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First Mixed Problem for a Parabolic Difference-Differential Equation

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ABSTRACT. The first mixed boundary value problem for a parabolic difference-differential equation with shifts with respect to the spatial variables is considered. The unique solvability of this problem and the smoothness of generalized solutions in some cylindrical subdomains are established. It is shown that the smoothness of generalized solutions can be violated on the interfaces of neighboring subdomains.

KEY WORDS: parabolic difference-differential equation, strongly elliptic operator, Sobolev spaces.

Introduction

Parabolic functional-differential equations arise in the investigation of nonlinear optic systems with two-dimensional feedback [1–3]. In contrast to parabolic differential equations, these equations have a number of principally new properties. For instance, the smoothness of a generalized solution can be violated inside the cylindrical domain even for an infinitely smooth right-hand side of the equation.

In the present paper, the first mixed problem for a parabolic difference-differential equation with shifts with respect to the spatial variables is considered. The unique solvability of this problem and the smoothness of generalized solutions (in the sense of distributions) in some cylindrical subdomains are established. It is also shown that the smoothness of generalized solutions can be violated on the interfaces of neighboring subdomains.

§1. Statement of the problem

Let $Q \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial Q = \bigcup_i \overline{M_i}$ $(i = 1, \dots, N_\circ)$, where M_i are $(n-1)$-dimensional manifolds of class C^∞ that are open and connected in the topology of ∂Q. Assume that, in a neighborhood of any point $g \in \partial Q \setminus \bigcup_i M_i$, the domain Q is diffeomorphic to an n-dimensional dihedral angle for $n \geq 3$ and to a plane angle for $n = 2$.

Denote by $W_2^k(Q)$ the Sobolev space of complex-valued functions in $L_2(Q)$ that have generalized derivatives (in the sense of distributions) belonging to $L_2(Q)$ up to the order k; this space is endowed with the norm

$$\|u\|_{W_2^k(Q)} = \left\{ \sum_{|\alpha| \leq k} \int_Q |\mathcal{D}^\alpha u(x)|^2 \, dx \right\}^{1/2}$$

Denote by $\overset{\circ}{W}{}_2^k(Q)$ the closure in $W_2^k(Q)$ of the set $\dot{C}^\infty(Q)$ of compactly supported infinitely differentiable functions and by $W_2^{-1}(Q)$ the space dual to $\overset{\circ}{W}{}_2^1(Q)$.

Introduce the bounded difference-differential operator $A_R \colon \overset{\circ}{W}{}_2^1(Q) \to W_2^{-1}(Q)$ by the formula

$$A_R u = \sum_{i,j=1}^n (R_{ijQ} u_{x_j})_{x_i} + \sum_{i=1}^n R_{iQ} u_{x_i} + R_{0Q} u. \tag{1.1}$$

Here $R_{ijQ} = P_Q R_{ij} I_Q$, $R_{iQ} = P_Q R_i I_Q$,

$$R_{ij} u(x) = \sum_{h \in M} a_{ijh}(x) u(x+h) \quad (i, j = 1, \dots, n),$$

$$R_i u(x) = \sum_{h \in M} a_{ih}(x) u(x+h) \quad (i = 0, 1, \dots, n),$$

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$M \subset \mathbb{R}^n$ is a finite set of vectors with integral-valued coefficients, a_{ijh}, $a_{ih} \in C^\infty(\overline{Q})$, $I_Q: L_2(Q) \to L_2(\mathbb{R}^n)$ is the operator of extending any function in $L_2(Q)$ by zero to $\mathbb{R}^n \backslash Q$, and $P_Q: L_2(\mathbb{R}^n) \to L_2(Q)$ is the operator of restricting a function in $L_2(\mathbb{R}^n)$ to Q.

Definition 1. An operator $-A_R$ is said to be *strongly elliptic* if there are constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$- \operatorname{Re}(A_R u, u)_{L_2(Q)} \geq c_1 \|u\|^2_{W_2^1(Q)} - c_2 \|u\|^2_{L_2(Q)} \tag{1.2}$$

for any $u \in \dot{C}^\infty(Q)$.

Necessary and sufficient conditions for strong ellipticity in algebraic form will be stated at the end of this section.

