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Interpolation and Integral Norms of Hyperbolic Polynomials

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ABSTRACT. The integral norm on the subspace of multivariate trigonometric polynomials with harmonics from the "hyperbolic cross" is equivalent to the interpolation norm taken on a finite set of points whose cardinality increases only slightly faster than the dimension of the subspace.

KEY WORDS: entropy numbers, ε-entropy, interpolation, p-quasimatrix orthonormal system.

Introduction

Suppose D is a bounded set in $\mathbb{R}^d$, $\Phi = \{\varphi_k(x)\}_1^\infty$, $x \in D$, is an bounded orthonormal system in $L^2(D)$, and

$$\Phi_N = \left\{ \sum_{k=1}^{N} c_k \varphi_k(x) \right\}$$

is the set of polynomials of order $\le N$ with respect to the system $\{\varphi_k\}$. We say that the system Φ is *p-quasimatrix*, $0 < p \le \infty$, if there exist positive constants K_1, K_2, K_3 such that for every $N = 1, 2, \dots$ there is a finite set $\Omega_N \subset D$ such that $|\Omega_N| \le K_1 N$ and for every $\varphi \in \Phi(N)$

$$K_2 \left(\frac{1}{|\Omega_N|} \sum_{x \in \Omega_N} |\varphi(x)|^p \right)^{1/p} \le \|\varphi\|_p \le K_3 \left(\frac{1}{|\Omega_N|} \sum_{x \in \Omega_N} |\varphi(x)|^p \right)^{1/p}$$

This definition for $p = \infty$ was introduced by B. Kashin [1]; for $0 < p < \infty$ see [2]. The quasimatrix property is important for discretization purposes when the values of a polynomial are known only at a finite number of points. It is shared by many concrete classical systems, e.g., trigonometric, Haar, Walsh, and Franklin. Considering trigonometric polynomials of several variables, we must specify the order of the exponentials. If the spectrum of a polynomial is, for example, in a cube with side n, then $N \simeq n^d$, and the system is p-quasimatrix for every p, $0 < p \le \infty$, as follows from the classical Marcinkiewicz theorem (see [3–5] or [6] for $0 < p < 1$).

We consider here polynomials with spectrum from the "hyperbolic cross." Such polynomials appear naturally when one approximates classes of functions with bounded mixed derivative (see, for example, [7]).

This paper is an application to this concrete problem of the empirical distribution method by G. Schechtman [8] and its refinement given in [9].

§1. Main result. Discussion

Let $\mathbf{r} = (r_1, r_2, \dots, r_d)$ be a vector whose coordinates are ordered by

$$0 < r_1 = r_2 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_d.$$

For $\mathbf{s} \in \mathbb{Z}_+^d$ define the set

$$\rho(\mathbf{s}) = \{\mathbf{k} \in \mathbb{Z}^d : 2^{s_j - 1} \le |k_j| < 2^{s_j},\ j = 1, \dots, d\}.$$

Let us denote

$$\delta_{\mathbf{s}}(x) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} c_{\mathbf{k}} e^{2\pi i (\mathbf{k}, \mathbf{x})}.$$

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Let $L_{p,n}$ be the space of trigonometric polynomials $T_n(x) = \sum_{1\le(\mathbf{s},\mathbf{r})<r_1 n} \delta_{\mathbf{s}}(\mathbf{x})$ defined on the unit cube $Q^d = [0,1]^d$, and let $B_{p,n}$ be the unit ball of this space:

$$\left\| \sum_{1\le(\mathbf{s},\mathbf{r})<r_1 n} \delta_{\mathbf{s}}(\mathbf{x}) \right\|_p \le 1.$$

Note that $\dim L_{p,n} \simeq 2^n n^{\nu-1}$. The question whether hyperbolic polynomials are quasimatrix was asked in [2]. In the following statement we prove that they are "almost quasimatrix" for $1 \le p < \infty$.

