt) + (vx, (x\phi)t) - ((tx)\phi, xv) - (tx, (x\phi)v)]\\
&= \frac{1}{2}[((vx)\phi, xt) - (vx, (xt)\phi)] + \frac{1}{2}[(vx, (x\phi)t) - (tx, (x\phi)v)]\\
&= \frac{1}{2}[(vx, (x\phi)t) - (tx, (x\phi)v)] = -\frac{1}{2}[((x\phi)t, xv) - ((x\phi)v, xt)].\end{aligned}$$

Whence $\frac{3}{2}[((x\phi)t, xv) - ((x\phi)v, xt)] = 0$. Therefore,

$$((x\phi)t, xv) - ((x\phi)v, xt) = 0. \tag{45}$$

Using the Jacobi identity, anticommutativity of A, (8), (7), and (45), we obtain

$$\begin{aligned}((x\phi)xt, v) &= ((x\phi)tx, v) - (xt(x\phi), v)\\
&= ((x\phi)t, xv) - (xt, (x\phi)v) = ((x\phi)t, xv) - ((x\phi)v, xt) = 0.\end{aligned}$$

In view of (7), from here we have $(v, (x\phi)xt) = 0$. Since v is an arbitrary element of A^2; applying (20) to this equality, we find that $(x\phi)xt = 0$. Since t is an arbitrary element in A^2; therefore,

$$(x\phi)x \in \operatorname{Ann} A^2. \tag{46}$$

The annihilator of an ideal in every Lie algebra is an ideal again. Since A is a prime algebra, we therefore have $\operatorname{Ann} A^2 = 0$. Using (46), we now obtain (44).

Linearizing (44), we come to the equality $(x\phi)y + (y\phi)x = 0$. From it and anticommutativity of A we obtain $x(y\phi) = (x\phi)y$. Applying this equality to the right-hand side of (43), we arrive at $(xy)\phi = (x\phi)y$. Hence, $R_y\phi = \phi R_y$ for every $y \in A$; i.e., $\phi \in \Gamma(A)$. The theorem is proven.

Theorem 6 implies the following

Corollary 3. *If ϕ is a $\frac{1}{2}$-derivation of a central prime Lie algebra A over a field Φ of characteristic $p \neq 2, 3$ with a nondegenerate symmetric invariant bilinear form then there is an $\alpha \in \Phi$ such that $x\phi = \alpha x$ for all $x \in A$.*

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