Let us consider a difference-differential equation

$$u_t(x,t) - A_R u(x,t) = f(x,t) \qquad \big((x,t) \in \Omega_T\big) \tag{1.3}$$

with boundary condition

$$u\big|_{\Gamma_T} = 0 \qquad \big((x,t) \in \Gamma_T\big) \tag{1.4}$$

and initial condition

$$u\big|_{t=0} = \varphi(x) \qquad (x \in Q), \tag{1.5}$$

where $\Omega_T = Q \times (0,T)$ and $\Gamma_T = \partial Q \times (0,T)$, $0 < T < \infty$.

Everywhere below we assume that the operator $-A_R$ is strongly elliptic. In this case it is natural to refer to problem (1.3)–(1.5) as the *first mixed problem* for a parabolic difference-differential equation. Without loss of generality we assume that $c_2 = 0$ in inequality (1.2). Indeed, the standard change of the unknown function, $u = \exp(c_2 t) w$, reduces Eq. (1.3) to the form $(-A_R + c_2 I)w + w_t = \exp(-c_2 t) f(x,t)$ $\big((x,t) \in \Omega_T\big)$.

To formulate conditions for strong ellipticity of the operator $-A_R$, we introduce an auxiliary notation. This notation will also be used in §4 in the investigation of the smoothness of the generalized solutions of problem (1.3)–(1.5). Denote by G the additive Abelian group generated by the set M and by Q_r the open connected components of the set

$$Q \Big\backslash \Big(\bigcup_{h \in G} (\partial Q + h) \Big).$$

Definition 2. The sets Q_r are called *subdomains*, and the collection $\mathcal{R}$ of all possible subdomains Q_r is called a *partition* of the set Q.

The partition $\mathcal{R}$ is naturally decomposed into disjoint classes as follows. We say that subdomains $Q_{r_1}, Q_{r_2} \in \mathcal{R}$ belong to the same class if there exists an $h \in G$ such that $Q_{r_2} = Q_{r_1} + h$. We denote each of the subdomains Q_r by Q_{sl}, where s is the index of the class $(s = 1, 2, \dots)$ and l is the index of the subdomain in the sth class. Since the domain Q is bounded, it follows that each class consists of finitely many $(N = N(s))$ subdomains Q_{sl}, and $N(s) \leq ([\operatorname{diam} Q] + 1)^n$.

To formulate necessary conditions for strong ellipticity in algebraic form, we introduce the matrices $R_{ijs}(x) (x \in \overline{Q}_{s1})$ of order $N(s) \times N(s)$ with the entries

$$r^{ijs}_{kl}(x) = \begin{cases} a_{ijh}(x + h_{sk}) & (h = h_{sl} - h_{sk} \in M), \\ 0 & (h_{sl} - h_{sk} \notin M). \end{cases} \tag{1.6}$$

By [3, Theorem 9.2], if an operator $-A_R$ is strongly elliptic, then the matrices

$$\sum_{i,j=1}^{n} \big(R_{ijs}(x) + R^*_{ijs}(x)\big) \xi_i \xi_j$$

are positive definite for any $s = 1, 2, \dots,\ x \in \overline{Q}_{s1}$, and $0 \neq \xi \in \mathbb{R}^n$.

Assume now that $x \in \overline{Q}_{s1}$ is an arbitrary point. Let us consider all points $x^l \in \overline{Q}$ such that $x^l - x \in G$. Since the domain Q is bounded, it follows that the set $\{x^l\}$ is finite; let it contain $I = I(s, x)$ points $(I \geq N(s))$. Let us enumerate the points x^l so that $x^l = x + h_{sl}$ for $l = 1, \dots, N = N(s),\ \ x^1 = x$, where h_{sl} satisfies the condition $Q_{sl} = Q_{s1} + h_{sl}$. Introduce matrices $A_{ijs}(x)$ of order $I \times I$ with entries $a_{lk}^{ijs}(x)$ by the formula

$$a_{lk}^{ijs}(x) = \begin{cases} a_{ijh}(x^l) & (h = x^k - x^l \in M), \\ 0 & (x^k - x^l \notin M). \end{cases}$$

By [3, Theorem 9.2], if the matrices

$$\sum_{i,j=1}^{n} (A_{ijs}(x) + A_{ijs}^*(x))\xi_i\xi_j$$

are positive definite for any $s = 1, 2, \dots,\ x \in \overline{Q}_{s1}$, and $0 \neq \xi \in \mathbb{R}^n$, then the operator $-A_R$ is strongly elliptic.