Theorem 1. *Let p $(1 \le p < \infty)$ be fixed. Then for every n there exist $N \ll 2^n n^{\max\{(d-1)(p-2),0\}+\nu+3}$ points $\mathbf{x}_j$ such that*

$$K_2\left(\frac{1}{N}\sum_{j=1}^{N} |T_n(\mathbf{x}_j)|^p\right)^{1/p} \le \|T_n\|_p \le K_3\left(\frac{1}{N}\sum_{j=1}^{N} |T_n(\mathbf{x}_j)|^p\right)^{1/p}$$

Remark 1. As we were informed by B. Kashin, for $p = 2$ there are general results for arbitrary orthonormal system with constant sum of squares of functions. The sufficient number of points in this case is $\approx N\log N$ for a system of N function.

Remark 2. For $p = \infty$, the uniform norm is equivalent to the discrete one only if the number of points is significantly larger than the number of harmonics of T_n (see [10]).

Remark 3. Is it possible to find a universal collection of points for all $1 < p < \infty$? It follows from the proof below that one can find a universal collection for any *finite* number of values of p.

Remark 4. This method works also for other nontrivial hyperbolic constructions, like hyperbolic wavelet polynomials [11].

§2. Finite-dimensional estimates of entropy numbers and other preliminary results

In this section we give preliminary results needed for the empirical distribution method due to G. Schechtman [8] and estimates of ε-entropy that are used in the scheme from [9].

Lemma 1 [9, p. 81]. *Let (Ω,μ) be a probability space, let $\mathcal{F}$ be a finite set in $L_1(\Omega,\mu)$ so that*

$$\|f\|_1 \le 1, \qquad \|f\|_\infty \le M, \qquad f \in \mathcal{F}$$

for some constant M. Let $0 < \varepsilon < 1$, and let N be an integer such that

$$2|\mathcal{F}| \le \exp\left(\frac{\varepsilon^2 N}{8M}\right).$$

Then there exist $\{t_j\}_{j=1}^N \in \Omega$ such that

$$\left| \|f\|_1 - \frac{1}{N}\sum_{j=1}^{N} |f(t_j)| \right| \le \varepsilon, \qquad f \in \mathcal{F}.$$

The following generalization of Lemma 1 is contained in its proof given in [9, p. 81].

Lemma 2. *Let $\mathcal{F}_1,\dots,\mathcal{F}_l$ such that*

$$\|f\|_1 \le 1, \qquad \|f\|_\infty \le M_j, \qquad f \in \mathcal{F}_j, \qquad j = 1,\dots,l,$$

and N satisfies to the condition

$$2\sum_{j=1}^{l} |\mathcal{F}_j| \exp\left(-\frac{\varepsilon_j^2 N}{8M_j}\right) < 1.$$

Then there exist $\{t_j\}_{j=1}^N \in \Omega$ such that

$$\left| \|f\|_1 - \frac{1}{N}\sum_{j=1}^{N} |f(t_j)| \right| \le \sum_{j=1}^{l} \varepsilon_j, \qquad f \in \bigcup_{j=1}^{l} \mathcal{F}_j.$$

The next lemma allows us to carry over estimates from a finite set to the whole space.

Lemma 3 [9]. *Let A be a bounded linear map from a Banach space X into a Banach space Y. Let $0 < \varepsilon < 1$ and assume that for an ε-net $\mathcal{F}$ of the unit ball of X there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|x\| \le \|Ax\| \le C_2\|x\|, \qquad x \in \mathcal{F}.$$

Then

$$\frac{C_1 - \varepsilon}{C_1 + \varepsilon}\|x\| \le \|Ax\| \le \frac{C_2}{1-\varepsilon}\|x\|,$$

for every $x \in X$.

The proof is straightforward.

An inequality involving different metrics for hyperbolic polynomials is given in the following statement.