Obviously, if $I = N$, then the matrix $R_{ijs}(x)$ coincides with the matrix $A_{ijs}(x)$. For $N < I$, the matrix $R_{ijs}(x)$ is obtained from A_{ijs} by deleting the last $I - N$ rows and columns.

§2. Existence and uniqueness of a generalized solution

Denote by $W_2^{k,0}(\Omega_T)$ the Sobolev space of complex-valued functions $u \in L_2(\Omega_T)$ that have the generalized derivatives $u_{x_i} \in L_2(\Omega_T)\ \ (i = 1, \dots, n)$, which is endowed with the norm

$$\|u\|_{W_2^{k,0}(\Omega_T)} = \left\{ \sum_{|\alpha| \leq k} \int_{\Omega_T} |\mathcal{D}^\alpha u(x,t)|^2\, dx\, dt + \int_{\Omega_T} |u(x,t)|^2\, dx\, dt \right\}^{1/2}$$

Set $\mathcal{V} = L_2\big((0,T);\, \overset{\circ}{W}{}_2^1(Q)\big)$. We can readily see that $\mathcal{V} = \big\{v \in W_2^{1,0}(\Omega_T) : v\big|_{\Gamma_T} = 0\big\}$, and the dual space is $\mathcal{V}' = L_2\big((0,T);\, W_2^{-1}(Q)\big)$. Introduce the bounded operator $L_R\colon \mathcal{V} \to \mathcal{V}'$ by the formula $L_R v(\,\cdot\,, t) = -A_R v(\,\cdot\,, t)$ for almost all $t \in (0, T)$.

We also introduce an unbounded operator $\Lambda_t\colon \mathcal{V}' \supset D(\Lambda_t) \to \mathcal{V}'$ that acts in the sense of distributions with values in $\mathcal{V}'$ by the formula $\Lambda_t v(\,\cdot\,, t) = \partial v(\,\cdot\,, t)/\partial t$ with the domain $D(\Lambda_t) = \big\{v \in \mathcal{V}' : \Lambda_t v \in \mathcal{V}'\big\}$.

Note that, if a function $u \in \mathcal{V}$ satisfies the operator equation

$$\Lambda_t u + L_R u = f, \tag{2.1}$$

where $f \in \mathcal{V}'$, then $u \in C^0\big([0,T];\, L_2(Q)\big)$ by [4, Chap. 1, Theorem 3.1 and Proposition 2.1]. Therefore, the relation $u\big|_{t=0}$ makes sense.

Let $f \in \mathcal{V}'$ and $\varphi \in L_2(Q)$.

Definition 3. A function $u \in \mathcal{V} \cap D(\Lambda_t)$ is called a *generalized solution of problem* (1.3)–(1.5) if this function satisfies the operator equation (2.1) and the initial condition (1.5).

Theorem 1. *Let a difference-differential operator $-A_R$ be strongly elliptic. In this case, for any $f \in \mathcal{V}'$ and $\varphi \in L_2(Q)$, problem* (1.3)–(1.5) *has a generalized solution $u \in \mathcal{V} \cap D(\Lambda_t)$, and this solution is unique.*

Proof. Obviously, the difference operators $R_{ijQ}\colon L_2(Q) \to L_2(Q)$ are bounded. Hence, there exists a constant $c_0 > 0$ such that

$$\big|(A_R u, v)_{L_2(Q)}\big| \leq c_0 \|u\|_{W_2^1(Q)} \|v\|_{W_2^1(Q)} \tag{2.2}$$

for any $u, v \in \dot{C}^\infty(Q)$. It follows from inequalities (1.2), (2.2) of the present paper and also from [4, Chap. 3, Theorem 4.1 and Remark 4.3] that a generalized solution of problem (1.3)–(1.5) exists and is unique. □

We assume now that $f \in L_2(\Omega_T)$ and $\varphi \in L_2(Q)$. In this case, we define a generalized solution of problem (1.3)–(1.5) in the sense of an integral identity.