Lemma 4 [4, p. 21]. *For $p' = p/(p-1)$,*

$$\|T_n\|_\infty \ll 2^{n/p} n^{(\nu-1)/p'} \|T_n\|_p, \qquad 1 \le p < \infty.$$

The next statements give some estimates of ε-entropy. Let us recall some definitions (cf. [12]). Let K be a compact set in the Banach space X. The ε*-entropy* $\mathcal{H}_\varepsilon(K; X)$ (or simply $\mathcal{H}(K, \varepsilon)$) is the logarithm to the base two of the number of points in the minimal ε-net. We use also the inverse quantities, the so-called *entropy numbers*, given by

$$\varepsilon_m(K; X) = \inf\left\{\varepsilon : K \subset \bigcup_{j=1}^{2^m}(x_j + \varepsilon B_X)\right\}.$$

The infimum is taken over all ε such that K can be covered by 2^m balls εB_X of radius ε.

Basic properties of ε-entropy or entropy numbers can be found, for example, in [12] and [13]. We state some of them below.

Let X denote the space $\mathbb{R}^n$ endowed with some Banach norm, and let X^* be its dual space. Let B^n and S^{n-1} be the Euclidean unit ball and unit sphere. The average of $\| \; \|_X$ on S^{n-1} is denoted by M_X, i.e.,

$$M_X = \int_{S^{n-1}} \|x\|\, d\sigma(x),$$

where σ is the normalized rotation invariant measure on S^{n-1}.

Lemma 5 [14], see also [13].

$$n \log\frac{1}{\varepsilon} \le \mathcal{H}_\varepsilon(B_X; X) \le n\log\left(1 + \frac{2}{\varepsilon}\right).$$

It is worth noting that $\mathcal{H}_\varepsilon(B_X; X) = 0$ if $\varepsilon \ge 1$.

The following lemma is Sudakov's classical result [15].

Lemma 6. *There exists an absolute constant C such that*

$$\mathcal{H}_\varepsilon(B_X; \mathbb{R}^n) \le Cn\left(\frac{M_{X^*}}{\varepsilon}\right)^2.$$

A dual version of this fact was first proved in [16], a different simple proof was given by A. Pajor and M. Talagrand in [9].

Lemma 7. *There exists an absolute constant C, such that*

$$\mathcal{H}_\varepsilon(B^n; X) \le Cn\left(\frac{M_X}{\varepsilon}\right)^2, \quad \varepsilon \ge M_X, \qquad \mathcal{H}_\varepsilon(B^n; X) \le Cn\log\frac{M_X}{\varepsilon}, \quad \varepsilon \le M_X.$$

Remark 5. The second estimate can be easily derived from the first one and Lemma 5. Indeed,

$$\mathcal{H}_\varepsilon(B^n\,;X) \le \mathcal{H}_{M_X}(B^n\,;X) + \mathcal{H}_{\varepsilon/M_X}(B_X\,;X),$$

and we need only use the preceding estimates.

The estimates of Lemma 7 can be rewritten for the entropy numbers as follows:

$$\varepsilon_m(B^n\,;X) \ll \begin{cases} \sqrt{n/m}\,M_X, & m \le n, \\ M_X e^{-m/n}, & m > n. \end{cases}$$

For a set $E \subset \mathbb{Z}^d$ of cardinality $|E|$, let us consider the Banach space $L_{p,E}$ of trigonometric polynomials with real coefficients

$$T(E\,;x) = \sum_{\mathbf{k}\in E} c_{\mathbf{k}} e^{2\pi i(\mathbf{k},\mathbf{x})}$$

endowed with the $L_p(Q^d)$ norm. We denote by $\deg E$ the largest degree of the exponentials $e^{2\pi i(\mathbf{k},\mathbf{x})}$, $\mathbf{k}\in E$, and $\deg e^{2\pi i(\mathbf{k},\mathbf{x})} = |k_1| + \cdots + |k_d|$. The following result was proved in [17].

Lemma 8.

$$M_{L_{q,E}} \ll \begin{cases} \sqrt{q}\,, & 2 < q < \infty, \\ \sqrt{\log \deg E}\,, & q = \infty. \end{cases}$$

Therefore, by Lemma 7

$$\varepsilon_m(B_{2,n}\,;L_q) \ll \begin{cases} \sqrt{2^n n^{\nu-1}/m}\,\sqrt{q}\,, & m \le 2^n n^{\nu-1}, \\ e^{-m/2^n n^{\nu-1}}\sqrt{q}\,, & m > 2^n n^{\nu-1}, \end{cases}$$

for any $2 < q < \infty$. Also,

$$\mathcal{H}_\varepsilon(B_{2,n}\,;L_\infty) \ll \begin{cases} \dfrac{2^n n^\nu}{\varepsilon^2}\,, & \varepsilon > \sqrt{n}\,, \\ 2^n n^{\nu-1}\log\dfrac{\sqrt{n}}{\varepsilon}\,, & \varepsilon < \sqrt{n}\,. \end{cases}$$

We need the following estimates of ε-entropy.

Lemma 9. *Let $B_{p,n}$ be the unit ball of the space of trigonometric polynomials described above. Then*

$$\mathcal{H}_\varepsilon(B_{p,n}\,;L_\infty) \ll \begin{cases} \dfrac{2^n n^\nu}{\varepsilon^p}\,, & 1 \le p \le 2, \\ \dfrac{2^n n^{(p-2)(d-1)+\nu}}{\varepsilon^p}\,, & 2 < p < \infty. \end{cases}$$

Proof. The estimate of $\mathcal{H}_\varepsilon(B_{2,n}\,;L_\infty)$ is given above. Let us find the estimate of $\mathcal{H}_\varepsilon(B_{p,n}\,;L_\infty)$ for $2 < p < \infty$. For every $\lambda > 0$ a trigonometric polynomial $T(x) \in B_{p,n}$ can be decomposed into the sum of two polynomials $T(x) = T_1(x) + T_2(x)$ such that

$$\|T_1(x)\|_2 \ll \lambda^{1/p-1/2}\|T(x)\|_p, \qquad \|T_2(x)\|_\infty \ll \lambda^{1/p} n^{d-1}\|T(x)\|_p.$$

To do this, we take the λ-cut $f_1(x)$ of $T(x)$, i.e. $f_1(x) = \lambda\,\mathrm{sign}\,T$ when $|T(x)| > \lambda$, $f_1(x) = T(x)$ elsewhere, and $f_2(x) = T(x) - f_1(x)$. We obtain a decomposition $T(x) = f_1 + f_2$ with functions f_1 and f_2 possesing the required properties. Then we apply the operator of the de la Vallée-Poussin hyperbolic means [4] to both sides of this equality. It is bounded in $L_2 \to L_2$ and its norm is $\simeq n^{d-1}$ in $L_\infty \to L_\infty$. Therefore,

$$\mathcal{H}_\varepsilon(B_{p,n}\,;L_\infty) \le \mathcal{H}_{\varepsilon/2}(\lambda^{1/p-1/2}B_{2,n}\,;L_\infty) + \mathcal{H}_{\varepsilon/2}(\lambda^{1/p}n^{d-1}B_{\infty,n}\,;L_\infty).$$

Choosing $\lambda = \varepsilon^p n^{-p(d-1)}$, we see that the second term is then equal to zero. For the first term, by the previous estimate,

$$\mathcal{H}_\varepsilon(B_{p,n}\,;\,L_\infty) \le \frac{\lambda^{2/p-1}2^n n^\nu}{\varepsilon^2} \le \frac{2^n n^{(p-2)(d-1)+\nu}}{\varepsilon^p}\,.$$

Let us proceed now to the case $1 < p < 2$. As in [9], we first estimate $\mathcal{H}_\varepsilon(B_{2,n}\,;\,L_r)$ for $2 < r < \infty$. By the Hölder inequality, for any $r < q$ we have

$$\|f-g\|_r \le \|f-g\|_2^{1-\theta}\|f-g\|_\infty^\theta \le 2\|f-g\|_\infty^\theta,$$

where $\theta = 1 - 2/r$. Therefore,

$$\mathcal{H}_\varepsilon(B_{2,n}\,;\,L_r) \le \mathcal{H}_{(\varepsilon/2)^{1/\theta}}(B_{2,n}\,;\,L_\infty) \le \frac{2^n n^\nu}{\varepsilon^{2/\theta}}\,.$$

To estimate the ε-entropy of the dual couple $\mathcal{H}_\varepsilon(B_{r',n}\,;\,L_2)$, we need the following statement [18].