Definition 4. A function $u \in \mathcal{V}$ is said to be a *generalized solution of problem* (1.3)–(1.5) if, for any $v \in \{v \in W_2^1(\Omega_T) : v|_{\Gamma_T} = 0, v|_{t=T} = 0\}$, the following integral identity holds:

$$\int_{\Omega_T} \left(-u\bar{v}_t + \sum_{i,j=1}^{n} R_{ijQ} u_{x_j} \bar{v}_{x_i} \right) dx\, dt = \int_{\Omega_T} f\bar{v}\, dx\, dt + \int_Q \varphi\bar{v}\big|_{t=0}\, dx. \tag{2.3}$$

Theorem 2. *Let an operator* $-A_R$ *be strongly elliptic and let* $f \in L_2(\Omega_T)$ *and* $\varphi \in L_2(Q)$. *In this case, Definitions* 3 *and* 4 *are equivalent.*

Proof. The equivalence of Definitions 3 and 4 follows from [4, Chap. 3, Theorems 4.1 and 4.2] provided that the set

$$\mathcal{V}_1 = \{v \in W_2^1(\Omega_T) : v|_{\Gamma_T} = 0, \ \ v|_{t=T} = 0\}$$

is dense in the space

$$\mathcal{V}_2 = \{v \in \mathcal{V} \cap D(\Lambda_t) : v|_{t=T} = 0\}.$$

Let us prove this fact.

Indeed, let $v \in \mathcal{V}_2$. Take a sequence of real-valued functions $\xi_n \in C^\infty[0,T]$ such that $\xi_n(t) = 1$ $(0 \le t \le T - 2/n)$, $\xi_n(t) = 0$ $(T - 1/n \le t \le T)$; $0 \le \xi_n(t) \le 1$, and $|\xi_n'(t)| \le Cn$ $(0 \le t \le T)$. It is clear that $\xi_n v \to v$ as $n \to \infty$ in the space $\mathcal{V}_2$. By smoothing the functions $\xi_n v$ with respect to t, we obtain a sequence of functions $v_n(t)$ with values in $\mathring{W}_2^1(Q)$ that are infinitely differentiable with respect to t and whose supports belong to $[0,T)$. By construction, we have $v_n \to v$ in the space $\mathcal{V}_2$, and $v_n \in \mathcal{V}_1$. □

§3. Analytic semigroups

Definition 5. Let X be a Banach space. A one-parameter family of bounded linear operators $T_t \colon X \to X$ $(t \ge 0)$ is said to be a *strongly continuous semigroup*, or a C_0*-semigroup*, provided that

1) $T_0 = I$;
2) $T_{t+s} = T_t T_s$ $(t, s \ge 0)$;
3) $\lim_{t\searrow 0} T_t x = x$ (for any $x \in X$).

Definition 6. A semigroup of class C_0 is said to be a *contraction* semigroup if $\|T_t\| \le 1$ $(t \ge 0)$.

Definition 7. The linear operator $A \colon X \supset D(A) \to X$ defined by the formula

$$Ax = \lim_{t\searrow 0} \frac{T_t x - x}{t} \quad \left(x \in D(A) = \left\{ x \in X : \lim_{t\searrow 0} \frac{T_t x - x}{t} \quad \text{exists} \right\} \right)$$

is called the *infinitesimal generator of the strongly continuous semigroup* $\{T_t\}$.

Introduce the unbounded operator $\mathcal{A}_R \colon L_2(Q) \supset D(\mathcal{A}_R) \to L_2(Q)$ that acts on the space of distributions $D'(Q)$ by the formula $\mathcal{A}_R u = A_R u$ $(u \in D(\mathcal{A}_R) = \{u \in \mathring{W}_2^1(Q) : A_R u \in L_2(Q)\})$.

Theorem 3. *Assume that an operator* $-A_R$ *is strongly elliptic. In this case, the operator* $\mathcal{A}_R$ *is an infinitesimal generator of a contraction semigroup* T_t *in* $L_2(Q)$.