Theorem. *Let X be a uniformly convex Banach space. Let $u\colon X \to Y$ be a compact operator, and u^*, its dual. Then, for all $m \in \mathbb{N}$ and $\alpha \in (0,\infty)$ we have*

$$C_0^{-1}\Big(\sum_{k=1}^m e_k(u^*)^\alpha\Big)^{1/\alpha} \le \Big(\sum_{k=1}^m e_k(u)^\alpha\Big)^{1/\alpha} \le C_0\Big(\sum_{k=1}^m e_k(u^*)^\alpha\Big)^{1/\alpha},$$

where C_0 depends only on X.

To apply this theorem, we rewrite the estimate of the entropy in terms of entropy numbers, i.e.,

$$\varepsilon_m(B_{2,n}\,;\,L_r) \ll \left(\frac{2^n n^\nu}{m}\right)^{1/2-1/r}$$

By the Marcinkiewicz multiplier theorem [7], for $1 < r < \infty$ the space $L^*_{r,n}$ is isomorphic to $L_{r',n}$, where $1/r + 1/r' = 1$. Let $r > 2$ and u be the identical embedding operator $u\colon L_{2,n} \to L_{r,n}$. Let us take $\alpha < 2r/(r-2)$. Then, by the monotonicity

$$e_{2m}(u^*)^\alpha m \le \sum_{k=m+1}^{2m} e_k(u^*)^\alpha \le \sum_{k=1}^{2m} e_k(u)^\alpha \le \sum_{k=1}^{2m}\frac{(2^n n^\nu)^{\alpha(1/2-1/r)}}{k^{\alpha(1/2-1/r)}} \ll \frac{(2^n n^\nu)^{\alpha(1/2-1/r)}}{m^{\alpha(1/2-1/r)-1}}\,,$$

or

$$e_{2m}(u^*) \le \left(\frac{2^n n^\nu}{m}\right)^{1/2-1/r},$$

and the estimate is proved, because $1/2 - 1/r = 1/r' - 1/2$. By the transitivity property of entropy numbers (see, for example, [13]), for each $1 < p < 2$

$$\varepsilon_{2m-1}(B_{p,n}\,;\,L_\infty) \le \varepsilon_m(B_{p,n}\,;\,L_2)\varepsilon_m(B_{2,n}\,;\,L_\infty).$$

Therefore,

$$\varepsilon_m(B_{p,n}\,;\,L_\infty) \ll \left(\frac{2^n n^\nu}{m}\right)^{1/p},$$

or

$$\mathcal{H}_\varepsilon(B_{p,n}\,;\,L_\infty) \ll \frac{2^n n^\nu}{\varepsilon^p}\,.$$

Let $p = 1$. Then by Lemma 2.7

$$\mathcal{H}_\varepsilon(B_{1,n}\,;\,L_2) \le 2^n n^{\nu-1}\left(\frac{M_{X^*}}{\varepsilon}\right)^2,$$

where X^* is the space dual to $L_{1,n}$. By [9] or [19],

$$M_{X^*} \le C(2^n n^{\nu-1})^{-1/2} K(X) \pi_2(i),$$

where $K(X)$ is the K-convexity constant of the Banach space X, and $\pi_2(i)$ is the 2-absolutely summing norm of the identical embedding operator $i\colon L_{1,n} \to L_{2,n}$. It is known [20] that $K_{1,n} \ll n^{1/2}$, and the method of [9] gives $\pi_2(i) \le (2^n n^{\nu-1})^{1/2}$. Substituting these estimates, we obtain

$$\mathcal{H}_\varepsilon(B_{1,n}; L_2) \le \frac{2^n n^\nu}{\varepsilon^2}.$$

Applying the transitivity property of entropy numbers again and the preceding estimate, we obtain

$$\mathcal{H}_\varepsilon(B_{1,n}; L_\infty) \le \frac{2^n n^\nu}{\varepsilon}.$$

The lemma is proved.

§3. Proof of Theorem 1

Let $\|T_n\| = 1$. To prove the theorem, we must find $0 < \varepsilon < 1$ and N points $\mathbf{x}_j$ such that

$$\left| \|T_n\|_p^p - \frac{1}{N} \sum_{j=1}^{N} |T_n(\mathbf{x}_j)|^p \right| \le \varepsilon.$$

Let $\mathcal{F}$ be an ε-net on the boundary of $B_{p,n}$, and $T \in \mathcal{F}$. By Lemma 4,

$$\|T\|_\infty \ll 2^{n/p} n^{(\nu-1)/p'}.$$

Denote this upper bound by M. For $j = 1, 2, \dots, l$ let $\mathcal{A}_j$ be $\varepsilon(1+\varepsilon)^j$-net of $B_{p,n}$ in L_∞, $\mathcal{A}_l$ contains only the zero element,

$$\log |\mathcal{A}_j| \le \mathcal{H}_{\varepsilon(1+\varepsilon)^j}(B_{p,n}, L_\infty).$$

Let $f_{j,T}$ be the nearest element to $T \in \mathcal{F}$ from $\mathcal{A}_j$, so that

$$\|T - f_{j,T}\|_\infty \le \varepsilon(1+\varepsilon)^j.$$

We will replace every polynomial $T \in \mathcal{F}$ by an almost simple function. Let us denote

$$C_{j,T} = \{x : |f_{j,T}| \ge (1+\varepsilon)^{j-1}\}, \quad D_{j,T} = C_{j,T} \setminus \bigcup_{k>j} C_{k,T}, \quad D_{0,T} = Q^d \setminus \bigcup_{k>0} C_{k,T}.$$

The simple function replacing T is the following:

$$\widehat{T} = T\chi_{D_{0,T}} + \sum_{j>0} (1+\varepsilon)^j \chi_{D_{j,T}},$$

where χ_A is the characteristic function of the set A. Let us prove that for every $D_{j,T}$, $j \ge 1$, we have

$$\frac{5}{12} |\widehat{T}| \le |T| \le \frac{13}{9} |\widehat{T}|,$$

while for $x \in D_{0,T}$ we have $|T| \le (1+\varepsilon)^2$. Indeed, if $x \in C_{j,T}$, then

$$|T(x)| \ge |f_{j,T}| - \varepsilon(1+\varepsilon)^j > (1+\varepsilon)^{j-1} - \varepsilon(1+\varepsilon)^j = (1+\varepsilon)^{j-1}(1 - \varepsilon(1+\varepsilon)).$$

If $x \notin C_{j,T}$ then

$$|T(x)| \le |f_{j,T}| + \varepsilon(1+\varepsilon)^j < (1+\varepsilon)^{j-1} + \varepsilon(1+\varepsilon)^j = (1+\varepsilon)^{j-1}(1+\varepsilon(1+\varepsilon)).$$

Therefore, if $x \in D_{j,T}$, then $x \in C_{j,T}$ but $x \notin C_{j+1,T}$, hence assuming $\varepsilon < 1/3$ we obtain

$$\frac{5}{12}(1+\varepsilon)^j \le |T| \le \frac{13}{9}(1+\varepsilon)^j.$$

Now, applying the empirical distribution method (Lemma 1 for $T\chi_{D_{0,T}}$ and Lemma 2 for the sum) with ε_j to be specified later), we obtain

$$\frac{1}{N}\sum_{k=1}^{N} |T(\mathbf{x}_k)|^p \chi_{D_{0,T}}(\mathbf{x}_k) - \int_{D_{0,T}} |T(\mathbf{x})|^p d\mathbf{x} \le \varepsilon_0,$$