Proof. It follows from inequality (1.2) with zero constant c_2 and from [5, Theorem 10.1] that the operator $\mathcal{A}_R$ is closed and its spectrum satisfies the relation $\sigma(\mathcal{A}_R) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. Moreover, for any $u \in D(\mathcal{A}_R)$ we have

$$-\operatorname{Re}(\mathcal{A}_R u, u)_{L_2(Q)} \ge c_1 \|u\|^2_{W_2^1(Q)}. \tag{3.1}$$

It follows from (3.1) and from the Cauchy–Schwarz–Bunyakovskii inequality that

$$\|(\lambda I - \mathcal{A}_R)u\|_{L_2(Q)} \|u\|_{L_2(Q)} \ge \lambda \|u\|^2_{L_2(Q)}$$

for $u \in D(\mathcal{A}_R)$ and $\lambda > 0$. Thus,

$$\|R(\lambda, \mathcal{A}_R)\| \leq \frac{1}{\lambda}.$$

Hence, by the Hille–Yosida theorem (see [6, §1.3]), the operator $\mathcal{A}_R$ is an infinitesimal generator of a contraction semigroup T_t in $L_2(Q)$. □

Write $\Delta = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2\}$, where $\varphi_1 < 0 < \varphi_2$.

Definition 8. A one-parameter family of bounded linear operators $T_z \colon X \to X \quad (z \in \Delta)$ is said to be a *holomorphic semigroup on* Δ if

1) the function $z \to T_z$ is analytic in Δ;
2) $T_0 = I$ and $\lim_{z\to 0, z\in\Delta} T_z x = x$ (for any $x \in X$);
3) $T_{z_1+z_2} = T_{z_1}T_{z_2}$ (for any $z_1, z_2 \in \Delta$).

A semigroup T_t is said to be *holomorphic* if it is analytic in a sector Δ that contains the positive real semiaxis.

Theorem 4. *Let an operator $-A_R$ be strongly elliptic; hence, the operator $\mathcal{A}_R$ is an infinitesimal generator of a contraction semigroup T_t in $L_2(Q)$. In this case, the semigroup T_t can be continued to a holomorphic semigroup T_z in a sector $\Delta_\delta = \{z \in \mathbb{C} : |\arg z| < \delta\}$.*

Proof. The theorem can be proved by the standard methods based upon the use of norm estimates for the resolvent of an infinitesimal generator of a semigroup. In turn, the above estimate can be derived from the Gårding-type inequality (1.2) (see [6, §7.2, Theorem 2.7] or [7, Chap. 14, §1, Theorem 1]). □

§4. Smoothness of the generalized solutions

Definition 9. A generalized solution of problem (1.3)–(1.5) for $f \in L_2(\Omega_T)$ and $\varphi \in L_2(Q)$ is said to be *strong* if $u_t \in L_2(\Omega_T)$.

To investigate the smoothness problem for generalized solutions of problem (1.3)–(1.5), we introduce an additional notation. Let us consider the sets

$$\mathcal{K} = \bigcup_{h_1, h_2 \in G} \{\overline{Q} \cap (\partial Q + h_1) \cap \overline{[(\partial Q + h_2) \setminus (\partial Q + h_1)]}\}, \qquad \mathcal{K}^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, \mathcal{K}) < \varepsilon\},$$

where $\varepsilon > 0$.

Let $D \subset \mathbb{R}^n$ be a bounded domain. Denote by $W_2^{2k,k}(D \times (0,T))$ the Sobolev space of complex-valued functions in $L_2(D \times (0,T))$ that have the generalized derivatives $\mathcal{D}_x^\alpha \mathcal{D}_t^\beta u \in L_2(D \times (0,T))$ ($|\alpha| + 2\beta \leq 2k$); let this space be endowed with the norm

$$\|u\|_{W_2^{2k,k}(D\times(0,T))} = \left\{ \sum_{|\alpha|+2\beta\leq 2k} \int_{\Omega_T} |\mathcal{D}_x^\alpha \mathcal{D}_t^\beta u(x,t)|^2 \, dx \, dt \right\}^{1/2}.$$