$$\sum_{j=1}^{l}\left(\frac{1}{N}\sum_{k=1}^{N}(1+\varepsilon)^{jp}\chi_{D_{j,T}}(\mathbf{x}_k) - \int_{Q^d}(1+\varepsilon)^{jp}\chi_{D_{j,T}}(\mathbf{x})\,d\mathbf{x}\right) \le \sum_{j=1}^{l}\varepsilon_j < \varepsilon$$

provided that the sum of probabilities satisfies the inequality

$$(\text{number of } \mathcal{D}_{0,T})\exp\left(-\frac{\varepsilon_0^2 N}{8}\right) + \sum_{j=1}^{l}(\text{number of } D_{j,T})\exp\left(-\frac{\varepsilon_j^2 N}{(1+\varepsilon)^{jp}8}\right) < \frac{1}{2}.$$

We can choose $\varepsilon_0 = \varepsilon/2$, then by Lemma 5 (the number of $\mathcal{D}_{0,T}) \le |\mathcal{F}| \le (1+2/\varepsilon)^{2^n n^{\nu-1}}$; therefore, estimating the first term, we must take

$$N \gg 2^n n^{\nu-1}.$$

Estimating the sum, we must estimate the number of sets $D_{j,T}$ for each j. We know that the number of sets $C_{j,T}$ is exactly $|\mathcal{A}_j|$. Therefore, the number of sets $D_{j,T}$ is less than or equal to $|\mathcal{A}_j|\cdot|\mathcal{A}_{j+1}|\cdot\ldots\cdot|\mathcal{A}_l|$. Applying the estimate for $|\mathcal{A}_j|$, we have

$$\sum_{j=1}^{l}\exp\left(\sum_{m=j}^{l}\mathcal{H}_{\varepsilon(1+\varepsilon)^m}(B_{p,n}, L_\infty) - \frac{\varepsilon_j^2 N}{(1+\varepsilon)^{jp}8}\right) < \frac{1}{2}.$$

Now we put $\varepsilon_j = \varepsilon/(jl)^{1/2}$ and require that

$$\sum_{m=j}^{l}\mathcal{H}_{\varepsilon(1+\varepsilon)^m}(B_{p,n}, L_\infty) < \frac{1}{2}\frac{\varepsilon_j^2 N}{(1+\varepsilon)^{jp}8}$$

for every j. Then, obviously ($e^{-t} < t^{-1}$ for positive t),

$$\sum_{j=1}^{l}\exp\left(\sum_{m=j}^{l}\mathcal{H}_{\varepsilon(1+\varepsilon)^m}(B_{p,n}, L_\infty) - \frac{\varepsilon_j^2 N}{(1+\varepsilon)^{jp}8}\right) < \sum_{j=1}^{l}\frac{(1+\varepsilon)^{jp}8jl}{\varepsilon^2 N} \ll \frac{(1+\varepsilon)^{lp}l^2}{\varepsilon^3 N} < \frac{1}{2}.$$

Or $N > M^p \log^2 M/\varepsilon^{3+p}$. Our choice must also satisfy the inequality

$$\sum_{m=j}^{l}\mathcal{H}_{\varepsilon(1+\varepsilon)^m}(B_{p,n}, L_\infty) < \frac{1}{2}\frac{\varepsilon^2 N}{(1+\varepsilon)^{jp}8jl},$$

which is true if

$$\frac{16l}{\varepsilon^2}\sum_{m=1}^{l}(1+\varepsilon)^{mp} m\mathcal{H}_{\varepsilon(1+\varepsilon)^m}(B_{p,n}, L_\infty) < N.$$

The latter condition is equivalent to the following integral one

$$N \gg \frac{\log M}{\varepsilon^3}\int_1^M \frac{\mathcal{H}_t(B_{p,n}, L_\infty)}{t^{1-p}}\log t\,dt.$$

To complete the proof, we carry over estimates from $\widehat{T}$ to T and apply Lemma 9.

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