Theorem 5. *Let $\partial Q \setminus \bigcup_i M_i \subset \mathcal{K}$, and let the difference-differential operator $-A_R$ be strongly elliptic. In this case, for any $\varphi \in D(\mathcal{A}_R)$ and $f \in L_2(\Omega_T)$ such that $f_t \in L_2(\Omega_T)$, problem (1.3)–(1.5) has a strong solution, which is defined by the formula*

$$u(x,t) = T_t\varphi(x) + \int_0^t T_{t-s} f(x,s) \, ds, \tag{4.1}$$

and this strong solution is unique. Moreover, $u(x,t) \in W_2^{2,1}((Q_{sl} \setminus \mathcal{K}^\varepsilon) \times (0,T))$ for any $\varepsilon > 0$ and any $s = 1, 2, \ldots$ and $l = 1, \ldots, N(s)$.

Proof. The existence and uniqueness of a solution of problem (1.3)–(1.5) and formula (4.1) follow from Theorem 3 of the present paper and from [6, §4.2, Corollary 2.10]. It follows from the definition of a strong solution and from Eq. (1.3) that

$$-A_R u(\cdot, t) = F(\cdot, t), \tag{4.2}$$

where $F(\cdot, t) = f(\cdot, t) - u_t(\cdot, t) \in L_2(Q)$ for almost all $t \in (0, T)$. By the theorem on the smoothness of generalized solutions of boundary value problems for strongly elliptic difference-differential equations [5, Theorem 11.2], for any $\varepsilon > 0$ we have

$$u(\cdot, t) \in W_2^2(Q_{sl} \setminus \mathcal{K}^\varepsilon)$$

and

$$\|u\|_{W_2^2(Q_{sl}\setminus\mathcal{K}^\varepsilon)} \le c\|F\|_{L_2(Q)} \tag{4.3}$$

for almost all $t \in (0, T)$ and $s = 1, 2, \ldots;\ l = 1, \ldots, N(s)$, where $c > 0$ does not depend on t.

Squaring inequality (4.3) and integrating from 0 to T, we obtain

$$\|u\|_{W_2^{2,0}((Q_{sl}\setminus\mathcal{K}^\varepsilon)\times(0,T))} \le c_1\big(\|f\|_{L_2(\Omega_T)} + \|u_t\|_{L_2(\Omega_T)}\big).$$

This yields $u \in W_2^{2,1}((Q_{sl} \setminus \mathcal{K}^\varepsilon) \times (0, T))$. □

Remark 1. It follows from Theorems 1 and 5 that, if the assumptions of Theorem 5 hold, then the generalized solution of problem (1.3)–(1.5) is strong, and therefore it must belong to the space $W_2^{2,1}((Q_{sl} \setminus \mathcal{K}^\varepsilon) \times (0, T))$.

The next example shows that the smoothness of strong solutions of problem (1.3)–(1.5) can be violated on the interface between neighboring cylinders $Q_{s_1 l_1} \times (0, T)$ and $Q_{s_2 l_2} \times (0, T)$, and also near the set $\mathcal{K} \times (0, T)$.

Example 1. Let us consider the first mixed problem (1.3)–(1.5) and assume that

$$Q = (0, 4/3) \times (0, 4/3), \quad A_R = \Delta R_Q, \quad R_Q = P_Q R I_Q,$$
$$Ru(x) = u(x) + au(x_1 + 1, x_2 + 1) + au(x_1 - 1, x_2 - 1), \qquad 0 < a < 1.$$

Obviously, the corresponding decomposition $\mathcal{R}$ of the domain Q consists of two classes of subdomains, 1) $Q_{11} = (0, 1/3) \times (0, 1/3)$, $Q_{12} = (1, 4/3) \times (1, 4/3)$ and 2) $Q_{21} = Q \setminus (\overline{Q}_{11} \cup \overline{Q}_{12})$. The set $\mathcal{K}$ belongs to the boundary ∂Q and consists of four points, $g^1 = (1/3, 0)$, $g^2 = (4/3, 1)$, $g^3 = (0, 1/3)$, and $g^4 = (1, 4/3)$.

The matrices $A_s(x)(x \in \overline{Q}_{s1};\ s = 1, 2)$, defined by formula (1.8), are

$$A_1(x) = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad (x \in \overline{Q}_{11}),$$
$$A_2(x) = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad (x \in \overline{Q}_{21} \cap \mathcal{K}), \qquad A_2(x) = (1) \quad (x \in \overline{Q}_{21} \setminus \mathcal{K}).$$

Thus, the matrices $A_s(x)(\xi_1^2 + \xi_2^2)$ $(x \in \overline{Q}_{s1};\ s = 1, 2)$ are positive definite (cf. (1.9)). Hence, the operator $-A_R$ is strongly elliptic.

Let us write

$$v_1(r, \varphi) = \xi(r) r^\lambda \sin \lambda\varphi, \qquad v_2(r, \varphi) = \xi(r) r^\lambda \sin \lambda\varphi\left(\frac{3\pi}{2} - \varphi\right),$$

where $\xi(r) \in \dot{C}^\infty(\mathbb{R})$, $0 \le \xi(r) \le 1$, $\xi(r) = 1$ for $r \le 1/8$, $\xi(r) = 0$ for $r \ge 1/6$, $\lambda = (2/\pi)\arccos(a/2)$, and r, φ are the polar coordinates.

Introduce the function $v(x)$ by the formula

$$v(x)=\begin{cases}\dfrac{v_1(x_1-1/3,x_2)-av_2(x_1-1/3,x_2)}{1-a^2}, & x\in Q_{11},\\ \dfrac{-av_1(x_1-4/3,x_2-1)+v_2(x_1-4/3,x_2-1)}{1-a^2}, & x\in Q_{12},\\ v_1(x_1-1/3,x_2)+v_2(x_1-4/3,x_2-1), & x\in Q_{21}.\end{cases} \tag{4.4}$$

Obviously,

$$R_Qv(x)=v_1\left(x_1-\frac{1}{3},x_2\right)+v_2\left(x_1-\frac{4}{3},x_2-1\right).$$

Since $0<\lambda<1$, we can readily see that $v\in {}^\circ W_2^1$, $-\Delta R_Qv\in L_2(Q)$; however, $v\notin W_2^2(Q_{s1}\cap S_\delta(g^1))$ for any $\delta>0$. Hence, the function $u(x,t)=tv(x)$ is a strong solution of problem (1.3)–(1.5) for $f(x,t)=v(x)-t\Delta R_Qv(x)\in L_2(\Omega_T)$ and $\varphi(x)=0$. However, $u\notin W_2^{2,0}((Q_{s1}\cap S_\delta(g^1))\times(0,T))$ for any $\delta>0$. Thus, Theorem 5 fails for $\varepsilon=0$ in general.

Let us show now that

$$u_{x_1}\big|_{x_1=1/3+0,\,x_2\le 1/8}\neq u_{x_1}\big|_{x_1=1/3-0,\,x_2\le 1/8}.$$

To this end, by (4.4) it suffices to show that

$$v_{1\varphi}\big|_{\varphi=\pi/2,\,r\le 1/8}\neq\frac{1}{1-a^2}\left\{v_{1\varphi}\big|_{\varphi=\pi/2,\,r\le 1/8}-av_{2\varphi}\big|_{\varphi=\pi/2,\,r\le 1/8}\right\}. \tag{4.5}$$

Relation (4.5) is equivalent to the following inequality:

$$\lambda\cos\frac{\lambda\pi}{2}\neq\frac{1}{1-a^2}\left(\lambda\cos\lambda\frac{\pi}{2}-\lambda\cos\lambda\pi\right). \tag{4.6}$$

Reducing similar terms and applying the relation $\cos(\lambda\pi/2)=a/2$, we can rewrite (4.6) in the form

$$a^3-a^2+2\neq 0. \tag{4.7}$$

Since the roots of the equation $a^3-a^2+2=0$ are of the form $a_{1,2}=1\pm i$ and $a_3=-1$, it follows that condition (4.7) holds for $0<a<1$. Therefore, $u\notin W_2^{2,0}(S_\sigma(y))$ for any $y=(y_1,y_2)$ and $\sigma>0$ such that $y_1=1/3$, $0<y_2<1/8$, and $\sigma<y_2$. Thus, the smoothness of strong solutions of problem (1.3)–(1.5) can be violated on the interface of neighboring subdomains $Q_{s_1l_1}\times(0,T)$ and $Q_{s_2l_2}\times(0,T)$.

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